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On character sums over flat numbers ☆

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ABSTRACT

Let $q \geqslant 2$ be an integer, χ be any non-principal character mod q, and $H = H(q) \leqslant q$. In this paper the authors prove some estimates for character sums of the form

$$\mathcal{W}(\chi, H; q) = \sum_{n \in \mathcal{F}(H)} \chi(n),$$

where

$$\mathcal{F}(H) = \{ n \in \mathbb{Z} \mid (n, q) = 1,\ 1 \leqslant n, \bar{n} \leqslant q,\ |n - \bar{n}| \leqslant H \},$$

$\bar{n}$ is defined by $n\bar{n} \equiv 1 \pmod{q}$.

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1. Introduction

Let $q \geqslant 2$ be an integer, χ be a non-principal character mod q. It is quite an important problem in analytic number theory to obtain a sharp estimate for the character sum

$$\sum_{x=N+1}^{N+H} \chi(f(x)),$$

where $f(x) \in \mathbb{Z}[x]$, N and H are positive integers. The classical result, due to Pólya and Vinogradov [P,V], is the estimate

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$$\sum_{n=N+1}^{N+H} \chi(n) \ll q^{1/2} \log q,$$

where $\ll$ is the Vinogradov's notation. About half a century later, Burgess' immortal work [B1,B2] showed that

$$\sum_{n=N+1}^{N+H} \chi(n) \ll H^{1-1/r} q^{(r+1)/4r^2+o(1)}$$

holds with $r = 1, 2, 3$ for any q and with arbitrary positive integer r if q is cube-free. Under Generalized Riemann Hypothesis, Montgomery and Vaughan [MV] sharpened the Pólya–Vinogradov bound to

$$\sum_{n=N+1}^{N+H} \chi(n) \ll q^{1/2} \log\log q.$$

Burgess' estimate is such a milestone that nobody can unconditionally beat the barrier in general by any advanced technology. (Some partial improvements and progresses can be found in [GS1,GS2,G], et al.) However, estimates for the character sums over special numbers and sequences, such as factorials, Beatty sequences, binomial coefficients and other combinatorial numbers, have attracted many scholars' interests. A complete list of the results and open problems are referred to [S].

In this paper, we shall deal with another kind of special numbers. It is known that the distribution of $\bar{n}$ is quite irregular, where $\bar{n}$ is the inverse of n mod q, i.e. $n\bar{n} \equiv 1 \pmod q$. How about the distribution of $|n - \bar{n}|$? In [Z], W. Zhang proved that

$$\sideset{}{^*}\sum_{\substack{n=1 \\ |n-\bar{n}| \leqslant \delta q}}^{q} 1 = \delta(2-\delta)\varphi(q) + O\big(q^{1/2}\tau^2(q)\log^3 q\big), \tag{1}$$

where $\delta \in (0, 1]$ is a constant, $\varphi(q)$ is the Euler function and $\tau(q)$ is the divisor function, $\sum^*$ denotes the summation over the integers that are coprime to q.

In fact, W. Zhang studied the number of the integers that are within a given distance to their inverses mod q. Now we consider the character sums over these integers. We shall present the problem of a more general case.

Let $q \geqslant 2$ be a fixed integer and $H = H(q) \leqslant q$. We put

$$\mathscr{F}(H) = \big\{ n \in \mathbb{Z} \,\big|\, (n, q) = 1,\ 1 \leqslant n, \bar{n} \leqslant q,\ |n - \bar{n}| \leqslant H \big\}.$$

Each element in $\mathscr{F}(H)$ is called an H-flat number mod q. Note that in the definition of $\mathscr{F}(H)$, the size of H is $O(q)$, not necessary being $H \asymp q$ as in (1).

In this paper, we shall study the character sums over such H-flat numbers mod q. That is we shall prove nontrivial upper bounds for

$$\mathcal{W}(\chi, H; q) = \sum_{n \in \mathscr{F}(H)} \chi(n). \tag{2}$$

It is obvious that $n \in \mathscr{F}(H)$ implies $q - n \in \mathscr{F}(H)$, thus $\chi(n) + \chi(q-n) = 0$ if $\chi(-1) = -1$, so $\mathcal{W}(\chi, H; q) = 0$. Hence we only deal with the case with $\chi(-1) = 1$ throughout this paper.

