Controllability results for nondensely defined evolution differential inclusions with nonlocal conditions

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Abstract

In this paper we prove controllability results for some semilinear nondensely defined evolution differential inclusions in Banach spaces with nonlocal conditions.

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1. Introduction

In this paper we prove controllability results for semilinear evolution differential inclusions with nonlocal conditions, of the form

\[ y'(t) \in Ay(t) + F(t, y(t)) + (\Theta u)(t), \quad t \in J := [0, b], \]

\[ y(0) + g(y) = y_0, \]

where \( A : D(A) \subset E \to E \) is a nondensely defined closed linear operator, \( F : J \times E \to \mathcal{P}(E) \setminus \emptyset \) is a multivalued map (\( \mathcal{P}(E) \) is the family of all subsets of \( E \)) and \( g \in C(C(J, E), E) \). Also the control function \( u(\cdot) \) is given in \( L^2(J, U) \), a Banach space of admissible control functions with \( U \) as a Banach space. Finally \( \Theta \) is a bounded linear operator from \( U \) to \( E \) and \( E \) is a separable Banach space with norm \( | \cdot | \).

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As indicated in [11,14] and references therein, the nonlocal condition \( y(0) + g(y) = y_0 \) can be applied in physics with better effect than the classical initial condition \( y(0) = y_0 \). For example, in [14], the author used

\[
g(y) = \sum_{k=1}^{p} c_i y(t_i),
\]

where \( c_i, i = 1, \ldots, p \), are given constants and \( 0 < t_1 < t_2 < \cdots < b \), to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, Eq. (3) allows the additional measurements at \( t_i, i = 1, \ldots, p \).

Differential equations or inclusions with nonlocal conditions where the operator \( A \) generates a \( C_0 \) semigroup, or equivalently, when a closed linear operator \( A \) satisfies

(i) \( \text{D}(A) = E \) (\( D \) means domain);

(ii) The Hille–Yosida condition that is, there exists \( M \geq 0 \) and \( \tau \in \mathbb{R} \) such that \((\tau, \infty) \subset \rho(A)\),

\[
\sup \{ (\lambda I - \tau)^n | (\lambda I - A)^{-n} | : \lambda > \tau, \, n \in \mathbb{N} \} \leq M,
\]

where \( \rho(A) \) is the resolvent operator set of \( A \) and \( I \) is the identity operator have been studied extensively.

Existence and uniqueness among other things, are derived. See [4,5,11,14].

For recent controllability results in the cases when the operator \( A \) generates a \( C_0 \) semigroup we refer to the papers by Benchohra and Ntouyas [6–8] and references cited therein. However, as indicated in [12], we sometimes need to deal with nondensely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on \([0, 1]\) and consider \( A = \partial^2 / \partial x^2 \) in \( C([0, 1], \mathbb{R}) \) in order to measure the solutions in the sup-norm, then the domain

\[
D(A) = \{ \phi \in C^2([0, 1], \mathbb{R}): \phi(0) = \phi(1) = 0 \}
\]

is not dense in \( C([0, 1], \mathbb{R}) \) with the sup-norm. See [12] for more examples and remarks concerning the nondensely defined operators.

Very recently in [3] Benchohra et al. studied existence results for nondensely defined impulsive semilinear differential inclusions. Our purpose here is to prove controllability results for nondensely defined semilinear differential inclusions with nonlocal conditions.

This paper will be organized as follows. In Section 2 we will recall some basic definitions and preliminary facts from multivalued analysis and integrated semigroups which will be used later. In Section 3 we shall present three results. In the first two we rely on Bohnenblust–Karlin’s fixed point theorem and for the third one on the Schaefer’s fixed point theorem combined with a selection theorem due to Bressan and Colombo [10] for lower semicontinuous multivalued operators with nonempty closed and decomposable values. Finally in Section 4 we present controllability results for the problem (1)–(2) for a special case of \( g \) given by (3).
2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper.

\( C(J, E) \) is the Banach space of continuous functions from \( J \) to \( E \) normed by
\[
\|y\|_\infty = \sup\{|y(t)| : t \in J\}
\]
and \( B(E) \) denotes the Banach space of bounded linear operators from \( E \) into \( E \), with norm
\[
\|N\|_{B(E)} = \sup\{|Ny| : |y| = 1\}.
\]

\( L^1(J, E) \) denotes the Banach space of measurable functions \( y : J \to E \) which are Bochner integrable normed by
\[
\|y\|_{L^1} = \int_0^b |y(t)| \, dt.
\]

Let \((X, d)\) be a metric space. We use the following notations:

- \( P(X) = \{ Y \in \mathcal{P}(X) : Y \neq \emptyset \} \)
- \( P_{\text{cl}}(X) = \{ Y \in \mathcal{P}(X) : Y \text{ closed} \} \)
- \( P_b(X) = \{ Y \in \mathcal{P}(X) : Y \text{ bounded} \} \)
- \( P_c(X) = \{ Y \in \mathcal{P}(X) : Y \text{ convex} \} \)
- \( P_{\text{cp}}(X) = \{ Y \in \mathcal{P}(X) : Y \text{ compact} \} \)

A multivalued map \( G : X \to \mathcal{P}(X) \) is convex (closed) valued if \( G(x) \) is convex (closed) for all \( x \in X \).

