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Stability of Singular Equilibria in Quasilinear Implicit Differential Equations

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This paper addresses stability properties of singular equilibria arising in quasilinear implicit ODEs. Under certain assumptions, local dynamics near a

and strong ones. Stability in the weak case is analyzed through certain linear matrix equations, a singular version of the Lyapunov equation being especially relevant in the study. Weak stable singularities include singular zeros having a spherical domain of attraction which contains other singular points. Regarding strong equilibria, stability is proved via a Lyapunov–Schmidt approach under additional hypotheses. The results are shown to be relevant in singular root-finding problems. © 2001 Academic Press

1. INTRODUCTION

Quasilinear implicit (also called *linearly implicit* or *quasilinear*) ordinary differential equations have been paid considerable attention in the last decade [32, 37–39, 43–45, 49]. These equations are defined by a differential system

$$A(x)\,\dot{x} = f(x), \tag{1}$$

where $A \in C^k(\mathbb{R}^n, \mathbb{R}^{n\times n})$, $\mathbb{R}^{n\times n}$ being the set of all $n\times n$ real matrices, and $f \in C^l(\mathbb{R}^n, \mathbb{R}^n)$, with $k, l \geqslant 2$.

From a mathematical point of view, it is of interest to investigate the dynamic behavior of (1) near singular points x^*, where $A(x^*)$ is non-invertible. If $A(x)$ has constant rank $r<n$ on a neighborhood of x^*, the equation can be reduced to a regular system in the theory of differential-algebraic equations (DAEs) [5, 7, 8, 23]. In this paper, the interest is focused on the case in which $A(x)$ is singular on a hypersurface including x^*. Under this hypothesis, existence of solutions, normal forms and phase

portraits have been analyzed in the neighborhood of some types of singularities [32, 37, 43, 49]. This case can also be reformulated as a singular semiexplicit DAE: in this framework, existence of solutions and certain qualitative properties have been studied in [48, 51, 52]. Computational aspects are addressed in [33, 39, 45, 48], and applications have been reported in fields such as nonlinear electrical circuits, bifurcation theory, phase transitions, and plasticity, among others [37–39, 43, 48].

Our purpose is to investigate the local behavior of (1) near singular equilibrium points. Non-singular equilibria of constantly rank-deficient DAEs have been studied in [42]. Relevant results can also be found in [31] and references therein. However, the qualitative analysis of singular equilibria addressed in this paper is not covered by the above-mentioned references, as will be shown later. Specifically, the characterization of local dynamics near singular equilibria through a continuous or directionally continuous vector field will be discussed. This study will motivate a classification of singular zeros (more generally, of geometric singularities [38]) into *weak* and *strong* ones. Stability of weak and strong equilibria will then be analyzed through certain singular matrix equations and a Lyapunov–Schmidt decomposition, respectively. The results in this paper show that certain systems display convergence to a singular equilibrium from n-dimensional invariant sets. This behavior is somewhat complementary to the one described by the *singularity induced bifurcation* theorem [51, 52], where, under certain conditions, convergence may be expected only from an invariant m-dimensional manifold with $m \leqslant n-1$.

The motivation for this study comes from the similarity between equation (1) and several continuous-time methods for singular root-finding problems. The continuous-time setting has been used to model analogues of root-finding algorithms [1, 3, 9, 18, 36, 46, 50, 54], and also to formulate new methods for this kind of problems [4, 25, 47]. These schemes have been further developed in the context of homotopy techniques [17, 26] and, more specifically, as *trajectory methods* (see [16] for a survey). Continuous models display a better behavior regarding global problems: specifically, difficulties arising when trajectories of Newton-based methods approach singular points in the search of regular roots can be overcome. In this framework, a unique continuous system may lead to different iterative techniques, including damped and accelerated versions of basic methods, through the use of different integration schemes. This approach shifts the convergence analysis of these iterations to a stability study of the continuous system and the discretization method, and therefore motivates a stability analysis of equilibria in the continuous-time context. The effect of discretization through some specific numerical schemes will also be mentioned in this work, although a complete discretization study is beyond its scope.

In the context of continuous-time techniques, the continuous Newton method is paradigmatic, as the aforementioned references illustrate. This method is defined by the quasilinear implicit equation

$$-J(x)\,\dot{x}=f(x), \tag{2}$$

the matrix $J \in C^{l-1}(\mathbb{R}^n, \mathbb{R}^{n\times n})$ being the Jacobian of f. The above-indicated stability results are therefore of interest when addressing singular root finding problems via continuous-time methods. Furthermore, the results should be significant regarding other continuous implicit methods and discretizations of (2). In particular, the classical Newton method can be obtained through forward Euler discretization of Eq. (2) with stepsize 1. Following this approach, stability of weak equilibria provides counterexamples to the usual assumption that the domain of convergence of Newton's method, when applied to singular roots, should always exclude other singularities [20, 21]. Directional stability of strong equilibria yields Reddien's theorem on local convergence to a singular zero from a cone-shaped region with vertex in the root [40]. Similarly, an accelerated modification of Newton's method reported in [29], yielding quadratic convergence to singular roots, may be modeled as a 3-stage explicit Runge–Kutta integration of the continuous Newton method (2). Further studies could extend these results to more general situations and other accelerated versions of the Newton method [10–13, 22, 27, 29, 41], as well as to other root-finding techniques for singular problems [14, 15, 28, 30].

The paper is structured as follows: in this section, we compile some algebraic facts and notational conventions, and the literature on quasilinear implicit differential equations is briefly reviewed. The relation with low-index singular DAEs is addressed. It is specifically shown that singularities studied in previous works [32, 37, 43, 44, 49] do not cover the equilibrium case analyzed here. In Section 2, the above-mentioned classification of singularities into weak and strong ones is carried out. The possibility of defining a continuous or directionally continuous vector field which describes the dynamics near a singular point is analyzed in light of this taxonomy. Necessary and sufficient conditions for the extended field to have an equilibrium point at the singularity are presented (Theorem 1). Stability of weak and strong singular equilibria is studied, together with some discretization issues and a few illustrative examples, in Sections 3 and 4. The weak case is analyzed through a singular version of the Lyapunov matrix equation (Theorem 2) and the study of a certain linear matrix equation (Theorem 3), while strong equilibria are analyzed via a Lyapunov–Schmidt approach (Theorem 4). Finally, concluding remarks appear in Section 5.

1.1. *Algebraic preliminaries*

We will make frequent use of some basic results involving matrices and determinants, which are compiled here for later reference. Let us first indicate that capital Latin letters (e.g., A, B) will usually be employed to denote matrices, while lower-case Latin letters (f, g, v, x) represent vectors, and, finally, Greek letters (α, β), as well as subscripted lower case Latin letters (x_1, x_2) correspond to scalars. To avoid violating some usual conventions, a few exceptions apply: the symbols i, j, k, l, m, n will be used for integers, while the time variables t, s are real scalars, and N as well as X denotes linear subspaces of $\mathbb{R}^n$.

A dotted symbol like $\dot{x}$ indicates time derivative, while a prime is used to represent differentiation (e.g., $f'(x)$, $f''(x)$). Sometimes the first differential will be represented by the Jacobian matrix, which can be given a distinguished letter, as in $J(x) = f'(x)$.

If A is a $n \times n$ matrix, we have

$$\det A\ I_n = \operatorname{Adj} A\ A = A\ \operatorname{Adj} A, \tag{3}$$

where Adj A is the adjoint matrix (transpose of the matrix of cofactors) of A. In particular, if A is a singular matrix, Adj A $v \in \operatorname{Ker} A$ for any vector $v \in \mathbb{R}^n$. The matrices A and Adj A also verify the relations

$$\operatorname{rk} A = n \Leftrightarrow \operatorname{rk} \operatorname{Adj} A = n \tag{4a}$$

$$\operatorname{rk} A = n-1 \Leftrightarrow \operatorname{rk} \operatorname{Adj} A = 1 \tag{4b}$$

$$\operatorname{rk} A \leqslant n-2 \Leftrightarrow \operatorname{rk} \operatorname{Adj} A = 0. \tag{4c}$$

If A has rank $n-1$, from (3) and (4b) we get that any non-zero column of Adj A expands the one-dimensional space Ker A, and that $v \in \mathbb{R}^n$ belongs to the space Rg A if and only if Adj A $v = 0$. Furthermore, if the zero eigenvalue of A is simple, then $\mathbb{R}^n = \operatorname{Ker} A \oplus \operatorname{Rg} A$, and multiplication by Adj A is equivalent to the projection onto Ker$A$ parallel to Rg A, except for a non-vanishing scalar.

Considering a matrix function $A(x)$, we can differentiate (3) to get

$$((\det A)'\ (x) \cdot\ )\ I_n = ((\operatorname{Adj} A)'\ (x) \cdot\ )\ A(x) + \operatorname{Adj} A(x)\ A'(x) \cdot, \tag{5}$$

where the dot $\cdot$ stands for the vector argument of the differential map. In particular, if a singular point x^* verifies $(\det A)'(x^*) \neq 0$, it follows that $\dim \operatorname{Ker} A(x^*) = 1$ or, equivalently, $\operatorname{rk} A(x^*) = n-1$ [37]. Also, if we consider a product

$$g(x) = A(x)\ f(x),$$

we have

$$g'(x)\cdot = (A'(x)\cdot)\ f(x) + A(x)\ f'(x)\cdot \tag{6}$$

$$g''(x)\cdot\cdot = (A''(x)\cdot\cdot)\ f(x) + 2(A'(x)\cdot)\ f'(x)\cdot + A(x)\ f''(x)\cdot\cdot, \tag{7}$$

the latter expression holding for symmetric argument. The following property is often useful in matrix stability analysis: if $\lambda_1, ..., \lambda_n, \alpha_1, ..., \alpha_n$ are the eigenvalues of the matrices H and $\frac{1}{2}(H + H^T)$, respectively, it can be shown that $\min_i \alpha_i \leqslant \mathrm{Re}(\lambda_j) \leqslant \max_i \alpha_i$ for all $j = 1, ..., n$ [34, Proposition 5.4.10]. In particular, if $\frac{1}{2}(H + H^T)$ is a negative definite (resp. semidefinite) matrix, then H must necessarily be a stable (resp. semistable [35]) matrix, that is, all of its eigenvalues have negative (resp. non-positive) real part.

