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Empirical Likelihood Confidence Intervals for Linear Regression Coefficients

SONG XI CHEN

Australian National University, Canberra, Australia

Nonparametric versions of Wilks' theorem are proved for empirical likelihood estimators of slope and mean parameters for a simple linear regression model. They enable us to construct empirical likelihood confidence intervals for these parameters. The coverage errors of these confidence intervals are of order n^{-1} and can be reduced to order n^{-2} by Bartlett correction.

1. INTRODUCTION

Empirical likelihood is a nonparametric technique for constructing confidence regions. It has sampling properties similar to those of bootstrap. However, instead of putting equal probability weight n^{-1} on each data value, empirical likelihood chooses the weights by profiling a multinomial likelihood supported on the sample. The use of empirical likelihood methods to construct confidence regions for β, which is the vector of unknown regression coefficients in a linear regression model, has been studied by Owen [1] and Chen [2]. Owen [1] pioneered this work by proving a nonparametric version of Wilks' theorem for the empirical likelihood ratio of β, which enables us to construct confidence regions for β using χ^2 tables. The second order properties of empirical likelihood confidence regions were discussed by Chen [2], showing that coverage errors are of order n^{-1} and Bartlett correction can be employed to reduce the coverage error to order n^{-2}.

However, it is not enough to just construct confidence regions for β. In practice, statisticians are often confronted with problems of constructing confidence intervals for a particular regression coefficient or certain linear combinations of β.

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In this paper we address the above problem under the simple linear regression model. A simple linear regression model is

$$y_i = a_0 + b_0 x_i + \varepsilon_i, \qquad 1 \leqslant i \leqslant n, \tag{1.1}$$

where all the variables appearing in (1.1) are scalars. Among them, x_i and y_i are the ith fixed design point and response, respectively, the ε_i's are independent and identically distributed random errors with mean zero and variance σ^2, and a_0 and b_0 are the unknown intercept and slope parameters, respectively.

There are two aims in this paper. First, we show how to construct empirical likelihood confidence intervals for the slope parameter b_0 and means $y_0 = a_0 + b_0 x_0$ for any fixed x_0, under model (1.1). Obviously the latter case includes the intercept parameter a_0 when one chooses $x_0 = 0$. Second, we study the coverage accuracy and Bartlett correctability of empirical likelihood confidence intervals for these parameters.

Analyses in Sections 3 and 4 show that both empirical likelihood confidence intervals for b_0 and y_0 have coverage errors of order n^{-1}, and that both confidence intervals are Bartlett correctable. Thus, simple scale adjustments can improve the coverage accuracy of those confidence intervals from order n^{-1} to order n^{-2}. A simulation study is presented in Section 5.

2. Preliminaries

In this section we introduce some notation and basic formulae which are used throughout this paper. We use $\hat{a}_0$ and $\hat{b}_0$ for the least squares estimates of a_0 and b_0, respectively, μ_j for the jth moment of ε_1 for $j = 1, 2$, and $\bar{x}$ and $\bar{y}$ for the means of x_i's and y_i's, respectively. We define auxiliary variables $z_i(a, b) = (1, x_i)^{\mathrm{T}} (y_i - a - bx_i)$ for $i = 1, ..., n$, where a and b are any candidate values for a_0 and b_0. Specifically we write z_i as $z_i(a_0, b_0)$. Furthermore, put

$$\sigma_x^2 = n^{-1} \sum (x_i - \bar{x})^2, \qquad m_j = n^{-1} \sum (x_i - \bar{x})^j, \qquad j = 3, 4$$

$$\hat{\sigma}^2 = n^{-1} \sum \hat{\varepsilon}_i^2, \qquad \hat{\mu}_j = n^{-1} \sum \hat{\varepsilon}_i^j, \qquad j = 3, 4,$$

$$\bar{\varepsilon} = \bar{y} - a_0 - b_0 \bar{x}, \qquad \text{where} \quad \hat{\varepsilon}_i = y_i - \hat{a} - \hat{b} x_i.$$

Let

$$V_n = \sigma^2 \begin{pmatrix} 1 & \bar{x} \\ \bar{x} & n^{-1} \sum x_i^2 \end{pmatrix}$$

be the average covariance matrix of auxiliary variables z_i's, let v_{1n} and v_{2n} be the largest and smallest eigenvalues of V_n, respectively, and let

$$U_n = \begin{pmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \end{pmatrix} = V_n^{-1/2}$$

be the inverse of the square root matrix of V_n. Moreover we define

$$g_{j_1 j_2 \cdots j_k}(x_i) = \prod_{l=1}^{k} (u_{j_l}^1 + u_{j_l}^2 x_i),$$

$$\bar{\alpha}^{j_1 j_2 \cdots j_k} = n^{-1} \sum E\{ g_{j_1 j_2 \cdots j_k}(x_i)\, \varepsilon_i^k \},$$

$$A^{j_1, j_2, \ldots, j_k}(a, b) = n^{-1} \sum g_{j_1 j_2 \cdots j_k}(x_i)(y_i - a - bx_i)^k - \bar{\alpha}^{j_1, j_2, \ldots, j_k}.$$

For simplicity of notation we write

$$A_0^{j_1, j_2, \ldots, j_k} = A^{j_1, j_2, \ldots, j_k}(a_0, b_0) \qquad \text{and} \qquad A^{j_1, j_2, \ldots, j_k} = A^{j_1, j_2, \ldots, j_k}(a, b).$$

We assume the following regularity conditions.

There exist positive constants C_1 and C_2 such that uniformly in n,

$$C_1 < v_{pn} \leqslant v_{1n} < C_2; \qquad \text{and} \qquad n^{-2} \sum_{j=1}^{n} E\, \|z_j\|^4 \to 0, \tag{2.1}$$

where $\| \ \|$ is the Euclidean norm. For any candidate values a and b or a_0 and b_0,

$$a = a_0 + O_p(n^{-1/2}) \qquad \text{and} \qquad b = b_0 + O_p(n^{-1/2}). \tag{2.2}$$

Let $l(a, b)$ be the empirical log likelihood ratio evaluated at (a, b). Write $p_1, \ldots, p_n$ for nonnegative numbers adding to unity. Then, according to the definition of empirical likelihood,

$$l(a, b) = -2 \min_{\sum p_i z_i(a, b) = 0} \sum_{i=1}^{n} \log (np_i).$$

Using the Lagrange method gives us

$$l(a, b) = 2 \sum \log \{1 + \lambda(1, x_i)^{\mathrm{T}} (y_i - a - bx_i)\},$$

and $\lambda = (\lambda_1, \lambda_2)$ satifies

$$\sum \frac{(1, x_i)^T (y_i - a - bx_i)}{1 + \lambda(1, x_i)^T (y_i - a - bx_i)} = 0.$$

Since the analytic solutions for both λ and $l(a, b)$ are difficult to obtain, we have to resort to expansions. Using (2.4) of Chen [2], under conditions (2.1) and (2.2), we have for $l(a, b)$ the Taylor expansion

$$\begin{aligned} n^{-1}l(a,b) &= A^jA^j - A^{jk}A^jA^k + \tfrac{2}{3}\bar{\alpha}^{jkl}A^jA^kA^l + A^{jl}A^{kl}A^jA^k \\ &\quad + \tfrac{2}{3}A^{jkl}A^jA^kA^l - 2\bar{\alpha}^{jkm}A^{lm}A^jA^kA^l \\ &\quad + (\bar{\alpha}^{jkn}\bar{\alpha}^{lmn} - \tfrac{1}{2}\bar{\alpha}^{jklm})\,A^jA^kA^lA^m + O_p(n^{-5/2}). \end{aligned} \tag{2.3}$$

Here we use the summation convention according to which, if an index occurs more that once in an expression, summation over the index is understood.

