Supersymmetric BKP systems and their symmetries

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Abstract

In this paper, we construct the additional symmetries of the supersymmetric BKP (SBKP) hierarchy. These additional flows constitute a B type $SW_{1+\infty}$ Lie algebra because of the B type reduction of the supersymmetric BKP hierarchy. Further we generalize the SBKP hierarchy to a supersymmetric two-component BKP (S2BKP) hierarchy equipped with a B type $SW_{1+\infty} \bigoplus SW_{1+\infty}$ Lie algebra. As a bosonic reduction of the S2BKP hierarchy, we define a new constrained system called the supersymmetric Drinfeld–Sokolov hierarchy of type D which admits a $N = 2$ supersymmetric Block type symmetry.
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1. Introduction

In the study of integrable hierarchies, it is interesting to find their symmetries and identify the algebraic structure of the symmetries. Among these symmetries, the additional symmetry is an important type which contains dynamic variables explicitly and these additional flows do not commute with each other. Additional symmetries of the Kadomtsev–Petviashvili (KP) hierarchy were introduced by Orlov and Shulman [1] which contain one important symmetry, i.e. the so-called Virasoro symmetry. These symmetries form a centerless $W_{1+\infty}$ algebra closely related to matrix models by means of the Virasoro constraint and string equations [2–5].

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Two sub-hierarchies as the BKP hierarchy and CKP hierarchy [6–11] have been shown to possess additional symmetries, with consideration of the reductions on the Lax operators.

Various generalizations and supersymmetric extensions [12] of the KP hierarchy have deep implications in mathematical physics, particularly in the theory of Lie algebras. In [13,14], the theory of the super-Lie algebras was surveyed by considering super-boson–fermion Correspondences. One important supersymmetric extension is the supersymmetric Manin–Radul Kadomtsev–Petviashvili (MR–SKP) hierarchy [15] which contains a lot of integrable super-solitary equations equipped with the super-pseudodifferential operators. Apart from the Manin–Radul one, Mulase supersymmetrize the KP hierarchy by constructing a hierarchy called the Jacobian SKP hierarchy which does not possess a standard Lax formulation [16]. This hierarchy has strict Jacobian flows, i.e. it preserves the super-Riemann surface about which one can also see [17]. The additional symmetries for super-hierarchies were firstly found in the paper [18] by constructing the standard Orlov–Schulman additional nonisospectral flows. Later the additional symmetry of the MR–SKP hierarchy was studied by Stanciu [19]. The ghost symmetries, Hamiltonian structures and extensions of the MR–SKP hierarchy were studied as well as reductions of the MR–SKP hierarchy [20,21]. Later the supersymmetric BKP (SBKP) hierarchy was constructed in [22]. After that these series of super-hierarchies were seldom studied in mathematical physics partly because of their extreme complexities.

For the symmetry of the two-component BKP hierarchy, there is a series of works such as [7,23–25]. In the paper [26], we construct the generalized additional symmetries of the two-component BKP hierarchy and identify its algebraic structure. Besides, the D type Drinfeld–Sokolov hierarchy was found to be a good differential model to derive a complete Block type infinite dimensional Lie algebra. About the Block algebra related to integrable systems, we did a series of works in [27–29]. In this paper, we will construct the additional symmetries of the supersymmetric BKP hierarchy. These additional flows constitute a B type $SW_{1+\infty}$ Lie algebra. Further we generalize the SBKP hierarchy to a supersymmetric two-component BKP hierarchy (S2BKP hierarchy) and derive its algebraic structure. As a reduction of the S2BKP hierarchy, in this paper a new supersymmetric Drinfeld–Sokolov hierarchy of type D will be constructed and proved to have a super-Block type additional symmetry.

In $(1 + 1)$ dimensional supersymmetric integrable systems, starting from 1980s, there is also a series of work about superversions of Korteweg–de Vries (KdV), Toda–KdV equations and so on [30–33]. Most of them are related to the conformal field theory and string theory [31,32]. The generalized KdV equations and Toda lattice equations are particularly interesting integrable nonlinear systems in connection with conformal field theories. Their Virasoro symmetry can be extended to a W_n algebra which is known to arise from the Hamiltonian structure of the generalized KdV equation [34] by incorporating conserved currents of higher spins. The supersymmetric version of the Drinfeld–Sokolov reduction of the Toda–KdV theories gives the generators of the related super-W-algebra with the commutation relations provided by the associated Hamiltonian structure [35]. In this paper we will extend the Lie algebraic method of Drinfeld and Sokolov [34] to the sypersymmetric case and develop a Lie superalgebraic method for a supersymmetric D type Drinfeld–Sokolov hierarchy. We further derive that the supersymmetric Drinfeld–Sokolov hierarchy of type D possesses a $N = 2$ supersymmetric Block type Lie algebraic structure.

This paper is arranged as follows. In the next section we recall some necessary facts of the SBKP hierarchy. In Sections 3, we will give the additional symmetries for the SBKP hierarchy. The ghost symmetry of the SBKP hierarchy will be devoted to Sections 4 and 5 using the techniques in [20]. Further in Sections 6 and 7, we generalize the SBKP hierarchy to a S2BKP hierarchy and derive its B type $SW_{1+\infty} \bigoplus SW_{1+\infty}$ algebra. As a bosonic reduction of the

supersymmetric two-component BKP hierarchy, we define a new constrained system called the supersymmetric Drinfeld–Sokolov hierarchy of type D which possesses a $N = 2$ supersymmetric Block type Lie algebra in the following two sections. Finally, we will give a short conclusion and a further discussion.

2. The supersymmetric BKP hierarchy

Let us firstly recall some basic facts [22] on the supersymmetric BKP system which is well defined by two Lax operators.

$\mathcal{A}$ is assumed as an algebra of smooth functions of a spatial coordinate x, a Grassmann variable θ and their super-derivation denoted as $D = \partial_\theta + \theta \partial$. This algebra $\mathcal{A}$ has the following multiplying rule

$$D^n \Phi = \sum_{i=0}^{\infty} \begin{bmatrix} n \\ n-i \end{bmatrix} (-1)^{|\Phi|(n-i)} \Phi^{[i]} D^{n-i}, \tag{2.1}$$

$$\begin{bmatrix} n \\ n-i \end{bmatrix} = \begin{cases} 0 & i < 0 \ \text{or} \ (n, i) = (0, 1) \ (\text{mod } 2); \\ \begin{pmatrix} \left[\frac{n}{2}\right] \\ \left[\frac{n-i}{2}\right] \end{pmatrix} & i \geq 0, \quad (n, i) \neq (0, 1) \ (\text{mod } 2). \end{cases} \tag{2.2}$$

Here the value $|\Phi|$ means the super-degree of the operator Φ which shows the operator Φ is fermionic or bosonic. The supersymmetric derivative D satisfies the supersymmetric analog of the Leibniz rule

$$D(ab) = D(a)b + (-1)^{|a|} a D(b), \tag{2.3}$$

where a is a homogeneous element of A. We introduce the even and odd time variables $(t_2, t_3, t_6, t_7, \cdot)$ and the following definition of even and odd flows

$$D_{4i-2} = \frac{\partial}{\partial t_{4i-2}}, \quad D_{4i-1} = \frac{\partial}{\partial t_{4i-1}} + \sum_{j=1}^{\infty} t_{4j-1} \frac{\partial}{\partial t_{4i+4j-2}}. \tag{2.4}$$

We recall that the supercommutator is defined as $[X, Y] = XY - (-1)^{|X||Y|} YX$. The bracket has a property as $[X, YZ] = [X, Y]Z + (-1)^{|X||Y|} Y[X, Z]$. Then $D^2 = \frac{1}{2}[D, D] = \partial$. This family of infinite odd and even flows satisfy a nonabelian Lie superalgebra whose commutation relations are

$$[D_{4i-2}, D_{4j-2}] = 0, \quad [D_{4i-2}, D_{4j-1}] = 0, \quad [D_{4i-1}, D_{4j-1}] = -2D_{4i+4j-2},$$

$$[D_{4i-2}, D] = 0, \quad [D_{4i-1}, D] = 0. \tag{2.5}$$

For any operator $A = \sum_{i \in \mathbb{Z}} f_i D^i \in \mathcal{A}$ and homogeneous operators P, Q, its nonnegative projection, negative projection, adjoint operator are respectively defined as

$$A_+ = \sum_{i \geq 0} f_i D^i, \quad A_- = \sum_{i < 0} f_i D^i, \quad A^* = \sum_{i \in \mathbb{Z}} (-D)^i \cdot f_i, \tag{2.6}$$

$$(PQ)^* = (-1)^{|P||Q|} Q^* P^*, \quad (P^{-1})^* = (-1)^{|P|} (P^*)^{-1}. \tag{2.7}$$

Also for the operator D^k, the adjoint operator is defined as

$$(D^k)^* = (-1)^{\frac{k(k+1)}{2}} D^k. \tag{2.8}$$

Basing on definitions in [22], the Lax operator of the supersymmetric BKP hierarchy has a form as

$$L = D + \sum_{i \geq 1} u_i D^{1-i}, \quad u_2 = -\frac{1}{2} u_1^{[1]}. \tag{2.9}$$

The supersymmetric BKP hierarchy is defined by the following Lax equations

$$D_{4k-2}L = [(L^{4k-2})_+, L], \qquad D_{4k-1}L = [(L^{4k-1})_+, L] - 2L^{4k}, \quad k \geq 1. \tag{2.10}$$

One can rewrite the operator L in a dressing form as

$$L = \Phi D \Phi^{-1}, \tag{2.11}$$

where

$$\Phi = 1 + \sum_{i \geq 1} a_i D^{-i}, \tag{2.12}$$

satisfy

$$\Phi^* = D \Phi^{-1} D^{-1}. \tag{2.13}$$

We call Eq. (2.13) the B type condition of the supersymmetric BKP hierarchy. Given L, the dressing operator Φ is determined uniquely up to a multiplication to the right by operators with constant coefficients. The dressing operator Φ takes values in a B type Volterra group. The supersymmetric BKP hierarchy (2.10) can also be redefined as

$$\frac{\partial \Phi}{\partial t_{4k-2}} = -(L^{4k-2})_- \Phi, \qquad \frac{\partial \Phi}{\partial t_{4k-1}} = -(L^{4k-1})_- \Phi, \tag{2.14}$$

with $k \geq 1$.