Finally, if we consider the set of *isotropic* vectors associated with a (symmetric) semidefinite matrix C

$$\mathrm{Is}\ C = \{v \in \mathbb{R}^n : v^T C v = 0\},$$

it may be easily shown, through the canonical form of C under congruence, that $\mathrm{Is}\ C$ is a linear space of codimension $\mathrm{rk}\ C$.

1.2. *Quasilinear Implicit ODEs and Singular Index*-0 *DAEs*

The dynamic behavior of system (1) around a point x^* is strongly determined by the local properties of the matrix $A(x)$. On a neighborhood of points with regular (invertible) A, the problem is locally reduced to the explicit system

$$\dot{x} = A(x)^{-1}\ f(x) \equiv h(x),$$

while the case in which $A(x)$ is singular with constant rank on a whole neighborhood of x^* leads to a standard problem in the theory of differential-algebraic equations, as indicated above. The attention in this work is restricted to cases in which $A(x)$ is singular on a hypersurface Ψ, with $x^* \in \Psi$. This occurs if x^* is a *noncritical singular point* [37], that is, if the condition $(\det A)'\ (x^*) \neq 0$ is satisfied. Singular points will be assumed to be noncritical in this paper. Note that, in this case, we have $\dim \mathrm{Ker}\ A(x^*) = 1$, as indicated in Subsection 1.1. This setting implies that every (sufficiently small) neighborhood of x^* includes an open dense subset where (1) may be written as an explicit ODE, defining this system as a singular index-0 DAE around x^* [5].

In Rabier's seminal work [37] it is shown that differentiation procedures together with the introduction of additional variables allows the reduction of general implicit differential equations to the quasilinear form indicated in (1). The properties of this equation are then related with those of the associated *canonical system*

$$\det A(x)\,\dot{x} = \operatorname{Adj} A(x)\, f(x),$$

which can be rewritten, with the notation $g(x) = \operatorname{Adj} A(x)\, f(x)$, $\omega(x) = \det A(x)$, as

$$\omega(x)\,\dot{x} = g(x). \tag{8}$$

It is easily proved that every solution of (1) is also a solution of the canonical system (8) and, conversely, every solution $x(t)$ of (8) such that $\omega(x(t)) \neq 0$ for t verifying $0 < |t| < |\tilde{t}|$ (for some $\tilde{t} \neq 0$) is also a solution of (1).

A taxonomy of singularities introduced in [38, 39] will help to clarify the rest of the exposition. Singular points x^* where $f(x^*) \in \operatorname{Rg} A(x^*)$ are called *geometric singularities*, while at *algebraic singularities* it is $f(x^*) \notin \operatorname{Rg} A(x^*)$. As indicated in Subsection 1.1, under the assumption $(\det A)'\,(x^*) \neq 0$ these conditions are equivalent to $\operatorname{Adj} A(x^*)\, f(x^*) = 0$ and $\operatorname{Adj} A(x^*)\, f(x^*) \neq 0$, respectively. Therefore, noncritical geometric singularities are those which satisfy $g(x^*) = 0$, while noncritical algebraic singularities verify $g(x^*) \neq 0$.

Rabier analyzes in [37] the dynamic behavior near *standard singular points*, namely, singular points x^* where $\omega'(x^*)\, g(x^*) \neq 0$ (implying that x^* is noncritical and algebraic). He shows that (8) has exactly two solutions $x(t) \in C^0([0, \tilde{t}\,], \mathbb{R}^n) \cap C^1((0, \tilde{t}\,], \mathbb{R}^n)$ verifying $x(0) = x^*$, both defined either for some $\tilde{t} > 0$ or $\tilde{t} < 0$ ($[0, \tilde{t}\,]$ standing in the latter case for $[\tilde{t}, 0]$). These solutions verify that $\|\dot{x}(t)\| \to \infty$ as $t \to 0$. The singularity x^* behaves as an *impasse point*, where solutions are no longer defined, being either a “repelling” (“inaccessible” in [38]) or an “attracting” (“accessible”) point if $\tilde{t} > 0$ or $\tilde{t} < 0$, respectively.

In this paper, we introduce a less restrictive definition of standard singular point, which also applies to geometric singularities. Noting that $g(x^*) = \operatorname{Adj} A(x^*) f(x^*) \in \operatorname{Ker} A(x^*)$, we can define an arbitrary singular point as standard if $(\det A)'\,(x^*)\, v \neq 0$ for any $v \in \operatorname{Ker} A(x^*) - \{0\}$. Remark that, in the algebraic case ($g(x^*) \neq 0$), this is reduced to Rabier's definition, since $\dim \operatorname{Ker} A(x^*) = 1$ and, therefore, $(\det A)'\,(x^*)\, v \neq 0 \Leftrightarrow (\det A)'\,(x^*)$ $g(x^*) \neq 0$.

Further works [43, 44] extend the results of Rabier to algebraic singularities which may be critical, that is, to situations in which $g(x^*) \neq 0$ but not necessarily $\omega'(x^*)\, g(x^*) \neq 0$, giving normal forms for system (1). Other authors [32, 49] address the geometric case ($g(x^*) = 0$) under the assumption of hyperbolicity of $g'(x^*)$.

Our interest is focused on singular equilibrium points of system (1). If $f(x^*) = 0$, the trajectory $x(t) = x^*$ satisfies equation (1) and, conversely, any steady state $\dot{x} = 0$ must necessarily correspond to a zero of f. Therefore, these equilibria are defined by the condition $f(x^*) = 0$, implying that $g(x^*) = 0$ (hence being geometric singularities) and

$$g'(x^*) = \text{Adj}\, A(x^*)\, J(x^*),$$

with $J(x^*) \equiv f'(x^*)$. Since $A(x^*)$ is singular, $\text{Adj}\, A(x^*)$ and therefore $g'(x^*)$ will also be singular and, hence, non-hyperbolic. This shows that the situation $f(x^*) = 0$ is not covered in the above-mentioned references and requires an specific study. The relevance of this case is apparent regarding singular root-finding problems, as discussed above.

In the specific case of the continuous Newton method (2), some authors [36, 53] report the presence of finite time trajectories which cease to exist beyond certain singular points. They remark the incidence of this effect in the global behavior of the system and, specifically, in the picture of attraction domains of regular roots. This phenomenon makes attraction domains more intricate in the discrete-time setting of the classical Newton method (obtained after Euler integration), due to fake numerical orbits not corresponding to any trajectory of the continuous system. This behavior can be properly identified as that of impasse points in the context of quasilinear equations, and provides additional motivation for the continuous-time study carried out here.

1.3. *Singular Index*-1 *Semiexplicit DAEs*

System (1) may be rewritten, after the introduction of an additional variable v, as

$$\dot{x} = v \tag{9a}$$

$$0 = A(x)\, v - f(x). \tag{9b}$$

This reformulation falls in the general framework of semiexplicit DAEs

$$\dot{x} = y(x, v) \tag{10a}$$

$$0 = z(x, v). \tag{10b}$$

Equation (10b) represents a *solution manifold* where the solutions of the DAE live. From a local point of view, if we consider a point (x^*, v^*) in the solution manifold, the assumption that $z_v(x^*, v^*)$ is an invertible matrix defines (10) as an *index*-1 problem. In this situation, (10b) defines locally a smooth manifold, and there exists a function $\tilde{z}$ verifying $z(x, v) = 0 \Leftrightarrow v = \tilde{z}(x)$ on a neighborhood of (x^*, v^*). The dynamics on this manifold may then be described, using x-coordinates, by $\dot{x} = y(x, \tilde{z}(x))$.

Singularities of semiexplicit index-1 DAEs occur at points (x^*, v^*) of the solution manifold such that $z_v(x^*, v^*)$ is singular but for which there exist arbitrarily close points where z_v is invertible. These points yield singularities also in the quasilinear *underlying ODE* [5], obtained after differentiation of the constraint (10b):

$$\dot{x} = y(x, v)$$

$$z_v(x, v)\,\dot{v} = -z_x(x, v)\,y(x, v).$$

The results of this paper are therefore of interest also in the context of singular index-1 semiexplicit DAEs. Specifically, singular equilibria of (10) are defined by the conditions $y(x^*, v^*) = z(x^*, v^*) = 0$ with $z_v(x^*, v^*)$ noninvertible, and encompass singular equilibrium points of the quasilinear problem (9) as a particular case. In this direction, the work of Venkatasubramanian *et al.* [51, 52] is particularly relevant, describing the qualitative behavior near certain singularities. However, the assumptions supporting the analysis there are not satisfied in the cases considered in the present paper, as discussed in Subsection 2.2, and lead to a behavior substantially different from the one discussed here.

2. GEOMETRIC SINGULARITIES AND SINGULAR EQUILIBRIA

In this section, we present a taxonomy of geometric singular points which describes different dynamic behaviors around these singularities. Since singular points are assumed noncritical throughout the paper, geometric singularities are those where $g(x) = 0$. The taxonomy is oriented to a local dynamic study of system (1) near a singular zero x^*, which motivates the introduction of a continuity requirement for the field

$$h(x) = \frac{g(x)}{\omega(x)} = \frac{\operatorname{Adj} A(x)\; f(x)}{\det A(x)} \tag{11}$$

at x^*. Note that h is a C^m vector field on the set of regular points, with $A \in C^k(\mathbb{R}^n, \mathbb{R}^{n\times n})$, $f \in C^l(\mathbb{R}^n, \mathbb{R}^n)$, and $m = \min\{k, l\}$.

