JOURNAL OF FUNCTIONAL ANALYSIS 57, 49–55 (1984)

Factoriality of Representations of the Group of Paths on $SU(n)$

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Communicated by Paul Malliavin

Received November 1983

Factoriality in the cyclic component of the vacuum for the energy representation of $SU(n)$-valued paths groups is proved. The main tool is a lemma concerning generic pairs of Cartan subalgebras in the Lie algebra $su(n)$ of $SU(n)$ groups.

1. INTRODUCTION

The energy representation of the Sobolev–Lie group of mappings from a manifold I into a compact semisimple Lie group G has been studied in a series of papers [1–4, 6, 7]. In particular the irreducibility has been shown in [6, 4, 2], for the cases where the manifold I has dimension $d \geqslant 5$, $d \geqslant 4$, and $d \geqslant 3$, respectively. The case $d = 2$ has been discussed in [2] and results about irreducibility (resp. reducibility) have been proven, according to the lengths of the roots involved. The case $d = 1$, in which the representation is reducible, has been studied in [1–3]. The basic observation in this case is that the representation (in the cyclic component of the vacuum) can be

* This work has been partially supported by the Deutsche Forschungsgemeinschaft and the Mathematical Seminar, Oslo.

0022-1236/84 $3.00

realized as the one given by left translations on the sample paths of Brownian motion with values in the Lie group G. This has led, in a recent paper [3], to the determination of irreducible components of the representation. Let us now briefly recall the situation with which we are confronted in the case $d = 1$, more precisely, in the case where I is either the real numbers $\mathbb{R}$ or the nonnegative real numbers $\mathbb{R}_+$ or the unit circle S^1 in $\mathbb{R}^2$. Let G^I be the group of C^∞ mappings from I into G, with pointwise multiplication and equal to the unit in G outside compacts. We call G^I the *G-valued path group*. Let $C(I, G)$ be the set of continuous mappings from I to G equipped with the Brownian motion measure μ_0, that is, the measure μ_0 giving the distribution of the standard Brownian motion on G; μ_0 is quasiinvariant with respect to left and right translations by elements of G^I. Let $\mathscr{H} = L^2(C(I, G), \mu_0)$, then we have a unitary representation U^L of G^I on the Hilbert space $\mathscr{H}$ given by

$$(U^L(\psi)f)(\eta) \equiv \left(\frac{d\mu_0(\psi^{-1}\eta)}{d\mu_0(\eta)}\right)^{1/2} f(\psi^{-1}\eta),$$

with $\psi \in G^I$, $\eta \in C(I, G)$, $f \in \mathscr{H}$.

This left-invariant representation is easily seen [1] to be unitarily equivalent to the cyclic component of the vacuum in the energy representation. The left-invariant representation U^{L} is unitary equivalent to the right-invariant representation U^{R} in $\mathscr{H}$,

$$(U^{\mathrm{R}}(\psi)f)(\eta) \equiv \left(\frac{d\mu_0(\eta\psi)}{d\mu_0(\eta)}\right)^{1/2} f(\eta\psi).$$

In [3] we proved in the case $G = SU(2)$ the factoriality of both representations U^{L}, U^{R}. We also proved that the factors generated by these representations are the commutant of each other. The method used for the proof of these results is the following:

We start with a diagonalization of the Abelian von Neumann algebra generated by $U^{\mathrm{R}}(T_{\mathrm{R}}^I)$, where T_{R}^I is the group of smooth compactly supported mappings from I to some maximal torus T_{R} in G. Since U^{L} commutes with U^{R} we then get a direct integral decomposition of U^{L}. In fact, we showed in [3] that

$$U^{\mathrm{L}} = \int^{\oplus} U^{(\alpha)} d\mu_{T_{\mathrm{R}}}(\alpha),$$

acting on

$$\mathscr{H} = \int^{\oplus} \mathscr{H}^{(\alpha)} d\mu_{T_{\mathrm{R}}}(\alpha),$$

where $\mu_{T_{\mathrm{R}}}$ is the Brownian motion measure on T_{R}, the integrals being over all values of α in $C(I, T_R)$. $\mathscr{H}^{(\alpha)}$ can be realized, for any α, as the space $L^2(C(I, G/T_R), \mu_1)$, with μ_1 the image of μ_0 by the canonical surjection $G \to G/T_R$, that is, μ_1 is the Brownian motion measure in the homogeneous space G/T_R. $U^{(\alpha)}$ is defined by

$$(U^{(\alpha)}f) \equiv \left(\frac{d\mu_1(\psi^{-1}\xi)}{d\mu_1(\xi)}\right)^{1/2} f(\psi^{-1}\xi) \exp i\langle \alpha^{-1}d\alpha, \phi^{-1}\psi^{-1}d\psi\phi\rangle,$$