With the above preparation, it is time to construct additional symmetries for the supersymmetric BKP hierarchy in the next section.

3. Additional symmetries of the supersymmetric BKP hierarchy

In this section, we are to construct additional symmetries for the supersymmetric BKP hierarchy by using the Orlov–Schulman operators whose coefficients depend explicitly on the time variables of the hierarchy. The Orlov–Schulman operators M_i and auxiliary operator $\mathbb{Q}$ are constructed in the following dressing structure

$$M_i = \Phi \Gamma_i \Phi^{-1}, \quad i = 0, 1; \quad \mathbb{Q} = \Phi Q \Phi^{-1},$$

where

$$\Gamma_0 = x + \frac{1}{2} \sum_{k \geq 1} (4k-2) t_{4k-2} D^{4k-4} + \frac{1}{2} (4k-1) t_{4k-1} D^{4k-3}$$

$$\quad - \frac{1}{2} \sum_{k \geq 1} t_{4k-1} \partial^{2k-2} Q + \sum_{i,j \geq 1} (i-j) t_{4i-1} t_{4j-1} \partial^{2i+2j-2}, \tag{3.1}$$

$$\Gamma_1 = \theta + \sum_{k \geq 1} t_{4k-1} \partial^{2k-1}, \tag{3.2}$$

where $Q = \partial_\theta - \theta \partial$.

Then one can get the following lemma.

Lemma 3.1. *The operators Γ_j, Q satisfy*

$$[D_{4i-2} - D^{4i-2}, \Gamma_j] = [D_{4i-1} - D^{4i-1}, \Gamma_j] = 0; \quad j = 0, 1, \tag{3.3}$$

$$[D_{4i-2} - D^{4i-2}, Q] = [D_{4i-1} - D^{4i-1}, Q] = 0, \tag{3.4}$$

$$[Q, \Gamma_0] = -\Gamma_1, \quad [Q, \Gamma_1] = 1, \quad [\partial, \Gamma_0] = 1. \tag{3.5}$$

Proof. For the proof, one can do the following direct calculation

$$
\begin{aligned}
[D_{4i-2} - D^{4i-2}, \Gamma_0] &= \frac{1}{2}(4i - 2)D^{4i-4} - [D^{4i-2}, x] \\
&= (2i - 1)\partial^{2i-2} - [\partial^{2i-1}, x] = 0,
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&[D_{4i-1} - D^{4i-1}, \Gamma_0] \\
&= [\frac{\partial}{\partial t_{4i-1}} - \sum_{j=1}^{\infty} t_{4j-1} \frac{\partial}{\partial t_{4i+4j-2}} - D^{4i-1}, \Gamma_0] \tag{3.7} \\
&= \frac{1}{2}(4i - 1)D^{4k-3} - \frac{1}{2}\partial^{2i-2}Q + 2\sum_{j \geq 1}(i - j)t_{4j-1}\partial^{2i+2j-2} \\
&\quad - \sum_{j=1}^{\infty}(2i + 2j - 1)t_{4j-1}D^{4i+4j-4} \\
&\quad - [D^{4i-1}, x - \frac{1}{2}\sum_{k \geq 1}(4k - 1)t_{4k-1}D^{4k-3}]. \tag{3.8}
\end{aligned}
$$

Because

$$[D^{4i-1}, x] = [\partial^{2i-1}D, x] = (2i - 1)D^{4i-3} + \partial^{2i-1}\theta\partial_x = \frac{1}{2}(4i - 1)D^{4k-3} - \frac{1}{2}\partial^{2i-2}Q,$$

$$[D^{4i-1}, \frac{1}{2}\sum_{k \geq 1}(4k - 1)t_{4k-1}D^{4k-3}] = -\sum_{j \geq 1}(4j - 1)t_{4j-1}D^{4i+4j-4}, \tag{3.9}$$

then

$$[D_{4i-1} - D^{4i-1}, \Gamma_0] = 0. \tag{3.10}$$

For Γ_1, we have

$$[D_{4i-2} - D^{4i-2}, \Gamma_1] = 0, \tag{3.11}$$

$$
\begin{aligned}
[D_{4i-1} - D^{4i-1}, \Gamma_1] &= [\frac{\partial}{\partial t_{4i-1}} - \sum_{j=1}^{\infty} t_{4j-1}\frac{\partial}{\partial t_{4i+4j-2}} - D^{4i-1}, \theta + \sum_{k \geq 1} t_{4k-1}\partial^{2k-1}] \\
&= \partial^{2i-1} - D^{4i-2} = 0. \tag{3.12}
\end{aligned}
$$

For Q, the following identities hold

$$[D_{4i-2} - D^{4i-2}, Q] = 0, \tag{3.13}$$

$$[D_{4i-1} - D^{4i-1}, Q] = [\frac{\partial}{\partial t_{4i-1}} - \sum_{j=1}^{\infty} t_{4j-1} \frac{\partial}{\partial t_{4i+4j-2}} - D^{4i-1}, Q] = 0, \tag{3.14}$$

$$[Q, \Gamma_0] = [Q, x] - [Q, \frac{1}{2} \sum_{k \geq 1} t_{4k-1} \partial^{2k-2} Q],$$

$$= -\theta - \sum_{k \geq 1} t_{4k-1} \partial^{2k-1} = -\Gamma_1, \tag{3.15}$$

$$[Q, \Gamma_1] = [Q, \theta] = 1. \qquad \square \tag{3.16}$$

Then it is easy to get the following lemma by dressing structures.

Lemma 3.2. *The operators* $M_j, \mathbb{Q}, L$ *satisfy*

$$[\mathbb{Q}, M_0] = -M_1, \quad [\mathbb{Q}, M_1] = 1, \quad [L^2, M_0] = 1, \tag{3.17}$$

$$D_k M_j = [(L^k)_+, M_j], \quad D_k \mathbb{Q} = [(L^k)_+, \mathbb{Q}], \quad k = 4i-2, 4i-1, \ i \in \mathbb{Z}_+. \tag{3.18}$$

Proof. The dressing structure

$$\Phi[D_{4i-1} - D^{4i-1}, \Gamma_1]\Phi^{-1} = 0; \tag{3.19}$$

will lead to

$$[\Phi D_{4i-1} \Phi^{-1} - \Phi D^{4i-1} \Phi^{-1}, M_1] = 0; \tag{3.20}$$

and further to

$$[D_{4i-1} - \Phi_{4i-1} \Phi^{-1} + \sum_{j=1}^{\infty} t_{4i-1} \Phi_{4i+4j-2} \Phi^{-1} - L^{4i-1}, M_1] = 0. \tag{3.21}$$

Then we get

$$[D_{4i-1} - (D_{4i-1}\Phi)\Phi^{-1} - L^{4i-1}, M_1] = 0; \tag{3.22}$$

and using Eq. (2.14) we can derive

$$[D_{4i-1} - (L^{4i-1})_+, M_1] = 0. \tag{3.23}$$

The other identities can be proved using the similar dressing techniques. $\square$

From now on, we will introduce the following operator B_{mklp} defined as

$$B_{mklp} = M_0^k M_1^l \mathbb{Q}^p L^{2m} - (-1)^{pl+m+p+l} L^{2m-1} (\mathbb{Q}^p) M_1^l M_0^k L, \tag{3.24}$$

where $k, m \geq 0$; $l, p = 0, 1$. This operator is the generator of the additional symmetry of the SBKP hierarchy which shows the difference between generators of the SKP and SBKP hierarchies. For the SKP case in [15], in the construction of B_{mklp}, it contains only one term.

Then the following proposition can be got.

Proposition 3.3. *The operator* B_{mnlp} *satisfies the following flow equations*

$$D_{4k-2} B_{mnlp} = -[(L^{4k-2})_-, B_{mnlp}], \quad D_{4k-1} B_{mnlp} = -[(L^{4k-1})_-, B_{mnlp}]. \tag{3.25}$$

Proof. The lemma can be proved by dressing the following identities by Φ

$$[D_{4k-2} - D^{4k-2}, \Gamma_0^n \Gamma_1^l Q^p \partial^m] = [D_{4k-1} - D^{4k-1}, \Gamma_0^n \Gamma_1^l Q^p \partial^m] = 0. \qquad \square \qquad (3.26)$$

To prove that B_{mnlp} satisfies the B type condition, we need the following lemma.

