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Submodule categories of wild representation type

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Abstract

Let Λ be a commutative local uniserial ring with radical factor field k. We consider the category $\mathcal{S}(\Lambda)$ of embeddings of all possible submodules of finitely generated Λ-modules. In case $\Lambda=\mathbb{Z}/\langle p^n\rangle$, where p is a prime, the problem of classifying the objects in $\mathcal{S}(\Lambda)$, up to isomorphism, has been posed by Garrett Birkhoff in 1934. In this paper we assume that Λ has Loewy length at least seven. We show that $\mathcal{S}(\Lambda)$ is controlled k-wild with a single control object $I \in \mathcal{S}(\Lambda)$. It follows that each finite dimensional k-algebra can be realized as a quotient $\mathrm{End}(X)/\mathrm{End}(X)_I$ of the endomorphism ring of some object $X \in \mathcal{S}(\Lambda)$ modulo the ideal $\mathrm{End}(X)_I$ of all maps which factor through a finite direct sum of copies of I.

MSC: Primary: 16G60; secondary: 20K27, 47A15

1. Introduction

Let Λ be a ring. An object in the submodule category $\mathcal{S}(\Lambda)$ is a pair $M = (M_0, M_1)$, sometimes written as $(M_1 \subseteq M_0)$, which consists of a finitely generated Λ-module M_0 together with a Λ-submodule M_1 of M_0. A morphism $f: M \to N$ in $\mathcal{S}(\Lambda)$ is given by a Λ-linear map $f: M_0 \to N_0$ which preserves the submodules, that is, $f(M_1) \subseteq N_1$ holds. In this paper, Λ always will be a commutative local uniserial ring of finite (Loewy-) length

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n. Usually, we will assume that $n \geqslant 7$. The radical factor field will be denoted by k and t will be a radical generator (thus $\Lambda/\langle t\rangle = k$).

We have the following two special cases in mind: in the first case, Λ is the ring $\mathbb{Z}/\langle p^n\rangle$, where p is a prime. Then the objects in $\mathscr{S}(\Lambda)$ are the embeddings of a subgroup in a p^n-bounded finite abelian group. The problem of classifying those embeddings, up to isomorphism, was raised by Birkhoff [2] in 1934. In the second case, Λ is the factor ring $k[T]/\langle T^n\rangle$ of the polynomial ring in one variable T over the field k. Then we consider all invariant subspaces of a nilpotent operator: the objects in $\mathscr{S}(k[T]/\langle T^n\rangle)$ may be written as triples (V, ϕ, U), where V is a k-space, $\phi: V \to V$ is a k-linear transformation with $\phi^n = 0$ and U is a subspace of V with $\phi(U) \subseteq U$.

Some remarks concerning notions of "wildness" of additive categories will be given in the last sections. According to Arnold [1], the category $\mathscr{S}(\Lambda)$ is "wild" if Λ is the ring $\mathbb{Z}/\langle p^n\rangle$ and $n=10$. In the case $\Lambda = k[T]/\langle T^n\rangle$ Simson has shown in [5] that $\mathscr{S}(k[T]/\langle T^7\rangle)$ is "wild" whereas $\mathscr{S}(k[T]/\langle T^6\rangle)$ is still tame, thus providing the precise bound for "wildness". It is not surprising that the special case $\Lambda = k[T]/\langle T^n\rangle$ is better understood, since in this case many powerful techniques are available (in particular covering theory). The main result presented here will not depend on Λ being an algebra over a field. In particular, it applies to the classical case of subgroups of finite abelian groups, as considered by Birkhoff, and it can be used in order to construct parametrized families of metabelian groups [4]. In case Λ is an algebra over a field, the last section shows in which way the main result can be strengthened.