\( G \) is bounded on bounded sets if \( G(B) = \bigcup_{x \in B} G(x) \) is bounded in \( X \) for all \( B \in P_b(X) \) (i.e., \( \sup_{x \in B} \{\sup|y| : y \in G(x)\} < \infty \)).

\( G \) is called upper semicontinuous (u.s.c.) on \( X \) if for each \( x_0 \in X \) the set \( G(x_0) \) is a nonempty, closed subset of \( X \), and if for each open set \( U \) of \( X \) containing \( G(x_0) \), there exists an open neighborhood \( V \) of \( x_0 \) such that \( G(V) \subseteq U \).

\( G \) is said to be completely continuous if \( G(B) \) is relatively compact for every \( B \in P_b(X) \). If the multivalued map \( G \) is completely continuous with nonempty compact values, then \( G \) is u.s.c. if and only if \( G \) has a closed graph (i.e., \( x_n \to x, y_n \to y, y_n \in G(x_n) \) imply \( y_n \in G(x) \)). \( G \) has a fixed point if there is \( x \in X \) such that \( x \in G(x) \).

The fixed point set of the multivalued operator \( G \) will be denoted by \( \text{Fix} G \).

A multivalued map \( N : J \to P_{\text{cl}}(X) \) is said to be measurable, if for every \( y \in X \), the function \( t \mapsto d(y, N(t)) = \inf\{|y - z| : z \in N(t)\} \) is measurable. For more details on multivalued maps see the books of Aubin and Cellina [2], Deimling [13], Górniewicz [16] and Hu and Papageorgiou [17].

**Definition 2.1** [1]. Let \( E \) be a Banach space. An integrated semigroup is a family of operators \((S(t))_{t \geq 0}\) of bounded linear operators \( S(t) \) on \( E \) with the following properties:

(i) \( S(0) = 0 \);
(ii) \( t \to S(t) \) is strongly continuous;
(iii) \( S(s)S(t) = \int_0^s (S(t + r) - S(r)) \, dr \) for all \( t, s \geq 0 \).

**Definition 2.2** [18]. An operator \( A \) is called a generator of an integrated semigroup if there exists \( \omega \in \mathbb{R} \) such that \( (\omega, \infty) \subseteq \rho(A) \) (\( \rho(A) \) is the resolvent set of \( A \)) and there exists
a strongly continuous exponentially bounded family $(S(t))_{t \geq 0}$ of bounded operators such that $S(0) = 0$ and $R(\lambda, A) := (\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t) dt$ exists for all $\lambda$ with $\lambda > \omega$.

**Proposition 2.1** [1]. Let $A$ be the generator of an integrated semigroup $(S(t))_{t \geq 0}$. Then for all $x \in E$ and $t \geq 0$,

$$
\int_0^t S(s)x \, ds \in D(A) \quad \text{and} \quad S(t)x = A \int_0^t S(s)x \, ds + tx.
$$

**Definition 2.3** [18].

(i) An integrated semigroup $(S(t))_{t \geq 0}$ is called locally Lipschitz continuous if, for all $\tau > 0$ there exists a constant $L$ such that

$$
|S(t) - S(s)| \leq L|t - s|, \quad t, s \in [0, \tau].
$$

(ii) An integrated semigroup $(S(t))_{t \geq 0}$ is called nondegenerate if $S(t)x = 0$, for all $t \geq 0$ implies that $x = 0$.

**Definition 2.4.** We say that the linear operator $A$ satisfies the Hille–Yosida condition if there exists $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and

$$
\sup \{ (\lambda - \omega)^n |(\lambda I - A)^{-n}| : n \in \mathbb{N}, \lambda > \omega \} \leq M.
$$

**Theorem 2.1** [18]. The following assertions are equivalent:

(i) $A$ is the generator of a nondegenerate, locally Lipschitz continuous integrated semigroup;

(ii) $A$ satisfies the Hille–Yosida condition.

If $A$ is the generator of an integrated semigroup $(S(t))_{t \geq 0}$ which is locally Lipschitz, then from [1], $S(\cdot)x$ is continuously differentiable if and only if $x \in D(A)$ and $(S(t))_{t \geq 0}$ is a $C_0$ semigroup on $D(A)$.

**Definition 2.5.** We say that $y : J \to E$ is an integral solution of (1)–(2) if

(i) $y \in C(J, E)$;

(ii) $\int_0^t y(s) \, ds \in D(A)$ for $t \in J$;

(iii) There exists a function $f \in L^1(J, E)$ such that $f(t) \in F(t, y(t))$ a.e. in $J$ and $y(t) = y_0 - g(y) + A \int_0^t y(s) \, ds + \int_0^t [f(s) + (\Theta u)(s)] \, ds$, $t \in J$.

From (ii) it follows that $y(t) \in \overline{D(A)}$, $\forall t \geq 0$. Also from (iii) it follows that $y_0 - g(y) \in \overline{D(A)}$. So, if we assume that $y_0 \in \overline{D(A)}$ we conclude that $g(y) \in \overline{D(A)}$. 

