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Erdős–Szekeres "happy end"-type theorems for separoïds

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To the memory of Víctor Neumann-Lara "mi abuelo huasteco"

Abstract

In 1935 Pál Erdős and György Szekeres proved that, roughly speaking, *any configuration of n points in general position in the plane have* $\log n$ *points in convex position — which are the vertices of a convex polygon*. Later, in 1983, Bernhard Korte and László Lovász generalised this result in a purely combinatorial context; the context of *greedoids*. In this note we give one step further to generalise this last result for arbitrary dimensions, but in the context of *separoids*; thus, via the geometric representation theorem for separoids, this can be applied to families of convex bodies. Also, it is observed that the existence of some *homomorphisms of separoids* implies the existence of not-too-small *polytopal* subfamilies — where each body is separated from its relative complement. Finally, by means of a probabilistic argument, it is settled, basically, that *for all d > 2, asymptotically almost all "simple" families of n "d-separated" convex bodies contains a polytopal subfamily of order* $\sqrt[d+1]{\log n}$.
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1. Introduction and statement of results

As pointed out by Morris and Soltan [23] "the known results on the Erdős–Szekeres problem have been proved using only some very simple combinatorial properties of the plane. It is natural to ask what the most general framework is for studying this problem". The first attempt in this direction seems to be the work of Korte and Lovász [19] which is based on the notion of a greedoid, a common generalisation of a matroid and of a convexity space (see Whitney [33]

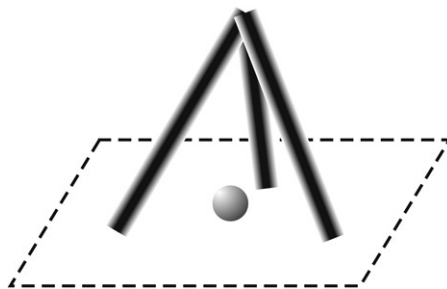*E-mail address:* strausz@math.unam.mx.

Fig. 1. A separoid S with $1 = \rho(S) < \kappa(S) < \gamma(S) = 3$.

and Levi [21]). More recently, the concept of a separoid was introduced [29–31], a common generalisation of a graph and of an oriented matroid (see e.g., Hell and Nešetřil [14] and Björner et al. [3]). Using the notion of a separoid a further step is given – small as it may be – to understand the Erdős–Szekeres theorem from a purely combinatorial point of view.

A *separoid* is a (finite) set S endowed with a symmetric relation $\dagger \subseteq \binom{2^S}{2}$ defined on its family of subsets, which satisfies the following two simple properties: for all $A, B \subseteq S$,

$$(*) \quad \begin{array}{ll} \circ & A \dagger B \implies A \cap B = \emptyset, \\ \circ\circ & A \dagger B \text{ and } B \subset B' \, (\subseteq S \setminus A) \implies A \dagger B'. \end{array}$$

A pair $A \dagger B$ is called a *Radon partition*, each part (A and B) is called a *component*, and the union $A \cup B$ is called the *support* of the partition. A pair of disjoint subsets $\alpha, \beta \subseteq S$ that is not a Radon partition is said to be *separated*, and denoted by $\alpha \mid \beta$. The *order* of the separoid is the cardinal $|S|$.

Separoids have several notions of dimension which are meaningful from a geometric perspective — they reflect basic results of combinatorial convexity (see [7,8]). Let us mention here a couple: the *Radon dimension* [27] is the minimum d such that each subset of S with at least $d + 2$ elements is the support of a Radon partition; the *Kirchberger dimension* [17] is the minimum d such that $\alpha \mid \beta$ if and only if for all $X \in \binom{S}{d+2}$ it follows that $(\alpha \cap X) \mid (\beta \cap X)$. We will denote these numbers by $\rho(S)$ and $\kappa(S)$, respectively.