where $f \in \mathscr{H}^{(\alpha)}$, $\xi \in C(I, G/T_R)$, ϕ is any element of ξ, and $\langle\, ,\rangle$ denotes the Killing form in the Lie algebra g of G. Remark that this representation $U^{(\alpha)}$ was denoted in [3] by $U^{\alpha^{-1}d\alpha}$. It is also proven in [3] that the representations $U^{(\alpha)}$ are irreducible in the case $G = SU(2)$ for almost all α with respect to the Brownian motion measure on $C(I, T_R)$. The restriction to $G = SU(2)$ in [3] was introduced in [3, Lemma 3.3], a lemma which states that the function $\phi \to \langle \delta, \phi^{-1}\psi^{-1}d\psi\phi\rangle$, where $\psi \in T_R^I$ and δ is a g-valued L^2-function on I, separates the points of $C(I, T_L \backslash G/T_R)$, with T_L, T_R maximal tori of G.

In this paper we prove a corresponding result for $G = SU(n)$ for all $n \geqslant 2$, with the consequence that all results of [3] can now be extended from $SU(2)$ to $SU(n)$. This is the natural boundary of the method (see the remark below following lemma 2.1).

2. A Lemma on the Lie Algebra of $SU(n)$

Let $su(n)$ be the Lie algebra of $SU(n)$, $n \geqslant 2$. We shall prove a lemma on $su(n)$ which gives a natural (and apparently new) extension of the corresponding result for $n = 2$ which can be reduced to the fact that given a vector x in $\mathbb{R}^3$, then for a generic set of vectors y in $\mathbb{R}^3$, not multiples of x, the vectors x, y and their exterior product form a basis of $\mathbb{R}^3$.

Lemma 2.1. *Let t_1 be a Cartan subalgebra of $su(n)$. For a generic set of Cartan subalgebras t_2 in $su(n)$ (generic being meant in a sense explained below (condition $G(t_1)$)) the subspaces t_1, t_2 and $[t_1, t_2] \equiv \{[u, v], u \in t_1, v \in t_2\}$ generate $su(n)$.*

Remark. One easily sees by direct inspection (see, e.g., [5]) that in the case of compact semisimple groups which are not $SU(n)$ the dimension of tori is not sufficiently big in order to expect that a corresponding lemma can be valid in such cases.

Let us now explain what is meant by "generic set" in the Lemma 2.1. We shall use the well-known fact that two Cartan subalgebras are conjugate via the adjoint representation of $SU(n)$ in $su(n)$. For a given Cartan subalgebra t_1, the set of Cartan subalgebras in $SU(n)$ can be identified with the quotient of $SU(n)$ by the normalizer $N(t_1)$ of t_1. We shall now express the condition of genericity of $SU(n)$ in an invariant way with respect to this quotient.

DEFINITION 2.2. Let us express any $u \in SU(n)$ in the basis in which t_1 is represented by diagonal $n \times n$ matrices (u_{ij}) $i, j = 1,..., n$. We say that u *satisfies the genericity condition* $G(t_1)$ if for any $A \subset \{1,..., n\}$, $A \neq \{1,..., n\}$ the vectors u_A^k, with coordinates $\{u_{ik}, i \in A\}$ are not orthogonal. Let now $\mathcal{O}(t_1)$ be the set of all elements of $SU(n)$ with the property $G(t_1)$. We shall say that a *Cartan subalgebra* t_2 *in* $su(n)$ *satisfies the genericity condition* $G(t_1)$ if $t_2 = ut_1u^*$ with u satisfying $G(t_1)$.

We remark that $\mathcal{O}(t_1)$ is a connected open subset of $SU(n)$, in fact, it is the complement in $SU(n)$ of a finite number of planes in the set of $n \times n$ complex matrices.

Proof of Lemma 2.1. Since t_i, $i = 1, 2$, are orthogonal to the linear space $[t_1, t_2]$ (with respect to the Killing form $\langle\ ,\ \rangle$) it is sufficient to show $t_1 \cap t_2 = \{0\}$ and $\dim[t_1, t_2] = (n-1)^2$. Let us first prove that $\dim[t_1, t_2] = (n-1)^2$. Since λ, $\mu \to [\lambda, \mu]$ is bilinear on $t_1 \times t_2$, it will be sufficient to prove that the conditions $\lambda \in t_1$, $\mu \in t_2$, $x \in SU(n)$, and

$$\operatorname{Tr}([\lambda, \mu]x) = 0 \tag{2.1}$$

imply $\lambda = 0$ or $\mu = 0$; t_1 will be assumed to be the set of diagonal matrices. Let λ_k be the diagonal matrix elements of λ; μ will be of the form uvu^* for some u satisfying $G(t_1)$ and v a diagonal matrix with elements v_k, $k = 1,..., n$. One has $\sum_{k=1}^n \lambda_k = \sum_{k=1}^n v_k = 0$. Assume $\lambda \neq 0$, then, if necessary, by reordering coordinates, one can assume that there exists a q, $1 \leqslant q < n$, such that $\lambda_1 = \lambda_2 = \cdots = \lambda_q \neq \lambda_j$, $j > q$.

