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Orthogonal Polynomials Suggested by a Queueing Model*

THEODORE S. CHIHARA

Department of Mathematical Sciences, Purdue University-Calumet, Hammond, Indiana 46323

AND

MOURAD E. H. ISMAIL

Department of Mathematics, Arizona State University, Tempe, Arizona 85287

The transition probabilities for the queueing model where potential customers are discouraged by queue length can be determined by orthogonal polynomials whose orthogonality relations have been found recently by E. A. van Doorn. Certain features of these polynomials are suggestive of the random walk polynomials studied by Karlin and McGregor. In this paper, we will study a class of orthogonal polynomials which includes the above random walk polynomials as well as the "duals" of the polynomials studied by van Doorn. The latter will yield a generalization of the van Doorn polynomials. Orthogonality relations will be obtained for these polynomials and the recurrence formulas in birth and death process form will be noted.

1. INTRODUCTION

A number of authors have studied the birth and death process for a queueing model where potential customers are discouraged by queue length (e.g., Natvig [13]). Recently, van Doorn [19] has explicitly found the orthogonality relations for the orthogonal polynomials which determine the transition probabilities for this process. Van Doorn did not determine the jump of the spectral function at the single limit point of the spectrum but he conjectured that it is 0. This will be confirmed in Section 4. These

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orthogonal polynomials are defined by the recurrence relation

$$-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x), \; n \geq 0,$$
$$Q_{-1}(x) = 0, Q_0(x) = 1, \tag{1.1}$$

where, for the above process

$$\lambda_n = \frac{\lambda}{n+1}, \; n \geqslant 0; \qquad \mu_0 = 0, \mu_n = \mu, \; n \geqslant 1,$$

with
$$\lambda > 0, \mu > 0. \tag{1.2}$$

van Doorn uses the theory and methods developed by Karlin and McGregor [10, 11] for their study of birth and death processes based upon (1.1).

In addition to their obvious importance for this application, these polynomials are of some interest as orthogonal polynomials per se. Their spectrum is a denumerable set with the single limit point μ. This fact is predicted by Krein's theorem (see [8, p. 117]) and there are a number of other examples with this property. But this is the first we have seen with infinitely many spectral points on both sides of the limit point which is not also symmetric with respect to the limit point and both the spectrum and the distribution function can be computed explicitly. The spectrum of the other known examples coincide with the zeros of certain transcendental functions, see Ismail [9] and Al-Salam and Ismail [2].

More significantly, these polynomials bear some resemblance to the orthogonal polynomials obtained by Karlin and McGregor [12] as the random walk polynomials associated with the queueing process with infinitely many servers. These polynomials were independently and simultaneously discovered by Carlitz [6]. This resemblance, which is most strikingly exhibited by their generating functions, suggests both sets might belong to a more general family of orthogonal polynomials.

In this paper, we will obtain a generating function for a family of orthogonal polynomials which will include the above random walk polynomials and the *kernel polynomials* of the queueing process polynomials studied by van Doorn. We will explicitly determine the orthogonality relation for this generalization and, to a certain extent, for the associated polynomials. Finally we will obtain a related class of orthogonal polynomials which satisfy a birth and death process recurrence relation (1.1) which generalizes (1.2). The orthogonality relations for the latter will also be obtained. Basic analogues of the polynomials contained in the present paper are under investigation [3]. They generalize the $U_n^{\alpha}(x)$'s of Al-Salam and Carlitz [1], the q-random walk polynomials of Askey and Ismail [4] and the polynomials in [2].

2. Generalization of the kernel Polynomials of $\{Q_n(x)\}$

We first consider the monic form of the polynomials defined by (1.1), (1.2):

$$\hat{Q}_n(x) = (-\lambda)^n (n!)^{-1} Q_n(x). \tag{2.1}$$

Recurrence relation (1.1), (1.2) then yields

$$\begin{aligned} \hat{Q}_{n+1}(x) &= \left[x - \left(\frac{\lambda}{n+1} + \mu\right)\right]\hat{Q}_n(x) - \frac{\lambda\mu}{n}\hat{Q}_{n-1}(x), \qquad n \geqslant 1, \\ \hat{Q}_0(x) &= 1, \qquad \hat{Q}_1(x) = x - \lambda. \end{aligned} \tag{2.2}$$

The fact that $\hat{Q}_1(x)$ cannot be obtained from (2.2) by taking $n = 0$ (with $\hat{Q}_{-1}(x) = 0$) suggests it might be simpler to study the corresponding kernel polynomials $K_n(x)$. These are the polynomials whose distribution function (measure) is x times the distribution function (measure) for $\{Q_n(x)\}$. The latter satisfy the recurrence relation

$$\begin{aligned} K_{n+1}(x) &= \left[x - \left(\frac{\lambda}{n+1} + \mu\right)\right]K_n(x) - \frac{\lambda\mu}{n+1}K_{n-1}(x), \qquad n \geqslant 0, \\ K_{-1}(x) &= 0, \qquad K_0(x) = 1, \end{aligned} \tag{2.3}$$

and are related to the $\hat{Q}_n(x)$ by

$$\begin{aligned} xK_{n-1}(x) &= \hat{Q}_n(x) + \frac{\lambda}{n+1}\hat{Q}_{n-1}(x), \\ \hat{Q}_n(x) &= K_n(x) + \mu K_{n-1}(x) \end{aligned} \tag{2.4}$$

(see [8, pp. 45, 46]). In the notation of Karlin and McGregor [10, Lemma 3], $K_n(x)$ is the monic form of the polynomials $H_{n+1}(x)/x$ which are associated with the dual process when the original birth and death process has a reflecting barrier at 0.

