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Counting words of minimum length in an automorphic orbit

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Abstract

Let u be a cyclic word in a free group F_n of finite rank n that has the minimum length over all cyclic words in its automorphic orbit, and let $N(u)$ be the cardinality of the set $\{v\colon |v| = |u| \text{ and } v = \phi(u) \text{ for some } \phi \in \operatorname{Aut} F_n\}$. In this paper, we prove that $N(u)$ is bounded by a polynomial function with respect to $|u|$ under the hypothesis that if two letters x, y with $x \neq y^{\pm 1}$ occur in u, then the total number of occurrences of $x^{\pm 1}$ in u is not equal to the total number of occurrences of $y^{\pm 1}$ in u. A complete proof without the hypothesis would yield the polynomial time complexity of Whitehead's algorithm for F_n.

1. Introduction

Let F_n be the free group of finite rank n on the set $\{x_1, x_2, \ldots, x_n\}$. We denote by Σ the set of *letters* of F_n, that is, $\Sigma = \{x_1, x_2, \ldots, x_n\}^{\pm 1}$. As in [1,5], we define a *cyclic word* to be a cyclically ordered set of letters with no pair of inverses adjacent. The *length* $|w|$ of a cyclic word w is the number of elements in the cyclically ordered set. For a cyclic word w in F_n, we denote the automorphic orbit $\{\psi(w)\colon \psi \in \operatorname{Aut} F_n\}$ by $\operatorname{Orb}_{\operatorname{Aut} F_n}(w)$.

The purpose of this paper is to provide a partial solution of the following problem raised by Myasnikov and Shpilrain [6]:

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Problem. Let u be a cyclic word in F_n which has the minimum length over all cyclic words in its automorphic orbit $\mathrm{Orb}_{\mathrm{Aut}\,F_n}(u)$, and let $N(u)$ be the cardinality of the set $\{v \in \mathrm{Orb}_{\mathrm{Aut}\,F_n}(u)\colon |v| = |u|\}$. Then is $N(u)$ bounded by a polynomial function with respect to $|u|$?

This problem was settled in the affirmative for F_2 by Myasnikov and Shpilrain [6], and Khan [3] improved their result by showing that $N(u)$ has the sharp bound of $8|u| - 40$ for F_2. The problem was motivated by the complexity of Whitehead's algorithm which decides whether, for given two elements in F_n, there is an automorphism of F_n that takes one element to the other. Indeed, a complete positive solution to the problem would yield that Whitehead's algorithm terminates in polynomial time with respect to the maximum length of the two words in question (see [6, Proposition 3.1]). Recently, Kapovich, Schupp and Shpilrain [2] proved that Whitehead's algorithm has strongly linear time generic-case complexity. In the present paper, we prove for F_n with $n \geqslant 2$ that $N(u)$ is bounded by a polynomial function with respect to $|u|$ under the following

Hypothesis 1.1.

(i) A cyclic word u has the minimum length over all cyclic words in its automorphic orbit $\mathrm{Orb}_{\mathrm{Aut}\,F_n}(u)$.
(ii) If two letters x_i (or x_i^{-1}) and x_j (or x_j^{-1}) with $i < j$ occur in u, then the total number of $x_i^{\pm 1}$ occurring in u is strictly less than the total number of $x_j^{\pm 1}$ occurring in u.

Before we state our theorems, we would like to establish several notation and definitions. As in [1,5], for $A, B \subseteq \Sigma$, we write $A + B$ for $A \cup B$ if $A \cap B = \emptyset$, and $A - B$ for $A \cap B^c$ if $B \subseteq A$, where B^c is the complement of B in Σ. We define a *Whitehead automorphism* σ of F_n as an automorphism of one of the following two types (cf. [4,7]):

(W1) σ permutes elements in Σ.
(W2) σ is defined by a set $A \subset \Sigma$ and a multiplier $a \in \Sigma$ with both $a, a^{-1} \notin A$ in such a way that if $x \in \Sigma$ then (a) $\sigma(x) = xa$ provided $x \in A$ and $x^{-1} \notin A$; (b) $\sigma(x) = a^{-1}xa$ provided both $x, x^{-1} \in A$; (c) $\sigma(x) = x$ provided both $x, x^{-1} \notin A$.

If σ is of the second type, then we write $\sigma = (A, a)$. By $(\bar{A}, a^{-1})$, we mean the Whitehead automorphism $(\Sigma - A - a^{\pm 1}, a^{-1})$. It is then easy to see that $(A, a)(w) = (\bar{A}, a^{-1})(w)$ for any cyclic word w in F_n.

For a Whitehead automorphism σ of the second type, we define the degree of σ as follows:

Definition 1.2. Let $\sigma = (A, a)$ be a Whitehead automorphism of F_n of the second type. Put $A' = \{i\colon \text{either } x_i \in A \text{ or } x_i^{-1} \in A, \text{ but not both}\}$. Then the *degree of* σ is defined to be $\max A'$. If $A' = \emptyset$, then the *degree of* σ is defined to be zero.

Let w be a fixed cyclic word in F_n that satisfies Hypothesis 1.1(i). For two letters $x, y \in \Sigma$, we say that x *depends on* y *with respect to* w if, for every Whitehead automorphism (A, a) of F_n such that

$$a \notin \{x^{\pm 1}, y^{\pm 1}\}, \quad \{y^{\pm 1}\} \cap A \neq \emptyset, \quad \text{and} \quad \exists v \in \mathrm{Orb}_{\mathrm{Aut}\,F_n}(w)\colon \big|(A, a)(v)\big| = |v| = |w|,$$

we have $\{x^{\pm 1}\} \subseteq A$. Then we have the following

Claim. *If x depends on y with respect to w, then y depends on x with respect to w.*

Proof. Suppose on the contrary that y does not depend on x. Then there exists a Whitehead automorphism (A, a) of F_n such that $a \notin \{x^{\pm 1}, y^{\pm 1}\}$, $x^{\pm 1} \cap A \neq \emptyset$, $|(A, a)(v)| = |v| = |w|$ for some $v \in \mathrm{Orb}_{\mathrm{Aut}\, F_n}(w)$, but such that $y^{\pm 1} \nsubseteq A$. Then $|(\bar{A}, a^{-1})(v)| = |v| = |w|$ and $y^{\pm 1} \cap \bar{A} \neq \emptyset$. Since x depends on y, $x^{\pm 1} \subseteq \bar{A}$. This gives $x^{\pm 1} \cap A = \emptyset$, which is a contradiction. □

We then construct the *dependence graph Γ_w of w* as follows: Take the vertex set as Σ, and connect two distinct vertices $x, y \in \Sigma$ by a non-oriented edge if either $y = x^{-1}$ or y depends on x with respect to w. Let C_i be the connected component of Γ_w containing x_i. Here, we make the following remark.

Remark.

(i) $\Gamma_w = \Gamma_v$ for any $v \in \mathrm{Orb}_{\mathrm{Aut}\, F_n}(w)$ with $|v| = |w|$.
(ii) If x_i depends on x_j, then $C_i = C_j$.
(iii) If $x_j^{\pm 1} \in C_i$ with $i \neq j$, then every Whitehead automorphism (A, a) such that either $x_i \in A$ or $x_i^{-1} \in A$ but not both and such that $|(A, a)(v)| = |v| = |w|$ for some $v \in \mathrm{Orb}_{\mathrm{Aut}\, F_n}(w)$ must have the multiplier a only in C_i, for otherwise $x_j^{\pm 1} \subseteq A$ but then $x_j^{\pm 1} \nsubseteq \bar{A}$, which is a contradiction because $x_i^{\pm 1} \cap \bar{A} \neq \emptyset$.

Clearly there exists a unique factorization

$$w = v_1 v_2 \cdots v_k \quad \text{(without cancellation)},$$

where each v_i is a non-empty (non-cyclic) word consisting of letters in C_{j_i} with $C_{j_i} \neq C_{j_{i+1}}$ (i mod k). The subword v_i is called a *C_{j_i}-syllable* of w. By the *C_i-syllable length of w* denoted by $|w|_{C_i}$, we mean the total number of C_i-syllables of w.

For Theorem 1.4, we suppose further that a cyclic word u satisfies the following

Hypothesis 1.3.

(i) The C_n-syllable length $|u|_{C_n}$ of u is minimum over all cyclic words in the set $\{v \in \mathrm{Orb}_{\mathrm{Aut}\, F_n}(u)\colon |v| = |u|\}$.
(ii) If the index j ($1 \leqslant j \leqslant n-1$) is such that $C_j \neq C_k$ for all $k > j$, then the C_j-syllable length $|u|_{C_j}$ of u is minimum over all cyclic words in the set $\{v \in \mathrm{Orb}_{\mathrm{Aut}\, F_n}(u)\colon |v| = |u| \text{ and } |v|_{C_k} = |u|_{C_k} \text{ for all } k > j\}$.

For an easy example, consider the cyclic words $u = x_1^2 x_2^3 x_3^4 x_4^5$ and $v = x_1 x_2^3 x_1 x_3^4 x_4^5$ in F_4. Clearly v is an automorphic image of u with $|v| = |u|$, so $\Gamma_u = \Gamma_v$. The dependence graph $\Gamma_u = \Gamma_v$ has four distinct connected components, each C_i of which contains only $x_i^{\pm 1}$. Then u satisfies Hypotheses 1.1 and 1.3, whereas v satisfies Hypotheses 1.1 and 1.3(i) but not Hypothesis 1.3(ii), because the C_1-syllable length of v can be decreased without changing $|v|$ and $|v|_{C_i}$ for all $i > 1$.

For another example, let $u = x_1^2 x_2^3 x_3^2 x_4 x_3^{-1} x_4 x_3 x_4^3$ and $v = x_1^2 x_3^2 x_2^3 x_4 x_3^{-1} x_4 x_3 x_4^3$ be cyclic words in F_4. Then v is an automorphic image of u with $|v| = |u|$, so $\Gamma_u = \Gamma_v$. In the dependence graph $\Gamma_u = \Gamma_v$, there are three distinct connected components $C_1, C_2, C_3 = C_4$. While u satisfies

Hypotheses 1.1 and 1.3, v does not satisfy Hypothesis 1.3(i), because the C_4-syllable length of v can be decreased without changing $|v|$.

Now we are ready to state our theorems, whose proofs will appear in Sections 3–4.

Theorem 1.4. *Let u be a cyclic word in F_n that satisfies Hypotheses* 1.1 *and* 1.3. *Let σ_i, $i = 1, \dots, \ell$, be Whitehead automorphisms of the second type such that $|\sigma_i \cdots \sigma_1(u)| = |u|$ for all i. Then there exist Whitehead automorphisms $\tau_1, \tau_2, \dots, \tau_s$ of the second type such that*

$$\sigma_\ell \cdots \sigma_2\sigma_1(u) = \tau_s \cdots \tau_2\tau_1(u),$$

where $\max_{1\leqslant i\leqslant \ell} \deg \sigma_i \geqslant \deg \tau_s \geqslant \deg \tau_{s-1} \geqslant \cdots \geqslant \deg \tau_1$, *and* $|\tau_j \cdots \tau_1(u)| = |u|$ *for all* $j = 1, \dots, s$.

Theorem 1.5. *Let u be a cyclic word in F_n that satisfies Hypothesis* 1.1, *and let $N(u)$ be the cardinality of the set* $\{v \in \mathrm{Orb}_{\mathrm{Aut}\, F_n}(u)\colon\ |v| = |u|\}$. *Then $N(u)$ is bounded by a polynomial function of degree $n(5n-7)/2$ with respect to $|u|$.*

The main idea of the present paper is to prove that the action of an automorphism on an element which satisfies Hypotheses 1.1 and 1.3 can be factored into a composition of automorphisms of ascending degrees, which will be achieved through Lemmas 3.1, 3.2 and Theorem 1.4. The proof of Theorem 1.4 will proceed by double induction on ℓ and r, where ℓ is the length of the chain $\sigma_\ell \cdots \sigma_2\sigma_1$ and $r = \max_{1\leqslant i\leqslant \ell} \deg \sigma_i$, with Lemma 3.1 (the case for $\ell = 2$ and any r) and Lemma 3.2 (the case for $r = 1$ and any ℓ) as the base steps of the induction.

Let $N_k(u)$ be the cardinality of the set $\{\phi(u)\colon \phi$ can be represented as a composition $\tau_s \cdots \tau_1$ ($s \in \mathbb{N}$) of Whitehead automorphisms τ_i of F_n of degree k such that $|\tau_i \cdots \tau_1(u)| = |u|$ for all $i = 1, \dots, s\}$. Then bounding $N(u)$ reduces to bounding each $N_k(u)$, as will be shown in the proof of Theorem 1.5 using the result of Theorem 1.4. Lemma 4.1 will be devoted to bounding $N_0(u)$, and Lemma 4.2 will show that $N_k(u)$ for $k \geqslant 1$ is at most $N_0(V_u)$, where V_u is a certain sequence of cyclic words constructed from u, thus bounding $N_k(u)$ for $k \geqslant 1$. Furthermore in Theorem 1.5 we will specifically give a bound for the degree of a polynomial bounding $N(u)$.