Applying (2.1) to the case where x is the root vector e_{ij} one gets for all $i \leqslant q$ and all $j > q$,

$$\sum_{k=1}^n u_{ik} \bar{u}_{jk} v_k = 0. \tag{2.2}$$

Multiplying (2.2) by u_{jl} and summing with respect to the index j one gets

$$\sum_{j=1}^n u_{jl} \sum_{k=1}^n u_{ik} \bar{u}_{jk} v_k = 0. \tag{2.3}$$

By the unitarity of u and (2.2) we get from (2.3),

$$\sum_{j=1}^n \alpha_{ij} u_{jl} = \mu_l u_{il}, \tag{2.4}$$

with

$$\alpha_{ij} = \sum_{k=1}^{n} u_{ik} \bar{u}_{jk} v_k. \tag{2.5}$$

Using the notations introduced in Definition 2.2, we have that the vectors $u_l^{\{1,\ldots,q\}}$ are eigenvectors of the self-adjoint matrix $((\alpha_{ij}))$, $i, j = 1,\ldots, q$. (Remark that $G(t_1)$ entails $u_{ij} \neq 0$ for all $i, j = 1,\ldots, n$.) If we now assume that μ_l is not constant we get a contradiction with the genericity property $G(t_1)$. By the above, the fact that μ_l is constant implies that $\dim[t_1, t_2] = (n-1)^2$.

To see that $t_1 \cap t_2 = \{0\}$ it suffices to remark that if v and uvu^* are diagonal we get (2.2) for $q = 1$ and proceed in the same way as above. This completes the proof of Lemma 2.1. ∎

Remarks. (i) The condition $G(t_1)$ for t_2 is, in general, strictly more restrictive than $t_1 \cap t_2 = \{0\}$. In fact, for $su(4)$, taking for t_1 the set of all self-adjoint diagonal matrices in $\mathbb{C}^4$, $t_2 = ut_1u^*$, and $u = U_1 \otimes U_2 \in SU(2) \otimes SU(2) \subset SU(4)$, where U_1 and U_2 have no zero matrix elements, we get $t_1 \cap t_2 = \{0\}$, but t_2 has not the property $G(t_1)$.

(ii) In the case of $SU(2)$, the lemma as noticed before is trivial even in a stronger version, the complement $\mathcal{O}(t_1)'$ of $\mathcal{O}(t_1)$ in $SU(n)$ being replaced by the torus generated by t_1. For $n \geq 3$, $\mathcal{O}(t_1)'$ has codimension $n^2 - 1 - 2$. So in any case we can assume the lemma to be valid for the complement of a set having at least codimension 2.

3. Applications to the Factoriality of the Energy Representation of the Paths Groups with Values in $SU(n)$

The lemma proven in Section 2 permits us to improve the result of [3] from the case of $SU(2)$ to the case of all $SU(n)$, $n \geq 2$. Let us denote by μ_0 (resp. μ) the Brownian motion measure on G (resp. on the double coset space $T_1 \backslash G / T_2$, where T_1 and T_2 are maximal tori in G); μ_0 is supported by $C(I, G)$ and μ is supported by $C(I, T_1 \backslash G / T_2)$; μ is the image of μ_0 by the canonical mapping $G \to T_1 \backslash G / T_2$. The following result extends Lemma 3.3 of [3] to the case of arbitrary $G = SU(n)$.

Lemma 3.1. *Let T_1, T_2 be maximal tori in $G = SU(n)$, $n \geq 2$. Then there exists a conul set $\mathcal{N}$ in $C(I, T_1 \backslash G / T_2)$ with respect to the Brownian motion measure μ such that the functions*

$$\xi \to \langle \phi^{-1} \alpha^{-1} \, d\alpha \phi, \beta^{-1} \, d\beta \rangle,$$

where ϕ is any representative of $\xi \in C(I, T_1\backslash G/T_2)$ and α, β are continuous functions with L^2-logarithmic derivatives and values in T_1 (resp. T_2) such that $\alpha(0) = \beta(0) = e$, e being the unit in T_1 (resp. T_2), generate the σ-algebra of Borel sets in $\mathcal{N}$.

