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Short Note

An $O(n^2)$ algorithm for maximum cycle mean of Monge matrices in max-algebra ☆

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Abstract

An $O(n^2)$ algorithm is described for computing the maximum cycle mean (eigenvalue) for $n \times n$ matrices, $A = (a_{ij})$ fulfilling Monge property, $a_{ij} + a_{kl} \leqslant a_{il} + a_{kj}$ for any $i < k$, $j < l$. The algorithm computes the value $\lambda(A) = \max(a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_k i_1})/k$ over all cyclic permutations $(i_1, i_2, \ldots, i_k)$ of subsets of the set $\{1, 2, \ldots, n\}$. A similar result is presented for matrices with inverse Monge property. The standard algorithm for the general case works in $O(n^3)$ time.

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1. Background of the problem

Let $G = (G, \otimes, \leqslant)$ be a linearly ordered, commutative group with neutral element $e = 0$. We suppose that G is radicable, i.e. for every integer $t \geqslant 1$ and for every $a \in G$, there exists a (unique) element $b \in G$ such that $b^t = a$. We denote $b = a^{1/t}$.

Throughout the paper, $n \geqslant 1$ is a given integer. The set of $n \times n$ matrices over G is denoted by M_n. We introduce a further binary operation $\oplus$ on G by the formula

$$a \oplus b = \max(a, b) \quad \text{for all } a, b \in G.$$

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The triple $(G,\oplus,\otimes)$ is called *max-algebra*. If $G=(G,\otimes,\leqslant)$ is the additive group of real numbers, then $(G,\oplus,\otimes)$ is called *max-plus algebra* (often used in applications).

The operations $\oplus,\otimes$ are extended to the matrix–vector algebra over G by the direct analogy to the conventional linear algebra. For $A=(a_{ij})\in M_n$, the problem of finding $x\in G^n$, $\lambda(A)\in G$, satisfying

$$A\otimes x=\lambda(A)\otimes x$$

is called an extremal eigenproblem corresponding to matrix A; here $\lambda(A)$ and x are usually called an extremal eigenvalue and an extremal eigenvector of A, respectively. Throughout the paper, we shall omit the word "extremal". This problem was treated by several authors during 1960s, e.g. [4,6,8,14], a survey of the results concerning this and similar eigenproblems can be found in [9,15,16]. Below, we summarize some of the main results.

First, we introduce the necessary notation. Let $N=\{1,2,\ldots,n\}$, and let C_n be the set of all cyclic permutations defined on nonempty subsets of N. For a cyclic permutation $\sigma=(i_1,i_2,\ldots,i_l)\in C_n$ and for $A\in M_n$, we denote number l, the *length* of σ, by $l(\sigma)$ and define $w_A(\sigma)$, the *weight* of σ, as

$$w_A(\sigma)=a_{i_1i_2}\otimes a_{i_2i_3}\otimes\cdots\otimes a_{i_li_1}.$$

Furthermore, define $\mu_A(\sigma)$, the *mean weight* of σ, by

$$\mu_A(\sigma)=w_A(\sigma)^{1/l(\sigma)}$$

and $\lambda(A)$, the *maximum cycle mean* (MCM) of A, as

$$\lambda(A)=\sum_{\sigma\in C_n}{}^{\oplus}\mu_A(\sigma),$$

where $\Sigma^{\oplus}$ denotes the iterated use of the operation $\oplus$.

The symbol D_A stands for a complete, arc-weighted digraph associated with A. The node set of D_A is N, and the weight of any arc (i,j) is a_{ij}. Throughout the paper, by a cycle in the digraph we mean an elementary cycle or a loop, and by path we mean a nontrivial elementary path, i.e. an elementary path containing at least one arc. Evidently, there is a one-to-one correspondence between cycles in D_A and elements of C_n. Therefore, we will use the same notation, as well as the concept of weight, for both cycles and cyclic permutations. A cycle $\sigma\in C_n$ is called optimal, if $\mu_A(\sigma)=\lambda(A)$, a node in D_A is called an eigennode if it is contained in at least one optimal cycle, E_A stands for the set of all eigennodes in D_A.