If we next set

$$K_n^{\#}(x) = (\lambda\mu)^{-n/2} K_n\left((\lambda\mu)^{1/2}x + \mu\right) \tag{2.5}$$

we obtain the recurrence

$$K_{n+1}^{\#}(x) = \left[x - \frac{\gamma}{n+1}\right]K_n^{\#}(x) - \frac{1}{n+1}K_{n-1}^{\#}(x), \tag{2.6}$$

where $\gamma = (\lambda/\mu)^{1/2}$. Thus we see that there is just one essential parameter for these polynomials. A similar transformation applied to $\hat{Q}_n(x)$ will result in two essential parameters remaining in $\hat{Q}_1(x)$.

From the recurrence relation (2.6), we can routinely derive the generating function

$$e^{w/x}(1 - xw)^{(1-\gamma x - x^2)/x^2} = \sum_{n=0}^{\infty} K_n^{\#}(x) w^n. \tag{2.7}$$

The latter is reminiscent of the generating function

$$e^{w/x}\left(1 - \frac{xw}{a}\right)^{a(1-x^2)/x^2} = \sum_{n=0}^{\infty} r_n(x, a)\frac{w^n}{n!}, \tag{2.8}$$

where the $r_n(x, a)$ are the random walk polynomials obtained by Karlin and McGregor [12, p. 117] (see also Carlitz [6]). The monic form of these polynomials

$$R_n(x) \equiv R_n(x, a) = \frac{a^n}{(a)_n} r_n(x, a) \tag{2.9}$$

satisfy

$$R_{n+1}(x) = xR_n(x) - \frac{an}{(n+a)(n+a-1)} R_{n-1}(x). \tag{2.10}$$

Note that when $a = 1$, (2.10) reduces to (2.6) for $\gamma = 0$.

Comparison of (2.7) and (2.8) suggests consideration of the generating function

$$\Phi(x, w) = e^{w/x}(1 - xw)^{-f(x)} = \sum_{n=0}^{\infty} P_n(x) w^n, \tag{2.11}$$

where $f(x) = (ax^2 + bx + c)/x^2$ and the parameters must be restricted so that $P_n(x)$ is a polynomial of degree n. We note that there is no essential gain in considering the more general appearing function

$$\Gamma(x, w) = e^{\alpha w/x}(1 - \beta xw)^{-f(x)} = \sum_{n=0}^{\infty} F_n(x) w^n$$

since the substitutions

$$x \to (\alpha/\beta)^{1/2} x, \; w \to (\alpha\beta)^{-1/2} w$$

transform $\Gamma(x, w)$ into (2.11).

Expanding the left side of (2.11) with the aid of the Cauchy product produces the formula

$$P_n(x) = \sum_{k=0}^{n} \binom{-f(x)}{k} \frac{x^{2k-n}(-1)^k}{(n-k)!}. \tag{2.12}$$

Since

$$\binom{-f(x)}{k} = \frac{(-1)^k (f(x))_k}{k!},$$

where

$$(a)_0 = 1, \qquad (a)_{n+1} = (a)_n (a+n), \qquad n \geqslant 0,$$

we obtain the explicit formula

$$P_n(x) = \frac{x^{-n}}{n!} \sum_{k=0}^{n} \binom{n}{k} \pi_k(x), \tag{2.13}$$

where

$$\pi_0(x) = 1, \pi_k(x) = \prod_{\nu=1}^{k} \left[(a+\nu-1)x^2 + bx + c \right], \qquad k > 0. \tag{2.14}$$

A necessary and sufficient condition for the coefficient of x^{-n} in the right side of (2.13) to vanish is

$$\sum_{k=0}^{n} \binom{n}{k} \pi_k(0) = (1+c)^n = 0$$

so we must take $c = -1$. It is also easy to verify that

$$\sum_{k=0}^{n} \binom{n}{k} \pi_k'(0) = -b \sum_{k=0}^{n} \binom{n}{k} (-1)^k k = 0$$

so that, in particular,

$$P_1(x) = ax + b.$$

However, it is not at all clear that $P_n(x)$ will reduce to a polynomial for $n > 1$. Therefore, we return to (2.11) and find that, when $c = -1$,

$$(1 - xw) \frac{\partial \Phi}{\partial w} - (ax + b - w)\Phi = 0, \tag{2.15}$$

and

$$\Phi(x, 0) = 1.$$

This yields the recurrence formula

$$(n+1)P_{n+1}(x) = [(n+a)x+b]P_n(x) - P_{n-1}(x), \qquad n > 0,$$
$$P_0(x) = 1, \qquad P_1(x) = ax + b. \tag{2.16}$$

This shows that $P_n(x)$ is indeed a polynomial of degree n if and only if $c = -1$.

Thus the only polynomial sequences $\{P_n(x)\}$ having a generating function of the form (2.12) are, essentially, the *orthogonal* polynomial sequences defined by (2.16). It is, of course, easy to verify that, conversely, the polynomials defined by (2.16) do satisfy (2.11) with $c = -1$.

Now writing

$$P_n(x) = P_n(x; a, b),$$

we note the special cases (of (2.3), (2.10))

$$K_n(x) = (\lambda\mu)^{-n/2} P_n\left((\lambda\mu)^{-1/2}(x-\mu); 1, -(\lambda/\mu)^{1/2}\right), \tag{2.17}$$

$$r_n(x, a) = a^{-n/2} n! P_n(xa^{-1/2}; a, 0). \tag{2.18}$$

In terms of the corresponding monic polynomials

$$\hat{P}_n(x; a, b) = \frac{n!}{(a)_n} P_n(x; a, b), \tag{2.19}$$

the recurrence relation reads

$$\hat{P}_{n+1}(x) = \left(x + \frac{b}{n+a}\right)\hat{P}_n(x) - \frac{n}{(n+a)(n+a-1)}\hat{P}_{n-1}(x). \tag{2.20}$$

We see that we will have orthogonality with respect to a positive measure if and only if b is real and $a > 0$. Krein's theorem [8, p. 117] shows that the spectrum is a denumerable set with 0 as its only limit point.

