JOURNAL OF MULTIVARIATE ANALYSIS 2, 96–114 (1972)

On the Distributions of a Class of Statistics in Multivariate Analysis

K. C. S. PILLAI[1] AND B. N. NAGARSENKER[2]

Department of Statistics, Purdue University, Lafayette, Indiana 47907

The noncentral distributions of $Y = \prod_{i=1}^{p} \theta_i^a(1 - \theta_i)^b$ are obtained, where a and b are known real numbers and θ_i's stand for latent roots of a matrix arising in each of three situations in multivariate normal theory, namely, test of equality of two covariance matrices, MANOVA, and canonical correlation. The study is extended to the complex case as well. The distributions are derived in terms of H-functions as a result of inverse Mellin transforms. Further, asymptotic expansions of the distribution of Y have been obtained in the case of two covariance matrices for selected values of (a, b).

1. INTRODUCTION

The noncentral distributions of statistics of the form $Y = \prod_{i=1}^{p} \theta_i^a(1 - \theta_i)^b$, where a and b are real numbers have been obtained in the following cases: (1) test of $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$, where, $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are the covariance matrices of two p-variate normal populations, (2) Manova, and (3) Canonical correlation, where θ_i's stand for latent roots of a matrix arising in each of the situations. The complex analog of the distributions also are treated. Among special cases of this statistic are (i) Wilks' $\Lambda = \prod_{i=1}^{p} (1 - \theta_i)$, (ii) Wilks–Lawley statistic, $U = \prod_{i=1}^{p} \theta_i$, (iii) the modified likelihood ratio criterion for test of (1) given by $\lambda = \prod_{i=1}^{p} \theta_i^{n_1/2} (1 - \theta_i)^{n_2/2}$ (See Section 3), (iv) $W = \prod_{i=1}^{p} \theta_i(1 - \theta_i)^{-1}$, and others. The density functions are given in terms of H-functions [2] as a result of employing inverse Mellin transforms. In sections 5–7, we give the asymptotic expansion of the distribution of Y in some special cases in connection with (1). The asymptotic

Received October 23, 1970; revised October 29, 1971.

AMS 1970 subject classifications: Primary 62H10; secondary 62E15.

Key words and phrases: Distributions, noncentral, likelihood ratio criterion, MANOVA, canonical correlation, equality of covariance matrices.

[1] Work of this author was supported by the National Science Foundation Grant No. GP-11473.

[2] Work of this author was supported by the National Science Foundation Grant No. GP-11473 and by David Ross Grant from Purdue Research Foundation.

expansion of the non-null distribution of a special case of Y was considered by Sugiura and Fujikoshi [8] for cases (2) and (3).

2. Noncentral Distributions of Y in the Real Case

Let us first consider test (1).

(a) *Equality of two covariance matrices.*

Let $\mathbf{X}_1$: $p \times n_1$ and $\mathbf{X}_2$: $p \times n_2$, $p \leqslant n_i$ $(i = 1, 2)$ be independent matrix variates with the columns of $\mathbf{X}_1$ independently distributed as $N(\mathbf{O}, \mathbf{\Sigma}_1)$ and those of $\mathbf{X}_2$ independently distributed as $N(\mathbf{O}, \mathbf{\Sigma}_2)$. Thus $\mathbf{S}_1 = \mathbf{X}_1\mathbf{X}_1'$ and $\mathbf{S}_2 = \mathbf{X}_2\mathbf{X}_2'$ are independently distributed as Wishart $(n_1, p, \mathbf{\Sigma}_i)$ $i = 1, 2$. Let $0 < f_1 \leqslant f_2 \leqslant \cdots \leqslant f_p < \infty$ be the characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ and $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_p < \infty$ be those of $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$. Here we proceed to obtain the distribution of

$$Y = \prod_{i=1}^{p} \theta_i^a(1 - \theta_i)^b, \tag{2.1}$$

where

$$\theta_i = f_i/(1 + f_i), \qquad i = 1, 2,..., p. \tag{2.2}$$

The density of $\theta_1, \theta_2, ..., \theta_p$ is given by (Khatri [5])