3. Empirical Likelihood Confidence Interval for b_0

In this section we show how to construct empirical likelihood confidence intervals for the slope parameter b_0 and analyse the coverage properties of these confidence intervals. We first prove a nonparametric version of Wilks' theorem for the empirical log likelihood ratio for b_0 (Theorem 3.1). Then we develop an Edgeworth expansion of the distribution of the empirical log likelihood ratio for b_0 (Theorem 3.2), which is used to show that the coverage errors of the confidence intervals are of order n^{-1}. Furthermore we demonstrate that the empirical likelihood confidence intervals are Bartlett correctable (Theorem 3.3). This means that simple scale adjustments can reduce the coverage errors from $O(n^{-1})$ to $O(n^{-2})$.

The empirical log likelihood ratio for b_0 may be obtained by minimizing $l(a, b_0)$ with respect to a, which is treated as a nuisance parameter in this section, since we are only interested in constructing confidence intervals for b_0. Let $\tilde{a}$ be the optimal a which minimizes $l(a, b_0)$. Then

$$l(b_0) = l(\tilde{a}, b_0) = \min_a l(a, b_0).$$

From (2.3), we know that

$$\begin{aligned} n^{-1}l(a,b_0) &= A^j(a,b_0)\,A^j(a,b_0) - A^{jk}(a,b_0)\,A^j(a,b_0)\,A^k(a,b_0) \\ &\quad + \{\tfrac{2}{3}\bar{\alpha}^{jkl}A^l(a,b_0) + A^{jl}(a,b_0)\,A^{kl}(a,b_0)\}\,A^j(a,b_0)\,A^k(a,b_0) \\ &\quad + \{\tfrac{2}{3}A^{jkl}(a,b_0) - 2\bar{\alpha}^{jkml}A^{lm}(a,b_0)\} \\ &\quad \times A^j(a,b_0)\,A^k(a,b_0)\,A^l(a,b_0) \\ &\quad + (\bar{\alpha}^{jkn}\bar{\alpha}^{lmn} - \tfrac{1}{2}\bar{\alpha}^{jklm})\,A^j(a,b_0)\,A^k(a,b_0)\,A^l(a,b_0)\,A^m(a,b_0) \\ &\quad + O_p(n^{-5/2}). \end{aligned} \tag{3.1}$$

Consider an expansion of $\tilde{a} = \hat{a} + a_1 + a_2 + a_3$, where $a_j = O_p(n^{-j/2})$, $j = 1, 2, 3$. We determine a_1, a_2, a_3 successively. Put

$$\gamma_j = n^{-1} \sum g_j(x_i), \qquad \gamma_{jk} = n^{-1} \sum g_{jk}(x_i),$$

$$\gamma_{jk,1}(a, b) = n^{-1} \sum g_{jk}(x_i)(y_i - a - bx_i),$$

$$\gamma_{jkl,2}(a, b) = n^{-1} \sum g_{jkl}(x_i)(y_i - a - bx_i)^2.$$

Some algebra shows that

$$a_1 = A^j(\hat{a})\,\gamma_j / \gamma_j \gamma_j = \bar{x}(\hat{b} - b_0),$$
$$a_2 = -(\gamma_i \gamma_i)^{-1}\, \gamma_j \{A^k(\hat{a}) - \gamma_k a_1\}[A^{jk}(\hat{a}) - \bar{\alpha}^{jkl}\{A^l(\hat{a}) - \gamma_l a_1\}]$$

and $a_3 = O_p(n^{-2})$. In summary we have

$$\tilde{a} = \hat{a} + \bar{x}(\hat{b} - b_0) - (\gamma_i \gamma_i)^{-1}\, \gamma_j \{A^k(\hat{a}) - \gamma_k a_1\}[A^{jk}(\hat{a}) - \bar{\alpha}^{jkl}\{A^l(\hat{a}) - \gamma_l a_1\}].$$

The above formula suggests using $\hat{a} + \bar{x}(\hat{b} - b_0)$ as an initial value for a in numerically searching for $\tilde{a}$. In the author's experience, this works well. Now, with $\tilde{a}$ substituted into (3.1), the empirical likelihood ratio statitstic at b_0 is given by

$$\begin{aligned}
n^{-1} l(b_0) = {} & \{A^j(\hat{a}) - \gamma_j a_1\}\{A^j(\hat{a}) - \gamma_j a_1\} \\
& - \{A^{jk}(\hat{a}) - 2\gamma_{jk,1}(\hat{a})\, a_1 + \gamma_{jk} a_1^2\}\{A^j(\hat{a}) - \gamma_j a_1\} \\
& \times \{A^k(\hat{a}) - \gamma_k a_1\} + \tfrac{2}{3}\bar{\alpha}^{jkl}\{A^j(\hat{a}) - \gamma_j a_1\} \\
& \times \{A^k(\hat{a}) - \gamma_k a_1\}\{A^l(\hat{a}) - \gamma_l a_1\} + A^{jl}(\hat{a})\, A^{kl}(\hat{a})\{A^j(\hat{a}) - \gamma_j a_1\} \\
& \times \{A^k(\hat{a}) - \gamma_k a_1\} - \gamma_j \gamma_j a_2^2 + \tfrac{2}{3}\{A^{jkl}(\hat{a}) - \gamma_{jkl,2}(\hat{a})\, a_1\} \\
& \times \{A^j(\hat{a}) - \gamma_j a_1\}\{A^k(\hat{a}) - \gamma_k a_1\}\{A^l(\hat{a}) - \gamma_l a_1\} - 2\bar{\alpha}^{jkm} A^{lm}(\hat{a}) \\
& \times \{A^j(\hat{a}) - \gamma_j a_1\}\{A^k(\hat{a}) - \gamma_k a_1\}\{A^l(\hat{a}) - \gamma_l a_1\} + \bar{\alpha}^{jkn}\bar{\alpha}^{lmn} \\
& \times \{A^j(\hat{a}) - \gamma_j a_1\}\{A^k(\hat{a}) - \gamma_k a_1\}\{A^l(\hat{a}) - \gamma_l a_1\}\{A^m(\hat{a}) - \gamma_m a_1\} \\
& - \tfrac{1}{2}\bar{\alpha}^{jklm}\{A^j(\hat{a}) - \gamma_j a_1\}\{A^k(\hat{a}) - \gamma_k a_1\}\{A^l(\hat{a}) - \gamma_l a_1\} \\
& \times \{A^m(\hat{a}) - \gamma_m a_1\} + O_p(n^{-5/2}).
\end{aligned}$$