Definition 2.6. If $y$ is an integral solution of (1)–(2), then it is given by

$$y(t) = S'(t)\left(y_0 - g(y)\right) + \frac{d}{dt} \int_0^t S(t-s)\left[f(s) + (\Theta u)(s)\right]ds.$$ 

Definition 2.7. The nonlocal problem (1)–(2) is said to be nonlocally controllable on the interval $J$, if for every $x_1 \in E$, there exists a control $u \in L^2(J, U)$, such that the solution $t \to y(t)$ of (1)–(2) satisfies $y(b) + g(y) = x_1$.

From the definition of the integrated solution we deduce that $x_1$ must necessarily belong in $\overline{D(A)}$.

Here and hereafter we assume that

(H1) $A$ satisfies the Hille–Yosida condition.

Let $B_\lambda = \lambda R(\lambda, A)$; then for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$.

For the proof of our first theorem we will use the following

Lemma 2.1 [19]. Let $X$ be a Banach space. Let $F : J \times X \to P_{c.p}(E)$; $(t, y) \mapsto F(t, y)$ be measurable with respect to $t$ for each $y \in X$, u.s.c. with respect to $y$ for each $t \in J$ and for each fixed $y \in C(J, X)$ the set

$$S_{F,y} = \{ g \in L^1(J, X) : g(t) \in F(t, y(t)) \text{ for a.e. } t \in J \}$$

is nonempty and let $\Gamma$ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$, then the operator

$$\Gamma \circ S_F : C(J, X) \to P_{c.p}(C(J, X)), \quad y \mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 2.2 (Bohnenblust and Karlin [9] see also [21, p. 452]). Let $X$ be a Banach space and $K \in P_{c.p}(X)$ and suppose that the operator $G : K \to P_{c.p}(K)$ is upper semicontinuous and the set $G(K)$ is relatively compact in $X$, then $G$ has a fixed point in $K$.

3. Main results

In this section we are concerned with the controllability for problem (1)–(2).

Let us list the following hypotheses:

(H2) Let $F : J \times E \to P_{c.p}(E)$; $(t, y) \mapsto F(t, y)$ be measurable with respect to $t$ for each $y \in E$, u.s.c. with respect to $y$ for each $t \in J$ and for each fixed $y \in C(J, E)$ the set $S_{F,y}$ is nonempty.

(H3) The operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$.

(H4) $y_0 \in \overline{D(A)}$ and $g : C(J, \overline{D(A)}) \to \overline{D(A)}$ is completely continuous and there exists $L > 0$ such that $|g(y)| \leq L$ for all $y \in C(J, \overline{D(A)})$. 

(H5) The linear operator $W : L^2(J, U) \to E$, defined by
\[
W u = \lim_{\lambda \to \infty} \int_0^b S'(b-s)B_{\lambda}(\Theta u)(s) \, ds,
\]
has an invertible operator $W^{-1}$ which takes values in $L^2(J, U) \setminus \ker W$ and there exist positive constants $M_1$ and $M_2$ such that $\|\Theta\| \leq M_1$ and $\|W^{-1}\| \leq M_2$.

(H6) There exists a continuous function $p : J \to \mathbb{R}^+$ and a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that
\[
|F(t, y)| \leq p(t)\psi(|y|), \quad t \in J, \ y \in E.
\]

Now, we are able to state and prove our main theorem in this section.

**Theorem 3.1.** Assume that assumptions (H1)–(H6) hold. Then the problem (1)–(2) is non-locally controllable on $J$.

**Proof.** Let $C := C(J, \overline{D(A)})$ denote the Banach space of continuous functions from $J$ to $\overline{D(A)}$ normed by
\[
\|y\|_C = \sup\{|y(t)| : t \in J\}.
\]
Using hypothesis (H5) for an arbitrary function $y(\cdot)$ and $x_1 \in \overline{D(A)}$ define the control
\[
u_y(t) = W^{-1}\left[x_1 - g(y) - S'(b)(y_0 - g(y)) - \lim_{\lambda \to \infty} \int_0^b S'(b-s)B_{\lambda}f(s) \, ds\right](t),
\]
where $f \in S_{r,y}$.

Consider the operator $N : C \to \mathcal{P}(C)$ defined by
\[
N(y)(t) = \left\{ h \in C : h(t) = S'(t)[y_0 - g(y)]
\right.
\quad + \frac{d}{dt} \int_0^t S(t-s)\left[f(s) + (\Theta u_y)(s)\right] \, ds, \ f \in S_{r,y} \left. \right\}.
\]
Clearly $x_1 - g(y) \in N(y)(b)$.

Let
\[
K = \{ y \in C : \|y\|_C \leq \alpha(t), \ t \in J \},
\]
where
\[
\alpha(t) = t^{-1}\left(\int_0^t m(s) \, ds\right).
\]
and

\[ I(z) = \int_c^z \frac{du}{\psi(u)}, \]

where

\[ m(t) = M^* e^{-\omega_0 t} p(t), \quad M^* = M \max\{e^{\omega b}, 1\}, \quad c' = M^* (|y_0| + L + b N^* c), \]

\[ c = M_1 M_2 \left[ |x_1| + L + M e^{\omega b} (|y_0| + L) + M e^{\omega b} \int_0^b e^{-\omega s} p(s) \psi (\alpha(s)) \, ds \right], \]

and \( N^* = \max\{1, e^{-\omega b}\}. \)

It is clear that \( K \) is closed convex and bounded set.

