Lacunary ideal convergence of multiple sequences

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Abstract An ideal $I$ is a family of subsets of $\mathbb{N} \times \mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In this article, the concept of lacunary ideal convergence of double sequences has been introduced. Also the relation between lacunary ideal convergent and lacunary Cauchy double sequences has been established. Furthermore, the notions of lacunary ideal limit point and lacunary ideal cluster points have been introduced and find the relation between these two notions. Finally, we have studied the properties such as solidity, monotonic.

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1. Introduction

The notion of statistical convergence for sequences of real numbers has been introduced by Steinhaus [1] and Fast [2] independently. Mursaleen and Edely [3] extended the above idea from single to double sequences of scalars and established relations between statistical convergence and strongly Cesáro summable double sequences. The notion of $I$-convergence was studied at initial stage by Kostyrko et al. [4]. Kostyrko et al. [5] gave some of basic properties of $I$-convergence and dealt with extremal $I$-limit points. Later on it was studied by Šalát et al. [6], Hazarika and Savas [7], Tripathy and Hazarika [8] and many others. In [9], Tripathy and Tripathy introduced the notion of ideal convergent double sequences. Fridy and Orhan [10] introduced the concept of lacunary statistical convergence. Some work on lacunary statistical convergence can be found in [11–16]. The notion of lacunary ideal convergence of real sequences was introduced in [17,18]. Hazarika [19–21] introduced the lacunary ideal convergent sequences of fuzzy real numbers and studied some basic properties of this notion. Hazarika [22] introduced the notion of lacunary ideal convergent double sequences of fuzzy real numbers. Bakery and Mohammed [23] introduced lacunary mean ideal convergence in generalized random $n$-normed spaces.

A family of sets $I \subseteq 2^\mathbb{N}$ (power sets of $\mathbb{N}$) is said to be an ideal if $I$ is additive i.e. $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I, B \subseteq A \Rightarrow B \in I$. A non-empty family of sets $F \subset 2^\mathbb{N}$ is a filter on $\mathbb{N}$ if and only if $\emptyset \notin F, A \cap B \in F$ for each $A, B \in F$, and any superset of an element of $F$ is in $F$. An ideal $I$ is called non-trivial if $I \neq \emptyset$ and $\mathbb{N} \notin I$. Clearly $I$ is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter

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in \( \mathbb{N} \), called the filter associated with the ideal \( I \). A non-trivial ideal \( I \) is called admissible if and only if \( \{ n : n \in \mathbb{N} \} \subset I \). A non-trivial ideal \( I \) is maximal if there cannot exist any non-trivial ideal \( J \neq I \) containing \( I \) as a subset (for details on ideals see [4]).

A lacunary sequence is an increasing integer sequence \( \theta = (k_r) \) such that \( k_0 = 0 \) and \( h_r = k_r - k_{r-1} \to \infty \). The intervals determined by \( \theta \) will be defined by \( J_r = (k_{r-1}, k_r] \) and the ratio \( \frac{k_r}{k_{r-1}} \) will be defined by \( \phi_r \) (for details on lacunary sequence see [24]).

2. Definitions and preliminaries

We denote \( w \) is the space of all sequences.

**Definition 1** [4]. A sequence \( (x_k) \in w \) is said to be \( I \)-convergent to the number \( L \) if for every \( \epsilon > 0 \), \( \{ k \in \mathbb{N} : |x_k - L| \geq \epsilon \} \in I \). We write \( I - \lim x_k = L \).

**Definition 2** [25]. A sequence \( (x_k) \in w \) is said to be \( I \)-null if \( L = 0 \). We write \( I - \lim x_k = 0 \).

**Definition 3** [25]. Let \( I \) be an admissible ideal of \( \mathbb{N} \). A sequence \( (x_k) \in w \) is said to be \( I \)-Cauhcy if for every \( \epsilon > 0 \) there exists a number \( m = m(\epsilon) \) such that \( \{ k \in \mathbb{N} : |x_k - x_m| \geq \epsilon \} \in I \).

**Definition 4** [17,18]. Let \( \theta = (k_r) \) be lacunary sequence. Then a sequence \( (x_k) \) is said to be lacunary \( I \)-convergent if for every \( \epsilon > 0 \) such that

\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| \geq \epsilon \right\} \in I.
\]

We write \( I_\theta - \lim x_k = L \).