Now, consider a family of convex sets $\mathcal{F} = \{K_1, \dots, K_n\}$ in some Euclidian space $\mathbb{E}^d$. A separoid $S(\mathcal{F})$ on $[n] = \{1, \dots, n\}$ can be defined by the following relation:

$$\alpha \mid \beta \iff \left\langle \bigcup_{i \in \alpha} K_i \right\rangle \cap \left\langle \bigcup_{j \in \beta} K_j \right\rangle = \emptyset,$$

where $\langle \cdot \rangle$ denotes the convex hull. Conversely, as proved in [1,5], every separoid can be represented in such a way by a family of convex sets in some Euclidian space. Therefore each separoid S has a minimum dimension where it can be represented; it is called the *geometric dimension* of S and denoted here by $\gamma(S)$. Furthermore, as proved in [6,30], if the separoid S is *acyclic* (i.e., if $\emptyset \mid S$) then $\gamma(S) \le |S| - 1$. It is easy to see (Fig. 1) that for all S, $\rho(S) \le \kappa(S) \le \gamma(S)$.

The separoid S is said to be in *general position* if no subset of $\rho(S) + 1$ elements is the support of a Radon partition. We say that S is *polytopal* [22] – or in *convex position* if you will – if every element is separated from its complement; that is, if for every $x \in S$ it follows that $x \mid (S \setminus x)$.

The following theorem is an easy corollary of the Erdős–Szekeres' "happy ending" theorem [9] when generalised to higher dimensions (see e.g. [15]; see also [23,28] and the references therein).

Theorem A. *For each $d > 1$, there is a function $\xi_d : \mathbb{N} \to \mathbb{N}$ such that if S is a separoid of order $|S| = \xi_d(n)$ in general position and $d = \rho(S) = \gamma(S)$, then S contains a polytopal subseparoid $P \subset S$ of order $|P| = n$.*

The best known bounds for ξ_d are

$$\text{(ESzTV)} \qquad 2^{n-2} + 1 \leq \xi_2(n) \leq \binom{2n-5}{n-2} + 2,$$

and

$$\text{(KV)} \qquad \Omega(C^{\,d-1\!\sqrt{n}}) \leq \xi_d(n) \leq \binom{2n-2d-1}{n-d} + d,$$

where $C = C(d)$ is a constant which only depends on d. The left inequality of (ESzTV) is due to Erdős and Szekeres [9] and is conjectured to be the best possible; the right one is due to Géza Tóth and Pavel Valtr [32]. The right inequality in (KV) is due to Gyula Károlyi [15] and the left one is due to Károlyi and Valtr [16].

Some generalisations of the Erdős–Szekeres theorem, to include convex sets instead of points, are due to Tibor Bisztriczky and Gabor Fejes Tóth [2], and János Pach and Géza Tóth [26]. In their works, the notion of "polytopal" is replaced by the following: a family of convex sets is said to be in "convex position" if none of its elements is **contained** in the convex hull of the union of the others. Also, their notion of "general position" is in terms of the notion of "convex position": every $(d+1)$-subset is required to be in convex position. Observe that these notions are not separoidal; that is, the separoid of a family of convex sets does not distinguish if an element intersects or if it is contained in the convex hull of others — they are simply non-separated. Furthermore, while for points the definitions of polytopal and convex position are equivalent, for convex sets we only have that polytopal implies convex position.

Due to the second condition in $(*)$, a separoid is determined by its *minimal* Radon partitions. Thus, to construct a separoid – and therefore a family of convex bodies – it is enough to define some Radon partitions and close the separoid to become a symmetric filter (in the canonical order $A \dagger B \preceq C \dagger D$ iff $A \subseteq C$ and $B \subseteq D$). In this way, it is easy to construct separoids in general position without non-trivial polytopal subseparoids; simply fix $d = \rho(S)$ and for each $(d+2)$-subset X of S choose a Radon partition of the form $x \dagger (X \setminus x)$ — clearly this indicates that for such separoids, $\rho(S) = \kappa(S) < \gamma(S)$ holds. So, it is natural to ask at this point for "meaningful" conditions which guarantee the existence of polytopal subseparoids. For example, we may add a Hadwiger-type hypothesis [13] which implies the following.