2. Preliminaries

We begin this section by setting some notation. Let w be a fixed cyclic word in F_n. As in [1], for $x, y \in \Sigma$, $x.y$ denotes the total number of occurrences of the subwords xy^{-1} and yx^{-1} in w. For $A, B \subseteq \Sigma$, $A.B$ means the sum of $a.b$ for all $a \in A$, $b \in B$. Then obviously $a.\Sigma$ is equal to the total number of $a^{\pm 1}$ occurring in w. For two automorphisms ϕ and ψ of F_n, by writing $\phi \equiv \psi$ we mean the equality of ϕ and ψ over all cyclic words in F_n, that is, $\phi(v) = \psi(v)$ for every cyclic word v in F_n.

We now establish two technical lemmas which will play a fundamental role in the proofs in Sections 3 and 4.

Lemma 2.1. *Let u be a cyclic word in F_n that satisfies Hypothesis* 1.1(i), *and let $\sigma = (A, a^{-1})$ and $\tau = (B, b)$ be Whitehead automorphisms of F_n such that $|\sigma(u)| = |\tau(u)| = |u|$. Put $A = C + E$ and $B = D + E$, where $E = A \cap B$. Then*

(i) *if $a^{-1} = b$, then $|(E, a^{-1})(u)| = |u|$;*
(ii) *if $a^{-1} \neq b$, $a^{\pm 1} \notin B$ and $b \notin A$, then $|(C, a^{-1})(u)| = |(D, b)(u)| = |u|$.*

Proof. It follows from [1, p. 255] that

$$\begin{cases} |\sigma(u)| - |u| = (A+a^{-1}).(A+a^{-1})' - a.\Sigma; \\ |\tau(u)| - |u| = (B+b).(B+b)' - b.\Sigma, \end{cases}$$

where $(A+a^{-1})' = \Sigma - (A+a^{-1})$ and $(B+b)' = \Sigma - (B+b)$. Since $|\sigma(u)| = |\tau(u)| = |u|$, we have $(A+a^{-1}).(A+a^{-1})' - a.\Sigma = (B+b).(B+b)' - b.\Sigma = 0$, so that

$$\big(A+a^{-1}\big).\big(A+a^{-1}\big)' + (B+b).(B+b)' - a.\Sigma - b.\Sigma = 0.$$

Following the notation in [1, p. 257], we write $A_1 = A + a^{-1}$, $A_2 = (A+a^{-1})'$, $B_1 = B + b$, $B_2 = (B+b)'$ and $P_{ij} = A_i \cap B_j$. Then as in [1, p. 257], we have

$$\begin{cases} P_{11}.P'_{11} + P_{22}.P'_{22} - a.\Sigma - b.\Sigma = 0; \\ P_{12}.P'_{12} + P_{21}.P'_{21} - a.\Sigma - b.\Sigma = 0, \end{cases} \tag{2.1}$$

where $P'_{ij} = \Sigma - P_{ij}$.

For (i), assume that $a^{-1} = b$. Then we have $a^{-1} \in P_{11}$ and $a \in P_{22}$. It follows from the first equality of (2.1) that

$$\begin{aligned} P_{11}.P'_{11} + P_{22}.P'_{22} - a.\Sigma - a.\Sigma &= \big(P_{11}.P'_{11} - a.\Sigma\big) + \big(P_{22}.P'_{22} - a.\Sigma\big) \\ &= \big|\big(P_{11} - a^{-1}, a^{-1}\big)(u)\big| - |u| + \big|(P_{22} - a, a)(u)\big| - |u| = 0. \end{aligned}$$

Since both $|(P_{11} - a^{-1}, a^{-1})(u)| - |u| \geqslant 0$ and $|(P_{22} - a, a)(u)| - |u| \geqslant 0$ by Hypothesis 1.1(i), we must have $|(P_{11} - a^{-1}, a^{-1})(u)| = |u|$, that is, $|(E, a^{-1})(u)| = |u|$, as required.

For (ii), assume that $a^{-1} \neq b$, $a^{\pm 1} \notin B$ and $b \notin A$. Then we have $a^{-1} \in P_{12}$, $a \notin P_{12}$, $b \in P_{21}$ and $b^{-1} \notin P_{21}$. Hence the second equality of (2.1) gives us that

$$\begin{aligned} P_{12}.P'_{12} + P_{21}.P'_{21} - a.\Sigma - b.\Sigma &= \big(P_{12}.P'_{12} - a.\Sigma\big) + \big(P_{21}.P'_{21} - b.\Sigma\big) \\ &= \big|\big(P_{12} - a^{-1}, a^{-1}\big)(u)\big| - |u| + \big|(P_{21} - b, b)(u)\big| - |u| = 0. \end{aligned}$$

As above, it follows from Hypothesis 1.1(i) that $|(P_{12} - a^{-1}, a^{-1})(u)| = |u|$ and $|(P_{21} - b, b)(u)| = |u|$. Since $P_{12} - a^{-1} = C$ and $P_{21} - b = D$, we have $|(C, a^{-1})(u)| = |(D, b)(u)| = |u|$, as desired. □

Lemma 2.2. *Let u be a cyclic word in F_n that satisfies Hypothesis* 1.1*, and let $\sigma = (A, a)$ be a Whitehead automorphism of F_n such that $|\sigma(u)| = |u|$. Then $a.\Sigma > b.\Sigma$ for every $b \in A$ with $b^{-1} \notin A$.*

Proof. In view of the assumption $|\sigma(u)| = |u|$ and [1, p. 255], we have $0 = |\sigma(u)| - |u| = (A+a).(A+a)' - a.\Sigma$, where $(A+a)' = \Sigma - (A+a)$, so that $(A+a).(A+a)' = a.\Sigma$. Now let $b \in A$ with $b^{-1} \notin A$. Then for the Whitehead automorphism $\tau = (A + a - b, b)$, we have $0 \leqslant |\tau(u)| - |u| = (A+a).(A+a)' - b.\Sigma$. Hence $(A+a).(A+a)' \geqslant b.\Sigma$; thus $a.\Sigma \geqslant b.\Sigma$. Here, the equality $a.\Sigma = b.\Sigma$ cannot occur by Hypothesis 1.1(ii); therefore $a.\Sigma > b.\Sigma$. □

Remark. By Lemma 2.2, if u is a cyclic word in F_n that satisfies Hypothesis 1.1 and $\sigma = (A, a)$ is a Whitehead automorphism of F_n such that $|\sigma(u)| = |u|$, then $\deg \sigma$ is at most $n - 1$.

3. Proof of Theorem 1.4

The aim of this section is to prove Theorem 1.4. The proof of Theorem 1.4 will proceed by double induction on ℓ and r, where ℓ is the length of the chain $\sigma_\ell \cdots \sigma_2\sigma_1$ and $r = \max_{1 \leqslant i \leqslant \ell} \deg \sigma_i$. Lemma 3.1 deals with the case for $\ell = 2$ and any r as one of the base steps of the induction. As the other base step, Lemma 3.2 deals with the case for $r = 1$ and any ℓ.

Lemma 3.1. *Let u be a cyclic word in F_n that satisfies Hypothesis* 1.1*, and let $\sigma_1 = (A, a)$ and $\sigma_2 = (B, b)$ be Whitehead automorphisms of F_n such that $|\sigma_2\sigma_1(u)| = |\sigma_1(u)| = |u|$. Suppose that* $\deg \sigma_1 > \deg \sigma_2$. *Then there exist Whitehead automorphisms $\tau_1, \ldots, \tau_s$ of F_n of the second type such that*

$$\sigma_2\sigma_1 \equiv \tau_s \cdots \tau_2\tau_1,$$

where $\deg \sigma_1 = \deg \tau_s \geqslant \cdots \geqslant \deg \tau_1$ *and $|\tau_i \cdots \tau_1(u)| = |u|$ for all $i = 1, \ldots, s$.*

Proof. It suffices to prove that there exist Whitehead automorphisms $\gamma_1, \ldots, \gamma_t$ of F_n such that

$$\sigma_2\sigma_1 \equiv \gamma_t \cdots \gamma_2\gamma_1,$$

where the index t is at most 3, $|\gamma_i \cdots \gamma_1(u)| = |u|$ for all $i = 1, \ldots, t$, and either $\deg \sigma_1 = \deg \gamma_t > \deg \gamma_j$ for all $j = 1, \ldots, t - 1$ or otherwise $\deg \sigma_1 = \deg \gamma_i$ for all $i = 1, \ldots, t$. Put $u' = \sigma_1(u)$; then $|\sigma_1^{-1}(u')| = |\sigma_2(u')| = |u|$, that is,

$$\bigl|\bigl(A, a^{-1}\bigr)(u')\bigr| = \bigl|(B, b)(u')\bigr| = |u|. \tag{3.1}$$

Also put $c = x_{\deg \sigma_1}$. Upon replacing (A, a), (B, b) by $(\bar{A}, a^{-1})$, $(\bar{B}, b^{-1})$, respectively, if necessary, where $\bar{A} = \Sigma - A - a^{\pm 1}$ and $\bar{B} = \Sigma - B - b^{\pm 1}$, we may assume that $c \in A$ and $c^{\pm 1} \notin B$ (clearly $c^{-1} \notin A$). By Lemma 2.2, we have $a.\Sigma > c.\Sigma$; hence either $a^{\pm 1} \notin B$ or $a^{\pm 1} \in B$, for otherwise $\deg \sigma_2 > \deg \sigma_1$, contrary to the hypothesis $\deg \sigma_1 > \deg \sigma_2$.

We first treat four cases for $a^{\pm 1} \notin B$ and then four cases for $a^{\pm 1} \in B$ according to whether b or b^{-1} belongs to A. For convenience, we write $A = C + E$ and $B = D + E$, where $E = A \cap B$.

Case 1. $a^{\pm 1} \notin B$ and $b^{\pm 1} \notin A$.

We consider two cases corresponding to whether or not E is the empty set.

Case 1.1. $E = \emptyset$.

Case 1.1.1. $a = b$.

It follows from [5, relation R2] that $\sigma_2\sigma_1 \equiv (A + B, a)$.

Case 1.1.2. $a \neq b$.

By [5, relation R3], we have $\sigma_2\sigma_1 \equiv (A, a)(B, b)$.

Case 1.2. $E \neq \emptyset$.

Case 1.2.1. $a = b$.

In view of (3.1) and Lemma 2.1(ii), we have $|(C, a^{-1})(u')| = |u|$. Since $(C, a^{-1})(u') = (E, a)(u)$, we have $|(E, a)(u)| = |u|$; hence

$$\begin{aligned}\sigma_2\sigma_1 &\equiv (B, a)\big[(C, a)(E, a)\big] \equiv \big[(B, a)(C, a)\big](E, a)\\ &\equiv (C + B, a)(E, a) \quad \text{by Case 1.1.1,}\end{aligned}$$

where $\deg \sigma_1 = \deg(C + B, a) > \deg(E, a)$.

Case 1.2.2. $a^{-1} = b$.

Lemma 2.1(i) together with (3.1) gives us that $|(E, a^{-1})(u')| = |u|$, so that $|(C, a)(u)| = |u|$; thus

$$\begin{aligned}\sigma_2\sigma_1 &\equiv \big(B, a^{-1}\big)\big[(E, a)(C, a)\big] \equiv \big[\big(B, a^{-1}\big)(E, a)\big](C, a) \equiv \big(D, a^{-1}\big)(C, a)\\ &\equiv (C, a)\big(D, a^{-1}\big) \quad \text{by Case 1.1.2,}\end{aligned}$$

where $\deg \sigma_1 = \deg(C, a) > \deg(D, a^{-1})$.

Case 1.2.3. $a^{\pm 1} \neq b$.

As in Case 1.2.1, we have $|(E, a)(u)| = |u|$; hence

$$\begin{aligned}\sigma_2\sigma_1 &\equiv (B, b)\big[(C, a)(E, a)\big] \equiv \big[(B, b)(C, a)\big](E, a)\\ &\equiv \big[(C, a)(B, b)\big](E, a) \quad \text{by Case 1.1.2,}\end{aligned}$$

where $\deg \sigma_1 = \deg(C, a) > \deg(B, b)$, $\deg(E, a)$.

Case 2. $a^{\pm 1} \notin B$, $b \notin A$ and $b^{-1} \in A$.

We consider this case dividing into two cases according to whether or not E is the empty set.

Case 2.1. $E = \emptyset$.

It follows from [5, relation R4] that $\sigma_2\sigma_1 \equiv (A + B, a)(B, b)$, where $\deg \sigma_1 = \deg(A + B, a) > \deg(B, b)$.

Case 2.2. $E \neq \emptyset$.

As in Case 1.2.1, we have $|(E, a)(u)| = |u|$; then

$$\begin{aligned}\sigma_2\sigma_1 &\equiv (B, b)\big[(C, a)(E, a)\big] \equiv \big[(B, b)(C, a)\big](E, a)\\ &\equiv \big[(C + B, a)(B, b)\big](E, a) \quad \text{by Case 2.1,}\end{aligned}$$

where $\deg \sigma_1 = \deg(C + B, a) > \deg(B, b)$, $\deg(E, a)$.

Case 3. $a^{\pm 1} \notin B$, $b \in A$ and $b^{-1} \notin A$.