2. Controlled wildness

Let $\mathscr{A}$ be an additive category and $\mathscr{C}$ a class of objects (or a full subcategory) in $\mathscr{A}$. Given objects A, A' in $\mathscr{A}$, we will write $\mathrm{Hom}(A, A')_{\mathscr{C}}$ for the set of maps $A \to A'$ which factor through a (finite) direct sum of objects in $\mathscr{C}$. Here we attach to $\mathscr{C}$ the ideal $\langle\mathscr{C}\rangle$ in $\mathscr{A}$ generated by the identity morphisms of the objects in $\mathscr{C}$. The same convention will apply to a single object C in $\mathscr{A}$: we denote by $\mathrm{Hom}(A, A')_C$ the set of maps $A \to A'$ which factor through a (finite) direct sum of copies of C. Given an ideal $\mathscr{I}$ of $\mathscr{A}$, we write $\mathscr{A}/\mathscr{I}$ for the corresponding factor category, as usual. It has the same objects as $\mathscr{A}$ and for any two objects A, A' of $\mathscr{A}$, the group $\mathrm{Hom}_{\mathscr{A}/\mathscr{I}}(A, A')$ is defined as $\mathrm{Hom}_{\mathscr{A}}(A, A')/\mathscr{I}(A, A')$. In particular, the category $\mathscr{A}/\langle\mathscr{C}\rangle$ has the same objects as $\mathscr{A}$ and

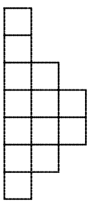
$$\mathrm{Hom}_{\mathscr{A}/\langle\mathscr{C}\rangle}(A, A') = \mathrm{Hom}_{\mathscr{A}}(A, A')/\mathrm{Hom}(A, A')_{\mathscr{C}}.$$

Definition. We say that $\mathscr{A}$ is *controlled k-wild* provided there are full subcategories $\mathscr{C} \subseteq \mathscr{B} \subseteq \mathscr{A}$ such that $\mathscr{B}/\langle\mathscr{C}\rangle$ is equivalent to $\mathrm{mod}\, k\langle X, Y\rangle$ where $k\langle X, Y\rangle$ is the free k-algebra in two generators. We will call $\mathscr{C}$ the *control class*, and in case $\mathscr{C}$ is given by a single object C then this object C will be the *control object*. We refer to [3] for a discussion of controlled wildness.

3. The setting

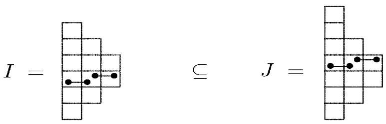
We are going to show that the category $\mathscr{S}(\Lambda)$ is controlled k-wild. In order to do so, we need to find suitable full subcategories $\mathscr{C} \subseteq \mathscr{B} \subseteq \mathscr{S}(\Lambda)$. In fact, $\mathscr{C}$ will consist of a single object I, whereas $\mathcal{B}$ will be a suitable subcategory of the “interval” in-between the object I and a related one J with $I \subset J$. Given two objects $I \subset J$ in $\mathcal{S}(\Lambda)$, we denote by the *interval* $[I, J]$ the class of all objects M of $\mathcal{S}(\Lambda)$ such that $I^m \subseteq M \subseteq J^m$ for some natural number m.

In order to exhibit objects in $\mathcal{S}(\Lambda)$, it is convenient to use some graphical description. It is well-known and easy to see that the indecomposable Λ-modules are up to isomorphism of the form $\Lambda/\langle t^i\rangle$ with $1 \leqslant i \leqslant n$, thus the indecomposable Λ-modules are characterized by the length ($\Lambda/\langle t^i\rangle$ has length i). The Krull–Remak–Schmidt theorem asserts that the isomorphism classes of the Λ-modules (of finite length) correspond bijectively to the partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ with all parts $\lambda_i \leqslant n$; the Λ-module corresponding to the partition $(\lambda_1, \ldots, \lambda_m)$ is just $\bigoplus_i \Lambda/\langle t^{\lambda_i}\rangle$, or, equivalently, the Λ-module with generators $x_1, \ldots, x_m$ and defining relations $t^{\lambda_i} x_i = 0$, for $1 \leqslant i \leqslant m$. We will attach to a partition its Young diagram using an arrangement of boxes, however we will deviate from the usual convention as follows: the various parts will be drawn vertically and not horizontally, and the parts will not necessarily be adjusted at the top or the socle. For example, we will consider below the partition $(7, 4, 2)$, and it will be suitable to draw the corresponding Young diagram as follows:



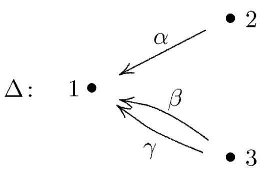
The left column has 7 boxes, the middle one 4 and the right column 2 boxes, as the partition $(7, 4, 2)$ asserts. The adjustment of these columns made here depends on the fact that we have in mind a particular submodule, and we want that there is a generating system for the submodule such that any of these generators is a linear combination of elements which belong to boxes at the same height.