Theorem 1. *Let $q \geqslant 2$, χ be a non-principal character mod q. Then we have*

$$\mathcal{W}(\chi, H; q) \ll q^{1/2}\tau^2(q)\log H.$$

The proof of Theorem 1 depends on the estimate for the general Kloosterman sums twisted by Dirichlet characters, and the upper bound in Theorem 1 is independent of H, to be precise, the result may be trivial if H is quite small. However if q is odd, and χ is the Jacobi symbol mod q (which reduces to Legendre symbol if q is a prime), we have corresponding calculation formulae for this Kloosterman sum, known as Salié sum, and we can obtain an upper bound depending on H, which can be stated as follows.

Theorem 2. *Let $q \geqslant 3$ be an odd square-free integer, χ be the Jacobi symbol mod q. Then we have*

$$\mathcal{W}(\chi, H; q) \ll H^{1-1/r}q^{(r+1)/4r^2}\tau(q)\log q,$$

where $r \geqslant 1$ is an arbitrary integer.

2. General Kloosterman sums and character sums

The classical Kloosterman sum is defined by

$$S(m, n; q) = \sum_{a \bmod q}^{*} e\left(\frac{ma + n\bar{a}}{q}\right),$$

where $e(x) = e^{2\pi ix}$. The well-known upper bound essentially due to A. Weil [W] is

$$S(m, n; q) \ll q^{1/2}(m, n, q)^{1/2}\tau(q),$$

where (m, n, q) denotes the greatest common divisor of m, n, q.

In the proof of the following sections, we require a general Kloosterman sum twisted by a Dirichlet character such as

$$S_\chi(m, n; q) = \sum_{a \bmod q}^{*} \chi(a)e\left(\frac{ma + n\bar{a}}{q}\right).$$

Taking $\chi = \chi^0$ as the principal character mod q, this reduces to $S(m, n; q)$.

We require an upper bound estimation for $S_\chi(m, n; q)$, the original proofs [W,E] carry over with minor modifications.

Lemma 1. *Let q be a positive integer, then we have*

$$S_\chi(m, n; q) \ll q^{1/2}(m, n, q)^{1/2}\tau(q).$$

Lemma 2. *Let $q \geqslant 2$, χ be a Dirichlet character mod q. For any d with $d|q$, we define*

$$T_\chi(m, n; d, q) = \sum_{a \bmod q}^{*} \chi(a)e\left(\frac{ma + n\bar{a}}{d}\right),$$

where $a\bar{a} \equiv 1 \pmod q$*. Then for any* d *with* $d\ell = q$, $(d, \ell) = 1$*, we have*

$$T_\chi(m,n;d,q) = \begin{cases} \varphi(\ell) S_{\chi_1}(m,n;d), & \text{if } \chi_2 = \chi_2^0, \\ 0, & \text{if } \chi_2 \neq \chi_2^0, \end{cases}$$

where $\chi_1 \bmod d$, $\chi_2 \bmod \ell$ *with* $\chi_1\chi_2 = \chi$*, and* χ_2^0 *is the principal character mod* ℓ*.*

Proof. Let $a = a_1\ell + a_2 d$, then

$$\begin{aligned} T_\chi(m,n;d,q) &= \sideset{}{^*}\sum_{a_1 \bmod d} \sideset{}{^*}\sum_{a_2 \bmod \ell} \chi(a_1\ell + a_2 d) e\left(\frac{ma_1\ell + n\overline{a_1\ell}}{d}\right) \\ &= \sideset{}{^*}\sum_{a_1 \bmod d} \sideset{}{^*}\sum_{a_2 \bmod \ell} \chi_1(a_1\ell)\chi_2(a_2 d) e\left(\frac{ma_1\ell + n\overline{a_1\ell}}{d}\right) \\ &= \chi_2(d) \sideset{}{^*}\sum_{a_1 \bmod d} \chi_1(a_1\ell) e\left(\frac{ma_1\ell + n\overline{a_1\ell}}{d}\right) \sideset{}{^*}\sum_{a_2 \bmod \ell} \chi_2(a_2), \end{aligned}$$

then the lemma follows from the orthogonality of Dirichlet characters. □

If q is odd, and χ is the Jacobi symbol mod q, we have a calculation formula of the general Kloosterman sums, known as Salié sums (see [I, Lemma 4.9]).