Since $\sigma_2\sigma_1 \equiv (B, b)(\bar{A}, a^{-1})$, we can apply Case 2.2 to get

$$\sigma_2\sigma_1 \equiv (B, b)\big(\bar{A}, a^{-1}\big) \equiv \big((\bar{A} \setminus B) + B, a^{-1}\big)(B, b)\big(\bar{A} \cap B, a^{-1}\big).$$

Here, since $(\bar{A} \setminus B) + B = \Sigma - C - a^{\pm 1}$ and $\bar{A} \cap B = D$, we have

$$\sigma_2\sigma_1 \equiv \big(\Sigma - C - a^{\pm 1}, a^{-1}\big)(B, b)\big(D, a^{-1}\big) \equiv (C, a)(B, b)\big(D, a^{-1}\big),$$

where $\deg \sigma_1 = \deg(C, a) > \deg(B, b), \deg(D, a^{-1})$.

Case 4. $a^{\pm 1} \notin B$ and $b^{\pm 1} \in A$.

By Case 1.2.3 applied to $\sigma_2\sigma_1 \equiv (B, b)(\bar{A}, a^{-1})$, we have

$$\sigma_2\sigma_1 \equiv (B, b)\big(\bar{A}, a^{-1}\big) \equiv \big(\bar{A} \setminus B, a^{-1}\big)(B, b)\big(\bar{A} \cap B, a^{-1}\big).$$

From the observation that $\bar{A} \setminus B = \Sigma - (C + B) - a^{\pm 1}$ and $\bar{A} \cap B = D$, it follows that

$$\sigma_2\sigma_1 \equiv \big(\Sigma - (C + B) - a^{\pm 1}, a^{-1}\big)(B, b)\big(D, a^{-1}\big) \equiv (C + B, a)(B, b)\big(D, a^{-1}\big),$$

where $\deg \sigma_1 = \deg(C + B, a) > \deg(B, b), \deg(D, a^{-1})$.

Case 5. $a^{\pm 1} \in B$ and $b^{\pm 1} \notin A$.

Since $\sigma_2\sigma_1 \equiv (\bar{B}, b^{-1})(A, a)$, we have $|(A, a^{-1})(u')| = |(\bar{B}, b^{-1})(u')| = |u|$. This implies by Lemma 2.1(ii) that $|(\bar{B} \setminus A, b^{-1})(u')| = |u|$, so that

$$\sigma_2\sigma_1 \equiv \big(\bar{B}, b^{-1}\big)(A, a) \equiv \big[\big(A \cap \bar{B}, b^{-1}\big)\big(\bar{B} \setminus A, b^{-1}\big)\big](A, a).$$

Here, by Case 1.1.2, we have $(\bar{B} \setminus A, b^{-1})(A, a) \equiv (A, a)(\bar{B} \setminus A, b^{-1})$; thus

$$\sigma_2\sigma_1 \equiv \big(A \cap \bar{B}, b^{-1}\big)(A, a)\big(\bar{B} \setminus A, b^{-1}\big).$$

Since $A \cap \bar{B} = C$ and $\bar{B} \setminus A = \Sigma - (C + B) - b^{\pm 1}$, we finally have

$$\sigma_2\sigma_1 \equiv \big(C, b^{-1}\big)(A, a)(C + B, b),$$

where $\deg \sigma_1 = \deg(C, b^{-1}) = \deg(A, a) = \deg(C + B, b)$.

Case 6. $a^{\pm 1} \in B$, $b \notin A$ and $b^{-1} \in A$.

Case 6.1. $c = b^{-1}$.

By Case 3 applied to $\sigma_2\sigma_1 \equiv (\bar{B}, b^{-1})(A, a)$, we get

$$\sigma_2\sigma_1 \equiv \big(\bar{B}, b^{-1}\big)(A, a) \equiv (A \setminus \bar{B}, a)\big(\bar{B}, b^{-1}\big)\big(\bar{B} \setminus A, a^{-1}\big).$$

Here, we see that $A \setminus \bar{B} = E + b^{-1}$ and $\bar{B} \setminus A = \Sigma - (C + B + b)$, so that

$$\sigma_2\sigma_1 \equiv \big(E + b^{-1}, a\big)(B, b)\big(C + B + b - a^{\pm 1}, a\big),$$

where $\deg \sigma_1 = \deg(E + b^{-1}, a) > \deg(B, b), \deg(C + B + b - a^{\pm 1}, a)$.

Case 6.2. $c \neq b^{-1}$.

In this case, $c.\Sigma > b.\Sigma$, since $\deg \sigma_1$ is determined by c. Apply Lemma 2.1(ii) to the equalities $|(\bar{A}, a^{-1})^{-1}(u')| = |(\bar{B}, b^{-1})(u')| = |u|$, that is, $|(\bar{A}, a)(u')| = |(\bar{B}, b^{-1})(u')| = |u|$, to obtain $|(\bar{B} \setminus \bar{A}, b^{-1})(u')| = |u|$. But since $c \in \bar{B} \setminus \bar{A}$ and $c^{-1} \notin \bar{B} \setminus \bar{A}$, we have $b.\Sigma > c.\Sigma$ by Lemma 2.2, which contradicts $c.\Sigma > b.\Sigma$. Hence this case cannot occur.

Case 7. $a^{\pm 1} \in B$, $b \in A$ and $b^{-1} \notin A$.

Case 7.1. $c = b$.

Applying Case 2.2 to $\sigma_2\sigma_1 \equiv (\bar{B}, b^{-1})(A, a)$, we get

$$\sigma_2\sigma_1 \equiv \big(\bar{B}, b^{-1}\big)(A, a) \equiv \big((A \setminus \bar{B}) + \bar{B}, a\big)\big(\bar{B}, b^{-1}\big)(A \cap \bar{B}, a).$$

From the observation that $(A \setminus \bar{B}) + \bar{B} = \Sigma - (D + b^{-1})$ and $A \cap \bar{B} = C - b$, it follows that

$$\sigma_2\sigma_1 \equiv \big(D + b^{-1} - a^{\pm 1}, a^{-1}\big)(B, b)(C - b, a),$$

where $\deg \sigma_1 = \deg(D + b^{-1} - a^{\pm 1}, a^{-1}) > \deg(B, b), \deg(C - b, a)$.

Case 7.2. $c \neq b$.

As in Case 6.2, $c.\Sigma > b.\Sigma$. By Lemma 2.1(ii) applied to the equalities $|(A, a^{-1})(u')| = |(\bar{B}, b^{-1})(u')| = |u|$, we get $|(\bar{B} \setminus A, b^{-1})(u')| = |u|$. But since $c^{-1} \in \bar{B} \setminus A$ and $c \notin \bar{B} \setminus A$, we must have $b.\Sigma > c.\Sigma$ by Lemma 2.2, contrary to the fact $c.\Sigma > b.\Sigma$. Hence this case cannot happen.

Case 8. $a^{\pm 1} \in B$ and $b^{\pm 1} \in A$.

Apply Lemma 2.1(ii) to the equalities $|(\bar{A}, a^{-1})^{-1}(u')| = |(\bar{B}, b^{-1})(u')| = |u|$, that is, $|(\bar{A}, a)(u')| = |(\bar{B}, b^{-1})(u')| = |u|$, to obtain $|(\bar{B} \setminus \bar{A}, b^{-1})(u')| = |u|$; then

$$\sigma_2\sigma_1 \equiv \big(\bar{B}, b^{-1}\big)\big(\bar{A}, a^{-1}\big) \equiv \big[\big(\bar{A} \cap \bar{B}, b^{-1}\big)\big(\bar{B} \setminus \bar{A}, b^{-1}\big)\big]\big(\bar{A}, a^{-1}\big).$$

Since $(\bar{B} \setminus \bar{A}, b^{-1})(\bar{A}, a^{-1}) = (\bar{A}, a^{-1})(\bar{B} \setminus \bar{A}, b^{-1})$ by Case 1.1.2, we have

$$\sigma_2\sigma_1 \equiv \big(\bar{A} \cap \bar{B}, b^{-1}\big)\big(\bar{A}, a^{-1}\big)\big(\bar{B} \setminus \bar{A}, b^{-1}\big).$$

It follows from $\bar{A} \cap \bar{B} = \Sigma - (C + B)$ and $\bar{B} \setminus \bar{A} = C - b^{\pm 1}$ that

$$\sigma_2\sigma_1 \equiv \big(C + B - b^{\pm 1}, b\big)(A, a)\big(C - b^{\pm 1}, b^{-1}\big),$$

where $\deg \sigma_1 = \deg(C + B - b^{\pm 1}, b) = \deg(A, a) = \deg(C - b^{\pm 1}, b^{-1})$.

The proof of the lemma is now completed. □

Remark. The proof of Lemma 3.1 can be applied without further change if we replace consideration of a single cyclic word u, the length $|u|$ of u, and the total number of occurrences of $x_i^{\pm 1}$ in u with consideration of a finite sequence $(u_1, \dots, u_m)$ of cyclic words, the sum $\sum_{i=1}^m |u_i|$ of the lengths of $u_1, \dots, u_m$, and the total number of occurrences of $x_i^{\pm 1}$ in $(u_1, \dots, u_m)$, respectively.

Lemma 3.2. *Let u be a cyclic word in F_n that satisfies Hypotheses* 1.1 *and* 1.3. *Let σ_i, $i = 1, \dots, \ell$, be Whitehead automorphisms of the second type such that $|\sigma_i \cdots \sigma_1(u)| = |u|$ for all i. Suppose that* $\max_{1 \leqslant i \leqslant \ell} \deg \sigma_i = 1$. *Then there exist Whitehead automorphisms $\tau_1, \tau_2, \dots, \tau_s$ of the second type such that*

$$\sigma_\ell \cdots \sigma_2 \sigma_1(u) = \tau_s \cdots \tau_2 \tau_1(u),$$

where $1 \geqslant \deg \tau_s \geqslant \deg \tau_{s-1} \geqslant \cdots \geqslant \deg \tau_1$, and $|\tau_j \cdots \tau_1(u)| = |u|$ for all $j = 1, \dots, s$.

Proof. We proceed by induction on ℓ. The case for $\ell = 2$ is already proved in Lemma 3.1. Now let σ_i, $i = 1, \dots, \ell + 1$, be Whitehead automorphisms of F_n such that $|\sigma_i \cdots \sigma_1(u)| = |u|$ for all i and such that $\max_{1 \leqslant i \leqslant \ell+1} \deg \sigma_i = 1$. Then by the induction hypothesis, there exist Whitehead automorphisms $\tau_1, \tau_2, \dots, \tau_s$ of F_n such that

$$\sigma_{\ell+1} \sigma_\ell \cdots \sigma_2 \sigma_1(u) = \sigma_{\ell+1} \tau_s \cdots \tau_2 \tau_1(u), \tag{3.2}$$

where $1 \geqslant \deg \tau_s \geqslant \deg \tau_{s-1} \geqslant \cdots \geqslant \deg \tau_1$, and $|\tau_j \cdots \tau_1(u)| = |u|$ for all $j = 1, \dots, s$.

Put $\tau_j = (A_j, a_j)$ for $j = 1, \dots, s$, and put $\sigma_{\ell+1} = (B, b)$. If $\deg \sigma_{\ell+1} = 1$ or $\deg \tau_j = 0$ for all j, then there is nothing to prove. So let $\deg \sigma_{\ell+1} = 0$, and let t $(1 \leqslant t \leqslant s)$ be such that $\deg \tau_s = \deg \tau_{s-1} = \cdots = \deg \tau_t = 1$ and $\deg \tau_{t-1} = \cdots = \deg \tau_2 = \deg \tau_1 = 0$. Upon replacing τ_i and $\sigma_{\ell+1}$ by $(\bar{A}_i, a_i^{-1})$ and $(\bar{B}, b^{-1})$, respectively, if necessary, we may assume that $x_1 \in A_i$ for all $t \leqslant i \leqslant s$ and that $x_1^{\pm 1} \notin B$. We may also assume without loss of generality that (B, b) cannot be decomposed into $(B_2, b)(B_1, b)$, where $B = B_1 + B_2$, $\deg(B_1, b) = \deg(B_2, b) = 0$ and $|(B_1, b)\tau_s \cdots \tau_1(u)| = |u|$.

Claim 1. *We may further assume that $\tau_s = (A_s, a_s)$ cannot be decomposed into $(A_{s2}, a_s)(A_{s1}, a_s)$, where $A_s = A_{s1} + A_{s2}$, $\deg(A_{s1}, a_s) = 0$, $\deg(A_{s2}, a_s) = 1$, $|(A_{s1}, a_s)\tau_{s-1} \cdots \tau_1(u)| = |u|$, and $a_i^{\pm 1} \notin A_{s1}$ for all i with $t \leqslant i < s$.*

Proof. Suppose that τ_s can be decomposed in the same way as in the statement of the claim. Then continuously applying Case 1 or Case 4 of Lemma 3.1 to $(A_{s1}, a_s)\tau_{s-1} \cdots \tau_t$ at most $1 + 2 + 2^2 + \cdots + 2^{s-t-1}$ times (here, note that if $s = t$, we do not need to apply Lemma 3.1), we get

$$(A_{s1}, a_j)\tau_{s-1} \cdots \tau_t = \tau'_{s-1} \cdots \tau'_t \varepsilon_p \cdots \varepsilon_1,$$

where $\tau'_{s-1}, \dots, \tau'_t$ are Whitehead automorphisms of degree 1 and $\varepsilon_p, \dots, \varepsilon_1$ are Whitehead automorphisms of degree 0, so that

$$(B, b)\tau_s \cdots \tau_t \cdots \tau_1(u) = (B, b)(A_{s2}, a_s)\tau'_{s-1} \cdots \tau'_t \varepsilon_p \cdots \varepsilon_1 \tau_{t-1} \cdots \tau_1(u), \tag{3.3}$$

where the length of u is constant throughout both chains. We then replace the chain on the right-hand side of (3.2) with that of (3.3). □

We consider three cases corresponding to whether or not $b = x_1^{\pm 1}$.