**Definition 5** [17,18]. Let \( \theta = (k_r) \) be lacunary sequence. Then a sequence \( (x_k) \) is said to be lacunary \( I \)-null if for every \( \epsilon > 0 \) such that

\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k| \geq \epsilon \right\} \in I.
\]

We write \( I_\theta - \lim x_k = 0 \).

**Definition 6** [17,18]. Let \( I \) be an admissible ideal of \( \mathbb{N} \) and let \( \theta = (k_r) \) be lacunary sequence. Then a sequence \( (x_k) \) is said to be lacunary \( I \)-Cauhcy if there exists a subsequence \( (x_{k'(r)}) \) of \( (x_k) \) such that \( k'(r) \in J_r \) for each \( r \), \( \lim_{r \to \infty} x_{k'(r)} = L \) and for every \( \epsilon > 0 \) such that

\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k - x_{k'(r)}| \geq \epsilon \right\} \in I.
\]

**Definition 7** [17,18]. A lacunary sequence \( \theta' = (k'(r)) \) is said to be a lacunary refinement of the lacunary sequence \( \theta = (k_r) \) if \( (k_r) \subset (k'(r)) \).

Throughout the paper, we shall denote by \( I \) is an admissible ideal of subsets of \( \mathbb{N} \times \mathbb{N} \) and \( \theta_{rs} = (k_{rs}) \) a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

3. Lacunary convergence of double sequences

By the convergence of a double sequence we mean the convergence in the Pringsheim’s sense [26]. A double sequence \( x = (x_{ij}) \) has a Pringsheim limit \( L \) (denoted by \( \lim x = L \)) provided that given an \( \epsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that \( |x_{ij} - L| < \epsilon \) whenever \( k, l > N \). We shall describe such an \( x = (x_{ij}) \) more briefly as “\( P \)-convergent”.

A double sequence \( \theta = \theta_{rs} = (k_{rs}) \) is called a double lacunary sequence if there exist two increasing sequence of integers \( (k_r) \) and \( (l_s) \) such that

\[
k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty
\]

and

\[
l_0 = 0, \overline{l}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.
\]

Let us denote \( k_{rs} = k_r l_s, h_{rs} = h_r \overline{l}_s \) and \( \theta_{rs} \) is determined by

\[
J_{rs} = \{(k, l) : k_{rs} - 1 < k \leq k_r \text{ and } l_{rs} - 1 \leq l \leq l_s\},
\]

\[
q_r = \frac{k_r}{k_{r-1}}, \overline{q}_s = \frac{l_s}{l_{s-1}} \text{ and } q_{rs} = q_r \overline{q}_s
\]

(for details on double lacunary sequences we refer to [27]).

**Definition 8.** A double sequence \( x = (x_{ij}) \) is said to be \( \theta_{rs} \)-convergent to \( L \in \mathbb{R} \) if for every \( \epsilon > 0 \) and there exist integers \( n_0 \in \mathbb{N} \) such that

\[
\frac{1}{h_{rs} J_{rs}} \sum_{(k, l) \in J_{rs}} |x_{kl} - L| < \epsilon
\]

for all \( r, s \geq n_0 \). In this case, we write \( \theta_{rs} - \lim x = L \).

**Theorem 1.** Let \( x = (x_{ij}) \) be a double sequence. If \( x = (x_{ij}) \) is \( \theta_{rs} \)-convergent then \( \theta_{rs} - \lim x = L \) is unique.

**Proof.** The proof of the theorem is straightforward, thus omitted. □

4. Lacunary ideal convergence of double sequences

**Definition 9.** Let \( \theta_{rs} = (k_{rs}) \) be a double lacunary sequence. Then a double sequence \( x_{ij} \) is said to be \( I_{\theta_{rs}} \)-convergent if for every \( \epsilon > 0 \) such that

\[
\left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs} J_{rs}} \sum_{(k, l) \in J_{rs}} |x_{kl} - L| \geq \epsilon \right\} \in I.
\]

We write \( I_{\theta_{rs}} - \lim x_{ij} = L \).

**Definition 10.** A double sequence \( x_{ij} \) is said to be \( I_{\theta_{rs}} \)-null if for every \( \epsilon > 0 \) such that

\[
\left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs} J_{rs}} \sum_{(k, l) \in J_{rs}} |x_{kl}| \geq \epsilon \right\} \in I.
\]
We write \( I_{0,y} \cap \lim x_{kl} = 0 \).