**Step 1**. \( N(K) \subseteq K. \)

For \( y \in K \) and \( h \in N(y) \) there exists a function \( f \in S_{F,Y} \) such that for every \( t \in J \) we have that

\[ h(t) = S'(t) (y_0 - g(y)) + \lim_{\lambda \to \infty} \int_0^t S'(t - s) B_2 \left[ f(s) + (\Theta u_\lambda)(s) \right] ds. \]

Thus

\[ |h(t)| \leq M^* (|y_0| + L) + M^* \int_0^t e^{-\omega s} p(s) \psi (\alpha(s)) \, ds \]

\[ + M^* N^* M_1 M_2 b \left[ |x_1| + L + M e^{\omega b} (|y_0| + L) \right] \]

\[ + M e^{\omega b} \int_0^b e^{-\omega s} p(s) \psi (\alpha(s)) \, ds \]

\[ \leq M^* (|y_0| + L) + \int_0^t m(s) \psi (\alpha(s)) \, ds \]

\[ + M^* N^* M_1 M_2 b \left[ |x_1| + L + M e^{\omega b} (|y_0| + L) \right] \]

\[ + M e^{\omega b} \int_0^b e^{-\omega s} p(s) \psi (\alpha(s)) \, ds \]

\[ = c' + \int_0^t \alpha'(s) \, ds = \alpha(t), \]
\[
\int_c^a \frac{du}{\psi(u)} = \int_0^t m(s) \, ds.
\]

Thus \(N(y) \in K\).

Step 2. \(N(K)\) is relatively compact.

Since \(K\) is bounded and \(N(K) \subset K\), it is clear that \(N(K)\) is bounded.

Let \(t \in (0, b]\) be a real number satisfying \(0 < \varepsilon < t\). For \(y \in K\) and \(h \in N(y)\) there exists a function \(f \in S_{F,y}\) such that

\[
h(t) = S'(t)(y_0 - g(y)) + \lim_{\lambda \to \infty} \int_0^{t - \varepsilon} S'(t - s) B_\lambda \left[ f(s) + (\Theta u_y)(s) \right] ds
\]

Define

\[
h_\varepsilon(t) = S'(t)(y_0 - g(y)) + \lim_{\lambda \to \infty} \int_0^{t - \varepsilon} S'(t - s) B_\lambda \left[ f(s) + (\Theta u_y)(s) \right] ds
\]

Since \(S'(t), \ t > 0,\) is compact, the set \(H_\varepsilon(t) = \{h_\varepsilon(t): h_\varepsilon \in N(y)\}\) is precompact in \(\overline{D(A)}\) for every \(\varepsilon, \ 0 < \varepsilon < t\). Moreover, for every \(h \in N(y)\),

\[
|h(t) - h_\varepsilon(t)| \leq M^* \int_{t - \varepsilon}^t e^{-\alpha s} \left[ p(s) \psi(\|y(s)\|) + c \right] ds
\]

Therefore there are precompact sets arbitrarily close to the set \(\{h(t): h \in N(y)\}\). Hence the set \(\{h(t): h \in N(y)\}\) is precompact in \(\overline{D(A)}\).

Step 3. \(N(K)\) is equicontinuous.

Let \(t_1, t_2 \in J, \ t_1 < t_2\). Let \(y \in K\) and \(h \in N(y)\), then there exists \(f \in S_{F,y}\) such that for each \(t \in J\) we have that

\[
h(t) = S'(t)(y_0 - g(y)) + \lim_{\lambda \to \infty} \int_0^t S'(t - s) B_\lambda \left[ f(s) + (\Theta u_y)(s) \right] ds.
\]
Then
\[ |h(\tau_2) - h(\tau_1)| \leq \left| \left[ S' (\tau_2) - S' (\tau_1) \right] (y_0 - g(y)) \right| \\
+ \left| \lim_{\lambda \to \infty} \int_0^{\tau_1} \left[ S' (\tau_2 - \tau_1 - s) B_k \left[ f (s) + (\Theta u_y) (s) \right] ds \right] \right| \\
+ \left| \lim_{\lambda \to \infty} \int_{\tau_1}^{\tau_2} S' (\tau_2 - \tau_1 - s) B_k \left[ f (s) + (\Theta u_y) (s) \right] ds \right| \\
\leq \left| \left[ S' (\tau_2) - S' (\tau_1) \right] (y_0 - g(y)) \right| \\
+ \left| \left[ S' (\tau_2 - \tau_1 - s) - I \right] \lim_{\lambda \to \infty} \int_0^{\tau_1} S' (\tau_1 - s) B_k \left[ f (s) + (\Theta u_y) (s) \right] ds \right| \\
+ Me^{\omega \tau_2} \int_{\tau_1}^{\tau_2} \left[ p (s) \psi (\alpha (s)) + c \right] ds.\]

The right-hand side tends to zero as \( \tau_2 - \tau_1 \to 0 \), since \( S' (t) \) is strongly continuous and the compactness of \( S' (t), t > 0 \), implies the continuity in the uniform operator topology.

As a consequence of Steps 2, 3, (H4) and the Arzelá–Ascoli theorem we deduce that \( N \) maps \( K \) into precompact sets in \( D(A) \).

Step 4. \( N \) has closed graph.