Case 1. $b \neq x_1^{\pm 1}$.

For all i with $t \leqslant i \leqslant s$, either $b^{\pm 1} \in A_i$ or $b^{\pm 1} \notin A_i$, since $\deg \tau_i = 1$. If $a_s^{\pm 1} \in B$, then the required result follows immediately from Case 5 or Case 8 of Lemma 3.1 applied to $(B, b)\tau_s$. So let $a_s^{\pm 1} \notin B$. If $b^{\pm 1} \notin A_s$ and $A_s \cap B = \emptyset$, then by Case 1.1.2 of Lemma 3.1 we have $(B, b)\tau_s \equiv \tau_s(B, b)$. Also if $b^{\pm 1} \in A_s$ and $B \subset A_s$, then Case 4 of Lemma 3.1 yields that $(B, b)\tau_s \equiv \tau_s(B, b)$. Hence, in either case, we have

$$(B, b)\tau_s \cdots \tau_t \cdots \tau_1(u) = \tau_s(B, b)\tau_{s-1} \cdots \tau_t \cdots \tau_1(u);$$

then the desired result follows by induction on $s - t$. Now suppose that either both $b^{\pm 1} \notin A_s$ and $A_s \cap B \neq \emptyset$ or both $b^{\pm 1} \in A_s$ and $B \not\subseteq A_s$. We argue two cases separately.

Case 1.1. $a_s^{\pm 1} \notin B$, $b^{\pm 1} \notin A_s$ and $A_s \cap B \neq \emptyset$.

By Case 1.2.3 of Lemma 3.1, we have $(B, b)\tau_s \equiv (A_s \setminus B, a_s)(B, b)(A_s \cap B, a_s)$; thus

$$(B, b)\tau_s \cdots \tau_t \cdots \tau_1(u) = (A_s \setminus B, a_s)(B, b)(A_s \cap B, a_s)\tau_{s-1} \cdots \tau_t \cdots \tau_1(u).$$

By Claim 1, there is j with $t \leqslant j < s$ such that $a_j^{\pm 1} \in A_s \cap B$. Let r be the largest such index.

First suppose that there exists a chain $\eta_m \cdots \eta_1$ of Whitehead automorphisms $\eta_i = (G_i, g_i)$ of degree 1 with $g_i^{\pm 1} \notin B$, $G_i \subset A_s$ and $G_i \cap B = \emptyset$ such that $|\eta_i \cdots \eta_1 \tau_s \cdots \tau_1(u)| = |u|$ for all $i = 1, \ldots, m$ and such that $|(H, a_r^{-1})\eta_m \cdots \eta_1 \tau_s \cdots \tau_1(u)| = |u|$ for some Whitehead automorphism (H, a_r^{-1}) of degree 1 with $H \subset A_s$. Then

$$\begin{aligned}(B, b)\tau_s \cdots \tau_1(u) &= (B, b)\eta_1^{-1} \cdots \eta_m^{-1}\eta_m \cdots \eta_1 \tau_s \cdots \tau_1(u)\\ &= \eta_1^{-1} \cdots \eta_m^{-1}(B, b)\eta_m \cdots \eta_1 \tau_s \cdots \tau_1(u) \quad \text{by Case 1.1.2 of Lemma 3.1.}\end{aligned}$$

Put $v = \eta_m \cdots \eta_1 \tau_s \cdots \tau_1(u)$. By Lemma 2.1(ii) applied to $|(\bar{B}, b^{-1})(v)| = |(H, a_r^{-1})(v)| = |u|$, we have $|(\bar{B} \setminus H, b^{-1})(v)| = |u|$. It follows from $\bar{B} \setminus H = \Sigma - (B \cup H) - b^{\pm 1}$ that $|(B \cup H, b)(v)| = |u|$, so that

$$(B, b)\tau_s \cdots \tau_1(u) = \eta_1^{-1} \cdots \eta_m^{-1}\big(H \setminus B, b^{-1}\big)(B \cup H, b)\eta_m \cdots \eta_1 \tau_s \cdots \tau_1(u),$$

where $\deg \eta_i^{-1} = \deg(H \setminus B, b^{-1}) = \deg(B \cup H, b) = \deg \eta_i = 1$, as required.

Next suppose that there does not exist such a chain $\eta_m \cdots \eta_1$ as above. Considering all the assumptions and the situations above, we can observe that this can possibly happen only in the case where all of a_s and a_s^{-1} that are lost in passing from $\tau_{s-1} \cdots \tau_1(u)$ to $\tau_s \cdots \tau_1(u)$ were newly introduced in passing from $\tau_{q-1} \cdots \tau_1(u)$ to $\tau_q \cdots \tau_1(u)$ for some $r < q < s$, and where for such $\tau_q = (A_q, a_s^{-1})$ (here note that $a_q = a_s^{-1}$),

$$\begin{aligned}&(B, b)\tau_s \cdots \tau_t \cdots \tau_1(u)\\ &\quad = (B, b)(A_s \setminus B, a_s)\tau_{s-1} \cdots \tau_{q+1}\big(A_q \setminus (A_s \cap B), a_s^{-1}\big)\tau_{q-1} \cdots \tau_t \cdots \tau_1(u),\end{aligned}$$

where the length of u is constant throughout the chain on the right-hand side. It then follows from Case 1.1.2 of Lemma 3.1 applied to $(B,b)(A_s \setminus B, a_s)$ that

$$\begin{aligned}&(B,b)\tau_s\cdots\tau_t\cdots\tau_1(u)\\&\quad=(A_s\setminus B,a_s)(B,b)\tau_{s-1}\cdots\tau_{q+1}\big(A_q\setminus(A_s\cap B),a_s^{-1}\big)\tau_{q-1}\cdots\tau_t\cdots\tau_1(u).\end{aligned}$$

Then induction on $s-t$ yields the desired result, which completes the proof of Case 1.1.

Case 1.2. $a_s^{\pm1}\notin B$, $b^{\pm1}\in A_s$ and $B\nsubseteq A_s$.

In this case, replace τ_i by $(\bar{A}_i, a_i^{-1})$ for all $t\leqslant i\leqslant s$ and then follow the arguments of Case 1.1.

Case 2. $b=x_1$.

We divide this case into two cases according to whether $a_s^{\pm1}\in B$ or not.

Case 2.1. $a_s^{\pm1}\in B$.

In this case, we have by Case 7.1 of Lemma 3.1 applied to $(B,x_1)\tau_s$ that

$$(B,x_1)\tau_s\cdots\tau_1(u)=\big(B\setminus A_s+x_1^{-1}-a_s^{\pm1},a_s^{-1}\big)(B,x_1)(A_s\setminus B-x_1,a_s)\tau_{s-1}\cdots\tau_1(u). \quad (3.4)$$

Here if $A_s\setminus B-x_1=\emptyset$, then

$$(B,x_1)\tau_s\cdots\tau_t\cdots\tau_1(u)=\big(B\setminus A_s+x_1^{-1}-a_s^{\pm1},a_s^{-1}\big)(B,x_1)\tau_{s-1}\cdots\tau_t\cdots\tau_1(u);$$

hence the desired result follows by induction on $s-t$.

So let $A_s\setminus B-x_1\neq\emptyset$. By Claim 1, there is j with $t\leqslant j<s$ such that $a_j^{\pm1}\in A_s\setminus B-x_1$. Let r be the largest such index. The following Claims 2–4 show that we may assume that a_r, a_s and x_1 belong to distinct connected components of the dependence graph Γ_u of u.

Claim 2. *a_r and x_1 belong to distinct connected components of Γ_u.*

Proof. Suppose on the contrary that a_r and x_1 belong to the same connected component C_1. Put $\mathcal{W}=\{\alpha\colon \alpha$ is a Whitehead automorphism of degree 0 such that $|\alpha(v)|=|v|=|u|$ for some $v\in \mathrm{Orb}_{\mathrm{Aut}\,F_n}(u)\}$. Then by (3.4), $(A_s\setminus B-x_1,a_s)\in\mathcal{W}$ and $(B,x_1)\in\mathcal{W}$. Since $x_1^{\pm1}\notin A_s\setminus B-x_1$ and $a_r^{\pm1}\in A_s\setminus B-x_1$, we see from the construction of Γ_u that a_s also belongs to C_1 and that every path from a_r or a_r^{-1} to x_1 or x_1^{-1} passes through a_s or a_s^{-1}. Also since $a_r^{\pm1}\notin B$ and $a_s^{\pm1}\in B$, every path from a_s or a_s^{-1} to a_r or a_r^{-1} passes through x_1 or x_1^{-1}, which contradicts the above fact that every path from a_r or a_r^{-1} to x_1 or x_1^{-1} passes through a_s or a_s^{-1}. □

Claim 3. *We may assume that a_s and x_1 belong to distinct connected components of Γ_u.*

Proof. Suppose that a_s and x_1 belong to the same connected component C_1. First consider the case where there exists a chain $\zeta_k\cdots\zeta_1$ of Whitehead automorphisms $\zeta_i=(E_i,e_i)$ of degree 1 with $e_i^{\pm1}\in B$ and $E_i\subset(B+x_1)$ such that $|\zeta_i\cdots\zeta_1\tau_s\cdots\tau_1(u)|=|u|$ for all $i=1,\dots,k$ and such that $|(H, a_r^{-1})\zeta_k \cdots \zeta_1 \tau_s \cdots \tau_1(u)| = |u|$ for some Whitehead automorphism (H, a_r^{-1}) of degree 1 with $H \subset A_s$. Then

$$\begin{aligned}(B, x_1)\tau_s \cdots \tau_1(u) &= (B, x_1)\zeta_1^{-1} \cdots \zeta_k^{-1}\zeta_k \cdots \zeta_1 \tau_s \cdots \tau_1(u)\\ &= \rho_k \cdots \rho_1 (B, x_1)\zeta_k \cdots \zeta_1 \tau_s \cdots \tau_1(u) \quad \text{by Case 7.1 of Lemma 3.1,}\end{aligned}$$

where $\rho_i = (B \setminus E_{k+1-i} + x_1^{-1} - e_{k+1-i}^{\pm 1}, e_{k+1-i}^{-1})$ for $i = 1, \dots, k$. Put $v = \zeta_k \cdots \zeta_1 \tau_s \cdots \tau_1(u)$. Then $|(B, x_1)(v)| = |(H, a_r^{-1})(v)| = |u|$, that is, $|(B, x_1)(v)| = |(\bar{H}, a_r)(v)| = |u|$. By Lemma 2.1(ii) applied to these equalities, we have $|(\bar{H} \setminus B, a_r)(v)| = |u|$, so that

$$\left|\left(H + (\bar{H} \setminus B), a_r\right)\left(H, a_r^{-1}\right)\zeta_k \cdots \zeta_1 \tau_s \cdots \tau_1(u)\right| = |u|.$$

It then follows from $H + (\bar{H} \setminus B) = \Sigma - (B \setminus H) - a_r^{\pm 1}$ that

$$\left|\left(B \setminus H, a_r^{-1}\right)\left(H, a_r^{-1}\right)\zeta_k \cdots \zeta_1 \tau_s \cdots \tau_1(u)\right| = |u|.$$

This implies that $(B \setminus H, a_r^{-1}) \in \mathcal{W}$, where $\mathcal{W}$ is defined in the proof of Claim 2. Since $a_s^{\pm 1} \in B \setminus H$ and $x_1^{\pm 1} \notin B \setminus H$, a_r must also belong to C_1 by the construction of Γ_u, which contradicts Claim 2.