Lemma 3. *If* $(q, 2n) = 1$*, and* χ *is the Jacobi symbol mod* q*, then we have*

$$S_\chi(m,n;q) = \varepsilon_q q^{1/2} \chi(n) \sum_{y^2 \equiv mn \pmod q} e\left(\frac{2y}{q}\right),$$

where ε_q *is a constant with* $|\varepsilon_q| = 1$*.*

We also require Burgess' classical result on character sums, see [B1, Theorem 2].

Lemma 4. *If* $q \geqslant 2$ *is square-free, then for any non-principal character* χ mod q*, we have*

$$\sum_{n=N+1}^{N+A} \chi(n) \ll A^{1-1/r} q^{(r+1)/4r^2} \log q,$$

where $r \geqslant 1$ *is an arbitrary integer.*

3. Proof of Theorem 1

It obvious that $\mathcal{W}(\chi, H; q)$ has the same essential bound with

$$\mathcal{W}^*(\chi, H; q) = \sum_{t \leqslant H} \sideset{}{^*}\sum_{\substack{n \leqslant q \\ n - \bar{n} \equiv t \pmod q}} \chi(n).$$

We denote $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$. Apply the identity

$$\sum_{n=1}^{q} e\left(\frac{an}{q}\right) = \begin{cases} q, & q \mid a, \\ 0, & q \nmid a, \end{cases}$$

together with Lemma 1 we can obtain that

$$\begin{aligned}
\mathcal{W}^*(\chi, H; q) &= \frac{1}{q} \sum_{m\leqslant q} \sum_{t\leqslant H} e\left(-\frac{mt}{q}\right) \sideset{}{^*}\sum_{n\leqslant q} \chi(n) e\left(m\frac{n-\bar{n}}{q}\right) \\
&\ll \frac{1}{q} \sum_{m\leqslant q-1} \min\left(H, \left\|\frac{m}{q}\right\|^{-1}\right) \left|S_\chi(m,-m;q)\right| \\
&\ll q^{-1/2}\tau(q) \sum_{m\leqslant q-1} (m,q)^{1/2} \min\left(H, \left\|\frac{m}{q}\right\|^{-1}\right) \\
&\ll q^{-1/2}\tau(q) \sum_{m\leqslant q-1} (m,q)^{1/2} \min\left(H, \frac{q}{m}\right) \\
&\ll Hq^{-1/2}\tau(q) \sum_{m\leqslant q/H} (m,q)^{1/2} + q^{1/2}\tau(q) \sum_{q/H<m\leqslant q-1} \frac{(m,q)^{1/2}}{m}.
\end{aligned}$$

By the following calculations,

$$\sum_{m\leqslant q/H} (m,q)^{1/2} \ll \sum_{d|q} d^{1/2} \sum_{\substack{m\leqslant q/H \\ d|m}} 1 \ll H^{-1} q\tau(q),$$

and

$$\begin{aligned}
\sum_{q/H<m\leqslant q-1} \frac{(m,q)^{1/2}}{m} &\ll \sum_{d|q} d^{1/2} \sum_{\substack{q/H<m\leqslant q-1 \\ d|m}} \frac{1}{m} \\
&\ll \sum_{d|q} d^{-1/2} \sum_{q/Hd<m\leqslant q/d} \frac{1}{m} \\
&\ll \tau(q)\log H,
\end{aligned}$$

we have

$$\mathcal{W}^*(\chi, H; q) \ll q^{1/2}\tau^2(q)\log H$$

and $\mathcal{W}(\chi, H; q)$ has the same bound. This completes the proof of Theorem 1.

4. Proof of Theorem 2

In this section, we shall deal with a special case of (2), that is q being an odd square-free integer, and $\chi = (\frac{\cdot}{q})$ being the Jacobi symbol mod q.