**Lemma 1.** Let \( x = (x_{kl}) \) be a double sequence. If \( \theta_{0,y} \cap \lim x = L \), then \( I_{0,y} \cap \lim x = L \).

**Proof.** Let \( \theta_{0,y} \cap \lim x = L \), then for every \( \varepsilon > 0 \) and there exists \( n_0 \in \mathbb{N} \) such that

\[
\frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |x_{kl} - L| < \varepsilon
\]

for all \( r,s \geq n_0 \). Therefore the set

\[
B = \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |x_{kl} - L| \geq \varepsilon \right\}
\]

\( \subseteq \{ (1,1), (2,2), \ldots, (n_0 - 1, n_0 - 1) \} \).

But, with \( I \) being admissible, we have \( B \in I \). Hence \( I_{0,y} \cap \lim x = L \). \( \square \)

**Theorem 2.** If \( (x_{kl}) \) is a double sequence such that \( I_{0,y} \cap \lim x_{kl} = \ell \) exists, then it is unique.

**Proof.** Proof of the theorem is straightforward, thus omitted. \( \square \)

Next theorem gives the algebraic characterization of lacunary ideal convergence.

**Theorem 3.** Let \( x = (x_{kl}) \) and \( y = (y_{kl}) \) are two sequences.

(a) If \( I_{0,y} \cap \lim_{k,l \to \infty, x_{kl} = \ell} \) and \( c(\neq 0) \in \mathbb{R} \), then \( I_{0,y} \cap \lim_{k,l \to \infty, x_{kl} = e} \).

(b) If \( I_{0,y} \cap \lim_{k,l \to \infty, x_{kl} = \ell} \) and \( I_{0,y} \cap \lim_{k,l \to \infty, y_{kl} = \ell} \), then \( I_{0,y} \cap \lim_{k,l \to \infty, (x_{kl} + y_{kl}) = \ell} \).

**Proof.** Proof of the theorem is straightforward, thus omitted. \( \square \)

**Definition 11.** Let \( I \) be an admissible ideal of \( \mathbb{N} \times \mathbb{N} \). A double sequence \( (x_{kl}) \) is said to be \( I_{0,y} \cap \text{Cauchy} \) if there exists a subsequence \( (x_{kl}(f(s))) \) of \( (x_{kl}) \) such that \( (k'(r), f(s)) \in J_{0,y} \) for each \( r,s \), \( \lim_{r,s \to \infty, x_{kl}(f(s)) = L} \) and for every \( \varepsilon > 0 \) such that

\[
\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |x_{kl} - x_{k'(r), f(s)}(f(i))| \geq \varepsilon \right\} \in I.
\]

**Theorem 4.** Let \( I \) be an admissible ideal of \( \mathbb{N} \times \mathbb{N} \). A double sequence \( (x_{kl}) \) is \( I_{0,y} \cap \text{convergent} \) if and only if it is \( I_{0,y} \cap \text{Cauchy} \) sequence.

**Proof.** Let \( (x_{kl}) \) be \( I_{0,y} \cap \text{convergent} \). Suppose that \( I_{0,y} \cap \lim x_{kl} = L \).

Write \( H_{ij} = \{ (r,s) \in \mathbb{N} \times \mathbb{N} : h_{i,j}^{-1} \sum_{k,j \in J_{i,j}} |x_{kl} - L| < \frac{\varepsilon}{2(2^j - 1)} \} \), for each \( i,j \in \mathbb{N} \).

Hence for each \( i,j \), \( H_{ij} \supseteq H_{i(j+1),j+1} \) and

\[
\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{i,j}} \sum_{k-j \in J_{i,j}} |x_{kl} - L| \geq \frac{\varepsilon}{2} \right\} \in I.
\]

We choose \( k_1, l_1 \) such that \( r \geq k_1, s \geq l_1 \), then

\[
\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |H_{i,j} \cap J_{i,j}| \geq \frac{\varepsilon}{2} \right\} \not\in I.
\]

Next we choose \( k_2 > k_1, l_2 > l_1 \) such that \( r \geq k_2, s \geq l_2 \), then

\[
\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |H_{i,j} \cap J_{i,j}| \geq \frac{\varepsilon}{2} \right\} \not\in I.
\]