Here are the objects I and J: The Λ-module J_0 is given by the partition $(7, 4, 2)$, say with generators x, y, z, annihilated by t^7, t^4, t^2 respectively, and I_0 is generated by tx, y, z, thus it corresponds to the partition $(6, 4, 2)$. The submodule J_1 is generated by $t^3x - ty$ and $ty - z$, and $I_1 = tJ_1$.



In these pictures, we have indicated the generators of the submodules using pairs of bullets which are connected by a horizontal line (and the shift of the columns was accomplished in such a way that the connecting lines become horizontal lines).

We are going to describe the interval $[I, J]$ in-between the objects I and J in terms of representations of a quiver Δ. The quiver Δ looks as follows:



It has three vertices: one sink (labelled 1) and two sources (labelled 2 and 3), thus there are three simple representations $S(1)$, $S(2)$, $S(3)$. The simple representation $S(1)$ is projective, the simple representations $S(2)$ and $S(3)$ are injective. Let us denote by $\mathrm{mod}_{\mathrm{e}}\, k\Delta$ the full subcategory of $\mathrm{mod}\, k\Delta$ given by all representations without a simple direct summand. Note that the representations of Δ without a simple injective direct summand are precisely the socle-projective representations (of course, a representation is said to be *socle-projective* provided the socle is projective). We denote by $\mathrm{mod}_{\mathrm{sp}}\, k\Delta$ the full subcategory of all socle-projective representations. The inclusion functors

$$\mathrm{mod}_{\mathrm{e}}\, k\Delta \subset \mathrm{mod}_{\mathrm{sp}}\, k\Delta \subset \mathrm{mod}\, k\Delta$$

allow us to identify the categories

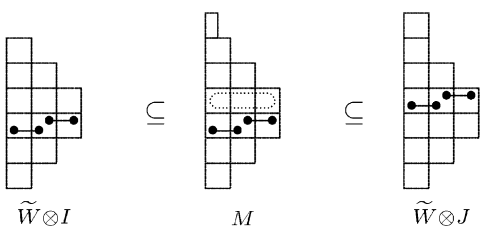
$$\mathrm{mod}_{\mathrm{e}}\, k\Delta = \mathrm{mod}_{\mathrm{sp}}\, k\Delta/\langle S(1)\rangle = \mathrm{mod}\, k\Delta/\langle S(1), S(2), S(3)\rangle,$$

since all the simple representations of Δ are projective or injective. Note that $\mathrm{mod}\, k\Delta/\langle S(1), S(2), S(3)\rangle$ is the factor category of $\mathrm{mod}\, k\Delta$ modulo the ideal of maps which factor through semisimple objects.

The key to proving the controlled wildness of $\mathscr{S}(\Lambda)$ is the following result.

Theorem 1. *Let Λ be a commutative local uniserial ring of length $n \geqslant 7$ and let k be its radical factor field. Then, the factor category $[I, J]/\langle I\rangle$ is equivalent to the category* $\mathrm{mod}_{\mathrm{e}}\, k\Delta$.

The definition of $[I, J]$ may be rephrased as follows: in order to form the direct sums I^m, J^m of copies of I and J, respectively, let $\widetilde{W}$ be a free Λ-module of rank m so that we may identify the inclusion $I^m \subset J^m$ with the map $\widetilde{W} \otimes I \to \widetilde{W} \otimes J$ induced by the inclusion $I \subset J$. Then each object M in $[I, J]$ can be visualized as the middle term in a sequence of inclusions of the following type:



Here, the dotted region represents the quotient $M_1/(\widetilde{W} \otimes_\Lambda I_1)$; and the half box on the top corresponds to the quotient $M_0/(\widetilde{W} \otimes_\Lambda I_0)$. Below we will use these two subspaces of $(\widetilde{W} \otimes J_1)/(\widetilde{W} \otimes I_1) \cong k^{2n}$ and of $(\widetilde{W} \otimes J_0)/(\widetilde{W} \otimes I_0) \cong k^n$, respectively, to reconstruct M. (Note that now the boxes no longer correspond to individual composition factors of the Λ-module M_0, but to suitable semisimple subfactors.)