Next consider the case where there does not exist such a chain $\zeta_k \cdots \zeta_1$ as above. Considering all the assumptions and the situations above, we can observe that this can possibly happen only in the case where all of a_s and a_s^{-1} that are lost in passing from $\tau_{s-1} \cdots \tau_1(u)$ to $\tau_s \cdots \tau_1(u)$ were newly introduced in passing from $\tau_{q-1} \cdots \tau_1(u)$ to $\tau_q \cdots \tau_1(u)$ for some $r < q < s$, and where for such $\tau_q = (A_q, a_s^{-1})$ (here note that $a_q = a_s^{-1}$),

$$\begin{aligned}&(B, x_1)\tau_s \cdots \tau_t \cdots \tau_1(u)\\ &\quad = (B, x_1)(A_s \cap B, a_s)\tau_{s-1} \cdots \tau_{q+1}\left(A_q \setminus (A_s \setminus B), a_s^{-1}\right)\tau_{q-1} \cdots \tau_t \cdots \tau_1(u),\end{aligned}$$

where the length of u is constant throughout the chain on the right-hand side. It then follows from Case 7.1 of Lemma 3.1 applied to $(B, x_1)(A_s \cap B, a_s)$ that

$$\begin{aligned}&(B, x_1)\tau_s \cdots \tau_t \cdots \tau_1(u)\\ &\quad = \left(B \setminus A_s + x_1^{-1} - a_s^{\pm 1}, a_s^{-1}\right)(B, x_1)\tau_{s-1} \cdots \tau_{q+1}\left(A_q \setminus (A_s \setminus B), a_s^{-1}\right)\tau_{q-1} \cdots \tau_t \cdots \tau_1(u).\end{aligned}$$

So in this case, apply induction on $s - t$ to get the desired result of the lemma, which completes the proof of Claim 3. □

Claim 4. *a_r and a_s belong to distinct connected components of Γ_u.*

Proof. Suppose on the contrary that a_r and a_s belong to the same connected component. Note that $a_r^{\pm 1} \notin B$, $a_s^{\pm 1} \in B$ and that $(B, x_1) \in \mathcal{W}$, where $\mathcal{W}$ is defined in the proof of Claim 2. It then follows from the construction of Γ_u that a_s and x_1 must belong to the same connected component, which contradicts Claim 3. □

So let C_1, $C_{r'}$ and $C_{s'}$ be the distinct connected components of Γ_u containing x_1, a_r, and a_s in that order. Here notice that C_1 consists of only $x_1^{\pm 1}$, since there exists a Whitehead automorphism

(A_s, a_s) of degree 1 such that $a_s \notin C_1$ and such that $|(A_s, a_s)(v)| = |v| = |u|$ for some $v \in \mathrm{Orb}_{\mathrm{Aut}\, F_n}(u)$ (see Remark (iii) in the introduction).

Put $u_1 = \tau_{t-1} \cdots \tau_1(u)$.

Claim 5. *We may assume that $\tau_i \tau_j \equiv \tau_j \tau_i$ for all $1 \leqslant i \neq j \leqslant t-1$.*

Proof. Put $\mathcal{M} = \{v\colon\ v = \phi(u)$ and $|v|_{C_i} = |u|_{C_i}$ for all $i = 1, \ldots, n$, where ϕ is a chain of Whitehead automorphisms of degree 0 throughout which the length of u is constant$\}$. Taking an appropriate $v \in \mathcal{M}$, we have Whitehead automorphisms $\delta_j = (D_j, d_j)$ of F_n of degree 0 such that

$$u_1 = \delta_h \cdots \delta_1(v), \tag{3.5}$$

where $|\delta_j \cdots \delta_1(v)| = |v|$ and $|\delta_j \cdots \delta_1(v)|_{C_{k_j}} > |v|_{C_{k_j}}$ for the connected component C_{k_j} containing d_j and for each $j = 1, \ldots, h$. Then for any $\delta_i = (D_i, d_i)$ and $\delta_j = (D_j, d_j)$ with $d_j \neq d_i^{\pm 1}$, if we replace δ_i and δ_j with $(\bar{D}_i, d_i^{-1})$ and $(\bar{D}_j, d_j^{-1})$, respectively, if necessary so that $d_i^{\pm 1} \notin D_j$ and $d_j^{\pm 1} \notin D_i$, then $D_i \cap D_j = \emptyset$. Hence by Case 1.1.2 of Lemma 3.1 that $\delta_j \delta_i \equiv \delta_i \delta_j$; thus (3.5) can be re-written as

$$u_1 = \delta_{pt_p}^{q_{pt_p}} \cdots \delta_{p1}^{q_{p1}} \cdots \delta_{1t_1}^{q_{1t_1}} \cdots \delta_{11}^{q_{11}}(v), \tag{3.6}$$

where $d_{ki} = d_{ki'}$ and $D_{ki} \neq D_{ki'}$ provided $i \neq i'$; $d_{k'i} \neq d_{ki}^{\pm 1}$ and $(\delta_{k't_{k'}}^{q_{k't_{k'}}} \cdots \delta_{k'1}^{q_{k'1}})(\delta_{kt_k}^{q_{kt_k}} \cdots \delta_{k1}^{q_{k1}}) \equiv (\delta_{kt_k}^{q_{kt_k}} \cdots \delta_{k1}^{q_{k1}})(\delta_{k't_{k'}}^{q_{k't_{k'}}} \cdots \delta_{k'1}^{q_{k'1}})$ provided $k \neq k'$. Here we may assume by Case 1.2.1 of Lemma 3.1 that $D_{ki} \subset D_{ki'}$ if $i < i'$. Then $\delta_{ki'} \delta_{ki} \equiv \delta_{ki} \delta_{ki'}$ by Case 1.2.1 of Lemma 3.1; hence $\delta_{k'i'} \delta_{ki} \equiv \delta_{ki} \delta_{k'i'}$ for any δ_{ki} and $\delta_{k'i'}$ in chain (3.6). Thus replace $\tau_{t-1} \cdots \tau_1(u)$ with the right-hand side of (3.6) to get our desired result. □

By Claim 5, we may write

$$u_1 = \tau_{t-1} \cdots \tau_p \tau_{p-1} \cdots \tau_1(u),$$

where τ_i has multiplier in $C_{r'}$ provided $p \leqslant i \leqslant t-1$; τ_i has multiplier not in $C_{r'}$ provided $1 \leqslant i \leqslant p-1$. Put

$$u_2 = \tau_{p-1} \cdots \tau_1(u).$$

Note that the number of $C_{r'}$-syllables of u remains unchanged throughout this chain.

Claim 6. *There exist Whitehead automorphisms $\varepsilon_i = (E_i, a_i)$, $t \leqslant i \leqslant s$, such that $|\varepsilon_i \cdots \varepsilon_t(u_2)| = |u|$ for all $i = t, \ldots, s$, where $E_i = \emptyset$ provided $a_i \in C_{r'}$; E_i is one of the three forms A_i, $A_i + C_{r'}$ and $A_i - C_{r'}$, whichever is smallest possible with priority given to lower i, provided $a_i \notin C_{r'}$.*

Proof. Suppose the contrary. It can possibly happen only when the number of $C_{r'}$-syllables of u_2 is decreased by $\tau_j \cdots \tau_t \tau_{t-1} \cdots \tau_p$ (for some $j \geqslant t$) followed by a chain of Whitehead automorphisms of degree 0 with multiplier in $C_{r'}$, where the length of u_2 is constant throughout the chain. Choosing the smallest such index j, put $\{j_1, \ldots, j_k\} = \{i\colon t \leqslant i \leqslant j$ and τ_i has multiplier in $C_{r'}\}$. Then we can observe that there is a chain $\zeta_m \cdots \zeta_1$ of Whitehead automorphisms of degree 0 with multiplier in $C_{r'}$ such that $|\zeta_m \cdots \zeta_1 \tau_{j_k} \cdots \tau_{j_1} \tau_{t-1} \cdots \tau_p(u_2)| = |u_2|$ and the number of $C_{r'}$-syllables of $\zeta_m \cdots \zeta_1 \tau_{j_k} \cdots \tau_{j_1} \tau_{t-1} \cdots \tau_p(u_2)$ is less than that of u_2. This is a contradiction, because through the chain $\zeta_m \cdots \zeta_1 \tau_{j_k} \cdots \tau_{j_1} \tau_{t-1} \cdots \tau_p$ only C_1-syllables and $C_{r'}$-syllables can mix and increasing the number of C_1-syllables cannot reduce the number of $C_{r'}$-syllables. □

For the chain $\varepsilon_s \cdots \varepsilon_t$, we consider two cases separately.

Case 2.1.1. $|(B, x_1)\varepsilon_s \cdots \varepsilon_t(u_2)| = |u|$.

For the Whitehead automorphisms $\delta_i = (D_i, d_i)$ $(p \leqslant i < t)$, where $D_i = A_i \setminus B$ and $d_i = a_i$ provided $x_1^{\pm 1} \notin A_i$; $D_i = \bar{A}_i \setminus B$ and $d_i = a_i^{-1}$ provided $x_1^{\pm 1} \in A_i$, and for the Whitehead automorphisms $\omega_j = (F_j, a_{t+s-j}^{-1})$ and $\nu_j = (H_j, a_j)$ $(t \leqslant j \leqslant s)$, where $F_j = \emptyset$ provided $a_{t+s-j} \in C_{r'} + B$; $F_j = E_{t+s-j} \setminus B$ provided $a_{t+s-j} \notin C_{r'} + B$; $H_j = \emptyset$ provided $a_j \in B$; $H_j = A_j \setminus B$ provided $a_j \notin B$, we have

$$(B, x_1)\tau_s \cdots \tau_1(u) = \nu_s \cdots \nu_t \delta_{t-1} \cdots \delta_p \omega_s \cdots \omega_t (B, x_1)\varepsilon_s \cdots \varepsilon_t \tau_{p-1} \cdots \tau_1(u), \tag{3.7}$$

where the length of u is constant throughout the chain on the right-hand side. By Case 1, it suffices to consider only the chain $(B, x_1)\varepsilon_s \cdots \varepsilon_t \tau_{p-1} \cdots \tau_1(u)$. Since for every j either $\deg \varepsilon_j = 1$ or $\varepsilon_j = 1$ and since $\varepsilon_r = 1$, the desired result follows by induction on $s - t$ from (3.7).

Case 2.1.2. $|(B, x_1)\varepsilon_s \cdots \varepsilon_t(u_2)| > |u|$.

We see that this case can possibly happen only when the cyclic word $\varepsilon_s \cdots \varepsilon_t(u_2)$ contains a subword of the form $(x_1 w_1 w_2 w_3)^\theta$, where $\theta = \pm 1$, w_1 (w_1 may be the empty word), w_2 and w_3 are words in B, $C_{r'}$ and $C_{s'}$, respectively, and not all of the letters in w_3 were newly introduced in passing from u_2 to $\varepsilon_s \cdots \varepsilon_t(u_2)$.

By Claim 5, we may write

$$u_1 = \tau_{t-1} \cdots \tau_q \tau_{q-1} \cdots \tau_1(u),$$

where τ_i has multiplier in $C_{s'}$ provided $q \leqslant i \leqslant t-1$; τ_i has multiplier not in $C_{s'}$ provided $1 \leqslant i \leqslant q-1$. Put

$$u_3 = \tau_{q-1} \cdots \tau_1(u).$$

Notice that the number of $C_{s'}$-syllables of u remains unchanged throughout this chain.

Claim 7. *There exist Whitehead automorphisms $\lambda_i = (J_i, a_i)$, $t \leqslant i \leqslant s$, such that $|\lambda_i \cdots \lambda_t(u_3)| = |u|$ for all $i = t, \ldots, s$, where $J_i = \emptyset$ provided $a_i \in C_{s'}$; J_i is one of the three forms A_i, $A_i + C_{s'}$ and $A_i - C_{s'}$, whichever is largest possible with priority given to lower i, provided $a_i \notin C_{s'}$.*

Proof. Suppose the contrary. In view of all the assumptions and the situations above, this can possibly happen only when the number of $C_{s'}$-syllables of u_3 is decreased by $\tau_j \cdots \tau_t \tau_{t-1} \cdots \tau_q$

(for some $j \geqslant t$) followed by a chain of Whitehead automorphisms of degree 0 with multiplier in $C_{s'}$, where the length of u_3 is constant throughout the chain. Choosing the smallest such index j, put $\{j_1, \ldots, j_k\} = \{i\colon\ t \leqslant i \leqslant j \text{ and } \tau_i \text{ has multiplier in } C_{s'}\}$. Then we can observe that there exists a chain $\delta_m \cdots \delta_1$ of Whitehead automorphisms of degree 0 with multiplier in $C_{s'}$ such that $|\delta_m \cdots \delta_1 \tau_{j_k} \cdots \tau_{j_1} \tau_{t-1} \cdots \tau_q(u_3)| = |u|$, and such that the number of $C_{s'}$-syllables of $\delta_m \cdots \delta_1 \tau_{j_k} \cdots \tau_{j_1} \tau_{t-1} \cdots \tau_q(u_3)$ is less than that of u_3. Reasoning as in Claim 6, we get a contradiction, which completes the proof of Claim 7. □

We then see that $|(B, x_1)\lambda_s \cdots \lambda_t(u_3)| = |u|$. Furthermore, for the Whitehead automorphisms $\delta_i = (D_i, d_i)$ $(q \leqslant i < t)$, where $D_i = A_i \cap B$ and $d_i = a_i$ provided $x_1^{\pm 1} \notin A_i$; $D_i = \bar{A}_i \cap B$ and $d_i = a_i^{-1}$ provided $x_1^{\pm 1} \in A_i$, and for the Whitehead automorphisms $\omega_j = (K_j, a_{t+s-j})$ and $\nu_j = (H_j, a_j^{-1})$ $(t \leqslant j \leqslant s)$, where $K_j = \emptyset$ provided $a_{t+s-j} \notin B - C_{s'}$; $K_j = B \setminus J_{t+s-j} + x_1^{-1} - a_{t+s-j}^{\pm 1}$ provided $a_{t+s-j} \in B - C_{s'}$; $H_j = \emptyset$ provided $a_j \notin B$; $H_j = B \setminus A_j + x_1^{-1} - a_j^{\pm 1}$ provided $a_j \in B$,

$$(B, x_1)\tau_s \cdots \tau_1(u) = \nu_s \cdots \nu_t \delta_{t-1} \cdots \delta_q \omega_s \cdots \omega_t (B, x_1)\lambda_s \cdots \lambda_t \tau_{q-1} \cdots \tau_1(u), \tag{3.8}$$

where the length of u is constant throughout the chain on the right-hand side. By Case 1, it suffices to consider only the chain $(B, x_1)\lambda_s \cdots \lambda_t \tau_{q-1} \cdots \tau_1(u)$. Since for every j either $\deg \lambda_i = 1$ or $\lambda_i = 1$ and since $\lambda_s = 1$, the desired result follows by induction on $s - t$ from (3.8). This completes the proof of Case 2.1.2.