Let \( y_n \to y_\ast, h_n \in N (y_n), y_n \in K \) and \( h_n \to h_\ast \). We shall prove that \( h_\ast \in N (y_\ast) \). \( h_n \in N (y_n) \) means that there exists \( v_n \in S_F, y_n \) such that for each \( t \in J \),

\[ h_n (t) = S' (t) \left[ y_0 - g (y_n) \right] + \lim_{\lambda \to \infty} \int_0^t S' (t - s) B_k \left[ v_n (s) + (\Theta u_{y_n}) (s) \right] ds.\]

We must prove that there exists \( v_\ast \in S_F, y_\ast \) such that for each \( t \in J \),

\[ h_\ast (t) = S' (t) \left[ y_0 - g (y_\ast) \right] + \lim_{\lambda \to \infty} \int_0^t S' (t - s) B_k \left[ v_\ast (s) + (\Theta u_{y_\ast}) (s) \right] ds.\]

Clearly since \( g \) is completely continuous we have that

\[ \left\| h_n - S' (t) \left[ y_0 - g (y_n) \right] - \lim_{\lambda \to \infty} \int_0^t S' (t - s) B_k (\Theta u_{y_n}) (s) ds \right\| C \to 0 \]

as \( n \to \infty \).
Consider the linear continuous operator
\[ \Gamma : L^1(J, E) \to C(J, E), \]
\[ v \mapsto \Gamma(v)(t) = \lim_{\lambda \to \infty} \int_0^t S'(t-s)B_\lambda v(s) \, ds. \]

From Lemma 2.1, it follows that \( \Gamma \circ S_F \) is a closed graph operator. Moreover, we have that
\[ h_n(t) - S'(t)[y_0 - g(y_n)] - \lim_{\lambda \to \infty} \int_0^t S'(t-s)B_\lambda(\Theta u y_n)(s) \, ds \in \Gamma(S_F, y_n). \]

Since \( y_n \to y^* \), it follows from Lemma 2.1 that
\[ h^*(t) - S'(t)[y_0 - g(y^*)] - \lim_{\lambda \to \infty} \int_0^t S'(t-s)B_\lambda(\Theta u y^*)(s) \, ds = \lim_{\lambda \to \infty} \int_0^t S'(t-s)B_\lambda v^*(s) \, ds \]
for some \( v^* \in S_F, y^* \).

As a consequence of Lemma 2.2 we deduce that \( N \) has a fixed point which gives rise to an integral solution of the problem (1)–(2) and therefore the system (1)–(2) is nonlocally controllable on \( J \).

In the previous theorem the assumption (H4) seems to be restrictive. In the next theorem we use a different approach, using again Bohnenblust–Karlin’s fixed point theorem.

**Theorem 3.2.** Assume that hypothesis (H1)–(H3) and (H5) hold. Also assume that the following hold:

(A1) \( y_0 \in \overline{D(A)}, g : C(J, \overline{D(A)}) \to \overline{D(A)} \) is completely continuous and
\[ \lim_{\| y \|_C \to \infty} \frac{|g(y)|}{\| y \|_C} = 0; \]

(A2) There exists a continuous function \( p : J \to \mathbb{R}^+ \) and a continuous nondecreasing function \( \psi : [0, \infty) \to [0, \infty) \) such that
\[ \forall r > 0, \sup \{ |F(t, y)| : |y| \leq r \} \leq p(t)\psi(|y|), \quad t \in J, \ y \in E, \]
and
\[ \lim_{r \to 0} \int_0^b e^{-aw} p(s)\psi(r) \, ds = 0. \]

Then the problem (1)–(2) is nonlocally controllable on \( J \).
Proof. Using hypothesis (H5) for an arbitrary function \( y(\cdot) \) and \( x_1 \in D(A) \) define the control and the operator \( N \) as in Theorem 3.1. For each positive integer \( n_0 \), let \( B_{n_0} = \{ y \in C : \| y \|_C \leq n_0 \} \). We have

Step 1. There exists a positive integer \( n_0 \geq 1 \) such that \( N(B_{n_0}) \subset B_{n_0} \).

Suppose that \( N(B_{n_0}) \not\subset B_{n_0} \). Then there exists \( y_n \in C, h_n \in N(y_n) \) such that \( \| y_n \|_C \leq n_0 \) and \( \| h_n \|_C > n \). Then we have for every \( n \geq 1 \) that

\[
h_n(t) = S'(t)(y_0 - g(y_n)) + \lim_{\lambda \to \infty} \int_0^t S'(t-s)B_\lambda[f_n(s) + (\Theta u_{y_n})(s)]ds
\]

for some \( f_n \in SF, y_n \). Then

\[
n < \| h \|_C \leq M^*(|y_0| + |g(y)|) + M^* \int_0^t e^{-\omega s}p(s)\psi(n)ds
\]

\[
+ M^*N^*M_1M_2b\left[|x_1| + |g(y)| + Me^{\omega b}(|y_0| + |g(y)|)\right]
\]

\[
+ Me^{\omega b}\int_0^b e^{-\omega s}p(s)\psi(n)ds.
\]

Thus

\[
1 < \frac{M^*}{n}(|y_0| + |g(y)|) + \frac{M^*}{n} \int_0^t e^{-\omega s}p(s)\psi(n)ds
\]

\[
+ \frac{1}{n}M^*N^*M_1M_2b\left[|x_1| + |g(y)| + Me^{\omega b}(|y_0| + |g(y)|)\right]
\]

\[
+ Me^{\omega b}\int_0^b e^{-\omega s}p(s)\psi(n)ds.
\]

Using (A1) and (A2) we conclude that \( 1 < 0 \) which is not true. Therefore there exists \( n_0 \in \mathbb{N} \) such that \( N(B_{n_0}) \subset B_{n_0} \).