4. The layer functors

Let us analyse the objects M in $[I, J]$. There are the *layer functors*

$$L_i \colon [I, J] \to \operatorname{mod} \Lambda$$

defined by

$$L_1 M = \operatorname{rad}^4 M_0 \cap \operatorname{soc} M_0,$$

$$L_2 M = \operatorname{rad}^3 M_0 \cap \operatorname{soc}^2 M_0,$$

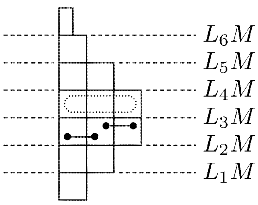
$$L_3 M = \operatorname{soc} M_0 + (\operatorname{rad}^2 M_0 \cap \operatorname{soc}^3 M_0),$$

$$L_4 M = \operatorname{soc}^2 M_0 + (\operatorname{rad} M_0 \cap \operatorname{soc}^4 M_0),$$

$$L_5 M = \operatorname{soc}^5 M_0,$$

$$L_6 M = \operatorname{soc}^6 M_0 \;=\; \widetilde{W} \otimes I_0,$$

where rad defines the radical and soc the socle. Note that the $L_i M$ are Λ-submodules of M_0. They form a filtration of M_0 and can be visualized as follows:



This definition of the submodules $L_i M$ only depends on M_0, it does not take M_1 into account.

Of special interest is the following observation:

Lemma. *The subobject* $(M_1 \cap L_3 M \subseteq L_6 M)$ *of* M *is a direct sum of copies of* I, *and any homomorphism from* I *to* M *maps into* $(M_1 \cap L_3 M \subseteq L_6 M)$.

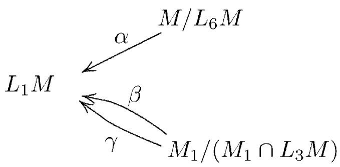
We call this subobject $(M_1 \cap L_3 M \subseteq L_6 M)$ the I*-socle* of M.

Proof. Let $\widetilde{W}$ be a free Λ-module such that the inclusions $\widetilde{W} \otimes I \subseteq M \subseteq \widetilde{W} \otimes J$ hold. The inclusion $\widetilde{W} \otimes I \subseteq M$ embeds $\widetilde{W} \otimes I$ into $(M_1 \cap L_3 M \subseteq L_6 M)$ and clearly

$\widetilde{W} \otimes I_0 = L_6 M$. But we also have $\widetilde{W} \otimes I_1 = \widetilde{W} \otimes tJ_1 = M_1 \cap L_3 M$. This shows that $\widetilde{W} \otimes I = (M_1 \cap L_3 M \subseteq L_6 M)$, thus $(M_1 \cap L_3 M \subseteq L_6 M)$ is a direct sum of copies of I. Given a map $I \to M$, it will send $I_0 = L_6 I$ into $L_6 M$ and $I_1 = I_1 \cap L_3 I$ into $M_1 \cap L_3 M$, thus it maps into $(M_1 \cap L_3 M \subseteq L_6 M)$. $\square$

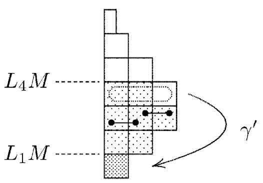
5. From $\mathscr{S}(\Lambda)$ to representations of Δ

Given M in $[I, J]$, let $F(M)$ be defined by



with $\alpha = t^6$, $\beta = t^3$, and a k-linear map γ which still has to be specified.