For the purpose easy analysis, we next express $l(b_0)$ in terms of powers of $(\hat{b} - b_0)$. Let us define $\eta_j = \sigma_x^2 u_j^2$, where u_j^2 is the $(j, 2)$ element in the matrix U_n. Using the facts that $\eta_j\eta_j = \sigma_x^2/\sigma^2$ and $\{A^j(\hat{a}) - \gamma_j a_1\}\{A^j(\hat{a}) - \gamma_j a_1\} = (\hat{b} - b_0)^2 \sigma_x^2/\sigma^2$, it may be shown that

$$
\begin{aligned}
n^{-1}l(b_0) = {} & \frac{\sigma_x^2}{\sigma^2}(\hat{b} - b_0)^2 - \eta_j\eta_k A_0^{jk}(\hat{b} - b_0)^2 + \tfrac{2}{3}\bar{\alpha}^{jkl}\eta_j\eta_k\eta_l(\hat{b} - b_0)^3 \\
& + \eta_j\eta_k(\gamma_{jk,1}\bar{\varepsilon} - \gamma_{jk}\bar{\varepsilon}^2 + A_0^{jl}A_0^{kl})(\hat{b} - b_0)^2 \\
& - (\gamma_l\gamma_l)^{-1}\gamma_j\gamma_m\eta_k\eta_n\{A_0^{jk}A_0^{mn} - 2\bar{\alpha}^{jkl}\eta_l A_0^{mn}(\hat{b} - b_0) \\
& + \bar{\alpha}^{jkl}\bar{\alpha}^{mnp}\eta_l\eta_p(\hat{b} - b_0)^2\}(\hat{b} - b_0)^2 \\
& + \tfrac{2}{3}\eta_j\eta_k\eta_l(A_0^{jkl} - 3\gamma_{jkl,2}\bar{\varepsilon} - 2\bar{\alpha}^{jklm}A_0^{lm})(\hat{b} - b_0)^3 \\
& + (\bar{\alpha}^{jkn}\bar{\alpha}^{lmn} - \tfrac{1}{2}\bar{\alpha}^{jklm})\eta_j\eta_k\eta_l\eta_m(\hat{b} - b_0)^4 \\
& + O_p(n^{-5/2}).
\end{aligned}
\tag{3.2}
$$

The following nonparametric version Wilk's theorem is a direct consequence of expansion (3.2).

THEOREM 3.1 (Wilks' theorem). *Assume conditions* (2.1). *Then,*

$$P\{l(b_0) < c\} = P(\chi_1^2 < c) + o(1), \qquad as \quad n \to \infty.$$

Proof. Since $\mathrm{Var}(\hat{b} - b_0) = n^{-1}\sigma^2/\sigma_x^2$, by the Central Limit Theorem, we know that $n^{1/2}(\hat{b} - b_0)\sigma_x/\sigma$ has asymptotically a standard normal distribution. Thus from (3.2),

$$l(b_0) = \frac{n\sigma_x^2}{\sigma^2}(\hat{b} - b_0)^2 + O_p(n^{-1/2}) = \chi_1^2 + o_p(1).$$

Hence the theorem is proved. ∎

From Theorem 3.1 an empricial likelihood confidence interval for b_0 with nominal coverage level α can be construced as follows. First find from χ_1^2 tables the value c_α such that $P(\chi_1^2 < c_\alpha) = \alpha$. Then $I_\alpha = \{b_0 \mid l(b_0) < c_\alpha\}$ is the α level confidence interval for b_0. Theorem 3.1 ensures that I_α has correct asymptotic coverage.

In the remainder of this section we investigate coverage accuracy of I_α. To do this, we decompose $l(b_0)$ from (3.2) as

$$l(b_0) = nR_b^2 + O_p(n^{-5/2}), \tag{3.3}$$

where $R_b = R_{b1} + R_{b2} + R_{b3}$ and $R_{bj} = O_p(n^{-j/2})$ for $j = 1, 2, 3$. Put

$$C_1 = -\tfrac{1}{2}\sigma^2\bar{\alpha}^{jkl}\bar{\alpha}^{mnp}\eta_k\eta_l\eta_n\eta_p\left(\gamma_j\gamma_m + \frac{1}{9\sigma_x^2}\eta_j\eta_m\right) + \eta_j\eta_k\eta_l\eta_m\left(\tfrac{1}{2}\bar{\alpha}^{jkn}\bar{\alpha}^{lmn} - \tfrac{1}{4}\bar{\alpha}^{jklm}\right).$$

Comparing (3.2) with (3.3) yields

$$R_{b1} = \frac{\sigma_x}{\sigma}(\hat{b} - b_0),$$

$$\frac{\sigma_x}{\sigma}R_{b2} = -\tfrac{1}{2}\eta_j\eta_k A_0^{jk}(\hat{b} - b_0) + \tfrac{1}{3}\bar{\alpha}^{jkl}\eta_j\eta_k\eta_l(\hat{b} - b_0)^2,$$

$$\begin{aligned}\frac{\sigma_x}{\sigma}R_{b3} = {} & \eta_j\eta_k\left(A_0^{jk}\bar{\varepsilon} - \tfrac{1}{2}\gamma_{jk}\bar{\varepsilon}^2 + \tfrac{1}{2}A_0^{jl}A_0^{kl}\right)(\hat{b} - b_0) + C_1(\hat{b} - b_0)^3 \\ & - \left(\frac{\sigma^2}{2}\gamma_k\gamma_n\eta_j\eta_m + \frac{\sigma^2}{8\sigma_x^2}\eta_j\eta_k\eta_m\eta_n\right)A_0^{jk}A_0^{mn}(\hat{b} - b_0) \\ & + \tfrac{1}{3}\eta_j\eta_k\eta_l A_0^{jkl}(\hat{b} - b_0)^2 \\ & + \left\{\sigma^2\bar{\alpha}^{jkl}\eta_k\eta_m\eta_l\left(\gamma_j\gamma_n + \frac{1}{6\sigma_x^2}\eta_j\eta_n\right) - \bar{\alpha}^{jkm}\eta_j\eta_k\eta_n\right)\Big\} \\ & \times A_0^{mn}(\hat{b} - b_0)^2. \end{aligned} \tag{3.4}$$