Proof. Let u_0 be such that $t_2 = u_0 t_1 u_0^*$ and $\mathcal{N}_0 \subset C(I, G)$ the set of paths ξ such that $\xi(t) \in u_0^* \mathcal{O}(t_1)$, with $\mathcal{O}(t_1)$ defined as in Section 2. Since the complement $\mathcal{O}(t_1)'$ of the set $\mathcal{O}(t_1)$ in $SU(n)$ is the intersection of $SU(n)$ with a finite number of planes, one has that $\mathcal{N}_0$ has full μ_0-measure in $C(I, G)$. Actually $\xi(0) = e \neq u_0^* \mathcal{O}(t_1)$. For any $n \in \mathcal{N}$ $n \geqslant 0$ the probability that for some $t \geqslant n$, $\xi(t) \in u_0^* \mathcal{O}(t_1)'$ can we written as the greatest lower bound of a countable set of probabilities of events "$\exists t$, $t \leqslant n$, $\xi(t) \in A$" for some sets A's sufficiently small (the A's are neighborhoods of sets which are in a countable partition of $u_0^* \mathcal{O}(t_1)'$). This shows that one can compute these probabilities using charts for $SU(n)$, independence of increments of each coordinate for the ordinary Brownian motion in $\mathbb{R}^{n^2-1}$, and by [8, Chap. 1, Propositions 2–5)] one has only to see that the codimension of $\mathcal{O}(t_1)'$ is at least 2, but this is the content of Remark (ii) at the end of Section 2.

The image $\mathcal{N}$ of $\mathcal{N}_0$ in $C(I, T_1\backslash G/T_2)$ is of full μ-measure. The functions on $u_0^* \mathcal{O}(t_1)$ defined by $g \to (g^{-1} t_1 g, t_2)$ have as derivatives functions of the type $x \to \langle [t_1, x], t_2 \rangle$. On the other hand, the same functions are constant on each double coset, which is a submanifold of dimension $(n-1)^2$. Lemma 2.1 asserts that these derivatives generate an $(n-1)^2$ subspace and the proposition is then proven. ∎

Proceeding now in exactly the same way as in [3], using the Lemma 3.1 instead of Lemma 3.3 of [3], we arrive at the extensions to $SU(n)$ of all results established for $SU(2)$ in [3]. In order to state these results, let us denote by $H(I, G)$ the Sobolev–Lie group obtained by closing G^I in the natural metric $|f^{-1}f'|$, with $|g|^2 \equiv (g, g) \equiv \int_I (dg(x), dg(x)) \rho(x)\, dx$, $\rho\, dx$ being the volume measure on I and $(dg(x), dg(x))$ being the natural scalar product.

THEOREM 3.2. *Let $G = SU(n)$, $n \geqslant 2$. The representation of $H(I, T_L \times T_R)$ in $L^2(C(I, G))$ given by $(\phi, \psi) \in H(I, T_L \times T_R) \to U^L(\phi)\, U^R(\psi)$ has simple spectrum. The representation space is $L^2(\mathcal{M}[I, t_L \times t_R], \nu_0)$, with $\mathcal{M}[I, t_L \times t_R]$ the measures on I with values in $t_L \times t_R$, and ν_0 the measure with Fourier transform*

$$\int e^{i(f, \psi_L)} e^{i(g, \psi_R)}\, d\nu_0(\psi_L, \psi_R)$$

$$= e^{-|f|^2/2} e^{-|g|^2/2} \int e^{-(f, \eta g \eta^{-1})}\, d\mu_0(\eta).$$

For almost all $\alpha \in C(I, T_L)$, $\alpha(0) = e$, with respect to μ_{T_R}, $U^{(\alpha_0)}$, where α_0 is a constant path, is in the von Neumann algebra generated by $U^{(\alpha)}(H(I, G))$; $U^{(\alpha)}(H(I, G))''$ contains all operators of multiplication by functions of ξ, $\xi \in C(I, G\backslash T_R)$.

The restriction of U^R to $H(I, T_R)$ has double commutant $U^R(H(I, T_R))''$ which is maximal Abelian in $U^R(H(I, G))''$; $U^L(H(I, T_L))''$ is maximal Abelian in $U^L(H(I, G))''$; $U^{(\alpha)}$ are irreducible for μ_{T_R}—almost all α; $(U^L)' = U^R$, $(U^R)' = U^L$, and U^L, U^R are factor representations. If $I = \mathbb{R}$, then the von Neumann algebras $U^L(H(I, G))''$ and $U^R(H(I, G))''$ generated by U^L, U^R are asymptotic Abelian factors of type III.

Acknowledgments

The first-named author would like to thank Prof. Mohammed Mebkhout and Prof. Ludwig Streit, the Université d'Aix-Marseille II, the Centre de Physique Théorique, CNRS, Marseille, ZiF, University of Bielefeld, and the Mathematics Department of Oslo University, for kind invitations which made this work possible. Part of this work was achieved during the stay of the third-named author at the Zentrum für interdisziplinäre Forschung, University of Bielefeld; he wants to express his thanks, in particular to Prof. Ludwig Streit, for his kind invitation and to all his colleagues in the Research Group for helpful discussions and the stimulating atmosphere they created at the ZiF.

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