Proceeding in this way inductively we can choose \( k_{p+1} > k_p, l_{p+1} > l_p \), such that \( r > k_{p+1}, s > l_{p+1} \) implies that \( H_{i,j} \cap J_{i,j} \not\in \phi \). Further for each \( r,s \) satisfying \( k_1 \leq r < k_2, l_1 \leq s < l_2 \), choose \( (k'(r), f(s)) \in H_{p,q} \cap J_{i,j} \) such that

\[
|x_{k'(r), f(s)}(f(i)) - L| < \frac{1}{pq}.
\]

This implies

\[
\lim_{(r,s) \to \mathbb{N} \times \mathbb{N}} x_{k'(r), f(s)} = L.
\]

Therefore, for every \( \varepsilon > 0 \), we have

\[
\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |x_{kl} - x_{k'(r), f(s)}(f(i))| \geq \varepsilon \right\} \subseteq \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |x_{kl} - L| \geq \varepsilon \right\}
\]

\( \cup \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |x_{k'(r), f(s)} - L| \geq \varepsilon \right\} \). i.e.

\[
\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |x_{kl} - x_{k'(r), f(s)}(f(i))| \geq \varepsilon \right\} \in I.
\]

Then \( (x_{kl}) \) is a \( I_{0,y} \cap \text{Cauchy} \) sequence.

Conversely suppose \( (x_{kl}) \) is a \( I_{0,y} \cap \text{Cauchy} \) sequence. Then for every \( \varepsilon > 0 \), we have

\[
\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |x_{kl} - L| \geq \varepsilon \right\} \subseteq \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |x_{k'(r), f(s)} - L| \geq \varepsilon \right\}
\]

\( \cup \left\{ (r,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{0,y}} \sum_{k,j \in J_{0,y}} |x_{k'(r), f(s)} - L| \geq \varepsilon \right\} \). It follows that \( (x_{kl}) \) is a \( I_{0,y} \cap \text{convergent} \) sequence. \( \square \)

Savas and Patterson [27] presented the definition of double lacunary refinement as follows.
Definition 12. The double index sequence $\rho = (k_r, l_s)$ is called a double lacunary refinement of the double lacunary sequence $\theta = (k_r, l_s)$ if $(k_r, l_s) \subseteq (\bar{k}_r, \bar{l}_s)$.

Theorem 5. If $(\rho_{rs})$ is a double lacunary refinement of $\theta_{rs}$ and $x_{rs} \to \ell(I_{\rho_{rs}})$, then $x_{rs} \to \ell(I_{\theta_{rs}})$.

Proof. Suppose each $I_{rs}$ of $\theta_{rs}$ contains the points $(\bar{k}_{rj}, \bar{l}_{sj})_{j \in J_{rs}}$ of $(\rho_{rs})$ so that

$$k_{r-1} < \bar{k}_{r1} < \bar{k}_{r2}, \ldots, \bar{k}_{rl} = k_r,$$

where $J_{rs} = (\bar{k}_{r1}, \bar{l}_{s1}], \bar{l}_{s1} < \bar{l}_{s2}, \ldots, \bar{l}_{sl} = l_s$ and $\bar{l}_{sl} < \bar{l}_{sl+1}$ for all $r, s$ and let $v(r), u(s) \geq 1$ this implies that $(k_r, l_s) \subseteq (\bar{k}_r, \bar{l}_s)$. Let $(J_{rs})_{j \in J_{rs}}$ be the sequence of abutting blocks of $(\bar{J}_{rs})$ ordered by increasing a lower right index points. Since $x_{rs} \to \ell(I_{\rho_{rs}})$, we have the following for each $\varepsilon > 0$,

$$\{(r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k, l) \in J_{rs}} |x_{kl} - \ell| \geq \varepsilon \}.$$

As before, we write $h_{rs} = h_r h_s; \bar{h}_{rs} = \bar{k}_{rj} - \bar{k}_{rj-1}; \bar{h}_{s} = \bar{l}_{sj} - \bar{l}_{sj-1}$.