Before we develop an Edgeworth expansion for $l(b_0)$ we introduce some notations. From (3.4) we see that there exists a smooth function H such that $R_b = H(\bar{U})$, where $\bar{U} = (\hat{b} - b_0, \bar{\varepsilon}, A_0^{11}, A_0^{12}, A_0^{22}, A_0^{111}, A_0^{112}, A_0^{122}, A_0^{222})$. Let

$$B_1 = \begin{pmatrix} V_{n1}^{-1/2} \otimes V_{n1}^{-1/2} \\ V_{n1}^{-1/2} \otimes V_{n2}^{-1/2} \\ V_{n2}^{-1/2} \otimes V_{n2}^{-1/2} \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} V_{n1}^{-1/2} \otimes V_{n1}^{-1/2} \otimes V_{n1}^{-1/2} \\ V_{n1}^{-1/2} \otimes V_{n1}^{-1/2} \otimes V_{n2}^{-1/2} \\ V_{n1}^{-1/2} \otimes V_{n2}^{-1/2} \otimes V_{n2}^{-1/2} \\ V_{n2}^{-1/2} \otimes V_{n2}^{-1/2} \otimes V_{n2}^{-1/2} \end{pmatrix}$$

be 3×4 and 4×8 matrices, respectively, where $\otimes$ is the Kronecker product of matrices and $V_{nj}^{-1/2}$ is the jth row of $V_n^{-1/2}$, $j = 1, 2$. From the definition of A_0^{jk} and A_0^{jkl}, $\bar{U}$ can be expressed as $\bar{U}=n^{-1}\sum U_i$, where U_i is a vector of nine dimensions having from

$$U_i=[\sigma_x^{-2}(x_i-\bar{x})\,\varepsilon_i,\ \varepsilon_i,\ \{(1x_i)\otimes(1x_i)\}\ B_1^T\varepsilon_i^2,\ \{(1x_i)\otimes(1x_i)\otimes(1x_i)\}\ B_2^T\varepsilon_i^3].$$

Put $T_n=n^{-1}\sum \mathrm{cov}(U_i)$ as the average covariance matrix of U_i's and g_1 as the density function of χ_1^2 distribution. Then, we have the following theorem.

THEOREM 3.2. *Assume that*

(i) *there exists positive constants* C_1, C_2 *such that uniformly in* n, $C_1\leqslant v_{2n}\leqslant v_{1n}\leqslant C_2$; (ii) $|x_i|$*'s for* $1\leqslant i\leqslant n$ *are uniformly bounded*; (ii) $E\,|\varepsilon_1|^{15}<\infty$; (iv) *for every positive* τ, $\lim_{n\to\infty}\int_{\|\varepsilon_1\|>\tau n^{1/2}}\|\varepsilon_1\|^{15}=0$; (v) *the smallest eigenvalue of* T_n *is bounded away from zero*; (vi) *the characteristic function* h *of* U_1 *satisfies* $\limsup_{|t|\to\infty}|h(t)|<1$. (3.5)

Then $P\{l(b_0)<c_\alpha\}=\alpha-(1+\frac{1}{2}t_1-\frac{1}{3}t_2)\,n^{-1}c_\alpha g_1(c_\alpha)+O(n^{-2})$, *where*

$$t_1=\frac{\mu_4}{\sigma^4\sigma_x^4}m_4,\qquad t_2=\frac{\mu_3^2}{\sigma^6\sigma_x^6}m_3^2,\quad m_j=n^{-1}\sum(x_i-\bar{x})^j,\quad \text{for}\quad j=3,4.$$

Proof. Let k_{bj} be the jth cumulant of $n^{1/2}R_b$. Calculations show that

$$\begin{aligned}
k_{b1}&=-\tfrac{1}{6}t_2^{1/2}n^{-1/2}+O(n^{-3/2}),\\
k_{b2}&=1+(1+\tfrac{1}{2}t_1-\tfrac{13}{36}t_2)\,n^{-1}+O(n^{-2}),\\
k_{bj}&=O(n^{-3/2}),\qquad j\geqslant 3.
\end{aligned}$$

A formal Edgeworth expansion for the distribution function of R_b can be constructed as

$$P(n^{1/2}R_b<x)=\int_{-\infty}^{x}\Psi(v)\,\phi(v)\,dv+O(n^{-3/2}),\tag{3.6}$$

where $\Psi(v)=1+\frac{1}{6}t_2^{1/2}vn^{-1/2}+\frac{1}{2}(1+\frac{1}{2}t_1-\frac{1}{3}t_2)(v^2-1)\,n^{-1}$. Accepting that expansion (3.6) may be justified, we establish an Edgewroth expansion for $l(b_0)$ as

$$\begin{aligned}
P\{l(b_0)<c\}&=P(-c^{1/2}<n^{1/2}R_b<c^{1/2})+O(n^{-3/2})\\
&=\int_{-c^{1/2}}^{c^{1/2}}\Psi(v)\,\phi(v)\,dv+O(n^{-3/2})\\
&=\alpha-(1+\tfrac{1}{2}t_1-\tfrac{1}{3}t_2)\,n^{-1}cg_1(c)+O(n^{-3/2}),
\end{aligned}$$

where g_1 is the density function of χ_1^2 distribution. By the evenness and oddness of the polynomials in the above Edgeworth expansion, it can be shown that the $O(n^{-3/2})$ term is actually $O(n^{-2})$. This leads us to Theorem 3.2.

It remains to check that expansion (3.6) is valid. Remember that $R_b = H(\bar{U})$, where H is a sufficient smooth function and $\bar{U}$ is the mean of independent but not identically distributed random variable U_i's. For this case, Bhattacharya and Rao [3, Theorem 20.2] have developed a valid Edgeworth expansion. It may be shown that conditions (3.5) imply the conditions of Theorem 20.2 of Bhattacharya and Rao [3]. Thus, a valid Edgeworth expansion for $\bar{U}$ can be obtained. Consequently, the Edgeworth expansion of $\bar{U}$ may be transformed by smooth function H to yield another valid Edgeworth expansion (3.6) for R_b, by using the results given by Skovgaard [4]. Therefore the theorem can be established. ∎

Theorem 3.2 states that the empirical likelihood confidence interval I_α has coverage error at order of n^{-1}. By looking at the coefficient of the n^{-1} term in the Edgeworth expansion for the distribution function of $l(b_0)$, we see that the coverage error is dominated by a combination of four factors: the moments of ε_i, the "moments" of the fixed design points, the nominal coverage level, and the sample size n. We should note that the conditions listed in (3.5) are just sufficient conditions for deriving the Edgeworth expansion given in Theorem 3.2.