Actually, let us define a surjective homomorphism $\gamma' \colon L_4 M \to L_1 M$ with kernel $L_3 M + tL_5 M$, the required map γ will be induced by the restriction of γ' to M_1 (note that $M_1 \subseteq L_4 M$). The map γ' will yield an isomorphism between the quotient of $L_4 M$ modulo the dotted region and the shaded box:



Here is the definition of $\gamma'(c)$ for $c \in L_4 M$ in a condensed form:

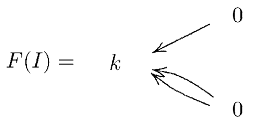
$$\gamma'(c) \quad = \quad t^2 \Big(\Big(\Big(\Big(tc + t^2 L_5 M \Big) \cap t^{-1} 0 \Big) + (M_1 \cap L_3 M) \Big) \cap t^3 L_6 M \Big)$$

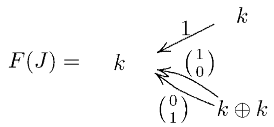
(note that γ' depends only on L_6 and $M_1 \cap L_3 M$, thus on the I-socle of M).

In order to understand the definition and to see that γ' is really a Λ-homomorphism, we proceed stepwise: thus, we start with $c \in L_4 M$. Take an element $c' \in (tc + t^2 L_5 M) \cap t^{-1} 0$, such an element exists since $tL_4 M = t^2 L_5 M + t^{-1} 0$. Next, take an element $c'' \in (c' + (M_1 \cap L_3 M)) \cap t^3 L_6 M$—again, we note that such an element exists, now we use that c' belongs to $L_3 M$ and that $L_3 M = (M_1 \cap L_3 M) + t^3 L_6 M$. The proposed definition of $\gamma'(c)$ amounts to $\gamma'(c) = t^2 c''$. Now c' is unique up the addition of elements from $t^2 L_5 M \cap t^{-1} 0 = t^{-1} 0 \cap L_2 M$, thus c'' is unique up to the addition of elements from $(M_1 \cap L_3 M) \cap t^3 L_6 M = L_2 M$, and the latter elements go to zero under the multiplication by t^2. This shows that $\gamma'(c)$ is a well-defined element and since c'' belongs to $L_3 M$, we see that $\gamma'(c)$ belongs to $L_1 M$.Of course, it is clear that such a construction yields a homomorphism γ'. One finally verifies that γ' is surjective and that its kernel is $L_3 M + tL_5 M$.

It also follows from the construction that a homomorphism $g: M \to N$ in $\mathscr{S}(\Lambda)$ between objects $M, N \in [I, J]$ commutes with γ'. Hence we obtain a functor $F: [I, J] \to \operatorname{mod} k\Delta$.

Example. Under this functor F, the object I is sent to $F(I) = S(1)$, whereas $F(J)$ is the injective envelope of $S(1)$:





Proposition 1. *The functor F is a full and dense functor from $[I, J]$ onto the category* $\operatorname{mod}_{sp} k\Delta$ *of socle-projective representations of Δ. The representation $F(I) = S(1)$ is simple projective, and the kernel of the induced functor $[I, J] \to \operatorname{mod}_{sp} k\Delta/\langle S(1)\rangle$ is just the ideal of all maps which factor through a direct sum of copies of I.*

The proof of Proposition 1 will be given at the end of the next section; Theorem 1 is an immediate consequence of Proposition 1.

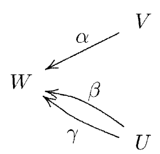
6. … and back to $\mathscr{S}(\Lambda)$

In order to show that the functor F is full and dense, we are going to present an inverse construction which we label Φ. We work in the homomorphism category $\mathscr{H}(\Lambda)$ for Λ. The objects in $\mathscr{H}(\Lambda)$ are the Λ-linear maps, say $A = (A_1 \xrightarrow{a} A_0)$, and a morphism between two such objects $A = (A_1 \xrightarrow{a} A_0)$ and $B = (B_1 \xrightarrow{b} B_0)$ consists of two homomorphisms $f_0: A_0 \to B_0$ and $f_1: A_1 \to B_1$ such that $f_0 a = b f_1$ holds. Clearly, $\mathscr{S}(\Lambda)$ is just the full exact subcategory of $\mathscr{H}(\Lambda)$ of those objects $A = (A_1 \xrightarrow{a} A_0)$ for which the map a is monic.