Theorem B. *Let S be a separoid and $d > 1$. If there exists a homomorphism $S \longrightarrow P$ onto a separoid of points $P \subset \mathbb{E}^d$ in general position (i.e., a mapping which preserves the minimal Radon partitions), then there exists a polytopal subseparoid of S of order $\xi_d^{-1}(|P|)$.*

The Hadwiger-type hypotheses are "geometric" in nature; they are geometric restrictions to the "position" of the convex sets that represent the separoid (see [7] for the early work on such a "special position" hypotheses, and see [8,11] for excellent updates on the subject). The following questions arise. How far can the hypothesis of Theorem B be weakened without changing the conclusion? Is there a purely combinatorial Erdős–Szekeres-type theorem?
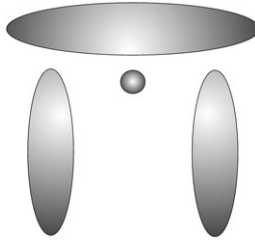
Fig. 2. A separoid in general position which is not dispersed.

We now introduce the following new concept. A separoid S is *dispersed* if it is in general position and it satisfies the following three conditions:

1. $\alpha \mid \beta$ and $x \in S \setminus (\alpha \cup \beta) \implies (\alpha \cup x) \mid \beta$ or $\alpha \mid (\beta \cup x)$.
2. $\alpha \mid \beta$ and $\alpha \neq \emptyset \implies \exists x \in \alpha : (\alpha \setminus x) \mid (\beta \cup x)$.
3. $x \dagger B$ minimal $\implies \forall y \notin B \cup x \; \exists! b \in B : x \dagger (B \cup y \setminus b)$.

These conditions can be interpreted as follows. Condition 1 guarantees that every separation extends to a *maximum* one, i.e., a separation of the form $H \mid (S \setminus H)$ — it allows to define the closure operator $[A] := \bigcap \{H \supset A : H \mid (S \setminus H)\}$. Condition 2 guarantees that the hyperplane which separates α from β can be moved in one of its "complementary directions" while traversing exactly one element — it implies that the closure operator $[\cdot]$ defines a *convex geometry* (or an *antimatroid*; see (AM) below). Finally, condition 3 is a special case of the Steinitz exchange lemma for Radon partitions (see (Z) below) — it is a key ingredient in the Korte and Lovász's generalisation of the Erdős–Szekeres theorem. Clearly, separoids of points in general position are dispersed (Fig. 2 represents a separoid which is not dispersed; furthermore, it does not satisfies any of the 3 conditions enumerated above).

With this definition at hand, we can prove the following

Theorem C. *For each $d > 1$, there exists a function $\xi_d : \mathbb{N} \to \mathbb{N}$ such that for each dispersed separoid S of order $|S| = \xi_d(n)$ and dimension $d = \kappa(S)$, there exists a polytopal subseparoid $P \subset S$ of order $|P| = n$.*

As pointed out by Patrice Ossona de Mendez [personal communication], a separoid that satisfies conditions 1 and 2 above, is an *antimatroid*; i.e., it satisfies that

$$(AM) \qquad x \mid B, \quad y \mid B \quad and \quad x \dagger (B \cup y) \implies y \mid (B \cup x).$$

Thus, Theorem C extends that of Korte and Lovász [19] to higher dimensions. Furthermore, as it is easy to see, uniform oriented matroids are also dispersed separoids, therefore

Corollary D. *For each $d > 1$, there exists a function $\xi_d : \mathbb{N} \to \mathbb{N}$ such that for each uniform oriented matroid M of order $|M| = \xi_d(n)$ and rank $d + 1$, there exists a polytopal oriented matroid $P \subset M$ of order $|P| = n$.*

Finally, by means of a probabilistic argument for random hypergraphs, it will be proved that