3. The Measure for $\{P_n(x)\}$

We next determine the orthogonality measure $d\theta$ for the $P_n(x)$. We first introduce the corresponding polynomials of the second kind. These are the polynomials $N_k(x)$ that satisfy the recurrence relation (2.16) for $n \geqslant 1$ but with the initial conditions

$$N_0(x) = 0, N_1(x) = a. \tag{3.1}$$

Let

$$\Psi(x,w) = \sum_{n=0}^{\infty} N_n(x) w^n.$$

Again using the recurrence relation (2.16) and taking (3.1) into account, we find

$$(1 - xw)\frac{\partial \Psi}{\partial w} - (ax + b - w)\Psi = a,$$
$$\Psi(x,0) = 0.$$

This yields

$$\Psi(x,w) e^{-w/x}(1 - xw)^{f(x)} = a\int_0^w e^{-u/x}(1 - xu)^{f(x)-1}\,du,$$

where, as before, $f(x) = (ax^2 + bx - 1)/x^2$. Thus

$$\Psi(x,w) = a\Phi(x,w)\int_0^w e^{-u/x}(1 - xu)^{f(x)-1}\,du \tag{3.2}$$

and this is valid for $|w| < |x^{-1}|$. We will be interested in (3.2) in the complex plane off the real axis.

For complex x such that $\mathrm{Re}(f(x)) > 0$, the integral in (3.2) exists as $w \to 1/x$. Rewriting (3.2) in the form

$$\sum_{n=0}^{\infty} N_n(z) w^n = a\sum_{n=0}^{\infty} P_n(z) w^n \int_0^w e^{-u/z}(1 - zu)^{f(z)-1}\,du$$

and applying Darboux' method [14, pp. 309–310] with the comparison function

$$g(w) = a\sum_{n=0}^{\infty} P_n(z) w^n \int_0^{1/z} e^{-u/z}(1 - zu)^{f(z)-1}\,du,$$

we obtain the asymptotic formula

$$N_n(z) \approx aP_n(z)\int_0^{1/z} e^{-u/z}(1 - zu)^{f(z)-1}\,du, \qquad \mathrm{Re}(f(z)) > 0. \tag{3.3}$$

The asymptotic behavior of $P_n(z)$ for $z \neq 0$ can also be determined by applying Darboux' method to the generating function (2.11). Indeed using the comparison function

$$h(w) = \exp(1/z^2)(1 - zw)^{-f(z)} = \exp(1/z^2)\sum_0^{\infty} \frac{(f(z))_n}{n!} w^n$$

we obtain

$$P_n(z) \approx \exp(1/z^2)\frac{(f(z))_n}{n!} = \exp(1/z^2)\frac{\Gamma(n+f(z))}{\Gamma(f(z))\Gamma(n+1)}, \tag{3.4}$$

since $(\sigma)_n = \Gamma(\sigma+n)/\Gamma(\sigma)$. We now apply

$$\Gamma(n+a)/\Gamma(n+b) \approx n^{a-b}, \qquad \text{as } n \to +\infty,$$

which follows from Stirling's formula, Rainville [16, p. 31], to express (3.4) in the form

$$P_n(z) \approx \exp(1/z^2) n^{f(z)-1}/\Gamma(f(z)), \qquad z \neq 0, f(z) \neq 0, -1, -2, \ldots. \tag{3.5}$$

For completeness we shall derive asymptotic formulas for $P_n(z)$ when $f(z) = 0, -1, 2, \ldots$ or $z = 0$ at the end of this section.

Consider next the $\mathcal{J}$-fraction corresponding to the recurrence relation (2.16):

$$\chi(z) = \frac{a}{|az+b} - \frac{1}{|(a+1)z+b} - \frac{2}{|(a+2)z+b} - \frac{3}{|(a+3)z+b} - \cdots. \tag{3.6}$$

We have

$$\chi(z) = \lim_{n\to\infty} \frac{N_n(z)}{P_n(z)}, \qquad \operatorname{Im}(z) \neq 0.$$

It follows from (3.3) that

$$\chi(z) = a\int_0^{1/z} e^{-u/z}(1-zu)^{f(z)-1}\,du.$$

Using the formula [16, p. 126],

$$\int_0^1 e^{st} t^{\alpha-1}(1-t)^{\beta-\alpha-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}\,{}_1F_1(\alpha;\beta;s),$$

$\operatorname{Re}(\beta) > \operatorname{Re}(\alpha) > 0$, we obtain

$$\chi(z) = \frac{a\Gamma(f(z))}{z\Gamma(1+f(z))}\,{}_1F_1(1; 1+f(z); -z^{-2}). \tag{3.7}$$

This gives us the expansion

$$\chi(z)=\frac{a}{z}\sum_{k=0}^{\infty}\frac{(-z^2)^{-k}}{(a+bz^{-1}-z^{-2})_{k+1}},\qquad \operatorname{Re}(a+bz^{-1}-z^{-2})>0. \tag{3.8}$$

Both sides of (3.8) are well defined for nonreal z since the equations

$$a+bz^{-1}-z^{-2}=-n,\qquad n=0,1,2,\ldots, \tag{3.9}$$

have only real solutions. Thus analytic continuation establishes the validity of (3.8) for all nonreal z.