The proofs of the other steps are similar to those in Theorem 3.1. Therefore we omit the details. □

By the help of the Schaefer’s fixed point theorem, combined with a selection theorem of Bressan and Colombo, for lower semicontinuous maps with decomposable values we shall present the second controllability result for the problem (1)–(2). Before this, let us introduce the following hypotheses which are assumed hereafter:

(B1) \( F : J \times E \to \mathcal{P}(E) \) is a nonempty compact valued multivalued map such that:

(a) \( (t, y) \mapsto F(t, y) \) is \( \mathcal{L} \otimes \mathcal{B} \) measurable;

(b) \( y \mapsto F(t, y) \) is lower semicontinuous for a.e. \( t \in J \).
For each $r > 0$, there exists a function $\varphi_r \in L^1(J, \mathbb{R}^+)$ such that
\[
\|F(t, y)\|_P := \sup \{ |v| : v \in F(t, y) \} \leq \varphi_r(t)
\]
for a.e. $t \in J$ and $y \in E$ with $|y| \leq r$.

In the proof of our theorem we will need the next auxiliary result.

**Lemma 3.1** [15]. Let $F : J \times E \to \mathcal{P}(E)$ be a multivalued map with nonempty, compact values. Assume (B1) and (B2) hold. Then $F$ is of l.s.c. type.

Next we state a selection theorem due to Bressan and Colombo.

**Theorem 3.3** [10]. Let $Y$ be separable metric space and let $N : Y \to \mathcal{P}(L^1(J, E))$ be a multivalued operator which is lower semicontinuous (l.s.c.) and has nonempty closed and decomposable values. Then $N$ has a continuous selection, i.e., there exists a continuous function ($\text{single-valued}$)
\[g : Y \to L^1(J, E)\]
such that $g(y) \in N(y)$ for every $y \in Y$.

**Theorem 3.4.** Suppose that hypotheses (H1), (H3)–(H6), (B1), (B2) hold. Then if
\[
\int_1^\infty \frac{ds}{s + \psi(s)} = \infty,
\]
the initial value problem (1)–(2) is nonlocally controllable on $J$.

**Proof.** Recall that $C := C(J, \overline{D(A)})$. (B1) and (B2) imply by Lemma 3.1 that $F$ is of lower semicontinuous type. Then from Theorem 3.3 there exists a continuous function $h : C \to L^1(J, E)$ such that $h(y) \in F(y)$ for all $y \in C$.

We consider the problem
\[
y'(t) = Ay(t) + h(y)(t) + (\Theta u)(t), \quad t \in J, \tag{4}
y(0) + g(y) = y_0. \tag{5}
\]
We remark that if $y \in C$ is a solution of the problem (4)–(5), then $y$ is a solution to the problem (1)–(2).

Transform the problem (4)–(5) into a fixed point problem by considering the operator $N_1 : C \to C$ defined by
\[
N_1(y) = S'(t)[y_0 - g(y)] + \frac{d}{dt} \int_0^t S(t-s)[h(y)(s) + (\Theta u_y)(s)]ds.
\]

**Step 1.** $N_1$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C$. Then
\[
\begin{align*}
|N_1(y_n)(t) - N_1(y)(t)| \\
\leq M^* |g(y_n) - g(y)| + M^* \int_0^t e^{-\alpha s} |h(y_n)(s) - h(y)(s)| \, ds \\
+ M^* N^* M_1 M_2 b \left[ (1 + M e^{\alpha b}) |g(y_n) - g(y)| \\
+ M e^{\alpha b} N^* \int_0^b |h(y_n)(s) - h(y)(s)| \, ds \right].
\end{align*}
\]

Since the functions \( h, g \) are continuous, then

\[ \|N_1(y_n) - N_1(y)\|_C \to 0 \quad \text{as} \quad n \to \infty. \]

Step 2. \( N_1 \) maps bounded sets into bounded sets in \( C \).

Indeed, it is enough to show that for any \( q > 0 \) there exists a positive constant \( \ell \) such that for each \( y \in B_q = \{ y \in C: \|y\|_C \leq q \} \) we have \( \|N(y)\|_C \leq \ell \). For each \( t \in J \) we have that

\[
\begin{align*}
|N_1(y)(t)| &= \left| S'(t)(y_0 - g(y)) + \frac{d}{dt} \int_0^t S(t-s)(h(y)(s) + (\Theta u_y)(s)) \, ds \right| \\
&\leq M^* (|y_0| + L) + M^* N^* \|\phi_q\|_{L^1} + M^* N^* b \alpha \alpha,
\end{align*}
\]

where

\[ c^* = M_1 M_2 \left( |x_1| + L + M e^{\alpha b} (|y_0| + L) + M N^* e^{\alpha b} \|\phi_q\|_{L^1} \right). \]

Thus

\[ \|N_1(y)\|_C \leq M^* (|y_0| + L) + M^* N^* \|\phi_q\|_{L^1} + M^* N^* b \alpha := \ell. \]

Step 3. \( N_1 \) maps bounded sets into equicontinuous sets of \( C \).