Theorem E. *Asymptotically almost all Radon separoids of order n in general position and dimension $d = \rho(S) > 2$, contains a polytopal subseparoid of order $\sqrt[d+1]{\log n}$.*
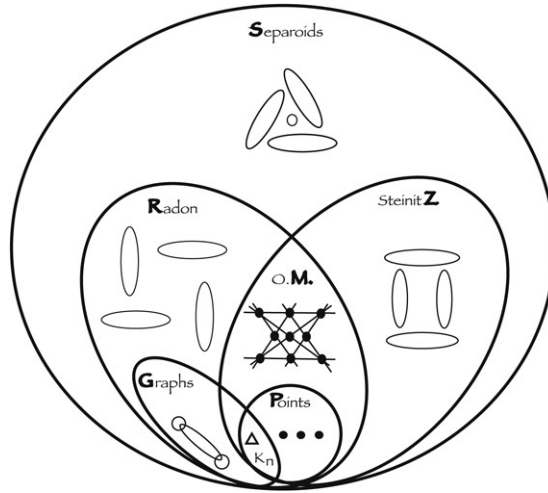
Fig. 3. Some classes of separoids.

2. Preliminaries and Theorems A and B

In order to be self-contained, we start with some basic notions and results on separoids. First, let us enumerate some important classes of separoids (see Fig. 3 for some small examples). Let S be a separoid. S is a *Radon* separoid (or a *simple* separoid) if each minimal Radon partition is unique in its support; i.e., iff for all minimal $A \dagger B$ and $C \dagger D$, it follows that

(R) $A \cup B \subseteq C \cup D \implies \{A, B\} = \{C, D\}$.

Clearly, if S is a Radon separoid, then $\rho(S) = \kappa(S)$.

S is a *Steinitz* separoid if it satisfies the so-called Steinitz exchange axiom; namely,

(Z) $A \dagger B$ minimal $\implies \forall x \notin A \cup B \; \exists y \in A \cup B : (A \setminus y) \dagger (B \cup x \setminus y)$.

An *oriented matroid* is a Radon separoid which satisfies the so-called weak elimination axiom; namely,

(OM) $A \dagger B, C \dagger D$ minimal, and $x \in B \cap C \implies$
$\exists E \dagger F$ minimal $: E \subseteq A \cup C \setminus x \quad$ and $\quad F \subseteq B \cup D \setminus x$.

As observed by Michel Las Vergnas [20], oriented matroids are Steinitz separoids.

The following result allows to study separation properties of convex sets from a purely combinatorial point of view (cf. [1,5,30]).

Theorem 1. *Every separoid S can be represented by a family of convex sets in some Euclidian space. Furthermore, the separoid is acyclic if and only if it can be represented by compact convex sets; in such a case, it can be represented in the $(|S| - 1)$-dimensional Euclidian space.*

Sketch of the proof. We consider here only the acyclic case (i.e., when $A \dagger B \implies |A||B| > 0$). Let S be identified with the set $\{1, \dots, n\}$. For each element $i \in S$ and each (minimal) Radon
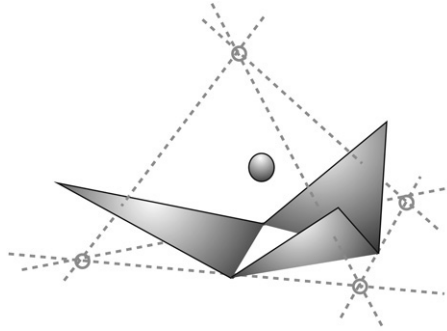
Fig. 4. A separoid of order 4 in $\mathbb{E}^3$.

partition $A \dagger B$ such that $i \in A$, consider the point

$$\rho^i_{A\dagger B} = \mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b - \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right],$$

where $\mathbf{e}_i$ denotes the ith vector of the canonical basis of $\mathbb{R}^n$. Then, each element $i \in S$ is represented by the convex hull of all such points:

$$i \mapsto \mathcal{K}_i = \left\langle \rho^i_{A\dagger B} : i \in A \text{ and } A \dagger B \right\rangle.$$

Observe that the convex sets $\mathcal{K}_i$ live in the affine hyperplane spanned by the basis (e.g., in Fig. 4 the separoid depicted in Fig. 1 is represented using this construction).