Theorem 1.1 (Cuninghame-Green [5]). *Let $A\in M_n$. Then $\lambda(A)$ is the unique eigenvalue of A.*

The MCM has several practical interpretations, e.g. in an industrial scheduling problem [5] or a ship-routing problem [6]. The problem of finding $\lambda(A)$ has been studied by several authors [2,5,6,10,11,12]. Various algorithms for solving this problem are known, that of Karp [10] having the best worst-case performance $O(n^3)$, and algorithms of a smaller complexity can be developed in special cases. One such case corresponds to matrices in which all elements of at least one cycle are equal to the maximum element of the matrix, e.g. if some diagonal element is maximal or if there are more than $\frac{1}{2}n(n-1)$ maximal elements. This property can be recognized in $O(n^2)$ operations via an algorithm for checking the existence of a cycle in digraph [11]. The procedure is *linear* in terms of the input, which is also $O(n^2)$.

2. The eigenvalue for Monge matrices

The aim of the present paper is to derive a linear procedure (in terms of the input) for a further special case, namely when A has the Monge property, or, more generally, when A is a Monge matrix.

Definition 2.1. We say that the matrix $A=(a_{ij})$ satisfies the Monge property and inverse Monge property if and only if

$$a_{ij} + a_{kl} \leqslant a_{il} + a_{kj} \quad \text{for any } i < k,\ \ j < l$$

and

$$a_{ij} + a_{kl} \geqslant a_{il} + a_{kj} \quad \text{for any } i < k,\ \ j < l,$$

respectively. A matrix A is called Monge (inverse Monge), if there is a permutation ϕ which, when applied to the rows and columns of A, gives a matrix A_ϕ with the Monge (inverse Monge) property.

The following theorem shows that, in computing the eigenvalue of a given matrix with Monge property, the computation may be restricted to cycles of lengths 1 and 2.

Theorem 2.1. *If $A=(a_{ij})$ has the Monge property, then*

$$\lambda(A) = \max_{i,j\in N} \left\{ a_{ii}, \frac{a_{ij}+a_{ji}}{2} \right\}.$$

The proof of Theorem 2.1 is based on two lemmas.

Lemma 2.2. *Let $C=(i_1,i_2,\ldots,i_k,i_1)$ be a cycle of length $k \geqslant 3$. Then there are arcs (i_j,i_{j+1}) and (i_l,i_{l+1}) in C such that*

$$i_j < i_l \quad \text{and} \quad i_{j+1} < i_{l+1}.$$

Proof. Let us define a sequence $u_1,\ldots,u_k$, containing all the nodes of the cycle C, in an increasing order. As C is a cycle, there are (uniquely determined) indices $s>1$ and $r<k$ such that (u_1,u_s) and (u_r,u_k) are two arcs of the cycle. If these arcs are distinct, i.e. if $s \neq k$ (and $1 \neq r$), then we are done, because necessarily $u_1 < u_r$ and $u_s < u_k$. If the two arcs coincide, then we have the situation that (u_1,u_k) is an arc of the cycle. We consider the arc (u_k,u_t) beginning in u_k. Since the length of C is greater than two, $t \neq 1$, and we have another arc (u_m,u_1) ending in u_1 (possibly $m=t$, if $k=3$). Then (u_m,u_1) and (u_k,u_t) are the desired pair of arcs. $\square$

Lemma 2.3. *Let $d_1, \ldots, d_k$ be any real numbers and $l \in \{1, 2, \ldots, k\}$. Then,*

$$\frac{d_1 + d_2 + \cdots + d_k}{k} \leqslant \max\left\{\frac{d_1 + \cdots + d_l}{l}, \frac{d_{l+1} + \cdots + d_k}{k-l}\right\}.$$

Proof. Let us denote $S = d_1 + d_2 + \cdots + d_k$, $S_1 = d_1 + \cdots + d_l$, $S_2 = d_{l+1} + \cdots + d_k$. Suppose that $S_1/l < S/k$ and $S_2/(k-l) < S/k$. Then, $S_1 k < Sl$ and $S_2 k < S(k-l)$. By adding these two inequalities we get $(S_1 + S_2)k < S(l + k - l)$, i.e. $Sk < Sk$, a contradiction. □