For each $\varepsilon > 0$ we have

$$\{(r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k, l) \in J_{rs}} |x_{kl} - \ell| \geq \varepsilon \} \subseteq \{(r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k, l) \in J_{rs}} |x_{kl} - \ell| \geq \varepsilon \}.$$

By (4.1), for each $\varepsilon > 0$ if we define

$$t_{ij} = \left( \frac{1}{h_{ij}} \sum_{(k, l) \in J_{ij}} |x_{kl} - \ell| \geq \varepsilon \right),$$

then $(t_{ij})$ is a Pringsheim null sequence. The transformation

$$(A_{\theta_{rs}})_{rs} = \frac{1}{h_{rs}} \sum_{(k, l) \in J_{rs}} \left( \frac{1}{h_{ij}} \sum_{(k, l) \in J_{ij}} |x_{kl} - \ell| \geq \varepsilon \right)$$
satisfies all conditions for a matrix transformation to map a Pringsheim null sequence into a Pringsheim null sequence. Therefore $x_{ij} \to \ell(I_{\theta_{ij}})$. This completes the proof of the theorem. □

Theorem 6. Let $\psi_{rs}$ be a set of all double lacunary sequences.

(a) If $\psi_{rs}$ is closed under arbitrary union, then $z^c_{\psi_{rs}} = \bigcap_{\theta_{rs} \in \psi_{rs}} z^c_{\theta_{rs}}$, where $\psi_{rs} = \bigcup_{\theta_{rs} \in \psi_{rs}} \theta_{rs}$.

(b) If $\psi_{rs}$ is closed under arbitrary intersection, then $z^c_{\psi_{rs}} = \bigcap_{\theta_{rs} \in \psi_{rs}} z^c_{\theta_{rs}}$, where $\psi_{rs} = \bigcap_{\theta_{rs} \in \psi_{rs}} \theta_{rs}$.

(c) If $\psi_{rs}$ is closed under union and intersection, then $z^c_{\psi_{rs}} \subseteq z^c_{\theta_{rs}} \subseteq z^c_{\psi_{rs}}$.

Proof.

(a) By hypothesis we have $\mu_{rs} \in \psi_{rs}$ which is a double refinement of each $\theta_{rs} \in \psi_{rs}$. Then from Theorem 5, we have if $(x_{rs}) \in z_{\mu_{rs}}$, implies that $(x_{rs}) \in z_{\psi_{rs}}$. Thus for each $\theta_{rs} \in \psi_{rs}$ we get $z^c_{\psi_{rs}} \subseteq z^c_{\theta_{rs}}$. The reverse inclusion is obvious. Hence $z^c_{\psi_{rs}} = \bigcap_{\theta_{rs} \in \psi_{rs}} z^c_{\theta_{rs}}$.

(b) By part (a) and Theorem 5, we have $z^c_{\psi_{rs}} \subseteq \bigcap_{\theta_{rs} \in \psi_{rs}} z^c_{\theta_{rs}}$. The reverse inclusion is obvious. Hence $z^c_{\psi_{rs}} = \bigcap_{\theta_{rs} \in \psi_{rs}} z^c_{\theta_{rs}}$.

(c) By part (a) and (b), we get $z^c_{\psi_{rs}} \subseteq z^c_{\theta_{rs}} \subseteq z^c_{\psi_{rs}}$. □

5. Lacunary ideal limit point and cluster point

Definition 13. Let $x = (x_{kl})$ be a double sequence. Then

1. An element $x_0$ is said to be $I_{\theta_{rs}}$-limit point of $x = (x_{kl})$ if there is a set $M = \{(k_1, l_1), (k_2, l_2), \ldots, (k_r, l_r), l_r < \cdots < l_s \} \subseteq \mathbb{N} \times \mathbb{N}$ such that the set

$M' = \{(r, s) \in \mathbb{N} \times \mathbb{N} : (k_r, l_s) \in J_{rs} \} \neq I$

and $\theta_{rs} - \lim_{x_{rs}} = x_0$.

2. An element $x_0$ is said to be $I_{\theta_{rs}}$-cluster point of $x = (x_{kl})$ if for every $\varepsilon > 0$ we have

$$\{(r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k, l) \in J_{rs}} |x_{kl} - x_0| \geq \varepsilon \} \subseteq I.$$
Theorem 9. Let \( x = (x_{k,j}) \) be a double sequence. Then the following statements are equivalent:

1. \( x_{k,j} \) is a \( I_{h_{k,j}} \)-limit point of \( x \).
2. There exist two double sequences \( y = (y_{k,j}) \) and \( z = (z_{k,j}) \) such that \( x = y + z \) and \( \theta_{k,j} - \lim y = x_0 \) and \( \{ (r, s) \in \mathbb{N} \times \mathbb{N} : (k, l) \in J_{r,s}, z_{k,l} \neq \emptyset \} \in I \). Hence \( x_0 \in \Gamma_{h_{r,s}}(x) \). □