Lemma 3.4. *The operators M_i satisfy the following conjugate identities,*

$$M_i^* = (-1)^i D L^{-1} M_i L D^{-1}, \quad \mathbb{Q}^* = -D L^{-1} \mathbb{Q} L D^{-1}. \qquad (3.27)$$

Proof. Using

$$\Phi^* = D \Phi^{-1} D^{-1}, \quad \Gamma_i^* = (-1)^i \Gamma_i, \quad Q^* = -Q, \qquad (3.28)$$

the following calculations

$$M_i^* = \Phi^{*-1} \Gamma_i^* \Phi^* = (-1)^i D \Phi D^{-1} \Gamma_i D \Phi^{-1} D^{-1} = (-1)^i D \Phi D^{-1} \Phi^{-1} M_i \Phi D \Phi^{-1} D^{-1},$$

will lead to this lemma. The anti adjoint property of $\mathbb{Q}$ can be proved in a similar way. $\square$

It is easy to check the following proposition holds basing on Lemma 3.4 above.

Proposition 3.5. *The operator B_{mklp} satisfies a B type condition, namely*

$$B_{mklp}^* = -D B_{mklp} D^{-1}. \qquad (3.29)$$

Proof. Using Proposition 3.4, the following calculation will lead to this proposition

$$
\begin{aligned}
B_{mklp}^* &= (M_0^k M_1^l \mathbb{Q}^p L^{2m} - (-1)^{pl+m+p+l} L^{2m-1} \mathbb{Q}^p M_1^l M_0^k L)^* \\
&= (-1)^{pl} L^{2m*} (\mathbb{Q}^p)^* M_1^{l*} M_0^{k*} + (-1)^{m+p+l} L^* M_0^{k*} M_1^{l*} (\mathbb{Q}^p)^* L^{2m-1*} \\
&= (-1)^{pl+m+p+l} D L^{2m-1} \mathbb{Q}^p M_1^l M_0^k L D^{-1} - D M_0^k M_1^l \mathbb{Q}^p L^{2m} D^{-1} \\
&= -D (M_0^k M_1^l \mathbb{Q}^p L^{2m} - (-1)^{pl+m+p+l} L^{2m-1} \mathbb{Q}^p M_1^l M_0^k L) D^{-1}. \qquad \square
\end{aligned}
$$

Basing on above proposition, it is reasonable to define additional flows of the supersymmetric BKP hierarchy as

$$D_{mklp} L = [-(B_{mklp})_-, \mathcal{L}], \quad k, m \geq 0; l, p = 0, 1. \qquad (3.30)$$

Proposition 3.6. *The flows (3.30) commute with the flows of the supersymmetric BKP hierarchy. Namely, one has*

$$[D_{mnlp}, D_k] = 0, \quad m, n \geq 0; l, p = 0, 1, \quad k = 4i-2, 4i-1, \ i \in \mathbb{Z}_+, \qquad (3.31)$$

which holds in the sense of acting on Φ.

Proof. The proposition can be checked case by case with the help of Eq. (3.25). For example,

$$
\begin{aligned}
[D_{mnlp}, D_k] \Phi \\
= D_{mnlp} D_k \Phi - (-1)^{(l+p)k} D_k D_{mnlp} \Phi \\
= (-1)^{(l+p)k} [(L^k)_-, (B_{mnlp})_-] \Phi \\
\quad + [(B_{mnlp})_-, L^k]_- \Phi + (-1)^{(l+p)k} [(L^k)_+, B_{mnlp}]_- \Phi = 0. \qquad \square \qquad (3.32)
\end{aligned}
$$

This proposition tells us that the additional flows of the supersymmetric BKP hierarchy are in fact its symmetries whose algebraic structure can be shown in the following proposition.

Proposition 3.7. *The algebra of additional symmetries of the SBKP hierarchy given by Eq.* (3.30) *is isomorphic to the Lie algebra* $SW_{1+\infty}$.

Proof. The isomorphism is given by

$$z \mapsto \partial, \quad \xi \mapsto Q + \Gamma_1 \partial, \tag{3.33}$$

$$\partial_z \mapsto \Gamma_0, \quad \partial_\xi \mapsto \Gamma_1, \tag{3.34}$$

which further lead to

$$z \mapsto L^2, \quad \xi \mapsto \mathbb{Q} + M_1 L^2, \tag{3.35}$$

$$\partial_z \mapsto M_0, \quad \partial_\xi \mapsto M_1. \tag{3.36}$$

One can find the above construction keeps ξ commuting with z. □

4. Ghost symmetry of supersymmetric BKP hierarchy

In this section, we will give another special symmetry which does not contain time variables explicitly. Before that, we firstly need to define the super-Baker–Akhiezer function (super-BA) and adjoint super-BA function as

$$\Phi_{BA} = \Phi e^\xi, \quad \Phi^*_{BA} = \Phi^{*-1} e^{-\xi}, \tag{4.1}$$

where

$$\xi(\lambda, \eta, \theta, t) = \sum_{k=1}^\infty \lambda^{4k-2} t_{4k-2} + \eta\theta + (\eta - \lambda\theta) \sum_{k=1}^\infty \lambda^{4k-2} t_{4k-1}, \ t_2 \equiv x. \tag{4.2}$$

The following property can be found

$$D_{4i-2} e^\xi = \partial^{2i-1} e^\xi, \quad D_{4i-1} e^\xi = D^{2i-1} e^\xi. \tag{4.3}$$

Then we can prove that

$$L^2 \Phi_{BA} = \lambda \Phi_{BA}, \quad L^{2*} \Phi^*_{BA} = -\lambda \Phi^*_{BA}, \tag{4.4}$$

$$D_{4i-2} \Phi_{BA} = (L^{4i-2})_+ \Phi_{BA}, \quad D_{4i-2} \Phi^*_{BA} = -(L^{4i-2})^*_+ \Phi^*_{BA}, \tag{4.5}$$

$$D_{4i-1} \Phi_{BA} = (L^{4i-1})_+ \Phi_{BA}, \quad D_{4i-1} \Phi^*_{BA} = -(L^{4i-1})^*_+ \Phi^*_{BA}. \tag{4.6}$$

The B type condition implies that the adjoint super-BA function Ψ_{BA} can be in fact the supersymmetric derivative of its corresponding super-BA function Φ_{BA}, i.e.

$$\Psi_{BA}(t, \lambda) = -\lambda^{-1} \Phi^{[1]}_{BA}(t, -\lambda). \tag{4.7}$$

Then we can also define super-eigenfunctions of the supersymmetric BKP hierarchy as

$$D_{4i-2} \phi = (L^{4i-2})_+ \phi, \quad D_{4i-2} \psi = -(L^{4i-2})^*_+ \psi, \tag{4.8}$$

$$D_{4i-1} \phi = (L^{4i-1})_+ \phi, \quad D_{4i-1} \psi = -(L^{4i-1})^*_+ \psi. \tag{4.9}$$

The super-eigenfunctions have the following spectral representation in term of integrals of Baker–Akhiezer functions as

$$\phi(t,\theta) = \int d\lambda d\eta \phi(\lambda,\eta) \Phi_{BA}(t,\theta;\lambda,\eta), \quad \psi(t,\theta) = \int d\lambda d\eta \bar{\psi}(\lambda,\eta) \Phi^*_{BA}(t,\theta;\lambda,\eta).$$

The B type condition also implies that the adjoint eigenfunction ψ can be chosen as the supersymmetric derivative of its corresponding eigenfunction ϕ, i.e.

$$\psi = \phi^{[1]}. \tag{4.10}$$

The supersymmetric tau function of the supersymmetric BKP hierarchy can be defined by the residue (the coefficient before D^{-1}) of supersymmetric Lax operators as

$$D_{4k-2} D \ln \tau = \text{res } L^{4k-2}, \quad D_{4k-1} D \ln \tau = \text{res } L^{4k-1}. \tag{4.11}$$

Define two eigenfunctions ϕ_1, ϕ_2 and the following operator

$$B_g = \phi_1 D^{-1} \phi_2^{[1]} - (-1)^{|\phi_2|} \phi_2 D^{-1} \phi_1^{[1]}, \quad |\phi_1| \equiv |\phi_2|, \tag{4.12}$$

which is used to generate the ghost flows. According to

$$D^{-1} \phi = (-1)^{|\phi|} [\phi D^{-1} - D^{-1} \phi^{[1]} D^{-1}], \tag{4.13}$$

one can find the operator B_g satisfies the B type condition, i.e.

$$B_g^* = -D B_g D^{-1}. \tag{4.14}$$

Then the ghost flow of the supersymmetric BKP hierarchy can be defined as following

$$D_Z L = [B_g, L] = [\phi_1 D^{-1} \phi_2^{[1]} - (-1)^{|\phi_2|} \phi_2 D^{-1} \phi_1^{[1]}, L], \tag{4.15}$$

where functions ϕ_1, ϕ_2 are the eigenfunction and adjoint eigenfunction of the supersymmetric BKP hierarchy. The following proposition will tell you the above flow is a symmetry of the supersymmetric BKP hierarchy.