Based on the expression for R_{bj}, $j = 1, 2, 3$ in (3.4), we may show that

$$E\{l(b_0)\} = n\{E(R_{b1})^2 + 2E(R_{b1}R_{b2}) + E(R_{b2})^2 + 2E(R_{b1}R_{b3})\} + O(n^{-2})$$
$$= 1 + (1 + \tfrac{1}{2}t_1 - \tfrac{1}{3}t_2)\, n^{-1} + O(n^{-2}).$$

We see that the difference between the means of $l(b_0)$ and limiting distribution χ_1^2 is of order n^{-1}. Next we show that Barlett correction can reduce the coverage errors of emprirical likelihood confidence intervals to order n^{-2}. Let $\rho_b = 1 + \frac{1}{2}t_1 - \frac{1}{3}t_2$ be the Bartlett correction for $l(b_0)$. We have the following theorem about the Bartlett correctability of confidence interval I_α:

THEOREM 3.3. *Assume condition* (3.5). *Then,*

$$P\{l(b_0) < c_\alpha(1 + \rho_b n^{-1})\} = \alpha + O(n^{-2}).$$

Proof. The method of proof is identical to that of Theorem 2.3 of Chen [2].

Theorem 3.3 implies that a simple Bartlett correction can increase the coverage accuracy of empirical likelihood confidence intervals for b_0 from $O(n^{-1})$ to $O(n^{-2})$. However, ρ_b is usually unknown because of unknown μ_3 and μ_4, the third and fourth moments of ε_1, in t_1 and t_2. An $n^{1/2}$-consistent estimate of ρ_b, denoted by $\hat{\rho}_b$, can be obtained by defining $\hat{\rho}_b = 1 + \frac{1}{2}\hat{t}_1 - \frac{1}{3}\hat{t}_2$, where $\hat{t}_1$ and $\hat{t}_2$ are obtained by replacing μ_3 and μ_4 in t_1 and t_2 by $\hat{\mu}_3$ and $\hat{\mu}_4$, respectively, where $\hat{\mu}_3$ and $\hat{\mu}_4$ are the moment estimators of μ_3 and μ_4. We may get the same order of accuracy by replacing ρ_b with $\hat{\rho}_b$ in Theorem 2.3, under moderate conditions such as: the joint distribution of components of the $l(b_0)$ and $\hat{\rho}_b$ admits multivariate Edgeworth expansions.

4. Empirical Likelihood Confidence Interval for Means

In this section we construct empirical likelihood confidence intervals for the mean value $y_0 = E(y \mid x = x_0) = a_0 + b_0 x_0$, for any fixed x_0. Since $y_0 = a_0$ when $x_0 = 0$, we may confine our attention to constructing empirical likelihood confidence intervals for a general y_0. The empirical log likelihood ratio for y_0, denoted as $l(y_0)$, may be obtained by minimizing $l(a, b)$ given in (2.3), under the constraint of $a + bx_0 = y_0$, that is,

$$l(y_0) = l(\tilde{a}, \tilde{b}) = \min_{a + bx_0 = y_0} l(a, b).$$

Suppose $\tilde{a}$ and $\tilde{b}$ have expansions $\tilde{a} = \hat{a} + a_1 + a_2 + a_3$ and $\tilde{b} = \hat{b} + b_1 + b_2 + b_3$, where $a_j, b_j = O_p(n^{-j/2})$, $j = 1, 2, 3$. Note that we use notations $\tilde{a}$ and a_j again here, but with meanings different from those in Section 3. In the following, a_j, b_j, $j = 1, 2, 3$, are determined successively. Put

$$\beta_j = n^{-1} \sum g_j(x_i)\, x_i, \qquad \beta_{jk} = n^{-1} \sum g_{jk}(x_i)\, x_i,$$

$$\beta_{jkl} = n^{-1} \sum g_{jkl}(x_i)\, x_i, \qquad \beta_{jk2} = n^{-1} \sum g_{jk}(x_i)\, x_i^2,$$

$$\beta_{jk,1}(a, b) = n^{-1} \sum g_{jk}(x_i)(y_i - a - bx_i),$$

$$\beta_{jkl,2}(a, b) = n^{-1} \sum g_{jkl}(x_i)(y_i - a - bx_i)^2.$$

Let $W_0 = \hat{a} + \hat{b}x_0 - y_0 = O_p(n^{-1/2})$ and $t^j = A^j(\hat{a}, \hat{b}) - \gamma_j a_1 - \beta_j b_1$. After some algebra we may show that

$$a_1 = -\frac{\sigma_x^2 + \bar{x}(\bar{x} - x_0)}{\sigma_x^2 + (\bar{x} - x_0)^2} W_0, \qquad b_1 = \frac{(\bar{x} - x_0)}{\sigma_x^2 + (\bar{x} - x_0)^2} W_0,$$

$$\begin{aligned} b_2 &= \{(\beta_p - \gamma_p x_0)(\beta_p - \gamma_p x_0)\}^{-1} \\ &\quad \times \{\bar{\alpha}^{jkl} t^j t^k (\beta_l - \gamma_l x_0) - A^{jk}(a, \hat{b})\, t^j (\beta_k - \gamma_k x_0)\}, \end{aligned}$$

$$a_2 = -b_2 x_0, \qquad a_3 = O_p(n^{-2}), \qquad b_3 = O_p(n^{-2}).$$

Now substitute $\tilde{a}$, $\tilde{b}$ into the formula for $n^{-1}l(\tilde{a}, \tilde{b})$, obtaining

$$
\begin{aligned}
n^{-1}l(y_0) = {} & t^j t^j - \{A^{jk}(\hat{a}, \hat{b}) - 2\gamma_{jk,1} a1 - 2\beta_{jk,1} b_1 \\
& + \gamma_{jk} a_1^2 + 2\beta_{jk} a_1 b_1 + \beta_{jk2} b_1^2\} t^j t^k \\
& + \tfrac{2}{3} \bar{\alpha}^{jkl} t^j t^k t^l - (\beta_j - \gamma_j x_0)(\beta_j - \gamma_j x_0) b_2^2 \\
& + A^{jl}(\hat{a}, \hat{b}) A^{kl}(\hat{a}, \hat{b}) t^j t^k \\
& + \tfrac{2}{3} \{A^{jkl}(\hat{a}, \hat{b}) - 3\gamma_{jkl,2} a_1 - 3\beta_{jkl,2} b_1\} t^j t^k t^l \\
& - 2\bar{\alpha}^{jkm} A^{lm}(\hat{a}, \hat{b}) t^j t^k t^l + (\bar{\alpha}^{jkn}\bar{\alpha}^{lmn} - \tfrac{1}{2} \bar{\alpha}^{jklm}) t^j t^k t^l t^m \\
& + O_p(n^{-5/2}).
\end{aligned} \tag{4.1}
$$