Note that the inclusion $I \to J$ gives rise to the short exact sequence in $\mathscr{H}(\Lambda)$

$$\varepsilon: \quad 0 \longrightarrow I \longrightarrow J \longrightarrow (k \oplus k \xrightarrow{0} k) \longrightarrow 0.$$

Let W be a vector space, V a subspace of W, U a subspace of $W \oplus W$, then we may consider the tripel (W, V, U) as a representation of the quiver Δ as follows:



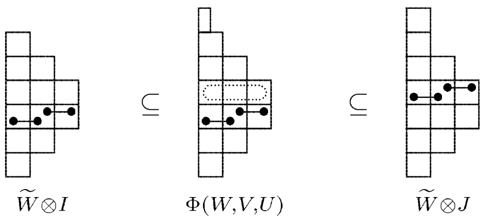
with α the inclusion map, and β the first, γ the second projection of U into W (thus $\beta(w_1, w_2) = w_1, \gamma(w_1, w_2) = w_2$, where $w_1, w_2 \in W$ and $(w_1, w_2) \in U$). Note that in this way, we obtain precisely all the representations of Δ which do not have a simple injective direct summand.

Let $\widetilde{W}$ be a free Λ-module with $\widetilde{W}/\mathrm{rad}\,\widetilde{W} = W$. In the category $\mathcal{H}(\Lambda)$, we consider the following fiber product construction of $\widetilde{W}\otimes_\Lambda \varepsilon$ along the inclusion $(U \xrightarrow{0} V) \longrightarrow (W \oplus W \xrightarrow{0} W)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{W}\otimes_\Lambda I & \longrightarrow & \Phi(W,V,U) & \longrightarrow & (U \xrightarrow{0} V) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{W}\otimes_\Lambda I & \longrightarrow & \widetilde{W}\otimes_\Lambda J & \longrightarrow & (W \oplus W \xrightarrow{0} W) & \longrightarrow & 0
\end{array}$$

In this way, we define the object $\Phi(W, V, U)$. Note that by the five lemma, the vertical map in the center of the above diagram is monic, so $\Phi(W, V, U)$, being a subobject of an object in $\mathcal{S}(\Lambda)$, also lies in $\mathcal{S}(\Lambda)$ and clearly $F\Phi(W, V, U)$ is the subobject (W, V, U) of $F(\widetilde{W} \otimes J) = (W, W, W \oplus W)$.

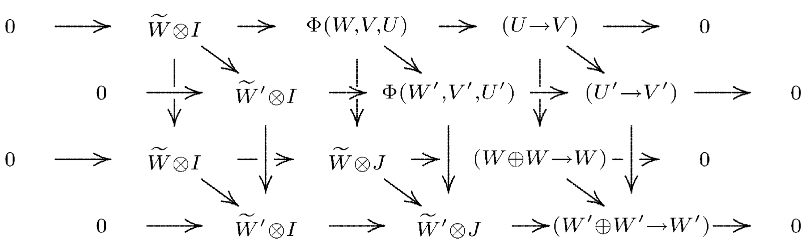
Let us look again at the visualization of the objects in $[I, J]$ considered above:



For such an object $\Phi(W, V, U)$, the dotted region represents U which is the quotient of $\Phi(W, V, U)_1$ modulo $\widetilde{W}\otimes_\Lambda I_1$; and the half box on the top corresponds to the subspace V of W, which is the quotient of $\Phi(W, V, U)_0$ modulo $\widetilde{W}\otimes_\Lambda I_0$.

Suppose (W, V, U) and (W', V', U') are two such triples (thus representations of the quiver Δ without simple injective direct summands). Let $\widetilde{W}$ and $\widetilde{W}'$ be free Λ-modules with $\widetilde{W}/\mathrm{rad}\,\widetilde{W} = W$ and $\widetilde{W}'/\mathrm{rad}\,\widetilde{W}' = W'$, respectively. A morphism $(W, V, U) \to (W', V', U')$ in the category $\mathrm{mod}\,\Delta$ is given by a map $g\colon W \to W'$ such that $g(V) \subseteq V'$ and $(g \oplus g)(U) \subseteq U'$.