Weak singularities are points where the field h is smoothly defined, while at strong singularities at most a directionally continuous extension of h may be expected. A characterization of singular zeros yielding equilibria of the field h, in either a continuous or directionally continuous sense, will be given in Theorem 1. This requirement is verified by all singular zeros in the particular case of the continuous Newton method.

2.1. *A Taxonomy of Geometric Singularities*

As indicated in Subsection 1.2, singular zeros, where $f(x^*) = 0$ and $\omega(x^*) = \det A(x^*) = 0$, are necessarily geometric singularities. There exists another type of geometric singular points, defined by

$$f(x^*) \neq 0, \; g(x^*) = \operatorname{Adj} A(x^*) \; f(x^*) = 0.$$

These points are termed *extraneous singularities* in the context of root-finding methods [4], but this concept can be naturally applied to general quasilinear implicit equations. In fact, singular points studied in [32, 49] fall in the framework of extraneous singularities, since the hyperbolicity assumption on $g'(x^*)$ excludes singular zeros, as shown in Subsection 1.2.

Another classification of geometric singularities can be considered. A geometric singular point x^* is said to be a *weak singularity* if there exists a neighborhood U^{x^*} where $\omega(x) = 0 \Rightarrow g(x) = 0$, that is, if there exists a singular neighborhood $U^{x^*} \cap \Psi$ of x^* entirely formed by geometric singularities. Geometric singular points which fail to satisfy this condition are called *strong singularities*, being accumulation points of the set of algebraic singularities. Since we are restricted to the noncritical case, a singular point is a weak singularity if and only if $f(x) \in \operatorname{Rg} A(x)$ for all x in a neighborhood of x^*.

The motivation for this taxonomy comes from the fact that, for $h(x)$ to be smoothly defined on an entire neighborhood of a singular point x^*, it must be $g(x) = 0$ at every singular point in this neighborhood, that is, x^* must necessarily be a weak singularity. This is due to the fact that, at algebraic singularities $\hat{x}$, it is $\lim_{x \to \hat{x}} \|h(x)\| = \infty$ and, therefore, these points must be excluded from any domain of smooth definition of h. The converse is also true in the noncritical case: it may be proved that h is defined as a C^{m-1} vector field on a neighborhood of x^* if this is a noncritical weak singularity [46]. It follows that weak singular points encompass situations where the domain of smooth definition of the field is larger than the domain of invertibility of A, against some common assumptions in the context of singular root-finding problems [20, 21].

No smooth extension of h on an entire neighborhood is possible if x^* is a strong singularity. We may however study if h can be continuously defined at x^* along some region excluding algebraic singularities. In the context of root-finding problems, different domains have been studied for this purpose, depending on the type of singularity [10, 21, 40]. Specifically, cone-shaped regions with vertex in the root have been proved specially relevant in the noncritical case [40]. The term *cone* will be used throughout the paper to represent a set

$$\mathscr{K}_\theta = \{x \in \mathbb{R}^n : \|P_X(x - x^*)\| \leqslant \theta \, \|P_N(x - x^*)\|\},$$

for some norm in $\mathbb{R}^n$ and some real number $\theta > 0$. In this expression, P_X (resp. P_N) denotes a projection onto a $(n-1)$-dimensional (resp. 1-dimensional) linear space X (resp. N) parallel to N (resp. X), N being transversal to X (that is, $N \cap X = \{0\}$). Intersections of balls and cones will be represented as

$$\mathscr{W}_{\theta,\rho} = \{x \in \mathbb{R}^n : \|P_X(x - x^*)\| \leqslant \theta \, \|P_N(x - x^*)\|, \ \|x - x^*\| \leqslant \rho\}.$$

Usually, N will also be transversal to the tangent space $T_{x^*}\Psi$, implying that $\mathscr{W}_{\theta,\rho} \cap \Psi = \{x^*\}$ for sufficiently small θ and ρ.

Note that we may consider $\mathscr{K}_\infty = \mathbb{R}^n$, and continuity of h along $\mathscr{K}_\infty$ corresponds to the weak case mentioned above. The expression *continuously defined* will be used, in a broad sense, for singularities x^* where the field h is defined along some cone $\mathscr{K}_\theta$. In the noncritical weak case, it is always possible to make this definition with $\theta = \infty$. On the contrary, strong singularities may or may not accept a continuous definition along some cone; when they do, it is necessarily with $\theta < \infty$ and the resulting field will be termed *directionally defined.*

Another important issue concerns the value that h takes at continuously defined singularities. This is addressed in the next subsection for the specific case of singular zeros.

2.2. *Continuously Defined Singular Equilibrium Points*

We address now the problem of the characterization of weak and strong singularities yielding continuously defined equilibria of h. Note that any kind of continuous extension of h with $h(x^*) = 0$ requires $f(x^*) = 0$. A converse result is shown in the next theorem.

THEOREM 1. *Let x^* be a noncritical singular equilibrium of system* (1), *where $f(x^*) = 0$ with $\omega(x^*) = \det A(x^*) = 0$ and $\omega'(x^*) = (\det A)'\,(x^*) \neq 0$. The limit*

$$\lim_{x \to x^*} h(x)$$

exists and equals 0 *along some cone $\mathscr{K}_\theta$ with $0 < \theta \leqslant \infty$ if and only if* $\operatorname{Rg} J(x^*) \subseteq \operatorname{Rg} A(x^*)$.

Proof. write

$$\lim_{x\to x^*} h(x) = \lim_{x\to x^*} \frac{g(x)}{\omega(x)} = \lim_{x\to x^*} \frac{g'(x^*)(x-x^*)+O(\|x-x^*\|^2)}{\omega'(x^*)(x-x^*)+O(\|x-x^*\|^2)}, \tag{12}$$

where g, ω are at least C^2 since $A \in C^k(\mathbb{R}^n, \mathbb{R}^{n\times n})$ and $f \in C^l(\mathbb{R}^n, \mathbb{R}^n)$, with $m = \min\{k, l\} \geqslant 2$. Note that $f(x^*) = 0 \Rightarrow g'(x^*) = \operatorname{Adj} A(x^*)\, J(x^*)$. We may take any cone transversal to the singular set, that is, with θ sufficiently small to assure $\omega'(x^*)\, v \neq 0$ for any direction v inside the cone. If we consider an open set of these directions, along the straight lines $x = x^* + tv$ it is

$$\lim_{x\to x^*} \frac{\operatorname{Adj} A(x^*)\, J(x^*)(x-x^*)+O(\|x-x^*\|^2)}{\omega'(x^*)(x-x^*)+O(\|x-x^*\|^2)}$$

$$= \lim_{t\to 0} \frac{t \operatorname{Adj} A(x^*)\, J(x^*)\, v + O(t^2)}{t\omega'(x^*)\, v + O(t^2)},$$

expression which equals zero for any v only if $\operatorname{Adj} A(x^*)\, J(x^*) = 0$ or, equivalently, if $\operatorname{Rg} J(x^*) \subseteq \operatorname{Rg} A(x^*)$. Conversely, the assumption $\operatorname{Rg} J(x^*) \subseteq \operatorname{Rg} A(x^*)$ yields an order 2 term in the numerator of (12), showing that $\lim_{x\to x^*} h(x) = 0$. ∎

It is remarkable that this result holds independently of the weak or strong nature of the singular zero x^*. If we consider standard strong equilibria (see Section 4), where $\omega'(x^*)\, v \neq 0$ for $v \in \operatorname{Ker} A(x^*) - \{0\}$, cones with axis $N = \operatorname{Ker} A(x^*)$ will be shown to be particularly relevant.

Note that, for h to have a continuously defined singular equilibrium at x^*, the Jacobian matrix $J(x^*)$ must also be singular. It is worth mentioning that the condition $\operatorname{Rg} J(x^*) \subseteq \operatorname{Rg} A(x^*)$ implies that $\lambda A(x^*) - J(x^*)$ is a singular matrix pencil, making the results of [31] non-applicable to this case. Similarly, this condition avoids system (1) from satisfying the hypotheses of the singularity induced bifurcation theorem [51, 52], which is based on the non-vanishing of the so-called *transformed field* at the equilibrium (see [51]). This suggests that a different behavior may be expected, as shown in the following sections.

In the specific case of the continuous Newton method, the condition $A(x) = -J(x)$ makes the following corollary immediate:

COROLLARY 1. *Noncritical singular zeros of f always lead to continuously defined equilibria of the continuous Newton method.*

Finally, another important issue concerning singular equilibrium points is that of their isolation. As pointed out by Keller [27], the condition

$$\operatorname{Adj} J(x^*) f''(x^*) vv \neq 0, \qquad v \in \operatorname{Ker} J(x^*) - \{0\},$$

suffices to prove that x^* verifying $f(x^*) = 0$ with $J(x^*)$ singular is an isolated zero of f. This condition is equivalent to the transversality of the vector $v \in \operatorname{Ker} J(x^*) - \{0\}$ and the tangent space $T_{x^*} \Psi_J$ (Ψ_J being the set of points where $J(x)$ is singular), since from (5) it follows that

$$((\det J)'(x^*) v) v = \operatorname{Adj} J(x^*) f''(x^*) vv \neq 0,$$

which implies $(\det J)'(x^*) v \neq 0$. Equilibrium points verifying this condition will be called *transversal*. Note that, in particular, it is $(\det J)'(x^*) \neq 0$ and, therefore, $\operatorname{Ker} J(x^*)$ is one-dimensional. In this case, the condition $\operatorname{Rg} J(x^*) \subseteq \operatorname{Rg} A(x^*)$ can be rewritten as $\operatorname{Rg} J(x^*) = \operatorname{Rg} A(x^*)$.