Let \( 0 < t_1 < t_2 \in J, \; t_1 < t_2 \) and \( B_q \) be a bounded set of \( C \) as in Step 2. Let \( y \in B_q \); then for each \( t \in J \) we have

\[
\begin{align*}
|N_1(y)(t_2) - N_1(y)(t_1)| \\
\leq \left| \left[ S'(t_2) - S'(t_1) \right] (y_0 - g(y)) \right| + \lim_{\lambda \to \infty} \int_0^{t_2} \left( S'(t_2 - s) - S'(t_1 - s) \right) B_\lambda h(y)(s) \, ds \\
+ \lim_{\lambda \to \infty} \int_0^{t_2} \left( S'(t_2 - s) - S'(t_1 - s) \right) B_\lambda (\Theta u_y)(s) \, ds \\
+ M e^{\alpha t_2} \int_{t_1}^{t_2} e^{-\alpha s} \left| \phi_q(s) + c^* \right| \, ds.
\end{align*}
\]
The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, since $S'(t)$ is strongly continuous and the compactness of $S'(t)$, $t > 0$, implies the continuity in the uniform operator topology.

As a consequence of Steps 1–3 and (H3), (H4) together with the Arzelá–Ascoli theorem we can conclude that $N_1 : C \to C$ is a completely continuous operator.

Step 4. The set

$$E(N_1) := \{y \in C : y = \sigma N_1(y) \text{ for some } 0 < \sigma < 1\}$$

is bounded.

Let $y \in E(N_1)$. Then $y = \sigma N_1(y)$ for some $0 < \sigma < 1$. Thus for each $t \in J$,

$$y(t) = \sigma \left( S'(t)(y_0 - g(y)) + \frac{d}{dt} \int_0^t S(t-s)[h(y)(s) + (\Theta uy)(s)] ds \right).$$

This implies by (H4)–(H6) that for each $t \in J$ we have

$$e^{-\omega t} |y(t)| \leq M(|y_0| + L) + M \int_0^t e^{-\omega s} p(s) \psi \left( |y(s)| \right) ds$$

$$+ M N^* M_1 M_2 b \left[ |x_1| + L + M e^{\omega b} (|y_0| + L) \right]$$

$$+ M e^{\omega b} \int_0^b e^{-\omega s} p(s) \psi \left( |y(s)| \right) ds.$$ 

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$v(t) \leq e^{\omega t} v(t) \quad \text{for all } t \in J$$

and

$$v(0) = M(|y_0| + L) + M N^* M_1 M_2 b \left[ |x_1| + L + M e^{\omega b} (|y_0| + L) \right]$$

$$+ M e^{\omega b} \int_0^b e^{-\omega s} p(s) \psi \left( |y(s)| \right) ds.$$ 

Then for each $t \in J$ we have

$$v'(t) = Me^{-\omega t} p(t) \psi \left( |y(t)| \right) \leq Me^{-\omega t} p(t) \psi \left( e^{\omega t} v(t) \right), \quad t \in J.$$ 

Then for each $t \in J$ we have

$$\left( e^{\omega t} v(t) \right)' = \omega e^{\omega t} v(t) + v'(t) e^{\omega t} \leq \omega e^{\omega t} v(t) + M p(t) \psi \left( e^{\omega t} v(t) \right)$$

$$\leq \max \{ \omega, M p(t) \} \left[ e^{\omega t} v(t) + \psi \left( e^{\omega t} v(t) \right) \right], \quad t \in J.$$ 

Thus

$$\int_{v(0)}^{e^{\omega t} v(t)} \frac{du}{u + \psi(u)} \leq \int_0^b \max \{ \omega, M p(t) \} ds < \infty.$$
Consequently, there exists a constant $d$ such that $v(t) \leq d$, $t \in J$, and hence $\|y\|_{C} \leq d$ where $d$ depends only on the constant $M, \omega$ and the functions $p$ and $\psi$. This shows that $\mathcal{E}(N_1)$ is bounded.

As a consequence of Schaefer’s fixed point theorem [20] we deduce that $N_1$ has a fixed point $y$ and therefore the problem (1)–(2) is nonlocally controllable on $J$.  

4. A special case

In this section, we suppose that the nonlocal condition is given by

$$y(0) + \sum_{k=1}^{p} c_k y(t_k) = y_0,$$  

where $c_k, k = 1, \ldots, p$, are nonnegative constants.

**Definition 4.1.** The nonlocal problem (1)–(6) is said to be nonlocally controllable on the interval $J$, if for every $x_1 \in E$, there exists a control $u \in L^2(J, U)$, such that the solution $t \mapsto y(t)$ of (1)–(6) satisfies $y(b) + \sum_{k=1}^{p} c_k y(t_k) = x_1$.