Define $\alpha^2(x_0) = \sigma_x^2/\{\sigma_x^2 + (\bar{x} - x_0)^2\}$ and $\xi^j = u_j^1 + u_j^2 x_0 j = 1, 2$. Then we have $t^j = \alpha^2(x_0)\, \xi^j W_0$, $t^j t^j = \alpha^2(x_0)\, \sigma^{-2} W_0^2$, and $(\beta_j - \gamma_j x_0)(\beta_j - \gamma_j x_0) = \sigma_x^2 \sigma^{-2} \alpha^{-2}(x_0)$. Substituting these formulae into (4.1), we obtain

$$
\begin{aligned}
n^{-1}l(y_0) = {} & \alpha^2(x_0)\, \sigma^{-2} W_0^2 - \alpha^4(x_0)\, \xi^j \xi^k A_0^{jk} W_0^2 \\
& + \tfrac{2}{3} \alpha^6(x_0)\, \bar{\alpha}^{jkl} \xi^j \xi^k \xi^l W_0^3 - \sigma_x^{-2} \sigma^2 \alpha^6(x_0) \\
& \times \{\bar{\alpha}^{jkl} \alpha^2(x_0)\, \xi^j \xi^k (\beta_l - \gamma_l x_0) W_0^2 - \xi^j (\beta_k - \gamma_k x_0) A_0^{jk} W_0\}^2 \\
& + 2\alpha^4(x_0)\, \xi^j \xi^k (\beta_{jk,1} - \gamma_{jk,1} x_0)(b_1 + \hat{b} - b_0) W_0^2 \\
& - \alpha^4(x_0)\, \xi^j \xi^k \{(\beta_{jk2} - 2\beta_{jk} x_0 + \gamma_{jk} x_0^2) \\
& \times (b_1 + \hat{b} - b_0)^2 - A_0^{jl} A_0^{kl}\} W_0^2 \\
& + \alpha^6(x_0)\, \xi^j \xi^k \xi^l \{\tfrac{2}{3} A_0^{jkl} - 2(\beta_{jkl} - \gamma_{jkl} \sigma^2 x_0) \\
& \times (b_1 + \hat{b} - b_0) - 2\bar{\alpha}^{jkm} A_0^{lm}\} W_0^3 \\
& - 2\alpha^6(x_0)\, \xi^j \xi^k \xi^l W_0^3 + \alpha^8(x_0) \\
& \times (\bar{\alpha}^{jkn} \bar{\alpha}^{lmn} - \tfrac{1}{2} \bar{\alpha}^{jklm})\, \xi^j \xi^k \xi^l \xi^m W_0^4 \\
& + O_p(n^{-5/2}).
\end{aligned} \tag{4.2}
$$

The following nonparametric version of Wilks' theorem is a conclusion of (4.2).

THEOREM 4.1 (Wilks' theorem). *Assume conditions* (2.1) *and* (2.2). *Then,*

$$P\{l(y_0) < c\} = P(\chi_1^2 < c) + o(1), \qquad n \to \infty.$$

Proof. From (4.2), we know that

$$
\begin{aligned}
l(y_0) &= n\alpha(x_0)^2 \sigma^{-2} \hat{W}_0^2 + O_p(n^{-1/2}) \\
&= n\sigma^{-2} \frac{\sigma_x^2}{\sigma_x^2 + (\bar{x} - x_0)^2} \hat{W}_0^2 + O_p(n^{-1/2}).
\end{aligned}
$$

Thus the theorem is proved by the fact that W_0 is asymptotically normal with mean zero and variance $n^{-1}\sigma^2\sigma_x^{-2}\{\sigma_x^2+(\bar{x}-x_0)^2\}$. ∎

An empirical likelihood confidence interval for y_0 with asymptotic coverage level α can be constructed as $J_\alpha=\{y_0 \mid l(y_0)<c_\alpha\}$ such that $P(\chi_1^2<c_\alpha)=\alpha$. In the rest of this section we investigate the second order properties of J_α.

A signed root decomposition of $l(y_0)$ can be obtained from (4.2) as

$$l(y_0)=nR_{y_0}^2+O_p(n^{-5/2}),$$

where $R_{y_0}=R_{y_0 1}+R_{y_0 2}+R_{y_0 3}$, and $R_{y_0 j}=O_p(n^{-j/2})$ for $j=1, 2, 3$. A little algebra shows that

$$\begin{aligned}
R_{y_0 1} &= \alpha(x_0)\,\sigma^{-1}W_0,\\
R_{y_0 2} &= \alpha^3(x_0)\,\sigma\xi^j\xi^k\{-\tfrac{1}{2}A_0^{jk}W_0+\tfrac{1}{3}\alpha^2(x_0)\,\bar{\alpha}^{jkl}\xi^l W_0^2\},\\
R_{y_0 1}R_{y_0 3} &= \alpha^4(x_0)\,\xi^j\xi^k(\beta_{jk,1}-\gamma_{jk,1}x_0)(b_1+\hat{b}-b_0)\,W_0^2+C_2W_0^4\\
&\quad -\tfrac{1}{2}\alpha^4(x_0)\,\xi^j\xi^k(\beta_{jk2}-2\beta_{jk}x_0+\gamma_{jk}x_0^2)(b_1+\hat{b}-b_0)^2\,W_0^2\\
&\quad -\alpha^6(x_0)\,\sigma^2\xi^j\xi^m\{\tfrac{1}{2}\sigma_x^{-2}(\beta_k-\gamma_l x_0)(\beta_n-\gamma_n x_0)+\tfrac{1}{8}\xi^k\xi^n\}\\
&\quad \times A_0^{jk}A_0^{mn}W_0^2+\alpha^8(x_0)\,\sigma^2\\
&\quad \times\{\sigma_x^{-2}\bar{\alpha}^{jkl}\xi^j\xi^k\xi^m(\beta_l-\gamma_l x_0)(\beta_n-\gamma_n x_0)\\
&\quad +\tfrac{1}{6}\bar{\alpha}^{jkl}\xi^j\xi^k\xi^l\xi^m\xi^n-\alpha^{-2}(x_0)\,\sigma^{-2}\bar{\alpha}^{jkm}\xi^j\xi^k\xi^n\}\,A_0^{mn}W_0^3\\
&\quad +\tfrac{1}{2}\alpha^4(x_0)\,\xi^j\xi^k A_0^{jl}A_0^{kl}W_0^2+\tfrac{1}{3}\alpha^6(x_0)\,\xi^j\xi^k\xi^l A_0^{jkl}W_0^3\\
&\quad -\alpha^6(x_0)\,\sigma^2\xi^j\xi^k\xi^l(\beta_{jkl,2}-\gamma_{jkl,2}x_0)(b_1+\hat{b}-b_0)\,W_0^3,
\end{aligned}$$

where

$$\begin{aligned}
C_2 = &-\tfrac{1}{2}\alpha^{10}(x_0)\,\sigma^2\sigma_x^{-2}\{\bar{\alpha}^{jkl}\xi^j\xi^k(\beta_l-\gamma_l x_0)\}^2\\
&-\tfrac{1}{18}\alpha^{10}(x_0)\,\sigma^2\{\bar{\alpha}^{jkl}\xi^j\xi^k\xi^l\}^2\\
&+\alpha^8(x_0)(\tfrac{1}{2}\bar{\alpha}^{jkn}\bar{\alpha}^{lmn}-\tfrac{1}{4}\bar{\alpha}^{jklm})\,\xi^j\xi^k\xi^l\xi^m.
\end{aligned}$$