Proof. Suppose that (1) holds. Then there exist sets \( M \) and \( M' \) as in Definition 13 such that \( M' \notin I \) and \( \theta_{r,s} - \lim x_{k,j} = x_0 \). Define the sequences \( y \) and \( z \) as follows:

\[
y_{k,j} = \begin{cases} x_{k,j}, & \text{if } (k, l) \in J_{r,s}; (r, s) \in M' \\ x_0, & \text{otherwise.} \end{cases}
\]

and

\[
z_{k,j} = \begin{cases} 0, & \text{if } (k, l) \in J_{r,s}; (r, s) \in M' \\ x_{k,j} - x_0, & \text{otherwise.} \end{cases}
\]

It suffices to consider the case \((k, l) \in J_{r,s} \) such that \( (r, s) \in \mathbb{N} \times \mathbb{N} \setminus M' \). Then for each \( \varepsilon > 0 \), we have

\[
\frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} |y_{k,j} - x_0| < \varepsilon.
\]

Hence \( \theta_{r,s} - \lim y = x_0 \).

Now we have

\[
\{ (r, s) \in \mathbb{N} \times \mathbb{N} : (k, l) \in J_{r,s}, z_{k,l} \neq \emptyset \} \subset \mathbb{N} \times \mathbb{N} \setminus M'
\]

and so \( \{ (r, s) \in \mathbb{N} \times \mathbb{N} : (k, l) \in J_{r,s}, z_{k,l} = \emptyset \} \in I \).

Now, suppose that (2) holds. Let \( M' = \{ (r, s) \in \mathbb{N} \times \mathbb{N} : (k, l) \in J_{r,s}, z_{k,l} = \emptyset \} \). Then, clearly \( M' \in F(I) \) and so it is an infinite set. Construct the set \( M = \{ (k_1, l_1) < (k_2, l_2) < \cdots < (k_j, l_j) < \cdots \} \subset \mathbb{N} \times \mathbb{N} \) such that \( k_i, l_i \in J_{r,s} \) and \( z_{k_i,l_i} = \emptyset \). Since \( x_{k_i,l_i} = y_{k_i,l_i} \) and \( \theta_{r,s} - \lim y = x_0 \) we obtain \( \theta_{r,s} - \lim x_{k,i,l_i} = x_0 \). This completes the proof. □

Theorem 8. Let \( x = (x_{k,j}) \) be a double sequence. Then the following statements are equivalent:

1. \( x_{k,j} \) is a \( I_{h_{k,j}} \)-limit point of \( x \).
2. There exist two double sequences \( y = (y_{k,j}) \) and \( z = (z_{k,j}) \) such that \( x = y + z \) and \( \theta_{k,j} - \lim y = x_0 \) and \( \{ (r, s) \in \mathbb{N} \times \mathbb{N} : (k, l) \in J_{r,s}, z_{k,l} = \emptyset \} \notin I \). Hence \( x_0 \in \Gamma_{h_{r,s}}(x) \).

Proof. Suppose that (1) holds. Then there exist sets \( M \) and \( M' \) as in Definition 13 such that \( M' \notin I \) and \( \theta_{r,s} - \lim x_{k,j} = x_0 \). Hence \( x_0 \in \Gamma_{h_{r,s}}(x) \).

For every \( \varepsilon > 0 \), we have

\[
\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} |y_{k,j} - \ell| \geq \varepsilon \} \subseteq \{ (k, l) \in \mathbb{N} \times \mathbb{N} : y_{k,j} \neq x_{k,j} \} \cup \{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} |y_{k,j} - \ell| \geq \varepsilon \}.
\]

As both the sets of right-hand side of the above relation is in \( I \), therefore we have that

\[
\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} |y_{k,j} - \ell| \geq \varepsilon \} \in I.
\]

This completes the proof of the theorem. □

6. Topological and algebraic properties

By \( e \) and \( e^{(n)} (n \in \mathbb{N}) \), we denote the sequences such that \( e_k = 1 \) for \( k = 0, 1, 2, \ldots \), and \( e^{(n)} = 1 \) and \( e^{(n)} = 0 (k \neq n) \). For any sequence \( x = (x_k)_{k=0}^\infty \), let \( x^{(n)} = \sum_{k=0}^n x_k e^{(k)} \) be its \( n \)-section.