Proposition 4.1. *The additional flow D_Z commutes with the supersymmetric BKP flows D_n, i.e.,*

$$[D_Z, D_n] L = 0, \quad n = 4i - 2, 4i - 1, \ i \in \mathbb{Z}_+. \tag{4.16}$$

Proof. Note that the derivatives $\phi_1^{[1]}, \phi_2^{[1]}$ are in fact adjoint supersymmetric eigenfunctions. The commutativity between ghost flows and supersymmetric BKP flows is in fact equivalent to the following zero-curvature equation which includes the following detailed proof

$$\begin{aligned}
&D_Z B_n - D_n(B_g) + [B_n, B_g] \\
&= [B_g, L^n]_+ - \phi_{1t_n} D^{-1} \phi_2^{[1]} - \phi_1 D^{-1} \psi_{2t_n}^{[1]} + (-1)^{|\phi_2|} [\phi_{2t_n} D^{-1} \phi_1^{[1]} + \phi_2 D^{-1} \phi_{1t_n}^{[1]}] \\
&\quad + [B_n, \phi_1 D^{-1} \phi_2^{[1]} - (-1)^{|\phi_2|} \phi_2 D^{-1} \phi_1^{[1]}] \\
&= (B_n \phi_1 D^{-1} \phi_2^{[1]})_- - (\phi_1 D^{-1} \phi_2^{[1]} B_n)_- - P_0(B_n \phi_1) D^{-1} \phi_2^{[1]} + \phi_1 D^{-1} P_0(B_n^* \phi_2^{[1]}) \\
&\quad - (-1)^{|\phi_2|} [(B_n \phi_2 D^{-1} \phi_1^{[1]})_- - (\phi_2 D^{-1} \phi_1^{[1]} B_n)_- \\
&\quad - P_0(B_n \phi_2) D^{-1} \phi_1^{[1]} + \phi_2 D^{-1} P_0(B_n^* \phi_1^{[1]})] = 0.
\end{aligned}$$

In the above proof the $P_0(A)$ means the coefficient over the term D^0 of the operator A. □

5. The supersymmetric two-component BKP hierarchy

Let us firstly define the supersymmetric two-component BKP hierarchy by two Lax operators. Now we introduce the even and odd time variables $(t_2, t_3, t_6, \cdot; \hat{t}_2, \hat{t}_3, \hat{t}_6, \cdot)$ and the following definition of even and odd flows

$$D_{4i-2} = \frac{\partial}{\partial t_{4i-2}}, \quad D_{4i-1} = \frac{\partial}{\partial t_{4i-1}} + \sum_{j=1}^{\infty} t_{4j-1} \frac{\partial}{\partial t_{4i+4j-2}}, \tag{5.1}$$

$$\hat{D}_{4i-2} = \frac{\partial}{\partial \hat{t}_{4i-2}}, \quad \hat{D}_{4i-1} = \frac{\partial}{\partial \hat{t}_{4i-1}} + \sum_{j=1}^{\infty} \hat{t}_{4j-1} \frac{\partial}{\partial \hat{t}_{4i+4j-2}}. \tag{5.2}$$

This two families of odd and even flows satisfy a nonabelian Lie superalgebra whose commutation relations are

$$[D_{4i-2}, D_{4j-2}] = 0, \quad [D_{4i-2}, D_{4j-1}] = 0, \quad [D_{4i-1}, D_{4j-1}] = -2D_{4i+4j-2},$$

$$[D_{4i-2}, D] = 0, \quad [D_{4i-1}, D] = 0, \tag{5.3}$$

$$[\hat{D}_{4i-2}, \hat{D}_{4j-2}] = 0, \quad [\hat{D}_{4i-2}, \hat{D}_{4j-1}] = 0, \quad [\hat{D}_{4i-1}, \hat{D}_{4j-1}] = -2\hat{D}_{4i+4j-2},$$

$$[\hat{D}_{4i-2}, D] = 0, \quad [\hat{D}_{4i-1}, D] = 0, \quad [\hat{D}_m, D_n] = 0. \tag{5.4}$$

The two Lax operators of the supersymmetric two-component BKP hierarchy will be defined in forms as

$$L = D + \sum_{i \geq 1} u_i D^{1-i}, \quad \hat{L} = D^{-1}\hat{u}_0 + \sum_{i \geq 1} \hat{u}_i D^{i-1}, \quad |u_i| = i, |\hat{u}_i| = i + 1, \tag{5.5}$$

such that

$$L^* = -DLD^{-1}, \quad \hat{L}^* = -D\hat{L}D^{-1}. \tag{5.6}$$

We call Eqs. (5.6) the B type condition of the supersymmetric two-component BKP hierarchy. The supersymmetric two-component BKP hierarchy is defined by the following Lax equations:

$$D_i L = -[(L^i)_-, L], \qquad \hat{D}_i \hat{L} = [(\hat{L}^i)_+, \hat{L}], \tag{5.7}$$

$$D_i \hat{L} = [(L^i)_+, \hat{L}], \qquad \hat{D}_i L = -[(\hat{L}^i)_-, L], \quad i = 4k-1, 4k-2, \ k \in \mathbb{Z}_+, \tag{5.8}$$

which is equivalent to the following equations

$$D_{4k-2} L = [(L^{4k-2})_+, L], \qquad \hat{D}_{4k-2} \hat{L} = -[(\hat{L}^{4k-2})_-, \hat{L}], \tag{5.9}$$

$$D_{4k-1} L = [(L^{4k-1})_+, L] - 2L^{4k}, \qquad \hat{D}_{4k-1} \hat{L} = -[(\hat{L}^{4k-1})_-, \hat{L}] + 2\hat{L}^{4k}, \tag{5.10}$$

$$D_{4k-2} \hat{L} = [(L^{4k-2})_+, \hat{L}], \qquad \hat{D}_{4k-2} L = -[(\hat{L}^{4k-2})_-, L], \tag{5.11}$$

$$D_{4k-1} \hat{L} = [(L^{4k-1})_+, \hat{L}], \qquad \hat{D}_{4k-1} L = -[(\hat{L}^{4k-1})_-, L], \tag{5.12}$$

$k \in \mathbb{Z}_+$.

One can write the operators L and $\hat{L}$ in a dressing form as

$$L = \Phi D \Phi^{-1}, \quad \hat{L} = \hat{\Phi} D^{-1} \hat{\Phi}^{-1}, \tag{5.13}$$

where

$$\Phi = 1 + \sum_{i \geq 1} a_i D^{-i}, \quad \hat{\Phi} = 1 + \sum_{i \geq 1} b_i D^i, \tag{5.14}$$

satisfy

$$\Phi^* = D\Phi^{-1}D^{-1}, \quad \hat{\Phi}^* = D\hat{\Phi}^{-1}D^{-1}. \tag{5.15}$$

Given L and $\hat{L}$, the dressing operators Φ and $\hat{\Phi}$ are determined uniquely up to a multiplication to the right by operators with constant coefficients. The dressing operators Φ and $\hat{\Phi}$ take values in two separated B type Volterra groups. The supersymmetric two-component BKP hierarchy can also be redefined as

$$D_{4k-2}\Phi = -(L^{4k-2})_-\Phi, \quad D_{4k-2}\hat{\Phi} = (L^{4k-2})_+\hat{\Phi}, \tag{5.16}$$

$$\hat{D}_{4k-2}\Phi = -(\hat{L}^{4k-2})_-\Phi, \quad \hat{D}_{4k-2}\hat{\Phi} = (\hat{L}^{4k-2})_+\hat{\Phi}, \tag{5.17}$$

$$\hat{D}_{4k-1}\Phi = -(L^{4k-1})_-\Phi, \quad D_{4k-1}\hat{\Phi} = (L^{4k-1})_+\hat{\Phi}, \tag{5.18}$$

$$\hat{D}_{4k-1}\Phi = -(\hat{L}^{4k-1})_-\Phi, \quad \hat{D}_{4k-1}\hat{\Phi} = (\hat{L}^{4k-1})_+\hat{\Phi}, \tag{5.19}$$

with $k \in \mathbb{Z}_+$.

Denote $t = (t_2, t_3, t_6, t_7, \ldots)$, $\hat{t} = (\hat{t}_2, \hat{t}_3, \hat{t}_6, \hat{t}_7, \ldots)$ and introduce two wave functions

$$w(z) = w(t, \hat{t}; z) = \Phi e^{\xi(t;z)}, \quad \hat{w}(z) = \hat{w}(t, \hat{t}; z) = \hat{\Phi} e^{xz+\hat{\xi}(\hat{t};-z^{-1})}, \tag{5.20}$$

where the functions $\xi, \hat{\xi}$ are defined as $\xi(t;z) = \sum_{k \in \mathbb{Z}_+} t_{4k-2}z^{4k-2} + t_{4k-1}z^{4k-1}$, $\hat{\xi}(\hat{t};z) = \sum_{k \in \mathbb{Z}_+} \hat{t}_{4k-2}\hat{z}^{4k-2} + \hat{t}_{4k-1}\hat{z}^{4k-1}$. It is easy to see $D^i e^{xz} = z^i e^{xz}$, $i \in \mathbb{Z}$ and

$$Lw(z) = zw(z), \quad \hat{L}\hat{w}(z) = z^{-1}\hat{w}(z). \tag{5.21}$$

With the above preparation, it is time to construct additional symmetries for the supersymmetric two-component BKP hierarchy in the next section.

6. Additional symmetries of the supersymmetric two-component BKP hierarchy

In this section, we are to construct additional symmetries for the supersymmetric two-component BKP hierarchy by using the Orlov–Schulman operators whose coefficients depend explicitly on the time variables of the hierarchy.