Case 2.2. $a_s^{\pm 1} \notin B$.

In this case, replace (B, x_1) and τ_i by $(\bar{B}, x_1^{-1})$ and $(\bar{A}_i, a_i^{-1})$ for all $t \leqslant i \leqslant s$, respectively, and then follow the arguments of Case 2.1.

Case 3. $b = x_1^{-1}$.

Replace (B, x_1^{-1}) by $(\bar{B}, x_1)$ and then repeat the arguments of Case 2. □

Remark. The proof of Lemma 3.2 can be applied without further change if we replace consideration of a single cyclic word u, the length $|u|$ of u, the total number of occurrences of $x_j^{\pm 1}$ in u, and the C_j-syllable length $|u|_{C_j}$ with consideration of a finite sequence $(u_1, \ldots, u_m)$ of cyclic words, the sum $\sum_{i=1}^m |u_i|$ of the lengths of $u_1, \ldots, u_m$, the total number of occurrences of $x_j^{\pm 1}$ in $(u_1, \ldots, u_m)$, and the sum $\sum_{i=1}^m |u_i|_{C_j}$ of the C_j-syllable lengths of $u_1, \ldots, u_m$, respectively.

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. The proof proceeds by double induction on ℓ and r, where ℓ is the length of the chain $\sigma_\ell \cdots \sigma_2 \sigma_1$ and $r = \max_{1 \leqslant i \leqslant \ell} \deg \sigma_i$. The base steps were already proved in Lemma 3.1 (the case for $\ell = 2$ and any r) and Lemma 3.2 (the case for $r = 1$ and any ℓ).

Let σ_i, $i = 1, \ldots, \ell + 1$ $(\ell + 1 \geqslant 3)$, be Whitehead automorphisms of F_n such that $|\sigma_i \cdots \sigma_1(u)| = |u|$ for all $i = 1, \ldots, \ell + 1$ and such that $\max_{1 \leqslant i \leqslant \ell + 1} \deg \sigma_i = r + 1 \geqslant 2$. By the induction hypothesis on ℓ, there exist Whitehead automorphisms $\tau_1, \tau_2, \dots, \tau_s$ of F_n such that

$$\sigma_{\ell+1}\sigma_\ell \cdots \sigma_2\sigma_1(u) = \sigma_{\ell+1}\tau_s \cdots \tau_2\tau_1(u),$$

where $r+1 \geqslant \deg\tau_s \geqslant \deg\tau_{s-1} \geqslant \cdots \geqslant \deg\tau_1$, and $|\tau_j \cdots \tau_1(u)| = |u|$ for all $j = 1, \dots, s$.

If either $\deg\sigma_{\ell+1} = r+1$ or both $\deg\tau_s \leqslant r$ and $\deg\sigma_{\ell+1} \geqslant r$, then there is nothing to prove. Also if $\deg\tau_s \leqslant r$ and $\deg\sigma_{\ell+1} < r$, then we are done by the induction hypothesis on r. So let t ($1 \leqslant t \leqslant s$) be such that $\deg\tau_i = r+1$ provided $t \leqslant i \leqslant s$ and $\deg\tau_i \leqslant r$ provided $1 \leqslant i < t$, and let $\deg\sigma_{\ell+1} \leqslant r$.

Put $\tau_j = (A_j, a_j)$ for $j = 1, \dots, s$ and $\sigma_{\ell+1} = (B, b)$. Upon replacing τ_i and $\sigma_{\ell+1}$ by $(\bar{A}_i, a_i^{-1})$ and $(\bar{B}, b^{-1})$, respectively, if necessary, we may assume that $x_{r+1} \in A_i$ for all $t \leqslant i \leqslant s$ and that $x_{r+1}^{\pm1} \notin B$. We may also assume without loss of generality that (B, b) cannot be decomposed to $(B_2, b)(B_1, b)$, where $B = B_1 + B_2$ and $|(B_1, b)\tau_s \cdots \tau_1(u)| = |u|$. We may further assume as in Claim 1 of Lemma 3.2 that $\tau_s = (A_s, a_s)$ cannot be decomposed to $(A_{s2}, a_s)(A_{s1}, a_s)$, where $A_s = A_{s1} + A_{s2}$, $\deg(A_{s1}, a_s) \leqslant r$, $\deg(A_{s2}, a_s) = r+1$, $|(A_{s1}, a_s)\tau_{s-1} \cdots \tau_1(u)| = |u|$, and $a_i^{\pm1} \notin A_{s1}$ for all i with $t \leqslant i < s$.

There are three cases to consider.

Case 1. $b = x_1$.

If $a_i^{\pm1} \notin B$ for all $t \leqslant i \leqslant s$, then continuous application of Cases 1–4 of Lemma 3.1 to $(B, x_1)\tau_s \cdots \tau_t$ at most $1 + 2 + 2^2 + \cdots + 2^{s-t}$ times together with the induction hypothesis on r yields the desired result. The following claim shows that it is indeed true that $a_i^{\pm1} \notin B$ for all $t \leqslant i \leqslant s$.

Claim. *$a_i^{\pm1} \notin B$ for all $t \leqslant i \leqslant s$.*

Proof. Suppose on the contrary that $a_i^{\pm1} \in B$ for some $t \leqslant i \leqslant s$. First let $a_s^{\pm1} \in B$. If either $x_1 \in A_s$ or $x_1^{-1} \in A_s$ but not both, then we have a contradiction by Cases 6.2 and 7.2 of Lemma 3.1, since $\deg\tau_s = r+1 \geqslant 2$. If $x_1^{\pm1} \in A_s$, then by Case 8 of Lemma 3.1,

$$(B, x_1)(A_s, a_s) \equiv \left(A_s \cup B - x_1^{\pm1}, x_1\right)(A_s, a_s)\left(A_s \setminus B - x_1^{\pm1}, x_1^{-1}\right),$$

but the existence of $(A_s \setminus B - x_1^{\pm1}, x_1^{-1})$ in this chain contradicts Lemma 2.2, because $x_{r+1} \in A_s \setminus B - x_1^{\pm1}$ and $x_{r+1}^{-1} \notin A_s \setminus B - x_1^{\pm1}$. If $x_1^{\pm1} \notin A_s$, then by Case 5 of Lemma 3.1,

$$(B, x_1)(A_s, a_s) \equiv \left(A_s \setminus B, x_1^{-1}\right)(A_s, a_s)(A_s \cup B, x_1),$$

but the existence of $(A_s \cup B, x_1)$ in this chain also contradicts Lemma 2.2, since $x_{r+1} \in A_s \cup B$ and $x_{r+1}^{-1} \notin A_s \cup B$.

Next let $a_s^{\pm1} \notin B$. Suppose that $a_i^{\pm1} \in B$ for some $t \leqslant i < s$. Let k be the largest such index. Put $v = \tau_{k-1} \cdots \tau_1(u)$. If $x_1 \in A_k$ and $x_1^{-1} \notin A_k$, then we can observe based on all the assumptions and the situations above that there exists a Whitehead automorphism (F, x_1) of degree $r+1$ with $(B \cup A_k - x_1) \subseteq F$ such that $|(F, x_1)\tau_k(v)| = |u|$. But this yields a contradiction to

Lemma 2.2, since $x_{r+1} \in F$ and $x_{r+1}^{-1} \notin F$. For a similar reason, the case where $x_1 \notin A_k$ and $x_1^{-1} \in A_k$ cannot happen, either. So A_k must contain either both of $x_1^{\pm 1}$ or none of $x_1^{\pm 1}$.

If there exists a chain $\zeta_p \cdots \zeta_1$ of Whitehead automorphisms of degree less than or equal to $r+1$ such that $|(B, x_1)\tau_k \zeta_p \cdots \zeta_1(v)| = |\tau_k \zeta_p \cdots \zeta_1(v)| = |\zeta_p \cdots \zeta_1(v)| = |u|$, then as in the case where $a_s^{\pm 1} \in B$ we reach a contradiction. Otherwise, choose chains $\zeta_p \cdots \zeta_1$ and $\omega_q \cdots \omega_1$ of Whitehead automorphisms of degree less than or equal to $r+1$ with q smallest possible such that $|\omega_j \cdots \omega_1 \tau_k \zeta_p \cdots \zeta_1(v)| = |\tau_k \zeta_p \cdots \zeta_1(v)| = |\zeta_p \cdots \zeta_1(v)| = |u|$ for all $j = 1, \ldots, q$, and such that $|(B, x_1)\omega_q \cdots \omega_1 \tau_k \zeta_p \cdots \zeta_1(v)| = |u|$. Clearly $q \leqslant s - k$.

Put $\omega_j = (G_j, g_j)$ for $j = 1, \ldots, q$. If $x_1^{\pm 1} \notin A_k$, then we see from the choice of k and the chain $\omega_q \cdots \omega_1$ that $g_1^{\pm 1} \notin A_k$. We also see that for the Whitehead automorphisms $\gamma_j = (H_j, g_j)$, $j = 1, \ldots, q$, where $H_j = G_j \setminus A_k$ provided $a_k^{\pm 1} \notin G_j$; $H_j = G_j \cup A_k$ provided $a_k^{\pm 1} \in G_j$, $|(B, x_1)\gamma_q \cdots \gamma_1 \tau_k \zeta_p \cdots \zeta_1(v)| = |\gamma_j \cdots \gamma_1 \tau_k \zeta_p \cdots \zeta_1(v)| = |u|$ for all $j = 1, \ldots, q$. Then by Case 1.1.2 or Case 5 of Lemma 3.1, we have $\gamma_1 \tau_k \equiv \tau_k \gamma_1$, which means the chain $\gamma_q \cdots \gamma_2$ of shorter length has the same property as $\omega_q \cdots \omega_1$ does, contrary to the choice of the chain $\omega_q \cdots \omega_1$. If $x_1^{\pm 1} \in A_k$, replace τ_k by $(\bar{A}_k, a_k^{-1})$. Then we get a contradiction in the same way, which completes the proof of the claim. □

Case 2. $b = x_1^{-1}$.

Repeat similar arguments to those in Case 1.

Case 3. $b \neq x_1^{\pm 1}$.

Let p ($1 \leqslant p \leqslant t$) be such that $\deg \tau_i = 0$ provided $1 \leqslant i < p$; $\deg \tau_i \geqslant 1$ provided $p \leqslant i \leqslant s$. As in Claim 5 of Lemma 3.2, we may assume that $\tau_i \tau_j \equiv \tau_j \tau_i$ for all $1 \leqslant i \neq j < p$. So there exists q with $1 \leqslant q \leqslant p$ such that τ_i has multiplier in C_1 provided $1 \leqslant i < q$; τ_i has multiplier not in C_1 provided $q \leqslant i < p$.

Put $w = \tau_{q-1} \cdots \tau_1(u)$. Notice that C_i-syllables remain unchanged throughout the chain $\tau_{q-1} \cdots \tau_1$ for all $i \geqslant 2$. Write

$$w = y_1 u_1 y_2 u_2 \cdots y_m u_m \quad \text{without cancellation,} \tag{3.9}$$

where for each $i = 1, \ldots, m$, $y_i = x_1$ or $y_i = x_1^{-1}$, and u_i is a (non-cyclic) subword in $\{x_2, \ldots, x_n\}^{\pm 1}$. Let F_{n+3} be the free group on the set

$$\{x_1, \ldots, x_n, x_{n+1}, x_{2n+1}, x_{3n+1}\}.$$

From (3.9) we construct a sequence $V_w = (v_1, v_2, \ldots, v_m)$ of cyclic words $v_1, v_2, \ldots, v_m$ in F_{n+3} with $\sum_{j=1}^m |v_j| = 2|u|$, where m is the total number of occurrences of $x_1^{\pm 1}$ in u, as follows: for each $j = 1, \ldots, m$,

if $y_j = x_1$ and $y_{j+1} = x_1$, then $v_j = x_1 u_j x_{3n+1} u_j^{-1}$;

if $y_j = x_1^{-1}$ and $y_{j+1} = x_1$, then $v_j = x_{n+1} u_j x_{3n+1} u_j^{-1}$;

if $y_j = x_1$ and $y_{j+1} = x_1^{-1}$, then $v_j = x_1 u_j x_{2n+1} u_j^{-1}$;

if $y_j = x_1^{-1}$ and $y_{j+1} = x_1^{-1}$, then $v_j = x_{n+1} u_j x_{2n+1} u_j^{-1}$,

where $y_{m+1} = y_1$.

Put $I = \{x_1, x_{n+1}, x_{2n+1}, x_{3n+1}\}^{\pm 1}$. From now on, when we say that (S, s) is a Whitehead automorphism of F_{n+3}, the following restrictions are imposed on S and s:

(1) $s \in \{x_2, \ldots, x_n\}^{\pm 1}$.
(2) S satisfies one of (i) $I \subseteq S$; (ii) $I \cap S = \{x_1, x_{2n+1}\}^{\pm 1}$; (iii) $I \cap S = \{x_{n+1}, x_{3n+1}\}^{\pm 1}$; (iv) $I \cap S = \emptyset$.