For the roots of (3.9), we write

$$\xi_n=\frac{-b-\left[b^2+4(n+a)\right]^{1/2}}{2(n+a)},\qquad \eta_n=\frac{-b+\left[b^2+4(n+a)\right]^{1/2}}{2(n+a)}. \tag{3.10}$$

We note that $\xi_n<0$ and $\{\xi_n\}$ is strictly increasing while $\eta_n>0$ and $\{\eta_n\}$ is strictly decreasing. Thus $\chi(z)$ has simple poles at ξ_n and η_n and an essential singularity at 0. Markoff's theorem [18, p. 57] implies that $\chi(z)$ is the Stieltjes transform $\int_{-\infty}^{\infty}(z-t)^{-1}\,d\theta(t)$ of the distribution function θ. Denote the residue of $\chi(z)$ at t by $\mathcal{J}(t)$. Then

$$\mathcal{J}(\xi_n)=\frac{a\xi_n^{1-2n}}{(n+a)n!(\xi_n-\eta_n)}\exp\left(-\xi_n^{-2}\right), \tag{3.11}$$

$$\mathcal{J}(\eta_n)=\frac{a\eta_n^{1-2n}}{(n+a)n!(\eta_n-\xi_n)}\exp\left(-\eta_n^{-2}\right). \tag{3.12}$$

The two formulas (3.11) and (3.12) can be combined into a single simpler formula (see 5.7)). These give the jumps of the spectral function θ at the spectral points ξ_n and η_n so there remains only the question of the behavior of θ at the remaining spectral point 0.

Again referring to (2.16), we note that

$$(n+1)P_{n+1}(0)=bP_n(0)-P_{n-1}(0).$$

Writing $f_n=2^{n/2}n!P_n(0)$, this becomes

$$f_{n+1}=2^{1/2}bf_n-2nf_{n-1},\qquad n>0.$$

Since $f_0=1$, $f_1=b\sqrt{2}$, this shows that

$$f_n=H_n(b/\sqrt{2}),$$

where $H_n(x)$ denotes the nth Hermite polynomial. Therefore

$$P_n(0; a, b) = 2^{-n/2}(n!)^{-1}H_n(b/\sqrt{2}). \tag{3.13}$$

In terms of the corresponding orthonormal polynomials

$$p_n(x; a, b) = \left[a^{-1}(n+a)n!\right]^{1/2}P_n(x; a, b) \tag{3.14}$$

we can write

$$p_n^2(0; a, b) = \frac{(n+a)}{a}\frac{H_n^2(b/\sqrt{2})}{2^n n!} = a^{-1}\sqrt{\pi}(n+a)h_n^2(b/\sqrt{2}), \tag{3.15}$$

where $h_n(x)$ is the orthonormal Hermite polynomial.

From the problem of moments [17, pp. 45, 46], it is known that the jump of θ at 0 will be

$$\rho(0) = \left\{\sum_{n=0}^{\infty} p_n^2(0; a, b)\right\}^{-1}.$$

Since the Hamburger moment problem associated with the Hermite polynomials is determined, and the corresponding solution is everywhere continuous, we have

$$\sum_{n=0}^{\infty} h_n^2(0) = \infty.$$

We conclude from (3.15) and the above relationship that θ is continuous at 0 because $\sum_0^\infty p_n^2(0; a, b)$ diverges, hence $\rho(0)$ is zero. In particular, taking $a = 1$, *this verifies van Doorn's conjecture* that for the polynomials in (1.1)–(1.2), there is no jump at 0.

The $P_n(x; a, b)$ are thus orthogonal with respect to the measure $d\theta(x)$, where $\theta(x) = \theta(x; a, b)$ has the jumps (3.11), (3.12) at the spectral points ξ_n and η_n and is continuous at 0. This measure has total mass 1 since (3.6) shows that

$$\chi(z) \approx \frac{1}{z} + 0(z^{-2}) \qquad (z \to \infty).$$

The precise orthogonality relation will be given in our summary of Section 5; see (5.5). For now we make the observation that the fact that $d\theta$ has total mass 1 yields the strange looking formula

$$\sum_{n=0}^{\infty} \frac{a}{(n+a)n!}\left\{\frac{\xi_n^{-2n}\exp(-\xi_n^{-2})}{1-\eta_n/\xi_n} + \frac{\eta_n^{-2n}\exp(-\eta_n^{-2})}{1-\xi_n/\eta_n}\right\} = 1. \tag{3.16}$$

When $b \to 0$, $\xi_n \to -(n+a)^{-1/2}$, $\eta_n \to (n+a)^{-1/2}$ so (3.16) reduces to

$$a \sum_{n=0}^{\infty} \frac{(n+a)^{n-1}}{n!} e^{-n-a} = 1,$$

which is a special case of Euler's formula

$$\alpha \sum_{n=0}^{\infty} \frac{(n+\alpha)^{n-1}}{n!} (ze^{-z})^n = e^{\alpha z}. \tag{3.17}$$

The full orthogonality relation in this limiting case is equivalent to Euler's formula (3.17), Carlitz [6].

Now we shall discuss briefly the asymptotics of $P_n(z)$ when $f(z) = 0, -1, \ldots$ or $z = 0$. We have proved already that the solutions of $f(z) = -m$ are ξ_m and η_m. Therefore

$$P_n(x_m) = \sum_{k=0}^{\min(m,n)} \binom{m}{k} \frac{(-1)^k}{(n-k)!} x_m^{2k-n}, \qquad x_m = \xi_m \text{ or } \eta_m. \tag{3.18}$$

As $n \to \infty$, the dominant term in (3.18) is the term when $k = m$. This implies

$$P_n(x_m) \approx \frac{(-1)^m x_m^{2m-n}}{(n-m)!}, \qquad x_m = \xi_m \text{ or } \eta_m,$$

which establishes the asymptotic relationship, [16, p. 31]

$$P_n(x_m) \approx \sqrt{2\pi}\, n^{n-m+1/2} e^{-n} x_m^{2m-n},\ x_m = \xi_m \text{ or } \eta_m. \tag{3.19}$$

Remark. It is possible to show by asymptotic arguments based on the Birkhoff–Trjitzinsky theory of difference equations, [5], that $\sum_0^\infty p_n^2(0, a, b) = \infty$ without reference to Hermite polynomials.