To prove that this construction is correct, two steps are needed. First, consider a Radon partition $A \dagger B$ and observe that the baricenters of $\left\langle \rho^a_{A\dagger B} : a \in A \right\rangle$ and $\left\langle \rho^b_{B\dagger A} : b \in B \right\rangle$ are equal. This implies that $\left\langle \bigcup_{a\in A}\mathcal{K}_a \right\rangle \cap \left\langle \bigcup_{b\in B}\mathcal{K}_b \right\rangle \neq \emptyset$.

Second, consider a separation $\alpha \mid \beta$ and define the affine extension $\psi_{\alpha|\beta}\colon \mathbb{R}^n \to \mathbb{R}$ of the equations

$$\psi_{\alpha|\beta}(\mathbf{e}_i) = \begin{cases} -1 & \text{if } i \in \alpha, \\ 1 & \text{if } i \in \beta, \\ 0 & \text{otherwise.} \end{cases}$$

A straight-forward argument shows that $\psi_{\alpha|\beta}(\rho^i_{A\dagger B}) < 0$ whenever $i \in \alpha \cap A$ and, analogously, $\psi_{\alpha|\beta}(\rho^i_{A\dagger B}) > 0$ whenever $i \in \beta \cap A$. This implies that the families $\{\mathcal{K}_a : a \in \alpha\}$ and $\{\mathcal{K}_b : b \in \beta\}$ are separated. $\bullet$

It is simple to verify that the implicit bound $\gamma(S) \leq n - 1$ is tight.

If the separoid can be represented by a family of points in some Euclidian space, it is called a *point separoid* [22] (also known as a *linear oriented matroid* [3] or as an *order type* [10]). The following characterisation appears in [5,6].

Theorem 2. *A separoid S in general position is a point separoid if and only if $\rho(S) = \gamma(S)$.*

Sketch of the proof. Let S be represented with a family of convex sets in $\mathbb{E}^d$, where $d = \rho(S) = \gamma(S)$. Choose a point in each convex set to construct the point separoid P, and let $\varphi\colon P \to S$ be the obvious bijection. Now, it is enough to prove that φ is an isomorphism of separoids (i.e., to

prove that $\alpha \mid \beta \iff \varphi(\alpha) \mid \varphi(\beta)$). One side is clear; for the other, let $A \dagger B$ be a minimal Radon partition of S. Since S is in general position, the support $A \cup B$ consists of $d + 2$ or more elements. Then its preimage $\varphi^{-1}(A \cup B)$ is a set of at least $d + 2$ points in $\mathbb{E}^d$ which, due to Radon's lemma, induces a partition $C \dagger D$ such that $C \cup D = \varphi^{-1}(A \cup B)$. The proof concludes by showing that every separoid S in general position such that $\rho(S) = \gamma(S)$ is a Radon separoid.   ●

Proof of Theorem A. Let S be a separoid in general position. If $\rho(S) = \gamma(S)$ then, due to Theorem 2, S is a point separoid and it can be represented with a family of points $P \subset \mathbb{E}^d$, where $d = \gamma(S)$. Therefore, the Erdős–Szekeres theorem applies (in its general form) and there exists a subfamily of $\xi_d^{-1}(|S|)$ elements in convex position.   ●

A function $\mu \colon S \to P$ is called a *homomorphism* if it preserves the minimal Radon partitions; i.e., iff $\mu(A) \dagger \mu(B)$ is a minimal Radon partition whenever $A \dagger B$ is (see Nešetřil and Strausz [24] where the concept was introduced as a generalisation of graph homomorphisms and studied from a structural point of view). Clearly, to check that a function $\mu \colon S \to P$ is a homomorphism it is enough to check it for all $(\kappa(S) + 2)$-subsets of S.