In order to assess the coverage accuracy of the confidence interval J_α we establish an Edgeworth expansion for the distribution of $l(y_0)$. To this end, we note from expressions for $R_{y_0 j}$ $j=1, 2, 3$, that there is a smooth function Q_1 such that $R_{y_0}=Q_1(\bar{S})$, where

$$\bar{S}=(W_0, b_1+\hat{b}-b_0, A_0^{11}, A_0^{12}, A_0^{22}, A_0^{111}, A_0^{112}, A_0^{122}, A_0^{222}).$$

Since $b_1=(\bar{x}-x_0)\{\bar{x}-x_0)^2\}^{-1}W_0$ and $W_0=\bar{\varepsilon}+(\bar{x}-x_0)(\hat{b}-b_0)$, there exists another smooth function Q_2 such that $\bar{S}=Q_2(\bar{U})$, where $\bar{U}$ was defined in Section 3. So, putting $Q=Q_1Q_2$, we have $R_{y_0}=Q(\bar{U})$. Define

$$s_1=\alpha^4(x_0)\,\sigma^{-4}\mu_4 q_1, \qquad s_2=\alpha^6(x_0)\,\sigma^{-6}\mu_3^2 q_2^2, \qquad s_3=\alpha^4(x_0)\,q_3,$$

where

$$q_1=1+6\frac{(\bar{x}-x_0)^2}{\sigma_x^2}-4\frac{(\bar{x}-x_0)^3}{\sigma_x^6}m_3+\frac{(\bar{x}-x_0)^4}{\sigma_x^8}m_4,$$

$$q_2=1+3\frac{(\bar{x}-x_0)^2}{\sigma_x^2}-\frac{(\bar{x}-x_0)^3}{\sigma_x^6}m_3,$$

$$q_3=1-3\frac{(\bar{x}-x_0)^2}{\sigma_x^2}+\frac{(\bar{x}-x_0)^2}{\sigma_x^6}m_4+\frac{(\bar{x}-x_0)^4}{\sigma_x^4}$$
$$+2\left\{\frac{(\bar{x}-x_0)^3}{\sigma_x^6}-\frac{(\bar{x}-x_0)}{\sigma_x^4}\right\}m_3.$$

Now the coverage accuracy of confidence interval J_α is discussed in the following theorem.

THEOREM 4.2. *Assume condition* (3.5). *Then,*

$$P\{l(y_0)<c_\alpha\}=\alpha-\left(\frac{s_1}{2}-\frac{s_2}{3}+s_3\right)n^{-1}c_\alpha g_1(c_\alpha)+O(n^{-3/2}). \tag{4.3}$$

Proof. Let $k_{y_0 j}$, $j=1,2,\ldots$, denote the jth cumulants of $n^{1/2}R_{y_0}$. Calculations show that

$$\begin{aligned} k_{y_0 1}&=-\tfrac{1}{6}s_2^{1/2}n^{-1/2}+O(n^{-3/2}),\\ k_{y_0 2}&=1+(\tfrac{1}{2}s_1-\tfrac{13}{36}s_2+s_3)\,n^{-1}+O(n^{-2}),\\ k_{y_0 j}&=O(n^{-3/2}), \qquad j\geqslant 3. \end{aligned} \tag{4.4}$$

A formal Edgeworth expansion for the distribution of $n^{1/2}R_{y_0}$ can be set up from (4.4) as

$$P(n^{1/2}R_b<x)=\int_{-\infty}^{x}\Pi(v)\,\phi(v)\,dv+O(n^{-3/2}), \tag{4.5}$$

where $\Pi(v)=1+\frac{1}{6}s_2^{1/2}vn^{-1/2}+\frac{1}{2}(\frac{1}{2}s_1-\frac{1}{3}s_2+s_3)(v^2-1)\,n^{-1}$. The validity of expansion (4.5) can be demonstrated in the same way that validity for expansion (3.6) in the proof of Theorem 3.2 was demonstrated. And the theorem can be obtained from expansion (4.5) in the same way that we derived Theorem 3.2. ∎

Theorem 4.2 states that the coverage errors of empirical likelihood confidence intervals for $y_0 = a + bx_0$ are of order of n^{-1}, provided that the x_0 is fixed and independent of sample size n. From the n^{-1} order term in (4.3) and the definitions of s_1, s_2, and s_3, we see that the coverage error is dominated by the combination of the following five factors: the moments of ε_i, the "moments" of the fixed design points, the nominal coverage level, the sample size n, and the size of $(\bar{x} - x_0)/\sigma_x$—the standard distance between x_0 and the centre, $\bar{x}$, of the design points.

In analogy with the Bartlett correction for the slope parameter b_0 developed in Theorem 3.3, we do the same thing here fore y_0. Calculations reveal that

$$E\{l(y_0)\} = n\{E(R_{y_0})^2 + O(n^{-2})\} = 1 + \left(\frac{s_1}{2} - \frac{s_2}{3} + s_3\right) n^{-1} + O(n^{-2}).$$

Put $\rho_{y_0} = (s_1/2 - s_2/3 + s_3)$, the Bartlett correction for $l(y_0)$. The Bartlett correction property for the empirical likelihood confidence intervals for y_0 is proved by the following theorem.

THEOREM 4.3. *Assume conditions* (3.5). *For any* $x > 0$ *and fixed* x_0,

$$P\{l(y_0) < c_\alpha(1 + \rho_{y_0} n^{-1})\} = \alpha + O(n^{-2}).$$

Thus a simple scale adjustment can increase the coverage accuracy of empirical likelihood confidence intervals for y_0 from $O(n^{-1})$ to $O(n^{-2})$. In practice, ρ_{y_0} is usually unknown, because μ_3 and μ_4 are unknown. However, an $n^{1/2}$-consistent estimate $\hat{\rho}_{y_0}$ of ρ_{y_0} can be obtained by replacing σ^2, μ_3, and μ_4 with $\hat{\sigma}^2$, $\hat{\mu}_3$, and $\hat{\mu}_4$, respectively, in s_1 and s_2, that is, $\hat{\rho}_{y_0} = (\hat{s}1/2 - \hat{s}2/3 + s_3)$, where $\hat{s}_1 = \alpha^4(x_0)\,\hat{\sigma}^{-4}\hat{\mu}_4 q_1$ and $\hat{s}_2 = \alpha^6(x_0)\,\hat{\sigma}^{-6}\hat{\mu}_3^2 q_2^2$. It may be shown that we may obtain the same order of accuracy by replacing ρ_{y_0} with $\hat{\rho}_{y_0}$ in Theorem 3.3.