4. Birth and Death Process Recurrence Relations

Since $P_n(x; a, b)$ reduces to the polynomials of (2.3) when $a = 1$, it is natural to look for a generalization of (1.1)–(1.2) for which the $P_n(x; a, b)$ are, after a suitable translation, the kernel polynomials. Guided by the formula (2.17), we define

$$G_n(x; a, \lambda, \mu) = \frac{(\lambda\mu)^{n/2} n!}{(a)_n} P_n\left(\frac{x-\mu}{\sqrt{\lambda\mu}}; a, b\right), \tag{4.1}$$

where $\lambda > 0$, $\mu > 0$, $a > 0$ and

$$b = -[\lambda - (a-1)\mu]/\sqrt{\lambda\mu}. \tag{4.2}$$

For these polynomials, we have the recurrence relation

$$\begin{aligned} G_{n+1}(x) &= \left[x - \mu - \frac{\lambda - (a-1)\mu}{n+a}\right] G_n(x) \\ &\quad - \frac{\lambda\mu n}{(n+a)(n+a-1)} G_{n-1}(x), \\ G_{-1}(x) &= 0, G_0(x) = 1. \end{aligned} \tag{4.3}$$

It can now be verified that if we set $\mu_0 = 0$ and

$$\lambda_n = \frac{\lambda}{n+a}, \qquad \mu_{n+1} = \frac{\mu(n+1)}{n+a}, \qquad n \geqslant 0, \tag{4.4}$$

then

$$\lambda_n + \mu_{n+1} = \mu + \frac{\lambda - (a-1)\mu}{n+a},$$

$$\lambda_n \mu_n = \frac{\lambda\mu n}{(n+a)(n+a-1)}, \qquad n \geqslant 1.$$

(Note that $\lambda_{n-1} > 0$, $\mu_n > 0$ for $n \geqslant 1$.)

It now follows that the $G_n(x)$ are the kernel polynomials for the polynomials $F_n(x; a, \lambda, \mu)$ which are defined by the recurrence formula

$$\begin{aligned} &F_{n+1}(x) = [x - (\lambda_n + \mu_n)] F_n(x) - \lambda_{n-1}\mu_n F_{n-1}(x), \qquad n \geqslant 0, \\ &F_{-1}(x) = 0, \qquad F_0(x) = 1. \end{aligned} \tag{4.5}$$

(See [8, pp. 45–47], where the notation is

$$\gamma_{2n} = \lambda_{n-1}, \gamma_{2n+1} = \mu_n, n \geqslant 1.)$$

It is now readily verified that the polynomials

$$Q_n(x; a, \lambda, \mu) = (-\lambda)^{-n}(a)_n F_n(x; a, \lambda, \mu) \tag{4.6}$$

satisfy the birth and death process recurrence relation (1.1) with λ_n and μ_n now given by (4.4). Also, it is routinely verified that the polynomials

$$H_{n+1}(x; a, \lambda, \mu) = (-1)^{n+1}\frac{\mu^{-n}x}{n!}(a)_n G_n(x; a, \lambda, \mu) \tag{4.7}$$

satisfy the birth and death process recurrence relation which is the dual of (1.1) (see (5.14)).

Assuming that the measure $d\Psi(x; a, \lambda, \mu)$ with respect to which the $Q_n(x; a, \lambda, \mu)$ are orthogonal has been normalized to have total mass 1, the corresponding orthonormal polynomials are

$$q_n(x; a, \lambda, \mu) = (\lambda/\mu)^{n/2}(n!)^{-1/2}Q_n(x; a, \lambda, \mu). \tag{4.8}$$

The polynomials defined by (4.5) are not the only orthogonal polynomials whose kernel polynomials are the $G_n(x; a, \lambda, \mu)$, of course. For each $H \geqslant 0$ there is a unique monic orthogonal polynomial sequence whose spectral function Ψ has jump H at 0 and such that $xd\Psi(x)$ is a constant multiple of $d\theta(x-\mu)/\sqrt{\lambda\mu}$. However, since the $Q_n(x; 1, \lambda, \mu)$ are the polynomials of (1.1)–(1.2), (4.5) is clearly the appropriate choice here. It is also the only case where we can calculate the coefficients in the recurrence formula explicitly. For more information concerning the other related polynomials, see [7, Sect. 3].

The spectrum for $\{Q_n(x; a, \lambda, \mu)\}$ consists of 0, μ, and the points

$$\sigma_k = \mu + (\lambda\mu)^{1/2}\xi_k, \qquad \tau_k = \mu + (\lambda\mu)^{1/2}\eta_k, \qquad k = 0, 1, 2, \ldots, \tag{4.9}$$

where ξ_k and η_k are given by (3.8) with b given by (4.2). Let $d\Psi(x) = d\Psi(x; a, \lambda, \mu)$ denote the corresponding measure normalized to have total mass 1. Then Ψ has a jump $H(0) > 0$ at 0, is continuous at μ, and has jumps $c\mathcal{J}(\xi_k)/\sigma_k$ and $c\mathcal{J}(\eta_k)/\tau_k$ at σ_k and τ_k, respectively. The constant c is determined by

$$H(0) + c\sum_{k=0}^{\infty}\left[\frac{\mathcal{J}(\xi_k)}{\sigma_k} + \frac{\mathcal{J}(\eta_k)}{\tau_k}\right] = 1. \tag{4.10}$$

To calculate $H(0)$, we note that $Q_n(0) = 1$. Hence by (4.8),

$$q_n^2(0) = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}.$$

Therefore

$$H(0) = \left\{\sum_{n=0}^{\infty} q_n^2(0)\right\}^{-1} = e^{-\lambda/\mu}. \tag{4.11}$$

Next observe that $\lambda_0 Q_1(x) = -x + \lambda_0$ $(\lambda = \lambda_0)$ and that for $x > 0$

$$xd\Psi(x) = cd\theta(t), \qquad t = (\lambda\mu)^{-1/2}(x-\mu).$$

Thus

$$0 = \int_{-\infty}^{\infty} \lambda_0 Q_1(x)\, d\Psi(x) = -c + \lambda_0,$$

hence,

$$c = \lambda/a.$$

The spectral function is thus completely determined and the orthogonality relation for the $Q_n(x;\, a, \lambda, \mu)$ is easily written (see (5.17)).