With the same dressing operators given in Eq. (5.14), Orlov–Schulman operators M_i, $\hat{M}_i$, $\mathbb{Q}$, $\hat{\mathbb{Q}}$ are constructed in the following dressing structure

$$M_i = \Phi\Gamma_i\Phi^{-1}, \quad \hat{M}_i = \hat{\Phi}\hat{\Gamma}_i\hat{\Phi}^{-1}, \ i = 0, 1; \quad \mathbb{Q} = \Phi Q \Phi^{-1}, \hat{\mathbb{Q}} = \hat{\Phi}Q\hat{\Phi}^{-1},$$

where

$$\Gamma_0 = x + \frac{1}{2}\sum_{k \geq 1}(4k-2)t_{4k-2}D^{4k-4} + \frac{1}{2}(4k-1)t_{4k-1}D^{4k-3}$$

$$- \frac{1}{2}\sum_{k \geq 1}t_{4k-1}\partial^{2k-2}Q + \sum_{i,j \geq 1}(i-j)t_{4i-1}t_{4j-1}\partial^{2i+2j-2}, \tag{6.1}$$

$$\hat{\Gamma}_0 = x + \frac{1}{2} \sum_{k \geq 1} (4k - 2)\hat{t}_{4k-2} D^{-4k} + \frac{1}{2}(4k - 1)\hat{t}_{4k-1} D^{-1-4k}$$

$$- \frac{1}{2} \sum_{k \geq 1} \hat{t}_{4k-1} \partial^{-1-2k} Q + \sum_{i,j \geq 1} (i - j)\hat{t}_{4i-1}\hat{t}_{4j-1} \partial^{-2i-2j}, \tag{6.2}$$

$$\Gamma_1 = \theta + \sum_{k \in \mathbb{Z}_+} t_{4k-1} \partial^{2k-1}, \tag{6.3}$$

$$\hat{\Gamma}_1 = \theta + \sum_{k \in \mathbb{Z}_+} \hat{t}_{4k-1} \partial^{-2k}, \tag{6.4}$$

where $Q = \partial_\theta - \theta\partial$.

Then one can derive the following lemma similarly as the case of the single-component supersymmetric BKP hierarchy.

Lemma 6.1. *The operators Γ_i and $\hat{\Gamma}_i$ satisfy*

$$[D_{4i-2} - D^{4i-2}, \Gamma_i] = [D_{4i-1} - D^{4i-1}, \Gamma_i] = 0, \tag{6.5}$$

$$[\hat{D}_{4i-2} + D^{-4i+2}, \hat{\Gamma}_i] = [\hat{D}_{4i-1} + D^{-4i+1}, \hat{\Gamma}_i] = 0; \tag{6.6}$$

$$[D_{4i-2}, \hat{\Gamma}_i] = [D_{4i-1}, \hat{\Gamma}_i] = [\hat{D}_{4i-2}, \Gamma_i] = [\hat{D}_{4i-1}, \Gamma_i] = 0; \tag{6.7}$$

where $i = 0, 1$.

Proof. For the proof, we do the following calculation

$$[\hat{D}_{4i-2} + D^{2-4i}, \hat{\Gamma}_0] = \frac{1}{2}(4k - 2)D^{-4k} + (1 - 2i)\partial^{-2i} = 0, \tag{6.8}$$

$$[\hat{D}_{4i-1} + D^{1-4i}, \hat{\Gamma}_0]$$

$$= [\frac{\partial}{\partial\hat{t}_{4i-1}} - \sum_{j=1}^{\infty} \hat{t}_{4j-1} \frac{\partial}{\partial\hat{t}_{4i+4j-2}} + D^{1-4i}, \hat{\Gamma}_0] \tag{6.9}$$

$$= \frac{1}{2}(4i - 1)D^{-1-4k} - \frac{1}{2}\partial^{-1-2i} Q + 2\sum_{j \in \mathbb{Z}} (i - j)\hat{t}_{4j-1}\partial^{-2i-2j}$$

$$- \sum_{j=1}^{\infty} (2i + 2j - 1)\hat{t}_{4j-1} D^{-4i-4j} + [D^{1-4i}, x + \frac{1}{2} \sum_{k \in \mathbb{Z}_+} (4k - 1)\hat{t}_{4k-1} D^{-1-4k}]. \tag{6.10}$$

Because

$$[D^{1-4i}, x] = [\partial^{-2i} \hat{D}, x] = -2i D^{-4i-1} + \partial^{-2i-1}\theta\partial = -\frac{1}{2}(4i - 1)D^{-1-4i} + \frac{1}{2}\partial^{-1-2i} Q,$$

$$[D^{1-4i}, \frac{1}{2} \sum_{k \in \mathbb{Z}_+} (4k - 1)\hat{t}_{4k-1} D^{-1-4k}] = \sum_{j \in \mathbb{Z}} (4j - 1)\hat{t}_{4j-1} D^{-4i-4j}, \tag{6.11}$$

then

$$[\hat{D}_{4i-1} + D^{4i-1}, \hat{\Gamma}_0] = 0. \tag{6.12}$$

For $\hat{\Gamma}_1$, we get

$$[\hat{D}_{4i-1} + D^{1-4i}, \hat{\Gamma}_1] = [\frac{\partial}{\partial \hat{t}_{4i-1}} - \sum_{j=1}^{\infty} \hat{t}_{4j-1} \frac{\partial}{\partial \hat{t}_{4i+4j-2}} + D^{1-4i}, \theta + \sum_{k \in \mathbb{Z}_+} \hat{t}_{4k-1} \partial^{-2k}]$$

$$= \partial^{-2i} - D^{-4i} = 0. \tag{6.13}$$

Further one can derive

$$[\hat{D}_k, \Gamma_i] = [D_k, \hat{\Gamma}_i] = 0. \tag{6.14}$$

For Q, we can get

$$[D_k, Q] = [\hat{D}_k, Q] = 0, \tag{6.15}$$

$$[Q, \hat{\Gamma}_0] = [Q, x] - [Q, \frac{1}{2} \sum_{k \in \mathbb{Z}_+} \hat{t}_{4k-1} \partial^{-1-2k} Q] \tag{6.16}$$

$$= -\theta - \sum_{k \in \mathbb{Z}_+} \hat{t}_{4k-1} \partial^{-2k} = -\hat{\Gamma}_1. \quad \square \tag{6.17}$$

Then it is easy to get the following lemma using the above lemma and dressing structures.

Lemma 6.2. *The operators* $M_j, \mathbb{Q}, L, \hat{M}_j, \hat{\mathbb{Q}}, \hat{L}$ *satisfy*

$$[\mathbb{Q}, M_0] = -M_1, \quad [\mathbb{Q}, M_1] = 1, \quad [L^2, M_0] = 1, \tag{6.18}$$

$$D_k M_j = [(L^k)_+, M_j], \quad D_k \mathbb{Q} = [(L^k)_+, \mathbb{Q}], \quad k = 4i - 2, 4i - 1, i \in \mathbb{Z}_+, \tag{6.19}$$

$$\hat{D}_k M_j = [-(\hat{L}^k)_-, M_j], \quad \hat{D}_k \mathbb{Q} = [-(\hat{L}^k)_-, \mathbb{Q}], \quad k = 4i - 2, 4i - 1, i \in \mathbb{Z}_+, \tag{6.20}$$

$$[\hat{\mathbb{Q}}, \hat{M}_0] = -\hat{M}_1, \quad [\hat{\mathbb{Q}}, \hat{M}_1] = 1, \quad [\hat{L}^{-2}, \hat{M}_0] = 1, \tag{6.21}$$

$$D_k \hat{M}_j = [(L^k)_+, \hat{M}_j], \quad D_k \hat{\mathbb{Q}} = [(L^k)_+, \hat{\mathbb{Q}}], \quad k = 4i - 2, 4i - 1, i \in \mathbb{Z}_+, \tag{6.22}$$

$$\hat{D}_k \hat{M}_j = [-(\hat{L}^k)_-, \hat{M}_j], \quad \hat{D}_k \hat{\mathbb{Q}} = [-(\hat{L}^k)_-, \hat{\mathbb{Q}}], \quad k = 4i - 2, 4i - 1, i \in \mathbb{Z}_+. \tag{6.23}$$

From now on, we will introduce the following two operators B_{mklp} and $\hat{B}_{mklp}$, given any pair of integers (m, k, l, p) with $m, k \geq 0, l, p = 0, 1$, as

$$B_{mklp} = M_0^k M_1^l \mathbb{Q}^p L^{2m} - (-1)^{pl+m+p+l} L^{2m-1} (\mathbb{Q}^p) M_1^l M_0^k L, \tag{6.24}$$

$$\hat{B}_{mklp} = \hat{M}_0^k \hat{M}_1^l \hat{\mathbb{Q}}^p \hat{L}^{-2m} - (-1)^{pl+m+p+l} \hat{L}^{-2m+1} (\hat{\mathbb{Q}}^p) \hat{M}_1^l \hat{M}_0^k \hat{L}^{-1}. \tag{6.25}$$

As a corollary, the following proposition can be got.

Proposition 6.3. *For any* $\bar{B}_{mklp} = B_{mklp}, \hat{B}_{mklp}$, *one has*

$$D_n \bar{B}_{mklp} = [(L^n)_+, \bar{B}_{mklp}],$$
$$\hat{D}_n \bar{B}_{mklp} = [-(\hat{L}^n)_-, \bar{B}_{mklp}], \quad n = 4i - 2, 4i - 1, \ i \in \mathbb{Z}_+. \tag{6.26}$$

To prove that B_{mklp} and $\hat{B}_{mklp}$ satisfy the B type condition, we need the following lemma.