In the particular case of the continuous Newton method, since $A(x) = -J(x)$, standard singular equilibrium points are transversal and vice-versa. In general, we will deal with standard singular equilibria, assuming transversality and the condition $\operatorname{Rg} J(x^*) = \operatorname{Rg} A(x^*)$.

3. STABILITY OF WEAK EQUILIBRIA

In this section, asymptotic stability of standard weak singular equilibria is studied under the transversality hypothesis and assuming $\operatorname{Rg} J(x^*) = \operatorname{Rg} A(x^*)$. Note that the weak condition defines h as a smooth vector field on a neighborhood of x^*, while the assumption $\operatorname{Rg} J(x^*) = \operatorname{Rg} A(x^*)$ leads to $h(x^*) = 0$.

Asymptotic stability of x^* will be analyzed through the matrix $H(x^*) = h'(x^*)$. The notation $A = A(x^*)$, $H = H(x^*)$, etc. will be used for abbreviation. Although the semistable case [35] does not necessarily imply stability of the equilibrium point x^*, it is also discussed for the sake of completeness. A singular counterpart of the Lyapunov matrix equation is central in the study, and sufficient conditions for semistability and stability of H are presented in Theorem 2. The results on the case $\operatorname{Ker} A = \operatorname{Ker} J$ can be extended through the spectral analysis of the linear matrix equation $AQ = J$, as it is done in Theorem 3.

In particular, weak singular equilibria of the continuous Newton method fulfill the hypotheses of Theorems 2 and 3, their stability being obtained as a simple corollary of these results. This corollary shows that domains of convergence of Newton method are not necessarily included in the domain of invertibility of the Jacobian matrix. Previous results in this direction can be seen in [46], where asymptotic stability is proved at weak singular zeros using the Lyapunov function $\| f(x) \|^2$. An example generalized from the continuous Newton method is provided with illustrative purposes. The ideas here presented will give some hints for the analysis of the more involved case of strong equilibria, which will be carried out in Section 4.

3.1. *Linear Stability of Weak Equilibria*

As mentioned above, our interest is focused on the local behavior of system (1) around a weak zero x^* with $h(x^*) = 0$. The smoothness of h gives

$$A(x)\, h(x) = f(x),$$

and, differentiating as in (6), the condition $h(x^*) = 0$ allows to write at x^* the matrix identity

$$AH = J. \tag{13}$$

Our goal is to derive stability properties of H from those of A and J. Recall that a matrix H is *stable* (resp. *semistable*) if all of its eigenvalues λ_i satisfy $\mathrm{Re}(\lambda_i) < 0$ (resp. $\mathrm{Re}(\lambda_i) \leqslant 0$) [24, 35], and note that stable matrices are also semistable.

Premultiplying (13) by A^T, we get

$$A^T A H = A^T J,$$

and, following Ostrowsky and Schneider [35], we obtain that the condition $\alpha_i < 0$ for every eigenvalue of the symmetric matrix $H + H^T$ (which in particular implies the stability of H, as indicated in Subsection 1.1) proves that H reverses semistability of symmetric matrices. Equivalently, if B is a positive semidefinite matrix, BH must be a semistable matrix. Therefore, assuming stability of $H + H^T$, the matrix $A^T J$ must be semistable. This fact provides a hint for the search of converse results giving sufficient conditions for stability of H, which will be obtained under the stronger assumption of $A^T J + J^T A$ being negative semidefinite.

To this end, let us write

$$H^T A^T = J^T, \tag{14}$$

and, from Eqs. (13) and (14),

$$A^T A H + H^T A^T A = A^T J + J^T A,$$

which can be rewritten, with $B = A^T A$, $C = A^T J + J^T A$, as

$$BH + H^T B = C. \tag{15}$$

This is a Lyapunov matrix equation with $B = A^T A$ singular. Note that we cannot impose $C = A^T J + J^T A$ being negative definite, since the eigenvalues of $A^T J$ would have negative real part and, hence, this matrix would be regular, against the hypothesis of A and J being singular. Under the requirement of $C = A^T J + J^T A$ being negative semidefinite, standard results [24] are not applicable in a straightforward manner, and therefore a specific analysis is mandatory. Theorem 2 below characterizes semistability of H and gives sufficient conditions for hyperbolicity. It is to be noted that (following the notational convention $A(x^*) = A$, etc.) all the hypotheses are stated at the point x^*. Note also that $\operatorname{Adj} A(x^*)\, v \in \operatorname{Ker} A(x^*)$ for any vector v and, taking $v \neq 0$ in the one-dimensional space $\operatorname{Ker} A(x^*)$, which satisfies $(\det A)'\,(x^*)\, v \neq 0$ since x^* is standard, we may write

$$\frac{\operatorname{Adj} A(x^*)\, f''(x^*)\, vv}{2(\det A)'\,(x^*)\, v} = \lambda v$$

for some real number λ. The significance of the parameter λ will be pointed out in Section 4, in the context of the Lyapunov–Schmidt reduction presented there.

THEOREM 2. *Let x^* be a (standard and transversal) weak equilibrium point of system* (1), *with* $\operatorname{Rg} J = \operatorname{Rg} A$. *Assume $C = A^T J + J^T A$ is negative semidefinite.*

(1) *If* $\operatorname{Ker} A \neq \operatorname{Ker} J$ *then H is semistable.*

(2) *If* $\operatorname{Ker} A = \operatorname{Ker} J$, *$H$ is semistable if and only if $\lambda < 0$.*

(3) *If* $\operatorname{rk} C = n - 1$ (*implying that* $\operatorname{Ker} A = \operatorname{Ker} J$), *then H is hyperbolic.*

Proof. Let $Hv = \lambda_i v$ for some $\lambda_i \in \mathbb{C}$, $v \in \mathbb{C}^n - \{0\}$. Denoting as $\bar{v}$ the conjugate transpose of v, from (1) we have

$$\lambda_i \bar{v} B v + \overline{\lambda_i} \bar{v} B v = 2\operatorname{Re}(\lambda_i)\, \bar{v} B v = \bar{v} C v. \tag{16}$$

Since $B = A^T A$ and $C = A^T J + J^T A$ are positive and negative semidefinite, respectively, it is $\bar{v} B v \geqslant 0$, $\bar{v} C v \leqslant 0$, and hence we get the implications

$$\operatorname{Re}(\lambda_i) > 0 \Rightarrow \bar{v} B v = 0 \equiv v \in \operatorname{Ker} A \tag{17a}$$

$$\operatorname{Re}(\lambda_i) = 0 \Rightarrow \bar{v} C v = 0 \equiv v \in \operatorname{Is} C. \tag{17b}$$

In the case (17a), we obtain from Eq. (13)

$$Jv = AHv = \lambda_i A v = 0$$

and therefore $v \in \text{Ker}\, J$. As $\text{Ker}\, A$ and $\text{Ker}\, J$ are one-dimensional linear spaces, we conclude that it must be $\text{Ker}\, A = \text{Ker}\, J$. Property (1) is then immediate.

If $v \in \text{Ker}\, A = \text{Ker}\, J$, we may explicitly compute Hv as

$$Hv = \lim_{t \to 0} \frac{h(x^* + tv) - h(x^*)}{t} = \lim_{t \to 0} \frac{\frac{1}{2} t^2 g''(x^*)\, vv + o(t^2)}{t^2 \omega'(x^*)\, v + o(t^2)}$$

but, operating as in (7) with $f(x^*) = 0$ and $v \in \text{Ker}\, J$, we get $g''(x^*)\, vv = \text{Adj}\, A(x^*)\, f''(x^*)\, vv$ and

$$Hv = \frac{\frac{1}{2}\, \text{Adj}\, A(x^*)\ f''(x^*)\, vv}{\omega'(x^*)\, v} = \lambda v.$$

It follows that $\lambda_i = \lambda \in \mathbb{R}$, and the condition $\lambda < 0$ avoids the existence of eigenvalues with positive real part, proving (2). Note that $\text{Ker}\, A = \text{Ker}\, J$ implies $\lambda \neq 0$, since

$$\begin{aligned} \text{Adj}\, A(x^*)\, f''(x^*)\, vv = 0 &\Leftrightarrow f''(x^*)\, vv \in \text{Rg}\, A(x^*) = \text{Rg}\, J(x^*) \\ &\Leftrightarrow \text{Adj}\, J(x^*)\, f''(x^*)\, vv = 0, \end{aligned}$$

which is not verified under the transversality hypothesis.

Finally, C being negative semidefinite implies that $\text{Is}\, C$ is a linear space (see Subsection 1.1), one-dimensional under the hypothesis $\text{rk}\, C = n - 1$. Since $C = A^T J + J^T A$, it is $\text{Ker}\, A \subseteq \text{Is}\, C$ and $\text{Ker}\, J \subseteq \text{Is}\, C$. The one-dimensionality of $\text{Is}\, C$ yields $\text{Is}\, C = \text{Ker}\, A = \text{Ker}\, J$. This space has a non-zero real eigenvalue, as shown above, and property (3) follows. ∎

The result on the case $\text{Ker}\, A = \text{Ker}\, J$ can be extended to situations in which C is not a negative semidefinite matrix, as shown next. Note that eigenvalues are enumerated without multiplicity, such that a multiple one yields $\lambda_i = \lambda_{i+1}$, etc.