Such a map g gives rise to a morphism $\Phi(g)\colon \Phi(W, V, U) \to \Phi(W', V', U')$ in the category $\mathcal{S}(\Lambda)$ which makes the following diagram commutative:



Indeed, one uses the projectivity of $\widetilde{W}$ as a Λ-module in order to obtain a lifting $\widetilde{g}$ of g which makes the diagram

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\tilde{g}} & \widetilde{W}' \\ {\scriptstyle \text{can}}\big\downarrow & & \big\downarrow{\scriptstyle \text{can}} \\ W & \xrightarrow[g]{} & W' \end{array}$$

commutative, then the bottom part and the right hand square in the three dimensional diagram commute. Now a fiber product construction in the category of morphisms in $\mathscr{H}(\Lambda)$ gives the forward pointing map in the middle of the top part of the diagram. This is the map $\Phi(g)$ we are looking for. Since the vertical maps in the middle are both monic, $\Phi(g)\colon \Phi(W, V, U) \to \Phi(W', V', U')$ is just the restriction of the map $\widetilde{W} \otimes_\Lambda J \overset{\widetilde{g}\otimes 1}{\to} \widetilde{W}' \otimes_\Lambda J$ and clearly $F\Phi(g) = g$.

Let us stress that the construction Φ is not functorial, since it depends on a choice of liftings: we had to write the vector space W as $W = \widetilde{W}/\mathrm{rad}\, \widetilde{W}$ for some free Λ-module $\widetilde{W}$ and given the linear transformation $g\colon W \to W'$, we used a lifting $\widetilde{g}\colon \widetilde{W} \to \widetilde{W}'$ of g.

Proof of Proposition 1. Note that $F(I) = (k, 0, 0)$ is the simple projective representation of Δ. Therefore, F induces a functor $[I, J]/\langle I \rangle$ into $\mathrm{mod}_{\mathrm{sp}}\, k\Delta/\langle S(1) \rangle$ and this functor $[I, J]/\langle I \rangle \longrightarrow \mathrm{mod}_{\mathrm{sp}}\, k\Delta/\langle S(1) \rangle$ is full and dense. It remains to determine its kernel. For this, let M, M' be in $[I, J]$ and consider a map $f\colon M \to M'$ such that $F(f)$ factors through a direct sum of copies of $S(1)$. It follows that $F(f)_2 = F(f)_3 = 0$. Now $F(M) = (L_1 M, M/L_6 M, M_1/(M_1 \cap L_3 M))$and $F(M') = (L_1 M', M'/L_6 M', M_1'/(M_1' \cap L_3 M'))$. The maps $F(f)_2\colon M/L_6 M \to M'/L_6 M'$ and $F(f)_3\colon M_1/(M_1 \cap L_3 M) \to M_1'/(M_1' \cap L_3 M')$ are induced by f; since these are zero maps, we see that

$$f(M_0) \subseteq L_6 M' \quad \text{and} \quad f(M_1) \subseteq M_1' \cap L_3 M',$$

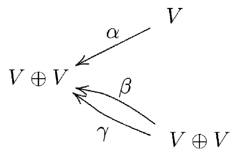
thus f maps into the I-socle of M'. By the lemma, the I-socle of M' is a direct sum of copies of I, thus f belongs to $\mathrm{Hom}(M, M')_I$, as we wanted to show. □

7. Conclusion

Our main result is

Theorem 2. *Let Λ be a commutative local uniserial ring of length $n \geqslant 7$ and let k be its radical factor. Then the category $\mathscr{S}(\Lambda)$ is controlled k-wild.*

For the proof, we need the (well-known) fact that the category $\mathrm{mod}\, k\Delta$ is strictly k-wild: recall that an additive category $\mathscr{A}$ is said to be *strictly k-wild* provided there exists a full embedding of the category $\mathrm{mod}\, k\langle X, Y\rangle$ into $\mathscr{A}$. The following embedding G of $\mathrm{mod}\, k\langle X, Y\rangle$ into the category of representations of Δ is known to be full and exact: consider a $k\langle X, Y\rangle$-module $(V; X, Y)$ (here, V is a k-space, and X and Y are linear transformations of V, they are given by the multiplication using the corresponding generators with the same names); under G we send it to the following representation of Δ:



with $\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 & X \\ 1 & Y \end{bmatrix}.$

Note that *no representation in the image of G has a simple direct summand*.