Then we can prove the following claim.

Claim 1. *For each Whitehead automorphism $\tau = (A, a)$ of F_n such that $a \neq x_1^{\pm 1}$ and $|\tau(w)| = |w|$, there exists a Whitehead automorphism α of F_{n+3} such that $\sum_{j=1}^{m} |\alpha(v_j)| = \sum_{j=1}^{m} |v_j|$ and $\alpha(V_w) = V_{\tau(w)}$.*

Proof. Given a Whitehead automorphism $\tau = (A, a)$, we define a Whitehead automorphism α of F_{n+3} as follows: If $x_1^{\pm 1} \in A$, then $\alpha = (A + x_{n+1}^{\pm 1} + x_{2n+1}^{\pm 1} + x_{3n+1}^{\pm 1}, a)$; if only $x_1 \in A$, then $\alpha = (A + x_1^{-1} + x_{2n+1}^{\pm 1}, a)$; if only $x_1^{-1} \in A$, then $\alpha = (A - x_1^{-1} + x_{n+1}^{\pm 1} + x_{3n+1}^{\pm 1}, a)$; if $x_1^{\pm 1} \notin A$, then $\alpha = (A, a)$.

Then each newly introduced letter $x_r^{\pm 1}$ in passing from w to $\tau(w)$ that remains in $\tau(w)$ produces two newly introduced letters $x_r^{\pm 1}$ in passing from V_w to $\alpha(V_w)$ that remain in $\alpha(V_w)$, and vice versa. Also each letter $x_r^{\pm 1}$ in w that is lost in passing from w to $\tau(w)$ produces two letters $x_r^{\pm 1}$ in V_w that are lost in passing from V_w to $\alpha(V_w)$, and vice versa. This yields that $\sum_{j=1}^{m} |\alpha(v_j)| = \sum_{j=1}^{m} |v_j|$.

Moreover it is clear that $\alpha(V_w) = V_{\tau(w)}$. □

The following claim is a converse of Claim 1.

Claim 2. *For each Whitehead automorphism $\alpha = (S, s)$ of F_{n+3} such that $\sum_{j=1}^{m} |\alpha(v_j)| = \sum_{j=1}^{m} |v_j|$, there exists a Whitehead automorphism $\tau = (A, a)$ of F_n such that $a \neq x_1^{\pm 1}$, $|\tau(w)| = |w|$ and such that $\alpha(V_w) = V_{\tau(w)}$.*

Proof. Given a Whitehead automorphism $\alpha = (S, s)$ of F_{n+3}, put $T = S \setminus I$. And define a Whitehead automorphism τ of F_n as follows: $\tau = (T + x_1^{\pm 1}, s)$ provided $I \subseteq S$; $\tau = (T + x_1, s)$ provided $I \cap S = \{x_1, x_{2n+1}\}^{\pm 1}$; $\tau = (T + x_1^{-1}, s)$ provided $I \cap S = \{x_{n+1}, x_{3n+1}\}^{\pm 1}$; $\tau = (T, s)$ provided $I \cap S = \emptyset$. Then reasoning in the same way as in Claim 1, we get a desired result. □

For each $\tau_i = (A_i, a_i)$, $q \leqslant i \leqslant s$, define a Whitehead automorphism α_i of F_{n+3} as in Claim 1. Also as in Claim 1, define a Whitehead automorphism β of F_{n+3} from $\sigma_{\ell+1} = (B, b)$. Then we have $\sum_{j=1}^{m} |\beta\alpha_s \cdots \alpha_q(v_j)| = \sum_{j=1}^{m} |\alpha_i \cdots \alpha_q(v_j)| = \sum_{j=1}^{m} |v_j|$ for all $i = q, \ldots, s$. Moreover, by the construction of α_i and β, the Whitehead automorphisms α_i and β of F_{n+3} are of degree at most $r + 1$, and each of defining sets of α_i and β contains either both of $x_1^{\pm 1}$ or none of $x_1^{\pm 1}$. This yields the same situation as for a chain of Whitehead automorphisms of F_{n+3} of maximum degree r.

Here we notice from Claims 1 and 2 that if Γ_u consists of g connected components, then either Γ_{V_w} consists of $g + 1$ connected components such that C_i equals C_i of Γ_u for all C_i's of Γ_{V_w} with $C_i \neq C_1$ and $C_i \neq C_{n+1}$, C_1 equals C_1 of Γ_u plus $x_{2n+1}^{\pm 1}$, and such that $C_{n+1} = \{x_{n+1}, x_{3n+1}\}^{\pm 1}$;

or Γ_{V_w} consists of g connected components such that C_i equals C_i of Γ_u for all C_i's of Γ_{V_w} with $C_i \neq C_1$ and such that C_1 equals C_1 of Γ_u plus $\{x_{n+1}, x_{2n+1}, x_{3n+1}\}^{\pm 1}$.

The sequence $V_w = (v_1, \ldots, v_m)$ satisfies neither Hypothesis 1.1 nor Hypothesis 1.3. However, this fact does not affect the proof of the base steps of the induction (that is, Lemmas 3.1 and 3.2) for the following four reasons: first each of the Whitehead automorphisms α_i and β has multiplier only in $\{x_2, \ldots, x_n\}^{\pm 1}$; second only the proof of Case 2.1 of Lemma 3.2 is concerned with the C_i-syllable length, but in the proof of Case 2.1 a_r or a_s cannot belong to the connected component C_1 of Γ_{V_w} (in fact, if a_r or a_s belonged to C_1, such a situation as Case 2.1 could not occur); third Claim 5 holds for V_w by replacing $\mathcal{M}$ with the set $\{\phi(V_w)\colon \phi$ is a chain of Whitehead automorphisms of degree 0 throughout which the length of V_w is constant, $|\phi(V_w)|_{C_i} = |V_w|_{C_i}$ for all C_i with $C_i \neq C_1$, and $|\phi(V_w)|_{C_1} \leqslant |\psi(V_w)|_{C_1}$ for every ψ which has the same property as $\phi\}$; finally the same arguments as used in Claims 6 and 7 in Case 2.1 of Lemma 3.2 are valid for V_w, since Hypothesis 1.3 holds for V_w if we only consider C_i's of Γ_{v_w} such that $x_1 \notin C_i$ and $x_{n+1} \notin C_1$.

This observation allows us to apply the induction hypothesis on r to $\beta\alpha_s \cdots \alpha_q(V_w)$. Hence, there exist Whitehead automorphisms $\gamma_1, \gamma_2, \ldots, \gamma_h$ of F_{n+3} such that

$$\beta\alpha_s \cdots \alpha_q(V_w) = \gamma_h \cdots \gamma_2\gamma_1(V_w), \tag{3.10}$$

where $r+1 \geqslant \deg\gamma_h \geqslant \deg\gamma_{h-1} \geqslant \cdots \geqslant \deg\gamma_1$ (here note that there is no γ_i of degree 1), and $\sum_{j=1}^m |\gamma_i \cdots \gamma_1(v_j)| = \sum_{j=1}^m |v_j|$ for all $i = 1, \ldots, h$.

As in Claim 2, from each γ_i we define a Whitehead automorphism ζ_i of F_n. Let k be such that $\deg\zeta_j \leqslant 1$ for $1 \leqslant j < k$ and $\deg\zeta_j \geqslant 2$ for $k \leqslant j \leqslant h$. Since $\beta\alpha_s \cdots \alpha_q(V_w) = V_{\sigma_{\ell+1}\tau_s\cdots\tau_q(w)}$ and $\gamma_h \cdots \gamma_2\gamma_1(V_w) = V_{\zeta_h\cdots\zeta_2\zeta_1(w)}$, we have by (3.10) that

$$\sigma_{\ell+1}\tau_s \cdots \tau_q(w) = \zeta_h \cdots \zeta_2\zeta_1(w),$$

where $r+1 \geqslant \deg\zeta_h \geqslant \deg\zeta_{h-1} \geqslant \cdots \geqslant \deg\zeta_k \geqslant 2$, and $|\zeta_i \cdots \zeta_1(w)| = |w|$ for $i = 1, \ldots, h$. Applying the base step for $r = 1$ (that is, Lemma 3.2) to $\zeta_{k-1} \cdots \zeta_1\tau_{q-1} \cdots \tau_1(u)$ completes the proof of Case 3. □

4. Proof of Theorem 1.5

The aim of this section is to prove Theorem 1.5. For a cyclic word w in F_n, let $N_k(w)$ denote the cardinality of the set $\Omega_k(w) = \{\phi(w)\colon \phi$ can be represented as a composition $\tau_s \cdots \tau_1$ $(s \in \mathbb{N})$ of Whitehead automorphisms τ_i of F_n of degree k such that $|\tau_i \cdots \tau_1(w)| = |w|$ for all $i = 1, \ldots, s\}$. Then bounding $N(u)$ reduces to bounding each $N_k(u)$, which is shown in the proof of Theorem 1.5 using the result of Theorem 1.4. In Lemma 4.1 we bound $N_0(u)$. In Lemma 4.2 we show that $N_k(u)$ for $k \geqslant 1$ is at most $N_0(V_u)$, where V_u is a certain sequence of cyclic words constructed from u, thus bounding $N_k(u)$ for $k \geqslant 1$.

Lemma 4.1. *Let u be a cyclic word in F_n. Then $N_0(u)$ is bounded by a polynomial function of degree $n-2$ with respect to $|u|$.*

Proof. Let m_i be the number of occurrences of $x_i^{\pm 1}$ in u for $i = 1, \ldots, n$. Clearly

$$N_0(u) \leqslant N_0\bigl(x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}\bigr).$$

So it suffices to show that $N_0(x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n})$ is bounded by a polynomial function of degree $n-2$ with respect to $|u|$. For a cyclic word v in F_n, define $|v|_s$ as

$$|v|_s=\sum_{i=1}^{n}|v|_{C_i}.$$

Noting that $|x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}|_s=n$, put $\mathcal{M}=\{v\colon\ |v|_s=n \text{ and } v=\Omega_0(x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n})\}$, and $\mathcal{L}=\{v\colon\ |v|_s>n \text{ and } v=\Omega_0(x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n})\}$. Obviously the cardinality of $\mathcal{M}$ is $(n-1)!$.

For the cardinality of $\mathcal{L}$, let $v\in\mathcal{L}$. Taking an appropriate $u'\in\mathcal{M}$ (note that u' can be chosen as follows: Write $v=x_{k_1}w_1x_{k_2}w_2\cdots x_{k_n}w_n$ (without cancellation), where w_i is a (non-cyclic) word in $\{x_{k_1},\dots,x_{k_i}\}$; then $u'=x_{k_1}^{m_{k_1}}x_{k_2}^{m_{k_2}}\cdots x_{k_n}^{m_{k_n}}$), we have Whitehead automorphisms $\tau_j=(A_j,a_j)$ of F_n of degree 0 such that

$$v=\tau_s\cdots\tau_1(u'), \tag{4.1}$$

where $|\tau_j\cdots\tau_1(u')|=|u'|$ and $|\tau_j\cdots\tau_1(u')|_s\geqslant|\tau_{j-1}\cdots\tau_1(u')|_s$ for all $j=1,\dots,s$. Then for any $\tau_i=(A_i,a_i)$ and $\tau_j=(A_j,a_j)$ with $a_j\neq a_i^{\pm1}$, if we replace τ_i and τ_j by $(\bar{A}_i,a_i^{-1})$ and $(\bar{A}_j,a_j^{-1})$, respectively, if necessary so that $a_i^{\pm1}\notin A_j$ and $a_j^{\pm1}\notin A_i$, then $A_i\cap A_j=\emptyset$. Hence by Case 1.1.2 of Lemma 3.1 that $\tau_j\tau_i\equiv\tau_i\tau_j$; thus (4.1) can be re-written as

$$v=\tau_{pt_p}^{q_{pt_p}}\cdots\tau_{p1}^{q_{p1}}\cdots\tau_{1t_1}^{q_{1t_1}}\cdots\tau_{11}^{q_{11}}(u'), \tag{4.2}$$

where $a_{ki}=a_{ki'}$ and $A_{ki}\neq A_{ki'}$ provided $i\neq i'$; $a_{k'i}\neq a_{ki}^{\pm1}$ and $(\tau_{k't_{k'}}^{q_{k't_{k'}}}\cdots\tau_{k'1}^{q_{k'1}})(\tau_{kt_k}^{q_{kt_k}}\cdots\tau_{k1}^{q_{k1}})\equiv(\tau_{kt_k}^{q_{kt_k}}\cdots\tau_{k1}^{q_{k1}})(\tau_{k't_{k'}}^{q_{k't_{k'}}}\cdots\tau_{k'1}^{q_{k'1}})$ provided $k\neq k'$. Here we may assume by Case 1.2.1 of Lemma 3.1 that $A_{ki}\subset A_{ki'}$ if $i<i'$. Then $\tau_{ki'}\tau_{ki}\equiv\tau_{ki}\tau_{ki'}$ by Case 1.2.1 of Lemma 3.1; hence $\tau_{k'i'}\tau_{ki}\equiv\tau_{ki}\tau_{k'i'}$ for any τ_{ki} and $\tau_{k'i'}$ in chain (4.2).