Finally, we mention that (4.10) and (4.11) lead to another strange looking formula:

$$\sum_{k=0}^{\infty} \left[\frac{\mathcal{J}(\xi_k)}{\mu + \sqrt{\lambda\mu}\, \xi_k} + \frac{\mathcal{J}(\eta_k)}{\mu + \sqrt{\lambda\mu}\, \eta_k} \right] = \frac{a}{\lambda}\left(1 - e^{-\lambda/\mu}\right). \tag{4.12}$$

When $b = 0$ ($\lambda/\mu = a - 1$), this also reduces to a special case of Euler's formula (3.17).

5. Formulas for $P_n(x;\, a, b)$ and $Q_n(x;\, a, \lambda, \mu)$

We summarize the properties of the two systems of orthogonal polynomials by listing the principal formulas derived above as well as some others that can be routinely obtained from them. Note that there is a lack of parallelism in the standardizations of $P_n(x)$ and $Q_n(x)$. The former has its spectral limit point at 0 while the latter has the corresponding point at μ.

$$P_n(x) = P_n(x;\, a, b) \qquad (a > 0,\, b \text{ real}) \tag{A}$$

Recurrence Relation

$$(n+1)P_{n+1}(x) = [(n+a)x + b]P_n(x) - P_{n-1}(x), \qquad n \geqslant 0. \tag{5.1}$$

The recurrence relation for the monic polynomials (2.19) is given by (2.20). For the corresponding birth and death process form, see (5.14).

Generating Function

$$e^{w/x}(1 - xw)^{(1 - bx - ax^2)/x^2} = \sum_{n=0}^{\infty} P_n(x) w^n. \tag{5.2}$$

Explicit Formulas

$$P_n(x) = x^{-n}(n!)^{-1}\sum_{k=0}^{n}\binom{n}{k}\pi_k(x), \tag{5.3}$$

where $\pi_0(x) = 1$, $\pi_k(x) = \prod_{\nu=0}^{k-1}[(a+\nu)x^2 + bx - 1]$, $\quad k \geqslant 1$,

$$P_n(x) = n!x^{-n}{}_2F_0(-n, a + bx^{-1} - x^{-2}; -; -x^2). \tag{5.4}$$

Orthogonality Relation

$$\sum_{k=0}^{\infty}\left[P_m(\xi_k)P_n(\xi_k)\mathcal{J}(\xi_k) + P_m(\eta_k)P_n(\eta_k)\mathcal{J}(\eta_k)\right] = \frac{a\delta_{mn}}{(n+a)n!}, \tag{5.5}$$

where

$$\xi_k = \frac{-b - \left[b^2 + 4(k+a)\right]^{1/2}}{2(k+a)},$$

$$\eta_k = \frac{-b + \left[b^2 + 4(k+a)\right]^{1/2}}{2(k+a)}, \qquad k = 0, 1, 2, \dots, \tag{5.6}$$

$$\mathcal{J}(\zeta_k) = \frac{a(k+a)\zeta_k^{-2k}}{\left[(k+a)^2 + 1\right]k!}\exp(-\zeta_k^{-2}), \tag{5.7}$$

where $\zeta_k = \xi_k$ or $\zeta_k = \eta_k$

$$P_n(\zeta_k) = \frac{1}{n!}\sum_{\nu=0}^{\min(k,n)}\binom{n}{\nu}\binom{k}{\nu}(-1)^{\nu}\nu!\zeta_k^{2\nu}. \tag{5.8}$$

Relations with Other Polynomials

$$xP_n\big((\lambda\mu)^{-1/2}(x-\mu); a, b\big)$$
$$= \frac{(-1)^{n+1}\lambda(\lambda/\mu)^{n/2}}{n+a}\left[Q_{n+1}(x; a, \lambda, \mu) - Q_n(x; a, \lambda, \mu)\right], \tag{5.9}$$

where $b = -[\lambda - (a-1)\mu]/\sqrt{\lambda\mu}$,

$$P_n(a^{-1/2}x; a, 0) = a^{n/2}(n!)^{-1}r_n(x, a), \tag{5.10}$$

where $\mathcal{F}(\xi)$ is given by (5.7) and

$$\sigma_k = \mu + (\lambda\mu)^{1/2}\xi_k$$
$$= \mu + \frac{\lambda - (a-1)\mu - \{[\lambda(a-1)\mu]^2 + 4(K+a)\lambda\mu\}^{1/2}}{2(k+a)},$$
$$\tau_k = \mu + (\lambda\mu)^{1/2}\eta_k$$
$$= \mu + \frac{\lambda - (a-1)\mu + \{[\lambda(a-1)\mu]^2 + 4(K+a)\lambda\mu\}^{1/2}}{2(k+a)},$$
$$(5.18)$$

$$Q_n(\rho_k) = (-1)^n\left(\frac{\mu}{\lambda}\right)^{n/2}\sum_{\nu=0}^{\min(k,n)}\binom{n}{\nu}\binom{k}{\nu}(-1)^\nu\rho_k^{2\nu}\left[1 + (n-k)(\mu/\lambda)^{1/2}\right], \tag{5.19}$$

where $\rho_k = \sigma_k$ or $\rho_k = \tau_k$.