**Lemma 4.1.** Assume that

(H7) There exists a bounded operator $D : E \to E$ such that

$$D = \left( I + \sum_{k=1}^{m+1} c_k S'(\eta_k) \right)^{-1}.$$

If $y$ is an integral solution of (1)–(6) then it is given by

$$y(t) = S'(t)D \left[ y_0 - \lim_{\lambda \to \infty} \sum_{k=1}^{p} \frac{\eta_k}{\lambda} S'(t_k - s)B_k \left[ f(s) + (\Theta uy)(s) \right] \right]$$

$$+ \frac{d}{dt} \int_{0}^{t} S(t-s) \left[ f(s) + (\Theta uy)(s) \right] ds, \quad t \in J.$$

**Proof.** Let $y$ be a solution of the problem (1)–(6). Define $w(s) = S(t - s)y(s)$. Then we have

$$w'(s) = -S'(t-s)y(s) + S(t-s)y'(s)$$

$$= -AS(t-s)y(s) - y(s) + S(t-s)y'(s)$$

$$= S(t-s)\left[ y'(s) - Ay(s) \right] - y(s)$$

$$= S(t-s)\left[ f(s) + (\Theta uy)(s) \right] - y(s).$$  

(7)
By integrating the previous equation we have
\[
\int_0^t w'(s) \, ds = \int_0^t S(t-s) \left[ f(s) + (\Theta u_y)(s) \right] \, ds - \int_0^t y(s) \, ds,
\]
\[
w(t) - w(0) = \int_0^t S(t-s) \left[ f(s) + (\Theta u_y)(s) \right] \, ds - \int_0^t y(s) \, ds,
\]
or
\[
\int_0^t y(s) = S(t)y(0) + \int_0^t S(t-s) \left[ f(s) + (\Theta u_y)(s) \right] \, ds.
\]

By differentiating the above equation we have that
\[
y(t) = S'(t)y(0) + \frac{d}{dt} \int_0^t S(t-s) \left[ f(s) + (\Theta u_y)(s) \right] \, ds.
\] (8)

In order to find \( y(t_k) \), we need to integrate Eq. (7) from 0 to \( t_k \). So, we have that
\[
\int_0^{t_k} w'(s) \, ds = \int_0^{t_k} S(t-s) \left[ f(s) + (\Theta u_y)(s) \right] \, ds - \int_0^{t_k} y(s) \, ds,
\]
\[
w(t_k) - w(0) = \int_0^{t_k} S(t-s) \left[ f(s) + (\Theta u_y)(s) \right] \, ds - \int_0^{t_k} y(s) \, ds,
\]
or
\[
S(t - t_k)y(t_k) - S(t)y(0) = \int_0^{t_k} S'(t-s)B_\lambda \left[ f(s) + (\Theta u_y)(s) \right] \, ds - \int_0^{t_k} y(s) \, ds.
\]

By differentiating the above equation we have that
\[
S'(t - t_k)y(t_k) - S'(t)y(0) = \lim_{\lambda \to \infty} \int_0^{t_k} S'(t-s)B_\lambda \left[ f(s) + (\Theta u_y)(s) \right] \, ds.
\] (9)

From this we conclude that
\[
y(t_k) - S'(t_k)y(0) = \lim_{\lambda \to \infty} \int_0^{t_k} S'(t-s)B_\lambda \left[ f(s) + (\Theta u_y)(s) \right] \, ds.
\] (10)
Then from Eqs. (8) and (10) we conclude that

\[ y(t) = S'(t)D \left[ y_0 - \lim_{\lambda \to \infty} \sum_{k=1}^{p} c_k \int_{0}^{t} S'(t_k - s) B_\lambda \left[ f(s) + (\Theta u_y)(s) \right] ds \right] \\
+ \frac{d}{dt} \int_{0}^{t} S(t - s) \left[ f(s) + (\Theta u_y)(s) \right] ds, \quad t \in J. \]

**Theorem 4.1.** Assume that hypothesis (H1)–(H3), (H5)–(H7) hold. Also assume that

(H8) \( y_0 \in \overline{D(A)} \) and the set \( \{ y_0 - \sum_{k=1}^{m+1} c_k y(\eta_k) \} \) is relatively compact.

Then the problem (1)–(6) is nonlocally controllable on \( J \).

**Proof.** Using hypothesis (H5) and (H7) for an arbitrary function \( y(\cdot) \) and \( x_1 \in \overline{D(A)} \) define the control

\[ u_y(t) = W^{-1} \left[ x_1 - \sum_{k=1}^{p} c_k y(\eta_k) - S'(b)Dy_0 \right. \]
\[ + S'(b) \lim_{\lambda \to \infty} \sum_{k=1}^{p} c_k D \int_{0}^{b} S'(t_k - s) B_\lambda f(s) ds \]
\[ - \lim_{\lambda \to \infty} \int_{0}^{b} S'(b - s) B_\lambda f(s) ds \left] \right(t), \]

where \( f \in SF_y \).

Consider the operator \( N_2 : C \to P(C) \) defined by

\[ N_2(y) = \left\{ h \in C: h(t) = S'(t)D \left[ y_0 - \lim_{\lambda \to \infty} \sum_{k=1}^{p} c_k \int_{0}^{t} S'(t_k - s) B_\lambda \left[ f(s) + (\Theta u)(s) \right] ds \right] + \frac{d}{dt} \int_{0}^{t} S(t - s) \left[ f(s) + (\Theta u)(s) \right] ds, \quad f \in SF_y, \right\} t \in J. \]

We need to show that \( N_2 \) has a fixed point. We omit the proof since it is similar to that given in Theorem 3.1, with obvious modifications. \( \square \)

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