Proof of Theorem B. Let S be a separoid and $P \subset \mathbb{E}^d$ a point separoid in general position. Suppose that in addition, there exists a homomorphism $\mu \colon S \to P$. Since P is in general position, the Erdős–Szekeres theorem applies and we can find a subfamily $Q \subset P$ of $|Q| = \xi_d^{-1}(|P|)$ points in convex position. Now, let $T = \{x \in \mu^{-1}(q) : q \in Q\}$ be a "choice" in $\mu^{-1}(Q)$. We claim that T is polytopal. For, suppose that there exists an $x \in T$ such that $x \dagger (T \setminus x)$. Then, there exists a $T' \subseteq T$ such that $x \dagger T'$ is minimal. This implies that $\mu(x) \dagger \mu(T')$ but this contradicts that $\mu(x) \mid (Q \setminus \mu(x))$.   ●

3. Theorem C

We say that the separoid S is *quasi-Radon* if it satisfies

$(qR) \qquad x \dagger B$ minimal and $b \in B \implies b \mid (B \setminus b \cup x)$.

Clearly, Radon separoids are quasi-Radon. Furthermore, we have that

Lemma 1. *Antimatroids in general position are quasi-Radon.*

Proof. Suppose that $x \dagger B$ is a minimal Radon partition of an antimatroid S in general position. The minimality and the general position implies that $b \mid (B \setminus b)$ and $x \mid (B \setminus b)$, whenever $b \in B$. It follows that (AM) implies (qR).   ●

We now prove the analogue of Ester Klein's observation for dispersed separoids.

Lemma 2. *Let S be a separoid with dimension $d = \rho(S)$. If S is dispersed, then*

$$(K) \qquad \forall A \in \binom{S}{d+3} \quad \exists P \in \binom{A}{d+2} : \quad \forall p \in P : p \mid (P \setminus p).$$

Proof. Let S be a dispersed separoid. Due to Lemma 1, S is a quasi-Radon separoid. With out loss of generality, suppose that $d = \kappa(S)$ and let A be a $(d + 3)$-subset of S. We may suppose that $A = B \cup \{x, y\}$, where $x \dagger B$ and $y \dagger B$.
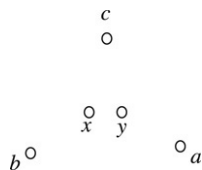
$$c$$
$$\circ$$

$$\overset{\circ}{\underset{x}{\times}} \quad \overset{\circ}{\underset{y}{\circ}}$$

$$\underset{b}{\circ} \qquad\qquad \underset{a}{\circ}$$

Fig. 5. Five points in the plane.

Since S is dispersed, there exist $a, b \in B$ such that $x \dagger (B \setminus a \cup y)$ and $y \dagger (B \setminus b \cup x)$. Observe that, since S is in general position and is quasi-Radon, such that a and b are unique; furthermore, they are different. Now, let $c \in B \setminus \{a, b\}$ and let $P = A \setminus c$. We will show that P is polytopal (see Fig. 5).

The uniqueness of a and b implies that $x \mid (P \setminus x)$ and $y \mid (P \setminus y)$. It also implies that $x \mid (B \setminus b \cup y)$. Since $x \dagger (B \cup y)$ and S is an antimatroid, it follows that $b \mid (A \setminus b)$ and therefore $b \mid (P \setminus b)$. Analogously $a \mid (P \setminus a)$, which concludes the case $d = 2$.