THEOREM 3. *Let x^* be a (standard and transversal) weak equilibrium point of system* (1), *with* $\text{Rg}\, A = \text{Rg}\, J$ *and* $\text{Ker}\, A = \text{Ker}\, J$. *Let Q_1, Q_2 be two solutions of $AQ = J$, with eigenvalues $\lambda_i^{(1)}$, $\lambda_i^{(2)}$, $i = 1, ..., n$, respectively. Then* $\text{Ker}\, A = \text{Ker}\, J$ *is an eigenspace of Q_j, $j = 1, 2$, which can be associated without loss of generality with $\lambda_1^{(j)}$ for $j = 1, 2$, and $\lambda_i^{(1)} = \lambda_i^{(2)}$, $i = 2, ..., n$. Therefore, the matrix H is stable (resp. semistable) if and only if $\lambda < 0$ (resp. $\lambda \leqslant 0$) and the eigenvalues of* any *solution of $AQ = J$ verify* $\text{Re}(\lambda_i) < 0$ *(resp.* $\text{Re}(\lambda_i) \leqslant 0$), *for $i = 2, ..., n$.*

Proof. Note first that coincidence of the one-dimensional spaces $\text{Ker}\, A$ and $\text{Ker}\, J$ implies that there exist real numbers $\lambda_1^{(j)}$, $j = 1, 2$, such that

$Q_j v_1 = \lambda_1^{(j)} v_1$ if $v_1 \in \text{Ker } A - \{0\}$, since $J v_1 = 0 = A Q_j v_1 \Rightarrow Q_j v_1 \in \text{Ker } A$ and therefore $Q_j v_1 = \lambda_1^{(j)} v_1$.

Let us then show that $\lambda_i^{(2)} = \lambda_i^{(1)}$ for $i = 2, ..., n$. Fixing i, we can assume that $\lambda_i^{(1)}$ has multiplicity m as an eigenvalue of Q_1 if $\lambda_i^{(1)} \neq \lambda_1^{(1)}$ (that is, $\lambda_i^{(1)} = \lambda_{i+1}^{(1)} = \cdots = \lambda_{i+m-1}^{(1)}$ if $m > 1$), and multiplicity $m+1$ if $\lambda_i^{(1)} = \lambda_1^{(1)}$. Then there exist m linearly independent generalized eigenvectors v_i, $v_{i+1}, ..., v_{i+m-1}$ and positive integers $m_1, ..., m_j$ with $0 < m_1 < m_2 < \cdots < m_j = m$, such that

$$(Q_1 - \lambda_i^{(1)} I_n)\, v_k = \begin{cases} \sigma v_1 & \text{if} \quad k = i \\ 0 & \text{if} \quad k = i + m_1,\ i + m_2, ..., i + m_{j-1} \\ v_{k-1} & \text{if} \quad k = i+1, ..., i + m_1 - 1,\ i + m_1 + 1, ..., i + m_2 - 1, \text{ etc.,} \end{cases}$$

where $\sigma = 0$ or 1, depending on $\lambda_1^{(1)}$.

We distinguish two cases in the analysis: the first one corresponds to eigenvalues which satisfy $\lambda_i^{(1)} \neq \lambda_1^{(2)}$, while the second one addresses the case in which $\lambda_i^{(1)} = \lambda_1^{(2)}$ for some values of the index i.

If $\lambda_i^{(1)} \neq \lambda_1^{(2)}$, we can construct m generalized eigenvectors w_i, $w_{i+1}, ..., w_{i+m-1}$ for Q_2 associated with $\lambda_i^{(1)}$, which implies that it is possible to rearrange the eigenvalues of Q_2 to get $\lambda_k^{(1)} = \lambda_k^{(2)}$, $k = i, ..., i+m-1$. For doing so, let $\Delta Q = Q_2 - Q_1$, and note that $A\, \Delta Q = 0$ and therefore $\text{Rg } \Delta Q \subseteq \text{Ker } A$, allowing to write $\Delta Q v_k = \xi_k v_1$. The vectors w_k can be constructed as $w_k = v_k + \gamma_k v_1$, where the parameters γ_k are defined recursively as

$$\gamma_k = \begin{cases} \dfrac{-\sigma - \xi_k}{\lambda_1^{(2)} - \lambda_i^{(1)}} & \text{if } k = i \\[2ex] \dfrac{-\xi_k}{\lambda_1^{(2)} - \lambda_{\text{i}}^{(1)}} & \text{if } k = i + m_1,\ i + m_2, ..., i + m_{j-1} \\[2ex] \dfrac{\gamma_{k-1} - \xi_k}{\lambda_1^{(2)} - \lambda_{\text{i}}^{(1)}} & \text{if } k = i+1, ..., i + m_1 - 1,\ i + m_1 + 1, ..., i + m_2 - 1, \text{ etc.} \end{cases}$$

With these definitions, we have

$$\begin{aligned} (Q_2 - \lambda_i^{(1)} I_n)\, w_i &= (Q_2 - \lambda_i^{(1)} I_n)\, v_i + \gamma_i (Q_2 - \lambda_i^{(1)} I_n)\, v_1 \\ &= (Q_1 - \lambda_i^{(1)} I_n)\, v_i + \Delta Q v_i + \gamma_i (Q_2 - \lambda_1^{(2)} I_n)\, v_1 \\ &\quad + \gamma_i (\lambda_1^{(2)} - \lambda_i^{(1)})\, v_1 \\ &= \sigma v_1 + \Delta Q v_i + \gamma_i (\lambda_1^{(2)} - \lambda_i^{(1)})\, v_1 \\ &= \sigma v_1 + \xi_i v_1 - \sigma v_1 - \xi_i v_1 = 0 \end{aligned}$$

and

$$
\begin{aligned}
(Q_2 - \lambda_i^{(1)} I_n)\, w_k &= (Q_2 - \lambda_i^{(1)} I_n)\, v_k + \gamma_k (Q_2 - \lambda_i^{(1)} I_n)\, v_1 \\
&= (Q_1 - \lambda_i^{(1)} I_n)\, v_k + \Delta Q v_k + \gamma_k (Q_2 - \lambda_1^{(2)} I_n)\, v_1 \\
&\quad + \gamma_k (\lambda_1^{(2)} - \lambda_i^{(1)})\, v_1 \\
&= \Delta Q v_k + \gamma_k (\lambda_1^{(2)} - \lambda_i^{(1)})\, v_1 = \xi_k v_1 - \xi_k v_1 = 0 \qquad (45)
\end{aligned}
$$

for $k = i + m_1,\ i + m_2$, etc. Finally

$$
\begin{aligned}
(Q_2 - \lambda_i^{(1)} I_n)\, w_k &= (Q_2 - \lambda_i^{(1)} I_n)\, v_k + \gamma_k (Q_2 - \lambda_i^{(1)} I_n)\, v_1 \\
&= v_{k-1} + \Delta Q v_k + \gamma_k (\lambda_1^{(2)} - \lambda_i^{(1)})\, v_1 = v_{k-1} + \gamma_{k-1} v_1 = w_{k-1}
\end{aligned}
$$

for $k = i+1, ..., i + m_1 - 1, i + m_1 + 1$, etc.

In the second case, defined by $\lambda_i^{(1)} = \lambda_1^{(2)}$, we will simply prove that $v_1, v_i, ..., v_{i+m-1} \in \operatorname{Ker}(Q_2 - \lambda_i^{(1)} I_n)^{m+1}$, implying that $\dim \operatorname{Ker}(Q_2 - \lambda_i^{(1)} I_n)^{m+1} = m+1$. From the analysis above we conclude that it must necessarily be $\dim \operatorname{Ker}(Q_2 - \lambda_i^{(1)} I_n)^{m+1} \leqslant m+1$, and therefore it suffices to show that $\dim \operatorname{Ker}(Q_2 - \lambda_i^{(1)} I_n)^{m+1} \geqslant m+1$. For doing so, let us write

$$
\begin{aligned}
(Q_1 - \lambda_i^{(1)} I_n)^m &= ((Q_2 - \lambda_i^{(1)} I_n) - \Delta Q)^m \\
&= (Q_2 - \lambda_i^{(1)} I_n)^m - \Delta Q (Q_2 - \lambda_i^{(1)} I_n)^{m-1} \\
&\quad + \Delta Q^2 (Q_2 - \lambda_i^{(1)} I_n)^{m-2} - \cdots \pm \Delta Q^m
\end{aligned}
$$

since $(Q_2 - \lambda_i^{(1)} I_n)\, v_1 = 0$ and then $(Q_2 - \lambda_i^{(1)} I_n)\, \Delta Q = 0$. Hence, we obtain

$$
(Q_2 - \lambda_i^{(1)} I_n)(Q_1 - \lambda_i^{(1)} I_n)^m = (Q_2 - \lambda_i^{(1)} I_n)^{m+1}. \qquad (18)
$$

Let us first suppose that $\lambda_i^{(1)} \neq \lambda_1^{(1)}$. Then we have that $(Q_1 - \lambda_i^{(1)} I_n)^m\, v_k = 0$ for $k = i, ..., i + m - 1$, and $(Q_1 - \lambda_i^{(1)} I_n)^m\, v_1 = (\lambda_1^{(1)} - \lambda_i^{(1)})^m\, v_1$, yielding $(Q_2 - \lambda_i^{(1)} I_n)(Q_1 - \lambda_i^{(1)} I_n)^m\, v_1 = 0$. Therefore, $v_1, v_i, ..., v_{i+m-1} \in \operatorname{Ker}(Q_2 - \lambda_i^{(1)} I_n)^{m+1}$.