Lemma 6.4. *Operators* M_i *and* $\hat{M}_i$ *satisfy the following conjugate identities,*

$$M_i^* = (-1)^i DL^{-1} M_i L D^{-1}, \quad \hat{M}_i^* = (-1)^i D\hat{L}\hat{M}_i \hat{L}^{-1} D^{-1}, \tag{6.27}$$

$$\mathbb{Q}^* = -DL^{-1}\mathbb{Q}LD^{-1}, \quad \hat{\mathbb{Q}}^* = -D\hat{L}\hat{\mathbb{Q}}\hat{L}^{-1}D^{-1}. \tag{6.28}$$

Proof. Using

$$\Phi^* = D\Phi^{-1}D^{-1}, \quad \hat{\Phi}^* = D\hat{\Phi}^{-1}D^{-1}, \tag{6.29}$$

the following calculations

$$M_i^* = \Phi^{*-1}\Gamma_i^*\Phi^* = (-1)^i D\Phi D^{-1}\Gamma_i D\Phi^{-1}D^{-1} = (-1)^i D\Phi D^{-1}\Phi^{-1}M_i \Phi D\Phi^{-1}D^{-1},$$

$$\hat{M}_i^* = \hat{\Phi}^{*-1}\hat{\Gamma}_i^*\hat{\Phi}^* = (-1)^i D\hat{\Phi} D^{-1}\hat{\Gamma}_i D\hat{\Phi}^{-1}D^{-1} = (-1)^i D\hat{\Phi} D^{-1}\hat{\Phi}^{-1}\hat{M}_i \hat{\Phi} D\hat{\Phi}^{-1}D^{-1},$$

will lead to this lemma. $\square$

It is easy to check the following proposition holds basing on Lemma 6.4 above.

Proposition 6.5. *Operators* B_{mklp} *and* $\hat{B}_{mklp}$ *satisfy the B type condition, namely*

$$B_{mklp}^* = -DB_{mklp}D^{-1}, \quad \hat{B}_{mklp}^* = -D\hat{B}_{mklp}D^{-1}. \tag{6.30}$$

Proof. Using Proposition 6.4, the following calculation will lead to the first identity of this proposition

$$\begin{aligned}
\hat{B}_{mklp}^* &= (\hat{M}_0^k \hat{M}_1^l \hat{\mathbb{Q}}^p \hat{L}^{2m} - (-1)^{pl+m+p+l}\hat{L}^{2m-1}\hat{\mathbb{Q}}^p \hat{M}_1^l \hat{M}_0^k \hat{L})^* \\
&= (-1)^{pl}\hat{L}^{2m*}(\hat{\mathbb{Q}}^p)^*\hat{M}_1^{l*}\hat{M}_0^{k*} + (-1)^{m+p+l}\hat{L}^*\hat{M}_0^{k*}\hat{M}_1^{l*}(\hat{\mathbb{Q}}^p)^*\hat{L}^{2m-1*} \\
&= (-1)^{pl+m+p+l}D\hat{L}^{2m-1}\hat{\mathbb{Q}}^p \hat{M}_1^l \hat{M}_0^k \hat{L} D^{-1} - D\hat{M}_0^k \hat{M}_1^l \hat{\mathbb{Q}}^p \hat{L}^{2m} D^{-1} \\
&= -D(\hat{M}_0^k \hat{M}_1^l \hat{\mathbb{Q}}^p \hat{L}^{2m} - (-1)^{pl+m+p+l}\hat{L}^{2m-1}\hat{\mathbb{Q}}^p \hat{M}_1^l \hat{M}_0^k \hat{L})D^{-1}.
\end{aligned}$$

The second identity can be proved in a similar way. $\square$

Now we can define the following additional equations as

$$D_{mklp}\Phi = -(B_{mklp})_-\Phi, \quad D_{mklp}\hat{\Phi} = (B_{mklp})_+\hat{\Phi}, \tag{6.31}$$
$$\hat{D}_{mklp}\Phi = -(\hat{B}_{mklp})_-\Phi, \quad \hat{D}_{mklp}\hat{\Phi} = (\hat{B}_{mklp})_+\hat{\Phi}. \tag{6.32}$$

These equations are equivalent to the following Lax equations

$$D_{mklp} L = [-(B_{mklp})_-, L], \quad D_{mklp} \hat{L} = [(B_{mklp})_+, \hat{L}], \tag{6.33}$$

$$\hat{D}_{mklp} L = [-(\hat{B}_{mklp})_-, L], \quad \hat{D}_{mklp} \hat{L} = [(\hat{B}_{mklp})_+, \hat{L}]. \tag{6.34}$$

Similarly, we will get the following proposition.

Proposition 6.6. *The flows* (6.33) *and* (6.34) *commute with the flows of the supersymmetric two-component BKP hierarchy. Namely, for any* $\bar{D}_{mnlp} = D_{mnlp}, \hat{D}_{mnlp}$ *and* $\bar{D}_k = D_k, \hat{D}_k$ *one has*

$$\left[\bar{D}_{mnlp}, \bar{D}_k \right] = 0, \quad m, n \in \mathbb{Z}_+; l, p = 0, 1; \quad k = 4i - 2, 4i - 1, i \in \mathbb{Z}_+, \tag{6.35}$$

which holds in the sense of acting on Φ *or* $\hat{\Phi}$.

Proof. The proposition can be checked case by case with the help of Eq. (6.26) and Eqs. (6.31)–(6.34). For example,

$$\begin{aligned}
\left[D_{mnlp}, \hat{D}_k \right] \Phi &= D_{mnlp} \hat{D}_k \Phi - (-1)^{(l+p)k} \hat{D}_k D_{mnlp} \Phi \\
&= (-1)^{(l+p)k} [(\hat{L}^k)_-, (B_{mnlp})_-] \Phi \\
&\quad - [(B_{mnlp})_+, \hat{L}^k]_- \Phi - (-1)^{(l+p)k} [(\hat{L}^k)_-, B_{mnlp}]_- \Phi = 0,
\end{aligned}$$

$$\begin{aligned}
\left[\hat{D}_{mnlp}, D_k \right] \hat{\Phi} &= (-1)^{(l+p)k} [(L^k)_+, (\hat{B}_{mnlp})_+] \hat{\Phi} + [-(\hat{B}_{mnlp})_-, L^k]_+ \hat{\Phi} \\
&\quad - (-1)^{(l+p)k} [(L^k)_+, \hat{B}_{mnlp}]_+ \hat{\Phi} = 0.
\end{aligned}$$

The other cases can be proved in similar ways. This is the end of this proposition. □

Similarly as the SBKP hierarchy, the algebraic structure of the additional symmetry of the S2BKP hierarchy will be talked about in the next proposition.

Proposition 6.7. *The algebra of additional symmetries of the two-component SBKP hierarchy is isomorphic to the Lie algebra of super-quasi-differential operators, which is isomorphic (as a Lie algebra) to* $SW_{1+\infty} \bigoplus SW_{1+\infty}$.

Proof. The isomorphism is given by

$$z \mapsto \partial, \quad \xi \mapsto Q + \Gamma_1 \partial, \tag{6.36}$$

$$\partial_z \mapsto \Gamma_0, \quad \partial_\xi \mapsto \Gamma_1, \tag{6.37}$$

$$\hat{z} \mapsto \partial, \quad \hat{\xi} \mapsto Q + \hat{\Gamma}_1 \partial, \tag{6.38}$$

$$\partial_{\hat{z}} \mapsto \hat{\Gamma}_0, \quad \partial_{\hat{\xi}} \mapsto \hat{\Gamma}_1, \tag{6.39}$$

which further lead to

$$z \mapsto L^2, \quad \xi \mapsto \mathbb{Q} + M_1 L^2, \tag{6.40}$$

$$\partial_z \mapsto M_0, \quad \partial_\xi \mapsto M_1, \tag{6.41}$$

$$\hat{z} \mapsto \hat{L}^{-2}, \quad \hat{\xi} \mapsto \hat{\mathbb{Q}} + \hat{M}_1 \hat{L}^{-2}, \tag{6.42}$$

$$\partial_{\hat{z}} \mapsto \hat{M}_0, \quad \partial_{\hat{\xi}} \mapsto \hat{M}_1. \tag{6.43}$$

One can find the above construction keeps ξ commuting with z and $\hat{\xi}$ commuting with $\hat{z}$. □

If we do a $(4n, 2)$-reduction from the supersymmetric two-component BKP hierarchy, a reduced hierarchy called the supersymmetric D type Drinfeld–Sokolov hierarchies with a supersymmetric Block type additional symmetry which will be discussed in the next section.

7. Supersymmetric D type Drinfeld–Sokolov hierarchy

Assume a new Lax operator $\mathcal{L}$ which has the following relation with two Lax operators of the supersymmetric two-component BKP hierarchy introduced in the last section

$$\mathcal{L} = L^{4n} = \hat{L}^2, \ n \geq 2. \tag{7.1}$$

Then the Lax operators of the supersymmetric two-component BKP hierarchy will be reduced to the following Lax operator of the supersymmetric D type Drinfeld–Sokolov hierarchy whose bosonic case can be seen in [24–26]

$$\mathcal{L} = D^{4n} + \sum_{i=1}^{n} D^{-1} \left(v_i D^{4i-1} + D^{4i-1} v_i \right) + D^{-1} \rho D^{-1} \rho; \ |v_i| = 0, |\rho| = 1. \tag{7.2}$$

One can easily find the Lax operator $\mathcal{L}$ of the supersymmetric D type Drinfeld–Sokolov hierarchy will satisfy the following B type condition

$$\mathcal{L}^* = D\mathcal{L}D^{-1}. \tag{7.3}$$

This Lax operator $\mathcal{L}$ of the supersymmetric D type Drinfeld–Sokolov hierarchy has the following dressing structure [25]

$$\mathcal{L} = \Phi D^{4n} \Phi^{-1} = \hat{\Phi} D^{-2} \hat{\Phi}^{-1}. \tag{7.4}$$

Here

$$\Phi = 1 + \sum_{i \geq 1} a_i D^{-i}, \quad \hat{\Phi} = 1 + \sum_{i \geq 1} b_i D^i \tag{7.5}$$

are pseudo supersymmetric differential operators that also satisfy the following B type condition

$$\Phi^* = D\Phi^{-1}D^{-1}, \quad \hat{\Phi}^* = D\hat{\Phi}^{-1}D^{-1}. \tag{7.6}$$