For $d > 2$, let $z \in P \setminus \{x, y, a, b\}$. The uniqueness of a implies that $x \mid (A \setminus \{x, z\})$, condition (qR) implies that $z \mid (A \setminus \{x, z\})$, and clearly $x \dagger (A \setminus x)$. Then, since S is an antimatroid, $z \mid (A \setminus z)$ and therefore $z \mid (P \setminus z)$.   •

We are ready to conclude the

Proof of Theorem C. Let $\xi = \xi_d(m) = \mathrm{R}_{d+2}(m, d + 3)$ be the Ramsey number which guarantees the existence of a monochromatic m-subset or a monochromatic $(d + 3)$-subset, in all two colourings of the $(d + 2)$-subsets of ξ elements. Then, given a d-dimensional dispersed separoid S of order ξ, if we colour its $(d+2)$-subsets green whenever they are in convex position, and red whenever they are not, by the Ramsey theorem there should exist either m elements all of whose $(d + 2)$-subsets are green, or $d + 3$ elements all of whose $(d + 2)$-subsets are red. Due to Lemma 2, this last is impossible and we are done.   •

4. Theorem E

Let $H = H_r(n, p)$ be the random uniform hypergraph of rank r and order n in which each edge appears with probability p (we will suppose that $\frac{1}{2} \leq p < 1$). Let $\omega(H)$ be its clique number.

Theorem 3. *While $n \to \infty$, $\mathbf{Pr}\left[\omega(H) \geq \sqrt[r-1]{\log n}\right] \to 1$.*

Sketch of the proof. We set

$$\mathbf{E}[X] = \binom{n}{k} p^{\binom{k}{r}}$$

the expected number of k-cliques of the random hypergraph H. The function $\mathbf{E}[X]$ drops under one at $k \sim \sqrt[r-1]{\log n}$ — roughly speaking it is like $n^k p^{k^r}$. So, let $k = k(n)$ satisfy $k \sim \sqrt[r-1]{\log n}$ and $\mathbf{E}[X] \to \infty$. We show now that

$$\frac{\mathbf{Var}[X]}{(\mathbf{E}[X])^2} \to 0.$$

A standard argument proves that it is enough to show that

$$\sum_{t=r}^{k} \frac{\binom{k}{t}\binom{n-k}{k-t}}{\binom{n}{k}} p^{-\binom{t}{r}} \to 0.$$

For, we simply bound the sum with the geometric series showing that

$$\frac{\binom{k}{t}\binom{n-k}{k-t}}{\binom{n}{k}} p^{-\binom{t}{r}} \le \left(\frac{k^2}{2^{k^{r-1}}}\right)^t.$$

The details are omitted since they add nothing to the present context. ●

Proof of Theorem E. Let $d > 2$ be a fixed number, and $S = [n]$ be an n-set. We define a separoid on S with the following minimal Radon partitions: for each $X \in \binom{S}{d+2}$, choose randomly and uniformly a subset $A \in 2^X \setminus \{\emptyset, X\}$ and let $A \dagger (X \setminus A)$. With this construction, S becomes a Radon separoid in general position with dimension $d = \rho(S) = \kappa(S)$.

Clearly, the probability that A is a singleton, or its complement, is

$$\mathbf{Pr}\left[|A| = 1 \quad \text{or} \quad |A| = n - 1\right] = \frac{2(d+2)}{2^{d+2} - 2} < \frac{1}{2}.$$

Thus, the probability that X is polytopal is $\frac{1}{2} \le p < 1$. Given such an S, we construct a uniform hypergraph H with vertex set $[n]$ and with an edge $e \in E(H) \subset \binom{[n]}{d+2}$ whenever e is polytopal as a subset of S. Therefore, due to Theorem 3, asymptotically almost always we can find a polytopal subseparoid $P \subset S$ of order $\sqrt[d+1]{\log n}$. ●

Remark. Clearly, any improvement on Theorem 3 would imply immediately an improvement on Theorem E — e.g., for the random graph G_n it is well known that $\omega(G_n) = \Theta(2 \log n)$. Also, the hypothesis on S seems to be just an artefact of the proof...

...let us close with a nice open question: Is the "random separoid" dispersed?

Acknowledgements

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