If $\lambda_i^{(1)} = \lambda_1^{(1)}$, we have $\dim \operatorname{Ker}(Q_1 - \lambda_i^{(1)} I_n)^{m+1} = m+1$. If $\dim \operatorname{Ker}(Q_1 - \lambda_i^{(1)} I_n)^m = m+1$, from (4) we immediately get $\dim \operatorname{Ker}(Q_2 - \lambda_i^{(1)} I_n)^{m+1} \geqslant m+1$. Finally, if $\dim \operatorname{Ker}(Q_1 - \lambda_i^{(1)} I_n)^m = m$, it must be $(Q_1 - \lambda_i^{(1)} I_n)^m\, v_k = 0$ for $k = 1, i, i+1, ..., i+m-2$, and $(Q_1 - \lambda_i^{(1)} I_n)^m\, v_{i+m-1} = v_1$. This implies that $(Q_2 - \lambda_i^{(1)} I_n)(Q_1 - \lambda_i^{(1)} I_n)^m\, v_{i+m-1} = 0$, proving that $v_1, v_i, ..., v_{i+m-1} \in \operatorname{Ker}(Q_2 - \lambda_i^{(1)} I_n)^{m+1}$. ∎

Note that $\lambda < 0$ in condition (2) of Theorem 2, together with condition (3), implies the stability of H. This is the case in the continuous Newton method, as shown below. Theorem 3 is also of application, giving the eigenvalues of the Newton field at standard weak zeros.

COROLLARY 2. *Standard weak equilibria x^* of the continuous Newton method are asymptotically stable, their eigenvalues being $-1/2$ (simple, with eigenspace* Ker $J(x^*)$*) and* -1.

Proof. x^* being a standard equilibrium point, we have that Ker $J(x^*)$ is one-dimensional and, therefore, $C = -2J^T(x^*)\, J(x^*)$ is a negative semi-definite matrix of rank $n-1$. Hence, stability only depends on the value of λ, which in this case is $-1/2$, since

$$\frac{\text{Adj}\, A(x^*)\, f''(x^*)\, vv}{2(\det A)'\,(x^*)\, v} = -\frac{\text{Adj}\, J(x^*)\, f''(x^*)\, vv}{2(\det J)'\,(x^*)\, v}$$

$$= -\frac{((\det J)'\,(x^*)\, v)\, v}{2(\det J)'\,(x^*)\, v} = -\frac{1}{2}\, v.$$

Application of Theorem 2 is then straightforward. Alternatively, we can use Theorem 3, with $Q = -I_n$, to prove that $\lambda_2 = \cdots = \lambda_n = -1$. ∎

This result generalizes the well-known behavior of the continuous Newton method in one-dimensional problems, where noncritical singular zeros yield asymptotically stable equilibria with eigenvalue $-1/2$.

Forward Euler discretization with stepsize 1 leads to the classical Newton iteration

$$x^{(k+1)} = x^{(k)} - J^{-1}(x^{(k)})\, f(x^{(k)}).$$

A spherical domain of attraction for weak singular zeros is also obtained in this discrete-time setting, contradicting the common assumption that the domain of convergence of singular roots must necessarily exclude other singularities [20, 21]. It is immediate to show that the eigenvalues $\tilde{\lambda}_i$ of this iteration at a weak equilibrium x^* are $1 + \lambda_i$, where λ_i, $i = 1 \ldots n$, stand for the eigenvalues of the Newton field at x^*. This fact yields linear convergence with asymptotic rate 1/2 in the direction of Ker $J(x^*)$.

This approach can also be used to model the following accelerated modification of Newton's method

$$y^{(k)} = x^{(k)} - J^{-1}(x^{(k)})\, f(x^{(k)})$$

$$z^{(k)} = y^{(k)} - J^{-1}(y^{(k)})\, f(y^{(k)})$$

$$x^{(k+1)} = z^{(k)} - 2J^{-1}(z^{(k)})\, f(z^{(k)}),$$

reported in [29] and yielding quadratic convergence to singular roots. This iteration may be seen as a 3-stage explicit Runge–Kutta integration of the continuous Newton method (2), and it can be easily shown that the eigenvalues are now defined as $\hat{\lambda}_i = (1+2\lambda_i)(1+\lambda_i)^2$, satisfying, since $\lambda_i = -1/2, -1$, the condition $\hat{\lambda}_i = 0$ required for superlinear convergence.

3.2. *Example*

The continuous Newton method applied to the fold $f(x) = (x_1^2, x_2)$, yielding the Newton field $h(x) = (-x_1/2, -x_2)$, provides a simple instance of the weak behavior studied in this section. The following example attempts to illustrate further the result above, in the case Ker $A =$ Ker J. Let us consider the system $A(x)\,\dot{x} = f(x)$ defined by

$$A(x) = \begin{pmatrix} 4x_1^3 - 2x_1 & 0 \\ -5x_1^4 & \alpha x_2 \end{pmatrix}, \qquad f(x) = \begin{pmatrix} \beta x_1^2 - x_1^4 \\ x_1^5 + x_2^2 - 1 \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{R} - \{0\}$, $\alpha < 0$. Note that the case $\alpha = -2$, $\beta = 1$ defines the continuous Newton method for the function f. We have $\omega(x) = \det A(x) = \alpha x_1 x_2 (4x_1^2 - 2)$ and

$$g(x) = \begin{pmatrix} \alpha x_1^2 x_2 (\beta - x_1^2) \\ -x_1^8 + (5\beta - 2)\, x_1^6 + 4x_1^3 x_2^2 - 4x_1^3 - 2x_1 x_2^2 + 2x_1 \end{pmatrix}$$

showing that points $x_1 = 0$, $x_2 \neq 0$ correspond to weak singularities where the field $h = g/\omega$ is smoothly defined. In particular, $(0, 1)$ and $(0, -1)$ are weak equilibria. We will fix $x^* = (0, 1)$, the results being entirely analogous for the point $(0, -1)$. We have

$$A(x^*) = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$$

and therefore Ker $A(x^*) = (v_1, 0)$. It is easily shown that $(\det A)'\,(x^*) = (-2\alpha, 0)$, and then

$$(\det A)'\,(x^*)\, v = -2\alpha v_1, \tag{19}$$

proving that x^* is a standard equilibrium point. The Jacobian matrix $J(x)$ is

$$J(x) = \begin{pmatrix} 2\beta x_1 - 4x_1^3 & 0 \\ 5x_1^4 & 2x_2 \end{pmatrix}$$

with $\det J(x) = 4x_1 x_2 (\beta - 2x_1^2)$. At the singularity x^*, we have

$$J(x^*) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

and $(\det J)'(x^*) = (4\beta, 0)$, $\operatorname{Ker} J(x^*) = (v_1, 0)$, showing that x^* is transversal. The condition $\operatorname{Rg} A(x^*) = \operatorname{Rg} J(x^*)$ is also trivially verified.

In this case, the matrix $C = A^T J + J^T A$ reads

$$C(x^*) = \begin{pmatrix} 0 & 0 \\ 0 & 4\alpha \end{pmatrix},$$

being negative semidefinite under the assumption $\alpha < 0$. Condition 3 of Theorem 2 is satisfied, and we are reduced to the study of the parameter λ. An easy computation gives

$$\operatorname{Adj} A(x^*)\ f''(x^*)\, vv = \begin{pmatrix} 2\alpha\beta v_1^2 \\ 0 \end{pmatrix}, \tag{20}$$

and this equation, together with (19), leads to $\lambda = -\beta/2$. The parameter β regulates the dynamic behavior in the direction of $\operatorname{Ker} A(x^*)$. This direction is "attracting" if and only if $\beta > 0$ and, therefore, H is stable if and only if $\beta > 0$. Note that this is verified in the particular case of the continuous Newton method, defined by $\alpha = -2$, $\beta = 1$.

This example can also be analyzed in light of Theorem 3: details are entirely analogous to the ones in the example to be discussed in Subsection 4.2 and, therefore, are not included here.

4. STABILITY OF STRONG EQUILIBRIA

We discuss in this section stability of standard strong equilibria, assuming transversality and the condition $\operatorname{Rg} J(x^*) = \operatorname{Rg} A(x^*)$. The study will be restricted to the case $\operatorname{Ker} A(x^*) = \operatorname{Ker} J(x^*)$, providing a strong analogue of Theorem 3. The proof of the main result (Theorem 4) can be seen as a Lyapunov–Schmidt approach to a reformulation of the problem in the framework of bifurcation theory. This result covers the continuous Newton method as a particular case, and may be considered as a continuous extension of Reddien's theorem on convergence of the classical Newton method to singular zeros [40].

In this situation, it is not possible to write $AH = J$ at x^*, since h is not a smooth vector field. However, the condition $\operatorname{Rg} A(x^*) = \operatorname{Rg} J(x^*)$ allows for the use of a similar approach to the one in Theorem 3, writing at x^* the identity $AQ = J$ for some matrix Q. An additional hypothesis to the ones in Theorem 3 suffices to prove directional stability when approaching a strong singular root from a locally cone-shaped region.

4.1. *Directional Stability of Strong Equilibria*

As in Subsection 3.1, let us consider $v \in \text{Ker } A(x^*) - \{0\}$, and write

$$\frac{\text{Adj } A(x^*)\ f''(x^*)\ vv}{2(\det A)'\ (x^*)\ v} = \lambda v \tag{21}$$

for some real number λ, which is non-zero under the assumption $\text{Ker } A = \text{Ker } J$. The relation of this parameter with the Lyapunov–Schmidt reduction of the problem will be pointed out later. In the following theorem, note that the condition $\text{Ker } A = \text{Ker } J$ implies the existence of some real number λ_1 such that $Qv = \lambda_1 v$ if $v \in \text{Ker } A - \{0\}$, as in Subsection 3.1. For simplicity in the proof, f and A are assumed to be C^3.

