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A Note on Rings of Endomorphisms

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1. In this note we present some methods for studying the behavior of regular elements in a ring with suitable chain conditions and an application to rings of endomorphisms.

In each case, a one-sided chain condition is supposed to hold on the right. For instance, stating that R is a Goldie ring, means that in the ring R direct sums of right ideals and right annihilators are each subject to the maximum condition. If S is a subset of R then $r_R(S)$, or briefly $r(S)$, denotes the right annihilator of S in R. Again $1_R(S)$, or $1(S)$, are used for the left annihilator. We shall use the existence of the maximal nilpotent ideal $N(R)$ in a Goldie ring R, which was established in [2].

2. THEOREM 1. *Let R be a Goldie ring, let $c \in R$ and $r(c) = 0$. Then $r(c + n) = 0$ for all $n \in N(R)$.*

Proof. Let $N = N(R)$ have exponent k, so that $N^k = 0$, $N^{k-1} \neq 0$. The chain

$$r(N^{k-1}c) \subset r(N^{k-1}c^2) \subset \cdots$$

must stop at $r(N^{k-1}c^{n_1}) = r(N^{k-1}c^{n_1+1}) = \cdots$, say.

Let $U = r(c + n)$, then

$$N^{k-1}(c + n)U = 0 = N^{k-1}cU,$$

and this implies that $N^{k-1}(U \cap c^{n_1}R) = 0$.

This is the first stage in a process of lowering the index of N at the expense of the identification of U. At the general stage the process gives

$$N^{k-r}(U \cap c^{n_1}R \cap c^{n_2}R \cap \cdots \cap c^{n_r}R) = 0$$

for $r = 1, 2,\ldots, k$.

At the last stage we have $U \cap c^{n_1}R \cap c^{n_2}R \cap \cdots \cap c^{n_k}R = 0$. Now the elements $c^{n_1},\ldots, c^{n_k}$ have zero right annihilators and thus generate right ideals which are essential in the module R_R. It follows that $U = 0$, so that $r(c + n) = 0$.

This theorem is illuminated by the concept of a *transfer ideal.* Let $\mathscr{C}$ be a subset of R, which is closed under multiplication. An ideal T of R is a *transfer ideal* of $\mathscr{C}$ if, whenever $c \in \mathscr{C}$ and $t \in T$ then $c + t \in \mathscr{C}$. When R has the maximum condition for ideals, the sum of the transfer ideals of $\mathscr{C}$ is a finite sum and is itself a transfer ideal of $\mathscr{C}$. Thus the notion of *maximal transfer ideal* of $\mathscr{C}$ is realized. Theorem 1 states that N is a transfer ideal for the set of half-regular elements. There may be others, because in the case of a commutative noetherian ring, the maximal transfer ideal of the set of regular elements ($\mathscr{C}$ in this instance) is the intersection of the maximal primes of zero and is usually larger than the nilpotent radical. Again when R is an arbitrary ring with unity and $\mathscr{C}$ is the group of units of R the maximal transfer ideal is the Jacobson radical which again may be much larger than any nilpotent ideal. In these examples, the maximal transfer ideal is an intersection of prime ideals and it is tempting to conjecture that this is always true under reasonable assumptions.

3. In order to apply this theorem to rings of endomorphisms we need an embedding into Goldie rings, which is carried out as follows.

Let R be a noetherian ring with unit element, M_R a finitely generated R-module and $S = \mathrm{End}_R M$. Taking elements of S to act on the left of M, this becomes an S-R-module which is S-faithful. Construct the triangular matrix ring

$$T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}.$$

LEMMA 1. *T is a Goldie ring.*

Proof. A right ideal I of T may be written in the form $\hat{I} + V$, where $\hat{I}$ is a right ideal of S and V is an R-submodule of $M \oplus R$, such that $IM \subset V$. Note that an object I of this form is in any case a right ideal of T. We examine left annihilators of T, prove that these satisfy the minimum condition and deduce that the right annihilators have the maximum condition.

Let $L = 1(I)$, then $1(I) = 1(\hat{I}) \cap 1(V)$. Now S satisfies the minimum condition for left annihilators (see[2]), hence there is a finite set $k_1, \ldots, k_t \in \hat{I}$ such that

$$1_S(\hat{I}) = 1_S\{k_1, \ldots, k_t\},$$

where 1_S denotes of course an annihilator taken in the ring S. Moreover,

$$1_T(\hat{I}) = 1_T\left\{\begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} k_t & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

As V is an R-submodule of $M \oplus R$ it is finitely generated, say

$$V = u_1 T + \cdots + u_n T.$$

Hence

$$1_T(I) = 1_T\left\{\begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} k_t & 0 \\ 0 & 0 \end{pmatrix}, u_1, \ldots, u_n\right\},$$

which shows that $1_T(I)$ is the annihilator of a finite set of elements of I and proves the minimum condition for left annihilators.

Now suppose that the right ideals $I_n = \hat{I}_n + V_n$, $n = 1, 2, 3, \ldots$ form an infinite direct sum of right ideals of T. Now

$$I_n\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \hat{I}_n M \\ 0 & 0 \end{pmatrix}$$

and $\hat{I}_n M$ is an R-submodule of M for $n = 1, 2, \ldots$. Since $I_n\left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right) \subset I_n$, the sum of these objects for $n = 1, 2, \ldots$ is also direct and hence so is the sum of the $\hat{I}_n M$. Hence $\hat{I}_n M = 0$ for large enough n and as M is a faithful S-module we obtain $\hat{I}_n = 0$ for large enough n. This leaves us with a direct sum of submodules V_n, which lies in the noetherian R-module $M \oplus R$. Hence $V_n = 0$ and $I_n = 0$ for large enough n. Hence T is a Goldie ring.

Examination of the result gives the following theorem.

THEOREM 2. *Let R be a noetherian ring, M_R a finitely generated R-module and $S = \mathrm{End}_R M$. Then $S = eTe$, where T is a Goldie ring and e is an idempotent in T.*

In passing we note that S need not be a Goldie ring as may be seen from the work of Björk [1].

We obtain also a generalisation of Levitski's theorem.

THEOREM 3. *Let R be a noetherian ring, M_R a finitely generated R-module and $S = \mathrm{End}_R M$. Then the nil subrings of S are nilpotent.*

It will be shown elsewhere that Theorem 3 can be generalised to endomorphism rings of noetherian modules over arbitrary rings.

Theorem 1 and the embedding theorem are brought together as follows.

THEOREM 4. *Let R, M, S be given as in Theorem 2. If $\phi \in S$ with* $\ker \phi = 0$ *and $\psi \in N(S)$ then* $\ker(\phi + \psi) = 0$.

Proof. The element $\begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix} \in T$ has zero right annihilator and $\begin{pmatrix} \psi & 0 \\ 0 & 0 \end{pmatrix} \in N(T)$. Thus by Theorem 1 we have

$$r_T \left\{ \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \psi & 0 \\ 0 & 0 \end{pmatrix} \right\} = 0,$$

and hence $\ker(\phi + \psi) = 0$.

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