Proof of Theorem 2.1. If a cycle C contains two arcs $(i_j, i_{j+1}), (i_l, i_{l+1})$ as described in Lemma 2.2, then the arcs will be substituted by arcs $(i_j, i_{l+1}), (i_l, i_{j+1})$. The substitution splits the cycle C into two smaller cycles. By a standard argument using the Monge property

$$a_{i_j i_{j+1}} + a_{i_l i_{l+1}} \leqslant a_{i_j i_{l+1}} + a_{i_l i_{j+1}},$$

the total weight of all the arcs used in C will not be diminished by the splitting of C. After repeating this procedure, every cycle will eventually be separated into a collection of cycles with lengths 1 or 2 and the total weight of the arcs in consideration will not decrease. By Lemma 2.3, the maximal cycle mean will not decrease, as well. The proof of Theorem 2.1 is complete. □

The following theorem shows that, in computing the eigenvalue of a given matrix with inverse Monge property, the computation may be restricted to cycles of length 1.

Theorem 2.4. *If $A = (a_{ij})$ has the inverse Monge property, then*

$$\lambda(A) = \max_{i \in N}\{a_{ii}\}.$$

Proof. It suffices to show that for any cycle $(i_1, i_2, \ldots, i_k)$ we have

$$a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_k i_1} \leqslant a_{i_1 i_1} + a_{i_2 i_3} + \cdots + a_{i_k i_2}.$$

Let us suppose w.l.o.g. that $i_1 = \min\{i_1, i_2, \ldots, i_k\}$. Then due to the inverse Monge property

$$a_{i_1 i_2} + a_{i_k i_1} \leqslant a_{i_1 i_1} + a_{i_k i_2},$$

we get

$$a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_k i_1} \leqslant a_{i_1 i_1} + a_{i_2 i_3} + \cdots + a_{i_k i_2}.$$

By using induction and Lemma 2.3 we have the claim. □

The assertion of the above theorem can be extended to Monge matrices by a modification of the algorithm described by Deineko and Filonenko in [7]. Their algorithm finds permutations ϕ and ψ such that ϕ applied to the rows and ψ applied to the columns of A give a matrix $A_{(\phi,\psi)}$, which has the Monge property, or shows that there is no such pair of permutations. In general, the permutations ϕ, ψ need not be equal. Rudolf [13] showed (see also Burkard et al. [1]) that there is an algorithm $\mathscr{A}_I$, which decides in time $O(n^2)$ whether a given $n \times n$ matrix A is Monge and finds a permutation ϕ such that A_ϕ has the Monge property, in the positive case. As it is noticed in [1], algorithm $\mathscr{A}_I$ can easily be extended to yield an $O(n^2)$ algorithm for inverse Monge matrices. So, we can formulate the following theorem.

Theorem 2.5. *There is an algorithm $\mathscr{A}$ which, for a given Monge (inverse Monge) matrix $A \in M_n$, computes the eigenvalue $\lambda(A)$ in $O(n^2)$ time.*

Proof. The first part of the algorithm tests whether matrix A is Monge (inverse Monge). This is done by the algorithm $\mathscr{A}_I$, or by its extension for inverse Monge matrices. In the positive case, $\mathscr{A}_I$ finds a permutation ϕ such that A_ϕ has the Monge (inverse Monge) property. The simultaneous permutation of rows and columns does not change the eigenvalue $\lambda(A) = \lambda(A_\phi)$. This part of the computation requires $O(n^2)$ time.

The second part of the algorithm uses the expression given in Theorems 2.1 and 2.4 for computing $\lambda(A_\phi)$. It is clear, that the computational complexity of this part is also $O(n^2)$. $\square$

Remark 2.1. The same result as in Theorem 2.4 can be obtained for matrices A satisfying the so-called weak Monge property

$$a_{ii} + a_{kl} \leqslant a_{il} + a_{ki} \quad \text{for any } i < k, l.$$

An $O(n^4)$ algorithm for recognition of $n \times n$ weak Monge matrices A (with A_ϕ satisfying the weak Monge property), is given in [3]. The authors mention that an improvement to $O(n^3 \log n)$ is possible but do not give further details in their paper. However, the computational complexity of this algorithm is too high for our purpose.

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