THEOREM 4. *Let x^* be a (standard and transversal) strong equilibrium point of system* (1), *with* $\text{Rg } A = \text{Rg } J$ *and* $\text{Ker } A = \text{Ker } J$. *Let Q satisfy $AQ = J$ at x^*, and write $Qv = \lambda_1 v$ for $v \in \text{Ker } A - \{0\}$. Assume that the remaining eigenvalues of Q satisfy* $\text{Re}(\lambda_2), ..., \text{Re}(\lambda_n) < \lambda < 0$. *Then there exists a cone $\mathscr{K}_\theta$ with $\theta > 0$ and axis $N = \text{Ker } A$, and a ball $\mathscr{B}_\rho$ with $\rho > 0$, such that $\mathscr{W}_{\theta, \rho} = \mathscr{K}_\theta \cap \mathscr{B}_\rho$ is an invariant set of system* (1) *converging to x^*.*

Proof. We proceed in two steps, first obtaining a decomposition of the field h valid in some set $\mathscr{W}_{\theta, \rho}$, as in the usual discrete-time approach, and then analyzing the resulting continuous-time dynamics.

Let us first remark that the equivalence of norms in finite-dimensional vector spaces proves that a cone with non-empty interior for some norm includes a cone with non-empty interior for *any* norm in $\mathbb{R}^n$; however, taking a specific norm detailed later will simplify the proof. Note also that we can assume, without loss of generality, that λ_1 is simple, in light of Theorem 3. We can then write $\mathbb{R}^n = N \oplus X$, with $N = \text{Ker } A$ and X being the invariant subspace associated with $\lambda_2, ..., \lambda_n$. Note that the linear operator $\tilde{Q} = Q|_X$ has eigenvalues $\lambda_2, ..., \lambda_n$. Let us define

$$\hat{x} = x - x^*, \qquad z = P_N(\hat{x}), \qquad y = P_X(\hat{x}).$$

Choose $\eta, \zeta \in \mathbb{R}$ such that

$$\max_{i=2, ..., n} \{\text{Re}(\lambda_i)\} < \eta < 2\zeta < \lambda < \zeta < 0.$$

From [2, Proposition 13.1], it follows that there exists a norm $\|\ \|_X$ in X such that

$$\|e^{\tilde{Q}t} y\|_X \leqslant e^{\eta t}\ \|y\|_X, \qquad \forall y \in X. \tag{22}$$

Take any norm $\| \|_N$ in the one-dimensional space N and, considering that $\hat{x}$ can be uniquely decomposed as $\hat{x} = z + y$, since $\mathbb{R}^n = N \oplus X$, define $\|\hat{x}\| = \|z\|_N + \|y\|_X$. Note that $\|z\| = \|z\|_N$ if $z \in N$ and $\|y\| = \|y\|_X$ if $y \in X$: therefore, only the notation $\| \|$ will be employed throughout the proof.

Consider the set

$$\mathscr{W}_{\theta,\rho} = \{x \in \mathbb{R}^n : \|P_X\hat{x}\| \leqslant \theta\, \|P_N\hat{x}\|,\ \|\hat{x}\| \leqslant \rho\},$$

where θ, $\rho > 0$ are chosen sufficiently small to satisfy the below-indicated requirements. We will prove that the interior $\mathrm{Int}(\mathscr{W}_{\theta,\rho})$ is an invariant set, which is sufficient to assure invariance of $\mathscr{W}_{\theta,\rho}$ [2, Proposition 16.3]. This will be done by showing that $\|y(s)\| < \theta\, \|z(s)\|$ and $\|y(s)\| + \|z(s)\| < \rho$, for all s such that $0 \leqslant s < t$, imply that $\|y(t)\| < \theta\, \|z(t)\|$ and $\|y(t)\| + \|z(t)\| < \rho$. The condition $x \to x^*$ will be naturally obtained.

To achieve so, note that, x^* being standard, it is $(\det A)'\,(x^*)\,v = \omega'(x^*)\,v \neq 0$ for any $v \in \mathrm{Ker}\,A(x^*) - \{0\}$. Take θ sufficiently small to assure that $\omega'(x^*)\,\hat{x} \neq 0$ for $x \in \mathscr{K}_\theta - \{x^*\}$, and write

$$h(x) = \frac{g(x)}{\omega(x)} = \frac{(1/2)\, g''(x^*)\,\hat{x}^2 + O(\|\hat{x}\|^3)}{\omega'(x^*)\,\hat{x} + O(\|\hat{x}\|^2)}, \qquad x \in \mathscr{W}_{\theta,\rho},$$

since $g(x^*) = g'(x^*) = \omega(x^*) = 0$. Operating as in (7) for $g(x) = \mathrm{Adj}\,A(x)\,f(x)$, we get

$$h(x) = \frac{((\mathrm{Adj}\,A)'\,(x^*)\,\hat{x})\,J(x^*)\,\hat{x} + (1/2)\,\mathrm{Adj}\,A(x^*)\,f''(x^*)\,\hat{x}^2}{\omega'(x^*)\,\hat{x}} + O(\|\hat{x}\|^2),$$

but from (5) it follows that

$$\begin{aligned}((\mathrm{Adj}\,A)'\,(x^*)\,\hat{x})\,J(x^*)\,\hat{x} &= ((\mathrm{Adj}\,A)'\,(x^*)\,\hat{x})\,A(x^*)\,Q\hat{x}\\ &= \omega'(x^*)\,\hat{x}Q\hat{x} - \mathrm{Adj}\,A(x^*)\,A'(x^*)\,\hat{x}Q\hat{x}\end{aligned}$$

and then

$$h(x) = Q\hat{x} + u(\hat{x}) + O(\|\hat{x}\|^2),$$

with

$$u(\hat{x}) = \frac{-(\mathrm{Adj}\,A(x^*)\,A'(x^*)\,\hat{x})\,Q\hat{x} + (1/2)\,\mathrm{Adj}\,A(x^*)\,f''(x^*)\,\hat{x}^2}{\omega'(x^*)\,\hat{x}}.$$

Let us then study the projections $P_N h$ and $P_X h$. Noting that $u(\hat{x}) \in N$, we can write

$$\begin{aligned} P_N h(x) &= P_N Q\hat{x} + u(\hat{x}) + O(\|\hat{x}\|^2) \\ &= QP_N \hat{x} + u(\hat{x}) + O(\|\hat{x}\|^2), \end{aligned} \tag{23}$$

while the condition $\|y\| < \theta \|z\|$ gives

$$\begin{aligned} u(\hat{x}) &= \frac{-(\text{Adj}\, A(x^*)\, A'(x^*)\, z)\, Qz + (1/2)\, \text{Adj}\, A(x^*)\, f''(x^*)\, zz + O(\theta)\, O(\|z\|^2)}{\omega'(x^*)\, z + O(\theta)\, O(\|z\|)} \\ &= \frac{-(\text{Adj}\, A(x^*)\, A'(x^*)\, z)\, Qz + (1/2)\, \text{Adj}\, A(x^*)\, f''(x^*)\, zz}{\omega'(x^*)\, z} + O(\theta)\, O(\|z\|). \end{aligned}$$

Since $Qz = \lambda_1 z$, it is

$$u(\hat{x}) = \frac{-\lambda_1 \omega'(x^*)\, zz + (1/2)\, \text{Adj}\, A(x^*)\, f''(x^*)\, zz}{\omega'(x^*)\, z} + O(\theta)\, O(\|z\|), \tag{24}$$

and, from (23) and (24) we obtain

$$\begin{aligned} P_N h(x) &= \frac{(1/2)\, \text{Adj}\, A(x^*)\, f''(x^*)\, zz}{\omega'(x^*)\, z} + O(\theta)\, O(\|z\|) + O(\|z\|^2) \\ &= \lambda z + O(\theta)\, O(\|z\|) + O(\|z\|^2), \end{aligned}$$

where we have used that $O(\|\hat{x}\|^2) = O(\|z\|^2)$. Correspondingly, we have

$$P_X h(x) = QP_X \hat{x} + O(\|\hat{x}\|^2) \equiv \tilde{Q} y + O(\|z\|^2).$$

We are then reduced to the study of the dynamics of

$$\begin{aligned} \dot{z} &= \lambda z + q(\theta, z) \\ \dot{y} &= \tilde{Q} y + r(z), \end{aligned}$$

where $q(\theta, z) = O(\theta)\, O(\|z\|) + O(\|z\|^2)$ and $r(z) = O(\|z\|^2)$. N being one-dimensional, θ and ρ can be chosen sufficiently small for $z(t)$ to satisfy

$$e^{2\zeta t}\, \|z(0)\| \leqslant \|z(t)\| \leqslant e^{\zeta t}\, \|z(0)\|, \tag{25}$$

while $y(t)$ may be written as [2, Proposition 15.2]

$$y(t) = e^{\tilde{Q}t} y(0) + \int_0^t e^{\tilde{Q}(t-s)} r(z(s))\, ds.$$

Choosing $\gamma > 0$ such that $\|r(z)\| \leqslant \gamma \|z\|^2$, from (22) and (25) we get the bound

$$\begin{aligned} \|y(t)\| &\leqslant e^{\eta t} \|y(0)\| + \int_0^t e^{\eta(t-s)} \gamma \|z(0)\|^2 e^{2\zeta s} \, ds \\ &= e^{\eta t} \|y(0)\| + \frac{\gamma \|z(0)\|^2}{2\zeta - \eta} (e^{2\zeta t} - e^{\eta t}). \end{aligned} \tag{26}$$

Considering that $\|z\| \leqslant \|\hat{x}\| + \|y\| < \|\hat{x}\| + \theta \|z\|$ and therefore $\|z\| < (1-\theta)^{-1} \|\hat{x}\|$ for $\theta < 1$, and taking $\rho(\theta)$ such that

$$\frac{\gamma \rho}{(1-\theta)(2\zeta - \eta)} \leqslant \theta,$$

we obtain

$$\|y(t)\| \leqslant e^{\eta t} \|y(0)\| + \theta \|z(0)\| (e^{2\zeta t} - e^{\eta t}) < \theta \|z(0)\| e^{2\zeta t} \leqslant \theta \|z(t)\|, \tag{27}$$

which proves invariance of the interior of the cone for sufficiently small ρ.