A sequence space \( X \) with a linear topology is called a \( K \)-space if each of the maps \( p_j : X \to \mathbb{C} \) defined by \( p_j(x) = x_j \) is continuous for all \( j \in \mathbb{N} \). A \( K \)-space is called an \( FK \)-space if \( X \) is complete linear metric space; a \( BK \)-space is a normed \( FK \)-space. An \( FK \)-space \( X \supseteq \varphi \) is said to have the \( AK \)-property if every sequence \( x = (x_k)_{k=0}^\infty \) in \( X \) has a unique representation \( x = \sum_{k=0}^{\infty} x_k e^{(k)} \), that is \( x = \lim_{n \to \infty} x^{(n)} \).

Freedman et al. [24] defined the space \( N_{\theta} \) in the following way: For any lacunary sequence \( \theta = (\theta_k) \),

\[
N_{\theta} = \left\{ (x_k) : \lim_{j \to \infty} \frac{1}{h_{j,k}} \sum_{k=j}^{\infty} |x_k - L| = 0, \text{ for some } L \right\}.
\]

The space \( N_{\theta} \) is a \( BK \)-space with the norm

\[
\| (x_k) \|_{\theta} = \sup_j \frac{1}{h_{j,k}} \sum_{k=j}^{\infty} |x_k|.
\]

\( N^0_{\theta} \) denote the subset of these sequences in \( N_{\theta} \) for which \( L = 0 \), \( (N^0_{\theta}, \| \cdot \|_{\theta}) \) is also a \( BK \)-space.

Definition 14. A sequence space \( E \) is said to be solid (or normal) if \( (x_k, x_n) \in E \), whenever \( (x_k) \in E \) and a sequence \( (x_k) \) of scalars with \( |x_k| \leq 1 \) for all \( k \in \mathbb{N} \).

Let \( K = \{ k_1 < k_2 < \ldots \} \subseteq \mathbb{N} \) and \( E \) be a sequence space. A \( K \)-step space of \( E \) is a sequence space

\[
\mathcal{E}_K = \{ (x_k) \in w : (k_n) \in E \}.
\]

A canonical preimage of a sequence \( \{ (x_k) \} \in \mathcal{E}_K \) is a sequence \( \{ y_k \} \in w \) defined as

\[
y_k = \begin{cases} x_k, & \text{if } n \in K \\ 0, & \text{otherwise.} \end{cases}
\]

A canonical preimage of a step space \( \mathcal{E}_K \) is a set of canonical preimages of all elements in \( \mathcal{E}_K \), i.e., \( y \) is in canonical preimage of \( \mathcal{E}_K \) if and only if \( y \) is canonical preimage of some \( x \in \mathcal{E}_K \).
Definition 15. A sequence space $E$ is said to be monotone if it contains the canonical preimages of its step spaces.

Lemma 2. Every solid space is monotone.

Theorem 10. The spaces $s^c_{h_{k,s}}$ and $(c^d_{h})_{h_{k,s}}$ are linear.

Proof. Proof of the theorem is straightforward, thus omitted. □

Theorem 11. $s^c_{h_{k,s}}$ and $(c^d_{h})_{h_{k,s}}$ are BK spaces with the norm

$$
\| \langle x_{k,l} \rangle \|_{h_{k,s}} = \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} |x_{k,l}|.
$$

Proof. The proof of the theorem is easy, thus omitted. □

Theorem 12. Let $\theta = (k_i)$ be a double lacunary sequence. Then the spaces $s^c_{h_{k,s}}$ and $(c^d_{h})_{h_{k,s}}$ are solid and monotone.

Proof. Let $(x_{k,l})$ be a sequence of scalars with $|x_{k,l}| \leq 1$ for all $k,l \in \mathbb{N}$. Then we have the space $(c^d_{h})_{h_{k,s}}$ is solid by the following relation

$$
\left\{ \begin{array}{l}
(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} |x_{k,l}| \geq \varepsilon \\
(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(k,l) \in J_{r,s}} |x_{k,l}| \geq \varepsilon
\end{array} \right\}.
$$

The space $(c^d_{h})_{h_{k,s}}$ is monotone by the Lemma 2. The other result follows similar way. □

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