The dressing structures inspire us to define two fractional operators as

$$\mathcal{L}^{\frac{1}{4n}} = D + \sum_{i \geq 1} u_i D^{-i}, \quad \mathcal{L}^{\frac{1}{2}} = D^{-1} \hat{u}_{-1} + \sum_{i \geq 1} \hat{u}_i D^i. \tag{7.7}$$

Two fractional operators $\mathcal{L}^{\frac{1}{4n}}$ and $\mathcal{L}^{\frac{1}{2}}$ can be rewritten in a dressing form as

$$\mathcal{L}^{\frac{1}{4n}} = \Phi D \Phi^{-1}, \quad \mathcal{L}^{\frac{1}{2}} = \hat{\Phi} D^{-1} \hat{\Phi}^{-1}. \tag{7.8}$$

The supersymmetric D type Drinfeld–Sokolov hierarchy being considered in this paper is defined by the following Lax equations:

$$D_k \mathcal{L} = [(\mathcal{L}^{\frac{k}{4n}})_+, \mathcal{L}], \quad \hat{D}_k \mathcal{L} = [-(\mathcal{L}^{\frac{k}{2}})_-, \mathcal{L}], \ k = 4i - 2, 4i - 1, \ i \in \mathbb{Z}_+. \tag{7.9}$$

The dressing operators Φ and $\hat{\Phi}$ are same as the ones of the supersymmetric two-component BKP hierarchy. Given $\mathcal{L}$, the dressing operators Φ and $\hat{\Phi}$ are uniquely determined up to a

multiplication to the right by operators of the form (7.5) and (7.6) with constant coefficients. The supersymmetric D type Drinfeld–Sokolov hierarchies can also be redefined as the following Sato equations

$$D_k \Phi = -(\mathcal{L}^{\frac{k}{4n}})_- \Phi, \quad D_k \hat{\Phi} = (\mathcal{L}^{\frac{k}{4n}})_+ \hat{\Phi}, \tag{7.10}$$

$$\hat{D}_k \Phi = -(\mathcal{L}^{\frac{k}{2}})_- \Phi, \quad \hat{D}_k \hat{\Phi} = (\mathcal{L}^{\frac{k}{2}})_+ \hat{\Phi}, \tag{7.11}$$

with $k = 4i - 2, 4i - 1, \ i \in \mathbb{Z}_+$.

After the above preparation, we will show that this supersymmetric D type Drinfeld–Sokolov hierarchy has a nice Block symmetry as its appearance in the bigraded Toda hierarchy [27].

8. Supersymmetric Block symmetries of supersymmetric D type Drinfeld–Sokolov hierarchies

In this section, we will put the constrained condition Eq. (7.1) into the construction of the flows of the additional symmetry which form a $N = 2$ supersymmetric extension of the well-known Block algebra [36].

With the dressing operators given in Eq. (7.8), we introduce two new Orlov–Schulman operators as following

$$\mathcal{M}_i = M_i L^{2-4n}, \quad \hat{\mathcal{M}}_i = \hat{M}_i \hat{L}^{-4}. \tag{8.1}$$

It is easy to see the following lemma holds.

Lemma 8.1. *The operators $\mathcal{M}_j$ and $\hat{\mathcal{M}}_j$ satisfy*

$$[\mathcal{L}, \mathcal{M}_0] = 1, \quad [\mathcal{L}, \hat{\mathcal{M}}_0] = 1; \quad [\hat{\mathbb{Q}}, \hat{\mathcal{M}}_0] = -\hat{\mathcal{M}}_1; \tag{8.2}$$

and

$$D_k \mathcal{M}_j = [(\mathcal{L}^{\frac{k}{4n}})_+, \mathcal{M}_j], \quad D_k \mathbb{Q} = [(\mathcal{L}^{\frac{k}{4n}})_+, \mathbb{Q}], \quad k = 4i - 2, 4i - 1, \ i \in \mathbb{Z}_+, \tag{8.3}$$

$$\hat{D}_k \mathcal{M}_j = [-(\mathcal{L}^{\frac{k}{2}})_-, \mathcal{M}_j], \quad \hat{D}_k \mathbb{Q} = [-(\mathcal{L}^{\frac{k}{2}})_-, \mathbb{Q}], \quad k = 4i - 2, 4i - 1, \ i \in \mathbb{Z}_+, \tag{8.4}$$

$$D_k \hat{\mathcal{M}}_j = [(\mathcal{L}^{\frac{k}{4n}})_+, \hat{\mathcal{M}}_j], \quad D_k \hat{\mathbb{Q}} = [(\mathcal{L}^{\frac{k}{4n}})_+, \hat{\mathbb{Q}}], \quad k = 4i - 2, 4i - 1, \ i \in \mathbb{Z}_+, \tag{8.5}$$

$$\hat{D}_k \hat{\mathcal{M}}_j = [-(\mathcal{L}^{\frac{k}{2}})_-, \hat{\mathcal{M}}_j], \quad \hat{D}_k \hat{\mathbb{Q}} = [-(\mathcal{L}^{\frac{k}{2}})_-, \hat{\mathbb{Q}}], \quad k = 4i - 2, 4i - 1, \ i \in \mathbb{Z}_+, \tag{8.6}$$

which can be simplified to

$$D_k \bar{\mathcal{M}}_j = [(\mathcal{L}^{\frac{k}{4n}})_+, \bar{\mathcal{M}}_j], \quad \hat{D}_k \bar{\mathcal{M}}_j = [-(\mathcal{L}^{\frac{k}{2}})_-, \bar{\mathcal{M}}_j], \tag{8.7}$$

where $\bar{\mathcal{M}}_j = \mathcal{M}_j$ or $\hat{\mathcal{M}}_j, k = 4i - 2, 4i - 1, \ i \in \mathbb{Z}_+$.

To make the operators used in the additional symmetry satisfying the B type condition, we need to prove the following B type property of $\mathcal{M}_i - \hat{\mathcal{M}}_i$ which is included in the following lemma.

Lemma 8.2. *The difference of two Orlov–Schulman operators $\mathcal{M}_0$ and $\hat{\mathcal{M}}_0$ for the supersymmetric D type Drinfeld–Sokolov hierarchy has the following D type property:*

$$\mathcal{L}^* (\mathcal{M}_0 - \hat{\mathcal{M}}_0)^* = -D \mathcal{L} (\mathcal{M}_0 - \hat{\mathcal{M}}_0) D^{-1}. \tag{8.8}$$

Proof. It is easy to find the two Orlov–Schulman operators $\mathcal{M}_0$ and $\hat{\mathcal{M}}_0$ of the supersymmetric D type Drinfeld–Sokolov hierarchy can be expressed by Orlov–Schulman operators M_0, $\hat{M}_0$ and Lax operators L, $\hat{L}$ of the supersymmetric two-component BKP hierarchy as

$$\mathcal{M}_0 = M_0 L^{2-4n}, \quad \hat{\mathcal{M}}_0 = -\hat{M}_0 \hat{L}^{-4}. \tag{8.9}$$

Using Lemma 6.4, putting Eq. (8.9) into $(\mathcal{M}_0 - \hat{\mathcal{M}}_0)^*$ can lead to

$$\begin{aligned}
(\mathcal{M}_0 - \hat{\mathcal{M}}_0)^* &= -DL^{1-4n} M_0 L D^{-1} + D\hat{L}^{-3} \hat{M}_0 \hat{L}^{-1} D^{-1} \\
&= -D(L^{1-4n} M_0 - L^{-4n})D^{-1} + D(\hat{L}^{-4} \hat{M}_0 + \hat{L}^{-2})D^{-1},
\end{aligned} \tag{8.10}$$

which can further lead to

$$\mathcal{L}^*(\mathcal{M}_0 - \hat{\mathcal{M}}_0)^* = -D(\mathcal{L}\mathcal{M}_0 - \mathcal{L}\hat{\mathcal{M}}_0)D^{-1}. \tag{8.11}$$

In the above calculation, the commutativity between $\mathcal{L}$ and $\mathcal{M}_0 - \hat{\mathcal{M}}_0$ is already used. Till now, the proof is finished. $\square$

One can also get

$$\mathcal{M}_i^* = -DL^{-1}M_i L D^{-1}, \quad \hat{\mathcal{M}}_i^* = -D\hat{L}\hat{\mathcal{M}}_i \hat{L}^{-1} D^{-1}. \tag{8.12}$$

For the supersymmetric D-type Drinfeld–Sokolov hierarchy, we define the additional operator $\mathcal{B}_{mk}^{lp\hat{l}\hat{p}}$ as

$$\begin{aligned}
\mathcal{B}_{mk}^{lp\hat{l}\hat{p}} = (\mathcal{M}_0 - \hat{\mathcal{M}}_0)^k (M_1^l \mathbb{Q}^p \hat{M}_1^{\hat{l}} \hat{\mathbb{Q}}^{\hat{p}} \\
- (-1)^{\prod + \sum + k} \hat{L}^{-1} \hat{\mathbb{Q}}^{\hat{p}} \hat{M}_1^{\hat{l}} \hat{L} L \mathbb{Q}^p M_1^l L^{-1}) \mathcal{L}^m,
\end{aligned} \tag{8.13}$$

where $l, p, \hat{l}, \hat{p} = 0, 1$; $\prod = \prod_{a,b=l,p,\hat{l},\hat{p}} ab$, $\sum = \sum_{a=l,p,\hat{l},\hat{p}} a$.

One can get the following proposition.