Relations with Other Polynomials

$$Q_n(\mu; a, \lambda, \mu) = \frac{(a-1)\mu}{\lambda}H_n(2^{-1/2}b) - \left(\frac{\mu}{\lambda}\right)^{1/2}H_{n+1}(2^{-1/2}b),$$
$$Q_n(\mu; 1, \lambda, \mu) = -\tfrac{1}{2}(2\mu/\lambda)^{(n+1)/2}H_{n+1}\left((\lambda/2\mu)^{1/2}\right). \tag{5.20}$$

Here $H_n(x)$ denotes the Hermite polynomial.

$$Q_n(x; a, \lambda, \mu) = (-1)^n(\mu/\lambda)^{n/2}n!\left\{P_n\left(\frac{x-\mu}{\sqrt{\lambda\mu}}; a, b\right)\right.$$
$$\left. + \left(\frac{\mu}{\lambda}\right)^{1/2}P_{n-1}\left(\frac{x-\mu}{\sqrt{\lambda\mu}}; a, b\right)\right\}. \tag{5.21}$$

Here again, $b = -[\lambda - (a-1)\mu]/\sqrt{\lambda\mu}$.

6. The Associated Polynomials

We discuss briefly the three parameter family of polynomials $P_n^c(x) = P_n^c(x; a, b)$ defined by

$$(n+c+1)P_{n+1}^c(x) = [(n+a+c)x + b]P_n^c(x) - P_{n-1}^c(x),$$
$$P_0^c(x) = 1,\ P_1^c(x) = \frac{(a+c)x+b}{c+1}. \tag{6.1}$$

Obviously

$$P_n^0(x;a,b) = P_n(x;a,b),$$
$$P_n^1(x;a,b) = a^{-1}N_{n+1}(x).$$

The polynomials $\{P_n^c(x)\}$ are orthogonal with respect to a positive measure if and only if

$$(n+c)(n+a+c)(n+a+c-1) > 0, \qquad n = 1,2,\ldots.$$

For simplicity we shall restrict ourselves to the case

$$c > -1, \qquad a + c > 0, \tag{6.2}$$

which suffices for orthogonality with respect to a positive measure.

As in the derivation of (3.2), we set

$$\Phi^c(x,w) = \sum_{n=0}^{\infty} P_n^c(x) w \tag{6.3}$$

and obtain

$$w(1-xw)\frac{\partial \Phi^c}{\partial w} - \left[w(ax+b+cx) - w^2 - c\right]\Phi^c = c,$$
$$\Phi^c(x,0) = 1,$$

whence

$$\Phi^c(x,w) = ce^{w/x}w^{-c}(1-xw)^{-f(x)}\int_0^w e^{-u/x}u^{c-1}(1-xu)^{f(x)}\,du, \tag{6.4}$$

where, as before, $f(x) = (ax^2+bx-1)/x^2$.

The polynomials of the second kind corresponding to $P_n^c(x;a,b)$ are

$$N_n^c(x) = \frac{c+a}{c+1}P_{n-1}^{c+1}(x). \tag{6.5}$$

The above procedure applied to the $N_n^c(x)$ gives

$$\sum_{n=0}^{\infty} N_n^c(x)w^n$$
$$= (c+a)e^{w/x}w^{-c}(1-xw)^{-f(x)}\int_0^w e^{-u/x}u^c(1-xu)^{-f(x)}\,du. \tag{6.6}$$

Next let

$$\chi^c(z) = \lim_{n\to\infty}\frac{N_n^c(x)}{P_n^c(x)}. \tag{6.7}$$

Once again using Darboux' method, we obtain

$$\chi^c(z) = \frac{(c+a)\int_0^{1/z} e^{-u/z} u^c (1-zu)^{f(z)-1}\,du}{c\int_0^{1/z} e^{-v/z} v^{c-1}(1-zv)^{f(z)-1}\,dv}, \tag{6.8}$$

for $\mathrm{Re}(f(z)) > 0$. With this restriction on z,

$$\chi^c(z) = \frac{(c+a)\Gamma(c+f(z))\,{}_1F_1(c+1; c+1+f(z); -z^{-2})}{z\Gamma(c+1+f(z))\,{}_1F_1(c; c+f(z); -z^{-2})} \tag{6.9}$$

and analytic continuation establishes this for $\mathrm{Im}(z) \neq 0$. Recall that when a and b are constants the function

$$w(z) = {}_1F_1(a; b; z)$$

is an entire transcendental function for $b \neq 0, -1, -2, \dots$, and satisfies the differential equation, Rainville [16, p. 124],

$$zw'' + (b-z)w' - aw = 0$$

and the differential recurrence relation

$$\frac{d}{dz}\,{}_1F_1(a; b; z) = \frac{a}{b}\,{}_1F_1(a+1; b+1; z).$$

Thus, ${}_1F_1(a; b; z)$ and ${}_1F_1(a+1; b+1; z)$ have no common zeros when a and b are independent of z. When $a = a(z)$, $b = b(z)$ depend on z the same conclusion holds because if $z = z_0$ is a common zero of ${}_1F_1(a(z); b(z); z)$ and ${}_1F_1(1 + a(z); 1 + b(z); z)$, $b(z_0) \neq 0, -1, -2, \dots,$ then ${}_1F_1(a(z_0); b(z_0); z)$ and ${}_1F_1(1 + a(z_0); 1 + b(z_0); z)$ have the common zero $z = z_0$, a contradiction. We are now in a position to prove

THEOREM 6.1. *The spectrum of the distribution function corresponding to the polynomials $\{P_n^c(x)\}$ is bounded and purely discrete. The jump points coincide with the zeros of ${}_1F_1(c; c + f(z); -z^{-2})$. The point $x = 0$ is the only limit point of the spectrum and is not a jump point.*

Proof. The boundedness of the spectrum follows from [8, p. 109]. Markoff's theorem [18, p. 57] identifies the continued fraction as the Stieltjes transform of the distribution function $\theta^c(t)$, that is

$$\chi^c(z) = \int_{-\infty}^{\infty} (z-t)^{-1}\, d\theta^c(t), \qquad z \notin \text{spectrum}. \tag{6.10}$$

The Stieltjes inversion formula, Widder [20, p. 339], implies the discreteness of the spectrum. The jumps of $\theta^c(z)$ occur at the poles of $\chi^c(z)$, hence the limit points of the spectrum are the essential singularities of $\chi^c(z)$. The relationship (6.9) shows that $z = 0$ is the only essential singularity of $\chi^c(z)$. The above facts can be alternatively established via Krein's theorem [8, p. 117].