Invariance of the ball $\mathrm{Int}(\mathscr{B}_\rho)$ may be proved by reducing ρ if necessary to verify

$$-\frac{\gamma \rho \eta}{(1-\theta)(2\zeta - \eta)} < -\zeta, \tag{28}$$

where it is to be noted that $\eta < 2\zeta < \zeta < 0$. From (25), (26), and (28), we get

$$\begin{aligned} \|y\| + \|z\| &\leqslant e^{\eta t} \|y(0)\| + \|z(0)\| \frac{\gamma \rho}{(1-\theta)(2\zeta - \eta)} (e^{2\zeta t} - e^{\eta t}) + e^{\zeta t}) \\ &\leqslant \|y(0)\| + \|z(0)\| < \rho, \end{aligned}$$

which relies on the condition $\eta < 0$ and the fact that the coefficient of $\|z(0)\|$ is decreasing since its derivative satisfies

$$\frac{\gamma \rho}{(1-\theta)(2\zeta - \eta)} (2\zeta e^{2\zeta t} - \eta e^{\eta t}) + \zeta e^{\zeta t} \leqslant -\frac{\gamma \rho}{(1-\theta)(2\zeta - \eta)} \eta e^{\eta t} + \zeta e^{\zeta t} < 0,$$

after multiplying the first and second member of (28) by $e^{\eta t}$ and $e^{\zeta t}$, respectively. Finally, convergence to x^* is immediate from (25) and (27). ∎

The first part of the reasoning above is inspired in Reddien's proof of convergence to singular zeros in the discrete Newton setting [40], and can be seen as a Lyapunov–Schmidt decomposition [19] of the bifurcation problem

$$p(h, x) = A(x)\, h - f(x) = 0,$$

where $p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. The field h is intended to be expressed as a function of the bifurcation parameter $x = (z, y)$ in a singular neighborhood $\mathscr{B}_{\hat{\rho}} \times \mathscr{W}_{\theta, \rho}$ of $(0, x^*)$. The following splitting yields this when the zero eigenvalue of $A(x^*)$ is simple,

$$p\colon (N \oplus X) \times \mathbb{R}^n \to N \oplus \operatorname{Rg} A(x^*),$$

with $N = \operatorname{Ker} A(x^*)$. Specifically, the regular variable $P_X h$ is given, up to the first order, by

$$\tilde{A}(x^*)\, P_X h - \tilde{J}(x^*)\, y = 0,$$

$\tilde{A}(x^*)$ (resp. $\tilde{J}(x^*)$) standing for $A(x^*)|_X\colon X \to \operatorname{Rg} A(x^*)$ (resp. $J(x^*)|_X\colon X \to \operatorname{Rg} A(x^*)$). This yields $P_X h = \tilde{Q} y + \text{h.o.t.}$

The first-order approximation of the reduced system is given by

$$\xi \operatorname{Adj} A(x^*)(A'(x^*)\, z P_N h - \tfrac{1}{2} f''(x^*)\, zz) = 0,$$

where $\xi \neq 0$ is such that $\xi \operatorname{Adj} A(x^*)$ defines the projection onto $\operatorname{Ker} A(x^*)$ parallel to $\operatorname{Rg} A(x^*)$ (see Subsection 1.1). The relation

$$((\det A)'\,(x^*)\, z)\, P_N h = \operatorname{Adj} A(x^*)\, A'(x^*)\, z P_N h,$$

together with (21), leads to $P_N h = \lambda z + \text{h.o.t.}$, clarifying the meaning of λ as a parameter which regulates the behavior of the reduced system. Note, however, that the proof of Theorem 4 encompasses more general situations since it does not rely on the hypothesis that A has a simple zero eigenvalue.

The behavior of the continuous Newton method falls in the framework described in Theorem 4, since in this case it is $\lambda = -1/2$, $\lambda_2 = \cdots = \lambda_n = -1$, as shown in Subsection 3.1:

Corollary 3. *Standard strong equilibria of the continuous Newton method locally have an invariant cone of convergence with vertex in x^*.*

It is worth mentioning that discretization issues similar to those of Section 3 apply in this case. However, a detailed discretization analysis is beyond the scope of this paper.

4.2. *Example*

Consider the system defined by

$$A(x) = \begin{pmatrix} x_1 & -x_2 \\ 0 & 1 \end{pmatrix}, \qquad f(x) = \begin{pmatrix} \gamma x_1^2 \\ -x_2 \end{pmatrix} \tag{29}$$

with $\gamma \in \mathbb{R} - \{-1, 0\}$. We have $\omega(x) = \det A(x) = x_1$ and

$$g(x) = \begin{pmatrix} \gamma x_1^2 - x_2^2 \\ -x_1 x_2 \end{pmatrix},$$

showing that $x^* = (0, 0)$ is a strong singular equilibrium point, which may be easily proved to be standard and transversal. At x^* we have

$$A(x^*) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and therefore Ker $A(x^*) = (v_1, 0)$. It is immediate that

$$(\det A)'\,(x^*)\,v = v_1. \tag{30}$$

The Jacobian matrix $J(x)$ is

$$J(x) = \begin{pmatrix} 2\gamma x_1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$J(x^*) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, we can write $AQ = J$ with

$$Q = \begin{pmatrix} \mu & \nu \\ 0 & -1 \end{pmatrix},$$

being $\mu,\ \nu \in \mathbb{R}$. We may in particular take $\mu \neq -1$, forcing Q to have simple eigenvalues $\lambda_1 = \mu$ (with eigenvector $(1, 0) \in \text{Ker } A$) and $\lambda_2 = -1$.

To compute λ, we have

$$\text{Adj } A(x^*)\ f''(x^*)\,vv = \begin{pmatrix} 2\gamma v_1^2 \\ 0 \end{pmatrix} \tag{31}$$

and, from (30) and (31), we obtain $\lambda = \gamma$. If $-1 < \gamma < 0$, Theorem 4 predicts the existence of an invariant cone with axis $(v_1, 0)$ converging to x^*. This result can be analytically contrasted, since the explicit solution of system (29) is

$$x_1(t) = \mathrm{sg}(x_1(0)) \sqrt{\left(x_1(0)^2 - \frac{x_2(0)^2}{\gamma+1}\right) e^{2\gamma t} + \frac{x_2(0)^2}{\gamma+1}\, e^{-2t}}$$

$$x_2(t) = x_2(0)\, e^{-t}.$$

The x_1-axis is invariant, leaving or approaching x^* if $\gamma > 0$ or $\gamma < 0$, respectively. Initial points $(x_1(0), x_2(0))$ with small $x_2(0)$ define a trajectory (x_1, x_2) which, for small t, behaves approximately as $(x_1(0)\, e^{\gamma t}, x_2(0)\, e^{-t})$, illustrating that these trajectories approach the direction of Ker $A(x^*)$ as they evolve towards x^*, if $-1 < \gamma < 0$. Note that for initial points $(x_1(0), x_2(0))$ close to the singular manifold $x_1 = 0$, that is, with $x_1(0)$ small enough, the trajectory $(x_1(t), x_2(t))$ ceases to exist for some $t > 0$, being attracted by an impasse point on the singular manifold.

Theorem 4 does not predict the behavior if $\gamma < -1$, but further studies might show the existence of other types of attraction domains. Such characterization is beyond the scope of this work.

Finally, the example introduced in Section 3 also presents, under the assumption $\beta = 1$, $\alpha = -2$, (which defines the continuous Newton method) a standard strong equilibrium located at (1, 0). It may be shown that transversality is also verified at this point, and that there exists an invariant cone of convergence in the direction of Ker $A = (0, 1)$.

5. CONCLUDING REMARKS

In this paper, stability properties of singular equilibria arising in quasilinear implicit differential equations $A(x)\, \dot{x} = f(x)$ have been analyzed. This study is motivated by the behavior of some root-finding methods when applied to singular problems. The continuous-time setting provides a framework from which different iterative methods can be derived through discretization.

Under certain geometrical conditions, a singular zero x^* of f has been shown to yield continuously defined equilibria if and only if there is a coincidence between the ranges of the coefficient matrix A and the Jacobian J at x^* (Theorem 1). In the weak case, this continuous definition can be performed on a spherical neighborhood of the singularity, providing examples in which the domain of definition of the field is larger than the domain of invertibility of A. Continuous definition at strong singularities may be expected only along certain domains, including cone-shaped regions with vertex in the root.

Stability results can be understood as an extension of Lyapunov's indirect method to quasilinear implicit equations. Sufficient conditions for stability in the weak case are given in Theorems 2 and 3, based on certain relations between the matrices $A(x^*)$ and $J(x^*)$, as well as on the parameter λ, which determines the behavior of the system in a specific one-dimensional subspace. Specifically, Theorem 3 shows that the eigenvalues of any solution Q of the matrix equation $A(x^*)\,Q = J(x^*)$, together with the parameter λ, regulate the qualitative local behavior of the system. This result can be extended to the strong case through the additional requirement of λ to be dominant, yielding convergence to a singular zero from a cone-shaped region with vertex in the root (Theorem 4). The significance of λ is further clarified through a Lyapunov–Schmidt approach to a reformulation of the problem in the framework of bifurcation theory.

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