Claim. *The length of the chain of Whitehead automorphisms on the right-hand side of* (4.2) *is at most $n-2$ without counting multiplicity, that is, $\sum_{i=1}^{p}t_i\leqslant n-2$.*

Proof. The proof proceeds by induction on the number of subwords of u' of the form $x_i^{m_i}$ which are fixed throughout chain (4.2). For the base step, suppose that u' has two such subwords $x_{r_1}^{m_{r_1}}$ and $x_{r_2}^{m_{r_2}}$ (note that u' must have at least two such subwords). The cyclic word u' can be written as $u'=x_{r_1}^{m_{r_1}}w$ (without cancellation), where w is a non-cyclic word that contains $x_i^{m_i}$ for all $i\neq r_1$. Upon replacing τ_{ij} by $(\bar{A}_{ij},a_{ij}^{-1})$ if necessary, we may assume that $x_{r_1}^{\pm1}\notin A_{ij}$ for all τ_{ij} in chain (4.2). Then the length of w is constant throughout the chain and only the subword $x_{r_2}^{m_{r_2}}$ of w is fixed in passing from w to $\tau_{pt_p}^{q_{pt_p}}\cdots\tau_{p1}^{q_{p1}}\cdots\tau_{1t_1}^{q_{1t_1}}\cdots\tau_{11}^{q_{11}}(w)$. It follows that the length of this chain is precisely $(n-1)-1=n-2$ without counting multiplicity. So the base step is done.

Now for the inductive step, suppose that u' has k subwords of the form $x_i^{m_i}$ which are fixed throughout chain (4.2), say $x_{r_1}^{m_{r_1}},\dots,x_{r_k}^{m_{r_k}}$. Write the cyclic word u' as $u'=x_{r_1}^{m_{r_1}}w$ (without cancellation), where w is a non-cyclic word that contains $x_i^{m_i}$ for all $i\neq r_1$. As above, upon replacing τ_{ij} by $(\bar{A}_{ij},a_{ij}^{-1})$ if necessary, we may assume that $x_{r_1}^{\pm1}\notin A_{ij}$ for all τ_{ij} in chain

(4.2). We then have that only the subwords $x_{r_2}^{m_{r_2}}, \ldots, x_{r_k}^{m_{r_k}}$ of w are fixed in passing from w to $\tau_{pt_p}^{q_{pt_p}} \cdots \tau_{p1}^{q_{p1}} \cdots \tau_{1t_1}^{q_{1t_1}} \cdots \tau_{11}^{q_{11}}(w)$, where the length of w is constant throughout the chain.

Let (w) be the cyclic word associated with w. If none of τ_{ij} in chain (4.2) is of the form either $(\Sigma - x_{r_1}^{\pm 1} - x_g^{\pm 1}, x_g)$ or $(\Sigma - x_{r_1}^{\pm 1} - x_g^{\pm 1}, x_g^{-1})$, then chain (4.2) can be applied to (w) with $\tau_{ij} \neq 1$ on (w) for every τ_{ij} in the chain. Then by the induction hypothesis applied to (w), the length of the chain is at most $(n-1)-2 = n-3$ without counting multiplicity, as desired. If one of τ_{ij} in chain (4.2) is of the form either $(\Sigma - x_{r_1}^{\pm 1} - x_g^{\pm 1}, x_g)$ or $(\Sigma - x_{r_1}^{\pm 1} - x_g^{\pm 1}, x_g^{-1})$, then we see that there can be only one of τ_{ij} of such a form, so that chain (4.2) can be applied to (w) with only one $\tau_{ij} = 1$ on (w). This together with the induction hypothesis applied to (w) yields that the length of chain (4.2) is at most $(n-1)-2+1 = n-2$ without counting multiplicity, as required. □

Obviously each multiplicity q_{ij} is less than the number of $a_{ij}^{\pm 1}$ occurring in u, so less than $|u|$. This together with the claim yields that the total number of chains of Whitehead automorphisms with the same properties as in (4.2) is less than $\binom{r}{n-2}|u|^{n-2}$, where r is the number of Whitehead automorphisms of F_n of degree 0. Thus the cardinality of $\mathcal{L}$ is less than $(n-1)!\binom{r}{n-2}|u|^{n-2}$, and therefore

$$N_0\left(x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}\right) = \#\mathcal{M} + \#\mathcal{L} \leqslant (n-1)! + (n-1)!\binom{r}{n-2}|u|^{n-2},$$

which completes the proof the lemma. □

Remark. The proof of Lemma 4.1 can be applied without further change if we replace consideration of a single cyclic word u, the length $|u|$ of u, and the total number of occurrences of $x_j^{\pm 1}$ in u with consideration of a finite sequence $(u_1, \ldots, u_m)$ of cyclic words, the sum $\sum_{i=1}^m |u_i|$ of the lengths of $u_1, \ldots, u_m$, and the total number of occurrences of $x_j^{\pm 1}$ in $(u_1, \ldots, u_m)$, respectively.

Lemma 4.2. *Let u be a cyclic word in F_n that satisfies Hypothesis* 1.1. *Then for each $k = 1, \ldots, n-1$, $N_k(u)$ is bounded by a polynomial function of degree $n + 3k - 2$ with respect to $|u|$* (*note that k is at most $n-1$ by the remark after Lemma* 2.2).

Proof. Let m_i be the number of occurrences of $x_i^{\pm 1}$ in u for $i = 1, \ldots, n$, and let $\ell_k = \sum_{j=1}^k m_j$ for $k = 1, \ldots, n-1$. Write

$$u = y_1 u_1 y_2 u_2 \cdots y_{\ell_k} u_{\ell_k} \quad \text{without cancellation,} \tag{4.3}$$

where for each $i = 1, \ldots, \ell_k$, $y_i = x_j$ or $y_i = x_j^{-1}$ for some $1 \leqslant j \leqslant k$, and u_i is a (non-cyclic) subword in $\{x_{k+1}, \ldots, x_n\}^{\pm 1}$. Let F_{n+3k} be the free group on the set

$$\{x_1, \ldots, x_n, x_{n+1}, \ldots x_{n+k}, x_{2n+1}, \ldots, x_{2n+k}, x_{3n+1}, \ldots, x_{3n+k}\}.$$

From (4.3) we construct a sequence $V_u = (v_1, \ldots, v_{\ell_k})$ of cyclic words $v_1, \ldots, v_{\ell_k}$ in F_{n+3k} with $\sum_{i=1}^{\ell_k} |v_i| = 2|u|$ as follows: for each $i = 1, \ldots, \ell_k$, if $y_i = x_j$ and $y_{i+1} = x_{j'}$, then $v_i = x_j u_i x_{3n+j'} u_i^{-1}$;

if $y_i = x_j^{-1}$ and $y_{i+1} = x_{j'}$, then $v_i = x_{n+j} u_i x_{3n+j'} u_i^{-1}$;

if $y_i = x_j$ and $y_{i+1} = x_{j'}^{-1}$, then $v_i = x_j u_i x_{2n+j'} u_i^{-1}$;

if $y_i = x_j^{-1}$ and $y_{i+1} = x_{j'}^{-1}$, then $v_i = x_{n+j} u_i x_{2n+j'} u_i^{-1}$,

where $y_{\ell_k+1} = y_1$.

Claim. *For each Whitehead automorphism σ of F_n of degree k such that $|\sigma(u)| = |u|$, there exists a Whitehead automorphism τ of F_{n+3k} of degree 0 such that $\sum_{i=1}^{\ell_k} |\tau(v_i)| = \sum_{i=1}^{\ell_k} |v_i|$ and $\tau(V_u) = V_{\sigma(u)}$.*

Proof. Let $\sigma = (S, a)$ be a Whitehead automorphism of F_n of degree k such that $|\sigma(u)| = |u|$. Upon replacing σ by $(\bar{S}, a^{-1})$, we may assume that $\sigma = (S, x_r)$. Note by Lemma 2.2 that the index r is bigger than k, since $\deg \sigma = k$. Put $S = T + P + Q$, where $T = S \cap \{x_{k+1}, \dots, x_n\}^{\pm 1}$, $P = S \cap \{x_1, \dots, x_k\}$ and $Q = S \cap \{x_1, \dots, x_k\}^{-1}$ (here note that $T = T^{-1}$, since $\deg \sigma = k$).

Then we consider the Whitehead automorphism $\tau = (T + P_1 + Q_1, x_r)$ of F_{n+3k} of degree 0, where $P_1 = \{x_i^{\pm 1}, x_{2n+i}^{\pm 1} \mid x_i \in P\}$ and $Q_1 = \{x_{n+i}^{\pm 1}, x_{3n+i}^{\pm 1} \mid x_i^{-1} \in Q\}$. If the sequence $V_u = (v_1, \dots, v_{\ell_k})$ of cyclic words $v_1, \dots, v_{\ell_k}$ in F_{n+3k} is constructed as above, then each newly introduced letter $x_r^{\pm 1}$ in passing from u to $\sigma(u)$ that remains in $\sigma(u)$ produces two newly introduced letters $x_r^{\pm 1}$ in passing from V_u to $\tau(V_u)$ that remain in $\tau(V_u)$, and vice versa. Also each letter $x_r^{\pm 1}$ in u that is lost in passing from u to $\sigma(u)$ produces two letters $x_r^{\pm 1}$ in V_u that are lost in passing from V_u to $\tau(V_u)$, and vice versa. This yields that $\sum_{i=1}^{\ell_k} |\tau(v_i)| = \sum_{i=1}^{\ell_k} |v_i|$. Moreover it is clear that $\tau(V_u) = V_{\sigma(u)}$. □

It is easy to see that if $u' \in \Omega_k(u)$ with $u' \neq u$, then $V_{u'} \neq V_u$. This together with the claim gives us that $N_k(u) \leqslant N_0((v_1, v_2, \dots, v_{\ell_k}))$. By the remark after Lemma 4.1, $N_0((v_1, v_2, \dots, v_{\ell_k}))$ is bounded by a polynomial function of degree $n + 3k - 2$ with respect to $2|u|$, which completes the proof of the lemma. □

Finally we give a proof of Theorem 1.5.

Proof of Theorem 1.5. Without loss of generality we may assume that u was chosen from the set $\{v \in \mathrm{Orb}_{\mathrm{Aut}\, F_n}(u)\colon |v| = |u|\}$ so that u satisfies Hypothesis 1.3. Let $v \in \mathrm{Orb}_{\mathrm{Aut}\, F_n}(u)$ be such that $|v| = |u|$. By Whitehead's theorem, there exist Whitehead automorphisms π of the first type and $\sigma_1, \dots, \sigma_\ell$ of the second type such that $v = \pi \sigma_\ell \cdots \sigma_1(u)$, where $|\sigma_i \cdots \sigma_1(u)| = |u|$ for all $i = 1, \dots, \ell$. Then by Theorem 1.4, there exist Whitehead automorphisms $\tau_1, \dots, \tau_s$ such that $v = \pi \tau_s \cdots \tau_1(u)$, where $n - 1 \geqslant \deg \tau_s \geqslant \deg \tau_{s-1} \geqslant \cdots \geqslant \deg \tau_1$, and $|\tau_j \cdots \tau_1(u)| = |u|$ for all $j = 1, \dots, s$ (here, note by the Remark after Lemma 2.2 that $\deg \tau_s \leqslant n - 1$). This implies that

$$N(u) \leqslant C N_0(u) N_1(u) \cdots N_{n-1}(u),$$

where C is the number of Whitehead automorphisms of the first type of F_n (which depends only on n). For each $k = 0, 1, \dots, n - 1$, $N_k(u)$ is bounded by a polynomial function of degree $n + 3k - 2$ with respect to $|u|$ by Lemmas 4.1 and 4.2. Therefore, $N(u)$ is bounded by a polynomial function of degree $n(5n - 7)/2$ with respect to $|u|$, as required. □

5. Limitations

We close this paper with a brief explanation why the presented technique is incapable of covering the entire problem domain (e.g. for $u = x_1^2 x_2^2 x_3^3 x_4^4$ the presented arguments cannot be applied). This amounts to explaining why condition (ii) of Hypothesis 1.1 cannot be dropped. As a matter of fact, in the presented arguments, condition (ii) of Hypothesis 1.1 played a most essential role, without which all of our arguments except Lemmas 2.1 and 4.1 would have broke down. Owing to Lemma 2.2 where we first used Hypothesis 1.1(ii), we were able to assume throughout the paper that

$$j > i \text{ when considering Whitehead automorphisms } \left(A, x_j^{\pm 1}\right) \text{ of degree } i. \tag{5.1}$$

This allowed us to exclude the worst case such as $a \in B$, $a^{-1} \notin B$, $b \in A$ and $b^{-1} \notin A$ in Lemma 3.1, for which case there does not exist a composition of Whitehead automorphisms of ascending degrees that equals $(B, b)(A, a)$. Also we proceeded with the proofs of Lemma 3.2 and Theorem 1.4 based on (5.1). For instance, Claim 1 in the proof of Lemma 3.2 yielded the existence r such that $a_r^{\pm 1} \in A_s \cap B$ in Case 1.1, where we did not have to worry about the case where $a_r \in A_s \cap B$ but $a_r^{-1} \notin A_s \cap B$. Furthermore, the equality in the claim in the proof of Lemma 4.2 would not have hold without (5.1).

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