Proposition 8.3. *The operator $\mathcal{B}_{mk}^{lp\hat{l}\hat{p}}$ satisfies a B type condition, namely*

$$(\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})^* = -D\mathcal{B}_{mk}^{lp\hat{l}\hat{p}} D^{-1}, \quad l, p, \hat{l}, \hat{p} = 0, 1. \tag{8.14}$$

Proof. Using the Proposition 6.4, the following calculation will lead to

$$\begin{aligned}
(\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})^* &= ((\mathcal{M}_0 - \hat{\mathcal{M}}_0)^k (M_1^l \mathbb{Q}^p \hat{M}_1^{\hat{l}} \hat{\mathbb{Q}}^{\hat{p}} - (-1)^{\prod + \sum + k} \hat{L}^{-1} \hat{\mathbb{Q}}^{\hat{p}} \hat{M}_1^{\hat{l}} \hat{L} L \mathbb{Q}^p M_1^l L^{-1}) \mathcal{L}^m)^* \\
&= \mathcal{L}^{m*}[(-1)^{\prod} (\hat{\mathbb{Q}}^{\hat{p}})^* \hat{M}_1^{\hat{l}*} \mathbb{Q}^{p*} M_1^{l*} \\
&\quad + (-1)^{\sum + k} L^{-1*} M_1^{l*} \mathbb{Q}^{p*} L^* \hat{L}^* \hat{M}_1^{\hat{l}*} \hat{\mathbb{Q}}^{\hat{p}*} \hat{L}^{-1*}](\mathcal{M}_0 - \hat{\mathcal{M}}_0)^{k*} \\
&= D\mathcal{L}^m[(-1)^{\prod + \sum + k} \hat{L}^{-1} \hat{\mathbb{Q}}^{\hat{p}} \hat{M}_1^{\hat{l}} \hat{L} L \mathbb{Q}^p M_1^l L^{-1} \\
&\quad - (M_1^l \mathbb{Q}^p \hat{M}_1^{\hat{l}} \hat{\mathbb{Q}}^{\hat{p}})](\mathcal{M}_0 - \hat{\mathcal{M}}_0)^k D^{-1} \\
&= -D(\mathcal{M}_0 - \hat{\mathcal{M}}_0)^k (M_1^l \mathbb{Q}^p \hat{M}_1^{\hat{l}} \hat{\mathbb{Q}}^{\hat{p}} \\
&\quad - (-1)^{\prod + \sum + k} \hat{L}^{-1} \hat{\mathbb{Q}}^{\hat{p}} \hat{M}_1^{\hat{l}} \hat{L} L \mathbb{Q}^p M_1^l L^{-1}) \mathcal{L}^m D^{-1}. \quad \square
\end{aligned}$$

That means it is reasonable to define the additional flow of the supersymmetric D type Drinfeld–Sokolov hierarchy as

$$\frac{\partial \mathcal{L}}{\partial c_{mk}^{lp\hat{l}\hat{p}}} = [-(\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})_-, \mathcal{L}], \quad l, p, \hat{l}, \hat{p} = 0, 1; m, k \in \mathbb{Z}_+. \tag{8.15}$$

Whether these additional flows are symmetries of the supersymmetric D type Drinfeld–Sokolov hierarchy will be answered in the next proposition.

Proposition 8.4. *The flows in Eq.* (8.15) *can commute with original flows of the supersymmetric Drinfeld–Sokolov hierarchy of type D, namely,*

$$\left[\frac{\partial}{\partial c_{mk}^{lp\hat{l}\hat{p}}}, D_n \right] = 0, \quad \left[\frac{\partial}{\partial c_{mk}^{lp\hat{l}\hat{p}}}, \hat{D}_n \right] = 0,$$

where $l, p, \hat{l}, \hat{p} = 0, 1; m, k \in \mathbb{Z}_+, n = 4i - 2, 4i - 1, i \in \mathbb{Z}_+$, *which hold in the sense of acting on* $\Phi, \hat{\Phi}$ *or* $\mathcal{L}$.

Proof. According to the definition,

$$[\partial_{c_{mk}^{lp\hat{l}\hat{p}}}, D_n]\Phi = \partial_{c_{mk}^{lp\hat{l}\hat{p}}}(D_n\Phi) - D_n(\partial_{c_{mk}^{lp\hat{l}\hat{p}}}\Phi),$$

and using the actions of the additional flows and the flows of the D type Drinfeld–Sokolov hierarchy on Φ, we have

$$[\partial_{c_{mk}^{lp\hat{l}\hat{p}}}, D_n]\Phi = -\partial_{c_{mk}^{lp\hat{l}\hat{p}}}\left((\mathcal{L}^{\frac{k}{4n}})_-\Phi\right) + D_n\left((\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})_-\Phi\right)$$

$$= -(\partial_{c_{mk}^{lp\hat{l}\hat{p}}}\mathcal{L}^{\frac{k}{4n}})_-\Phi - (\mathcal{L}^{\frac{k}{4n}})_-(\partial_{c_{mk}^{lp\hat{l}\hat{p}}}\Phi)$$

$$+ [D_n(\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})]_-\Phi + (\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})_-(D_n\Phi).$$

Using Eq. (7.9) and Eq. (8.7), it equals

$$[\partial_{c_{mk}^{lp\hat{l}\hat{p}}}, D_n]\Phi = [\left(\mathcal{B}_{mk}^{lp\hat{l}\hat{p}}\right)_-, \mathcal{L}^{\frac{k}{4n}}]_-\Phi + (\mathcal{L}^{\frac{k}{4n}})_-\left(\mathcal{B}_{mk}^{lp\hat{l}\hat{p}}\right)_-\Phi$$

$$+ [(\mathcal{L}^{\frac{k}{4n}})_+, \mathcal{B}_{mk}^{lp\hat{l}\hat{p}}]_-\Phi - (\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})_-(\mathcal{L}^{\frac{k}{4n}})_-\Phi$$

$$= [(\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})_-, \mathcal{L}^{\frac{k}{4n}}]_-\Phi - [\mathcal{B}_{mk}^{lp\hat{l}\hat{p}}, (\mathcal{L}^{\frac{k}{4n}})_+]_-\Phi$$

$$+ [(\mathcal{L}^{\frac{k}{4n}})_-, (\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})_-]\Phi$$

$$= 0.$$

The other cases of this proposition can be proved in similar ways. $\quad\Box$

The above proposition indicates that Eq. (8.15) is a symmetry of the supersymmetric D type Drinfeld–Sokolov hierarchy. Further we can prove that the following identities hold true

$$\frac{\partial \mathcal{M}_i}{\partial c_{mk}^{lp\hat{l}\hat{p}}} = [-(\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})_-, \mathcal{M}_i], \quad \frac{\partial \hat{\mathcal{M}}_i}{\partial c_{mk}^{lp\hat{l}\hat{p}}} = [(\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})_+, \hat{\mathcal{M}}_i], \tag{8.16}$$

$$\frac{\partial w(z^{\frac{1}{4n}})}{\partial c_{mk}^{lp\hat{l}\hat{p}}} = -(\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})_- w(z^{\frac{1}{4n}}), \quad \frac{\partial \hat{w}(z^{\frac{1}{2}})}{\partial c_{mk}^{lp\hat{l}\hat{p}}} = (\mathcal{B}_{mk}^{lp\hat{l}\hat{p}})_+ \hat{w}(z^{\frac{1}{2}}). \tag{8.17}$$

Using same techniques used in [27], the following theorem can be derived.

Theorem 8.5. *The flows in Eq.* (8.15) *about additional symmetries of supersymmetric D type Drinfeld–Sokolov hierarchy compose a supersymmetric type Block Lie algebra which contains the following Block Lie algebra while* $l = p = \hat{l} = \hat{p} = 0$

$$[\partial_{c_{ml}^{0000}}, \partial_{c_{sk}^{0000}}] = (km - sl)\partial_{c_{m+s-1,k+l-1}^{0000}}, \quad m, s, k, l \in \mathbb{Z}_+, \tag{8.18}$$

which holds in the sense of acting on Φ, $\hat{\Phi}$ *or* $\mathcal{L}$.

Proof. The similar proof for Eq. (8.18) can be found in our paper [26]. $\square$

This is one kind of supersymmetric extensions of the Block algebra because it is involved with the supersymmetric variables and supersymmetric derivative D. However, its algebra structure is still not clear now, which deserves further study in the future.

9. Conclusions and discussions

Our earlier papers show that the Block type algebras appear not only in Toda type difference systems but also in differential systems such as two-BKP hierarchy, D type Drinfeld–Sokolov hierarchy [26]. The above results show that in their corresponding supersymmetric systems, there also exists the hidden $N = 2$ Block type supersymmetric algebraic structures. These results further show that the Block type infinite dimensional Lie algebra has a certain of universality in integrable hierarchies.

Although the supersymmetric two-component BKP hierarchy and supersymmetric Drinfeld–Sokolov hierarchy of type D might be not available in the fermionic string theory now comparing with the application of the classical KP hierarchy to the bosonic string, they have a great advantage of showing their superconformal structures. In this paper, we also show the structure of a super-Block algebra of the supersymmetric two-BKP hierarchy and its reduced hierarchy. The superconformal algebra may appear in the related Hamiltonian structures by considering the reductions of super-BKP hierarchies like reductions of the super-KP system to super-KdV system. This may be an interesting subject for our future study which may relate the supersymmetric BKP systems in this paper to problems in physics. There are also some other interesting subjects such as the relation of the hierarchies introduced in this paper with the quantum spin chains as in [37]. These directions might be included in our future study.

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