The nonzero poles of the right sides of (6.10) are all real and simple, hence the nonzero poles of $\chi^c(z)$ must also be real and simple. These poles coincide with the zeros of ${}_1F_1(c; c + f(z); -z^{-2})$ because the numerator and denominator ${}_1F_1$'s in (6.9) have no common zeros. It remains to analyse the jump at $z = 0$. Since the moment problem is determined (bounded spectrum) and the orthonormal set is

$$\left\{\frac{n+c+a}{c+a}(c+1)_n\right\}^{1/2} P_n^c(x; a, b) \tag{6.11}$$

the point $x = 0$ is a jump point if and only if

$$\sum_0^\infty \frac{n+c+a}{c+a}(c+1)_n\left(P_n^c(0; a, b)\right)^2 = \infty, \tag{6.12}$$

Shohat and Tamarkin [17, pp. 45–46]. The polynomials $P_n^c(0; a, b)$ satisfy the recursion

$$(c+n+1)P_{n+1}^c(0; a, b) = bP_n^c(0; a, b) - P_{n-1}^c(0; a, b), \qquad n > 0, \tag{6.13}$$

with $P_0^c(0; a, b) = 1$, $P_1^c(0; a, b) = b/c$. The Birkhoff–Trjitzinsky theory of difference equations, see [5], can be applied to (6.13) to prove

$$n(c+1)_n\{P_n^c(0; a, b)\}^2 = 0\left(n\cos^2\left(n\frac{\pi}{2} - b\sqrt{n}\right)\right.$$

with the 0-term $\neq 0$ for $b \neq 0$. This establishes (6.12) and the proof is complete.

In this process of proving theorem 6.1 we actually proved

THEOREM 6.2. *All the zeros of* ${}_1F_1(c; c + f(z^{-1}) - z^2)$ *are real and simple when* $c > -1$, $a + c > 0$. *There are infinitely many such zeros and* $\pm\infty$ *are their only limit points.*

The polynomials $\{P_n^c(0; a, b)\}$ generated by (6.13) are orthogonal polynomials as functions of b. They are orthogonal with respect to a positive measure if and only if $c > 0$. They are scalar multiples of the associated Hermite polynomials $\{H_n^c(b/\sqrt{2})\}$. The $H_n^c(x)$'s satisfy

$$H_0^c(x) = 1, \qquad H_1^c(x) = 2x,$$
$$H_{n+1}^c(x) = 2xH_n^c(x) - 2(n+c)H_{n-1}^c(x), \qquad n > 0, \tag{6.14}$$

as could be seen from replacing n by $n + c$ in the three term recurrence relation satisfied by the Hermite polynomials [16, p. 188]. The determinancy of the moment problem associated with the H_n^c's follows from Carleman's criterion [17, p. 59] but the corresponding distribution function is not known. One way of deriving the orthogonality relation for the Hermite polynomials is to obtain it as a limiting case of the corresponding relation for the ultraspherical polynomials. Unfortunately it is not clear how this would work at the associated level. Pollaczek [15] computed the weight function for the associated ultraspherical polynomials but it is just not clear how to take limits in his formulas. Another interesting related set are the associated Laguerre polynomials. These questions are still under investigation.

Finally we include a slight simplification of the generating function (6.4). In (6.4) replace u by $w(1 - u)$ to get

$$\begin{aligned}\Phi^c(x, w) &= c(1 - xw)^{-1}\int_0^1 e^{uw/x}(1 - u)^{c-1}\left(1 - \frac{ux}{wx - 1}\right)^{f(x)-1} du \\ &= c(1 - xw)^{-1}\sum_0^\infty \frac{(w/x)^k}{k!}\int_0^1 u^n(1 - u)^{c-1}\left(1 - \frac{ux}{wx - 1}\right)^{f(x)-1} du \\ &= c(1 - xw)^{-1}\sum_0^\infty \frac{(w/x)^k}{(c)_{k+1}}\,{}_2F_1\left(\begin{matrix}1 - f(x), k + 1; \\ c + k + 1;\end{matrix}\ \frac{xw}{xw - 1}\right),\end{aligned}$$

where we used the integral form of Theorem 16, p. 47 in [16], when $\mathrm{Re}f(x) > 0$. We then use analytic continuation. Apply the Pfaff-Kummer transformation (Theorem 20, p. 60 in [16]) to obtain

$$\Phi^c(x, w) = (1 - xw)^{-f(x)}\sum_0^\infty \frac{(w/x)^k}{(c + 1)_k}\,{}_2F_1\left(\begin{matrix}c, 1 - f(x); \\ c + k + 1;\end{matrix}\ xw\right). \tag{6.15}$$

Formula (6.15) was established under the assumption $c \neq 0$. It is easy to see that (6.15) when $c = 0$ agrees with the generating function for the P_n's.

The generating function (6.15) is probably as simple as we can get. It can also be expressed as a double series in the form

$$\Phi^c(x, w) = (1 - xw)^{-f(x)} \sum_{k, j \geq 0}^\infty \frac{(1)_k (c)_k (1 - f(x))_j}{(c + 1)_{k+j} k! j!} x^{k+j} w^{j-k},$$

which is a confluent form of the Appell Function F_3; see [16, p. 265].

Note Added in Proof. After writing the present paper we became aware of Jet Wimp's very interesting forthcoming work [21] where the weight functions of the associated Hermite and Laguerre polynomials have been computed. In a private communication (and independently) Richard Askey outlined how a delicate limiting process applied to Pollaczek's work [15] also gives the weight functions of the associated Hermite and Laguerre polynomials. These remarks are relevant to section 6. Otto Ruehr kindly showed us how to establish (3.16) using Lagrange inversion and Laplace transform theory.

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