Proof of Theorem 2. Let $\mathscr{B}$ be the full subcategory of all objects M in $[I, J]$ such that either $M = I$ or $F(M)$ lies in the image of the functor G: $\operatorname{mod} k\langle X, Y\rangle \to \operatorname{mod}_{\mathrm{sp}} k\Delta$. The required equivalence $\mathscr{B}/\langle I\rangle \to \operatorname{mod} k\langle X, Y\rangle$ is given by the restriction of the functor F in Proposition 1 to $\mathscr{B}$. □

Let us mention some more details of this equivalence

$$\mathscr{B}/\langle I\rangle \longrightarrow \operatorname{mod} k\langle X, Y\rangle.$$

The $k\langle X, Y\rangle$-module $(V; X, Y)$ corresponds to $\Phi(V \oplus V, V \oplus 0, U_{XY})$ in $[I, J]$, where

$$U_{XY} = \{(v_1, v_2, Xv_2, v_1 + Yv_2) \mid v_1, v_2 \in V\} \subseteq V \oplus V \oplus V \oplus V,$$

the Λ-module $\Phi(V \oplus V, V \oplus 0, U_{XY})_0$ is given by the partition $(7^d, 6^d, 4^{2d}, 2^{2d})$ with $d = \dim V_k$, and its submodule $\Phi(V \oplus V, V \oplus 0, U_{XY})_1$ is given by the partition $(4^{2d}, 2^{2d})$.

The above equivalence has the following consequence:

Corollary. *Let R be a finite-dimensional k-algebra. There exists M in $\mathscr{S}(\Lambda)$ such that* $\operatorname{End}(M)/\operatorname{End}(M)_I$ *is isomorphic to R.*

On the other hand, we stress the following (clearly also well-known) fact:

Proposition 2. *The category $\mathscr{S}(\Lambda)$ is not strictly K-wild, for any field K.*

Proof. Assume $\mathscr{S}(\Lambda)$ is strictly K-wild for some field K, possibly different from the radical factor field k of Λ. There are infinitely many isomorphism classes of finite length $K\langle X, Y\rangle$-modules M with endomorphism ring K, and there are pairs M, M' of such modules with $\operatorname{Hom}(M, M') = 0 = \operatorname{Hom}(M', M)$; for example, just take for M and M' two non-isomorpic one-dimensional representations. Then the endomorphism ring of the direct sum is $\operatorname{End}(M \oplus M') = K \times K$. Thus, a full embedding of $\operatorname{mod} K\langle X, Y\rangle$ into $\mathscr{S}(\Lambda)$ yields an object $(A \subseteq B)$ in $\mathscr{S}(\Lambda)$ with endomorphism ring $K \times K$. Note that the multiplication with the radical generator t of Λ gives a nilpotent endomorphism of any object $(A \subseteq B)$. Thus, if $\operatorname{End}(A \subseteq B) = K \times K$, then t has to act by zero on B. However, there are only two indecomposables $(A \subseteq B)$ in $\mathscr{S}(\Lambda)$ such that t acts as zero on B, namely

$$S_1 = (0 \subseteq k) \quad \text{and} \quad S_2 = (k \subseteq k),$$

where k is the field $\Lambda/\operatorname{rad}\Lambda$ as above. As there are nonzero maps from S_1 to S_2, it follows that $K \times K$ cannot be realized as an endomorphism ring. □

8. K-Algebras

If Λ is a K-algebra (for any field K, not necessarily isomorphic to the radical factor k of Λ) then the assignments $\widetilde{W} = W \otimes_K \Lambda$ and $\widetilde{g} = g \otimes_K \Lambda$ make Φ: mod $K\Delta \to \mathcal{S}(\Lambda)$ into a functor.

Proposition 3. *Assume that Λ is a K-algebra.*

1. *The functor Φ is exact (and additive) and hence naturally equivalent to the tensor functor $- \otimes_{K\Delta} \Phi(K\Delta)$.*
2. *The composition $F \circ \Phi$ is naturally equivalent to the identity functor on* mod $K\Delta$, *and hence Φ preserves indecomposables and reflects isomorphisms.*
3. *The exact embedding* mod $K\langle X, Y\rangle \to \mathrm{mod}_{\mathrm{sp}}\, K\Delta \to \mathcal{S}(\Lambda)$, *makes the category $\mathcal{S}(\Lambda)$ K-wild in the sense of Drozd.*

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