Following the similar arguments in Section 3, we have

$$\begin{aligned}\mathcal{W}^*(\chi, H; q) &= \frac{1}{q}\sum_{m\leqslant q}\sum_{t\leqslant H} e\left(-\frac{mt}{q}\right)S_\chi(m,-m;q)\\ &= \frac{1}{q}\sum_{d|q}\sum_{\substack{m=1\\(m,q)=d}}^{q}\sum_{t\leqslant H} e\left(-\frac{mt}{q}\right)S_\chi(m,-m;q)\\ &= \frac{1}{q}\sum_{d|q}\sum_{\substack{m=1\\(m,q/d)=1}}^{q/d}\sum_{t\leqslant H} e\left(-\frac{mdt}{q}\right)S_\chi(md,-md;q)\\ &= \frac{1}{q}\sum_{d|q}\sideset{}{^*}\sum_{m\leqslant d}\sum_{t\leqslant H} e\left(-\frac{mt}{d}\right)\sideset{}{^*}\sum_{a \bmod q}\chi(a)e\left(\frac{ma-m\bar{a}}{d}\right)\\ &= \frac{1}{q}\sum_{d|q}\sideset{}{^*}\sum_{m\leqslant d}\sum_{t\leqslant H} e\left(-\frac{mt}{d}\right)T_\chi(m,-m;d,q).\end{aligned}$$

We write $q = d\ell$, where $(d,\ell) = 1$ since q is square-free. We also write χ_1 mod d, χ_2 mod ℓ with $\chi_1\chi_2 = \chi$. Note that χ is a real primitive character mod q, so from Lemma 2 we know that, χ_1 and χ_2 must be real primitive characters mod d and ℓ respectively, thus $T_\chi(m,-m;d,q) = 0$ if $\ell > 1$. Applying Lemmas 2 and 3, we can deduce that

$$\begin{aligned}\mathcal{W}^*(\chi, H; q) &= \frac{1}{q}\sideset{}{^*}\sum_{m\leqslant q}\sum_{t\leqslant H} e\left(-\frac{mt}{q}\right)S_\chi(m,-m;q)\\ &= \frac{\varepsilon_q}{q^{1/2}}\sum_{t\leqslant H}\sum_{m\leqslant q}\chi(m)e\left(-\frac{mt}{q}\right)\sum_{y^2\equiv -m^2 \ (\mathrm{mod}\ q)} e\left(\frac{2y}{q}\right)\\ &= \frac{\varepsilon_q}{q^{1/2}}\sum_{t\leqslant H}\sum_{m\leqslant q}\chi(m)e\left(-\frac{mt}{q}\right)\sum_{\delta^2\equiv -1 \ (\mathrm{mod}\ q)} e\left(\frac{2\delta m}{q}\right).\end{aligned}$$

Thus

$$\mathcal{W}^*(\chi, H; q) = \frac{\varepsilon_q}{q^{1/2}}\tau(\chi)\sum_{\delta^2\equiv -1 \ (\mathrm{mod}\ q)}\sum_{t\leqslant H}\chi(2\delta - t),$$

where $\tau(\chi) = \sum_{n \bmod q}\chi(n)e(\frac{n}{q})$ is the Gauss sum.

Applying Lemma 4 and $|\tau(\chi)| = q^{1/2}$, we obtain that

$$\mathcal{W}^*(\chi, H; q) \ll H^{1-1/r}q^{(r+1)/4r^2}\tau(q)\log q$$

for any integer $r \geqslant 1$. And $\mathcal{W}(\chi, H; q)$ has the same bound. This completes the proof of Theorem 2.

5. Final remarks

Each positive integer can be represented as the product of two coprime parts, one is square-free and the other is square-full. In fact, the method in Section 4 can lead to a nontrivial estimate for the modulo q, whose square-full part is quite small in comparison with q. The estimate depends mainly on a corresponding result to Lemma 2 for such q.

Theorem 3. *Let $q \geqslant 3$ be an odd integer, χ be the Jacobi symbol mod q. Then we have*

$$\mathcal{W}(\chi, H; q) \ll \left(Hq_2^{-1}\right)^{1-1/r} \varphi^2(q_2) q_1^{(r+1)/4r^2} 2^{\omega(q)} \log q_1,$$

where q_1 is the square-free part of q, q_2 is square-full, $\omega(q)$ denotes the number of distinct prime factors of q, and $r \geqslant 1$ is an arbitrary integer.

We should point out again that Theorem 3 is nontrivial when the square-full part of q is quite small in comparison with q.

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