5. SIMULATION RESULTS

This section describes simulation experiments carried out to examine the coverage properties of the empirical likelihood confidence intervals for b_0 and y_0 proposed in the previous sections. The following simple linear regression model was treated:

$$y_i = 1 + x_i + \varepsilon_i, \qquad i = 1, ..., n.$$

The data set x_i was the one which has been displayed in Chen [2]. We chose sample sizes $n = 15$, 30, 50 and nominal coverage level $\alpha = 0.90$, 0.95.

We assigned two error patterns for ε_i. One was $\varepsilon_i = N(0, 1)$ and another was $\varepsilon_i = E(1.00) - 1.00$, where $N(0, 1)$ and $E(1.00)$ were random variables with the standard normal distribution and the exponential distribution with unit mean, respectively. The normal and exponential random variables were generated by the routines of Press *et al.* [5].

For each combination of n, α, and ε_i we display in Table I the coverages of the uncorrected confidence intervals and two Bartlett corrected confidence intervals based on 10,000 simulations. One of the corrected confidence intervals used the theoretical Bartlett correction; another used the empirical Bartlett correction. Standard errors are given for each of the simulated coverages. To empirically justify the expansions developed in Theorems 3.2 and 4.2, we also calculated theoretical coverages up to second order in Edgeworth expansions for $l(b_0)$ and $l(y_0)$. Since the coverages can be obtained without simulation, they are called "predicted coverages".

The following broad conclusions may be drawn from the results summarized in Table I. First, the differences between the uncorrected coverages and their corresponding "predicted coverages" converge to zero as n increases. This gives empirical justification for Theorems 3.2 and 4.2. Second, substantial improvements on coverage accuracy have been made by implementing Bartlett corrections. This can be observed by looking at

TABLE I

Estimated True coverages, from 10,000 Simulations, of α-level Empirical Likelihood Confidence Regions for b_0 and y_0's

	ε_i	$N(0, 1)$		$E(1.00) - 1.00$	
n	α	0.90	0.95	0.90	0.95
		(1) Coverages for slope parameter b_0			
15	predic.	0.840	0.909	0.750	0.849
	uncorr.	0.803 (0.40)	0.860 (0.35)	0.789 (0.41)	0.858 (0.35)
	ρ_{b_0}	0.859 (0.35)	0.911 (0.28)	0.904 (0.30)	0.950 (0.22)
	$\hat{\rho}_{b_0}$	0.853 (0.35)	0.906 (0.29)	0.856 (0.35)	0.916 (0.28)
30	predic.	0.878	0.935	0.845	0.913
	uncorr.	0.862 (0.35)	0.919 (0.27)	0.840 (0.37)	0.902 (0.30)
	ρ_{b_0}	0.884 (0.32)	0.935 (0.25)	0.880 (0.31)	0.931 (0.24)
	$\hat{\rho}_{b_0}$	0.883 (0.32)	0.934 (0.25)	0.871(0.34)	0.928 (0.26)
50	predic.	0.888	0.939	0.870	0.930
	uncorr.	0.882 (0.32)	0.9386 (0.24)	0.860 (0.35)	0.926 (0. 26)
	ρ_{b_0}	0.896 (0.31)	0.948 (0.22)	0.887 (0.32)	0.944 (0.23)
	$\hat{\rho}_{b_0}$	0.896 (0.31)	0.948 (0.22)	0.880 (0.33)	0.938 (0.24)

(*Table continued*)

TABLE 1–*Continued*

ε_i		$N(0, 1)$		$E(1.00) - 1.00$	
n	α	0.90	0.95	0.90	0.95
	(2) Coverages for intercept parameter a_0				
15	predic.	0.858	0.921	0.802	0.884
	uncorr.	0.822 (0.38)	0.884 (0.32)	0.805 (0.40)	0.868 (0.34)
	ρ_{y0}	0.861 (0.35)	0.918 (0.27)	0.883 (0.32)	0.927 (0.26)
	$\hat{\rho}_{y0}$	0.857 (0.35)	0.915 (0.28)	0.848 (0.36)	0.900 (0.30)
30	predic.	0.880	0.937	0.865	0.921
	uncorr.	0.864 (0.34)	0.922 (0.27)	0.840 (0.37)	0.901 (0.30)
	ρ_{y0}	0.888 (0.32)	0.937 (0.24)	0.874 (0.33)	0.933 (0.25)
	$\hat{\rho}_{y0}$	0.884 (0.32)	0.936 (0.24)	0.863 (0.34)	0.922 (0.27)
50	predic.	0.887	0.941	0.871	0.931
	uncorr.	0.883 (0.32)	0.933 (0.25)	0.860 (0.35)	0.920 (0.27)
	ρ_{y0}	0.894 (0.31)	0.942 (0.23)	0.884 (32)	0.942 (0.23)
	$\hat{\rho}_{y0}$	0.894 (0.31)	0.942 (0.23)	0.877 (0.33)	0.933 (0.25)
	(3) Coverages for mean parameter y_0 with $x_0 = 5.00$				
15	predic.	0.865	0.926	0.840	0.909
	uncorr.	0.837 (0.37)	0.899 (0.30)	0.815 (0.39)	0.869 (0.34)
	ρ_{y0}	0.871 (0.34)	0.924 (0.27)	0.868 (0.34)	0.908 (0.29)
	$\hat{\rho}_{y0}$	0.867 (0.34)	0.922 (0.27)	0.846 (0.36)	0.893 (0.31)
30	predic.	0.885	0.940	0.875	0.934
	uncorr.	0.882 (0.32)	0.936 (0.25)	0.861 (0.35)	0.922 (0.27)
	ρ_{y0}	0.897 (0.30)	0.946 (0.23)	0.884 (0.32)	0.938 (0.24)
	$\hat{\rho}_{y0}$	0.897 (0.30)	0.946 (0.23)	0.876 (0.33)	0.932 (0.25)
50	predic.	0.889	0.943	0.879	0.936
	uncorr.	0.887 (0.32)	0.937 (0.24)	0.871 (0.33)	0.923 (0.27)
	ρ_{y0}	0.898 (0.30)	0.945 (0.23)	0.891 (0.31)	0.939 (0.25)
	$\hat{\rho}_{y0}$	0.897 (0.30)	0.944 (0.23)	0.884 (0.32)	0.933 (0.25)

Note. Rows headed "predic.," "uncorr.," "b_0" or "y_0" and "$\hat{b}_0$" or "$\hat{y}_0$" give the predicted, uncorrected, and Bartlett-corrected coverages, respectively. The figures in parentheses are 10^2 times the standard errors associated with the simulated coverages.

both the standard errors and the absolute errors. Third, the empirical Bartlett correction performs similarly to its theoretical Bartlett correction counterpart, except for small sample skewed case.

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