$$f(\theta_1, \theta_2, ..., \theta_p) = C(p, n, \mathbf{\Lambda})[\pi^{p^2/2}/\Gamma_p(p/2)] \mid \mathbf{\theta} \mid^{(n_1-p-1)/2} \cdot \mid \mathbf{I}_p - \mathbf{\theta} \mid^{(n_2-p-1)/2} \prod_{i>j} (\theta_i - \theta_j)\, {}_1F_0(n/2, \mathbf{M}, \mathbf{\theta}), \tag{2.3}$$

where

$$\mathbf{\theta} = \operatorname{diag}(\theta_1, \theta_2, ..., \theta_p), \qquad \mathbf{M} = I_p - \mathbf{\Lambda}^{-1},$$
$$\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_p), \qquad n = n_1 + n_2, \tag{2.4}$$

and

$$C(p, n, \mathbf{\Lambda}) = \Gamma_p(n/2)[\Gamma_p(n_1/2)\, \Gamma_p(n_2/2)]^{-1} \mid \mathbf{\Lambda} \mid^{-n_1/2}. \tag{2.5}$$

Now using the density (2.3), we get

$$E(Y^h) = C(p, n, \mathbf{\Lambda})\, \pi^{p(p-1)/4} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa\, C_\kappa(\mathbf{M})}{k!} \cdot \frac{\prod_{i=1}^{p} \Gamma[n_1/2 + ah + k_i - (i-1)/2] \prod_{i=1}^{p} \Gamma[n_2/2 + bh - (i-1)/2]}{\prod_{i=1}^{p} \Gamma[n/2 + (a+b)h + k_i - (i-1)/2]}. \tag{2.6}$$

TABLE I

Table of Constants for Different Sets of Values (a, b)
$(i = 1, 2, \ldots, p)$

	$a > 0, b > 0$	$a > 0,$ $b = 0$	$a = 0, b > 0$	$a > 0, b = -a$	$a > 0, b < 0,$ $a + b > 0$	$a > 0, b < 0,$ $a + b < 0$	$a < 0, b < 0$
$\alpha =$	$\pi^{p(p-1)/4}$	1	1	$\pi^{p(p-1)/2}$	$\pi^{p(p-1)/4}$	$\rightarrow$ same	$\rightarrow$ same
$\delta =$	1	$\Gamma_p(n_2/2)$	$\Gamma_p(n_1/2, \kappa)$	$[\Gamma_p(n/2, \kappa)]^{-1}$	1	1	1
$r =$	$2p$	p	p	p	p	p	0
$s =$	0	0	0	p	p	p	$2p$
$t =$	p	p	p	p	$2p$	p	$2p$
$u =$	$2p$	p	p	p	p	$2p$	p
$a_i =$	$(n/2)+k_i-(i-1)/2$	$\rightarrow$ same	same	$1-(n_2/2)+(i-1)/2$	$\rightarrow$ same	$\rightarrow$ same	$1-(n_1/2)-k_i+(i-1)/2$
$a_{p+i} =$	—	—	—	—	$(n/2)+k_i-(i-1)/2$	—	$1-(n_2/2)+(i-1)/2$
$\alpha_i =$	$a+b$	a	b	a	b	b	a
$\alpha_{p+i} =$	—	—	—	—	$a+b$	—	b
$b_i =$	$(n_1/2)+k_i-(i-1)/2$	$\rightarrow$ same	$(n_2/2)-(i-1)/2$	$(n_1/2)+k_i-(i-1)/2$	$\rightarrow$ same	$\rightarrow$ same	$1-(n/2)-k_i+(i-1)/2$
$b_{p+i} =$	$(n_2/2)-(i-1)/2$	—	—	—	—	$1-(n/2)+(i-1)/2$	—
$\beta_i =$	a	a	b	a	a	a	$a+b$
$\beta_{p+i} =$	b	—	—	—	—	$a+b$	—

Making use of the inverse Mellin transform, we have the density of Y as

$$f(Y) = C(p, n, \mathbf{\Lambda})\, \pi^{p(p-1)/4} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa C_\kappa(\mathbf{M})}{k!}\, Y^{-1}$$

$$\cdot \frac{1}{2\pi i} \int_C Y^{-h} \frac{\prod_{i=1}^{p} \Gamma[n_1/2 + ah + k_i - (i-1)/2] \cdot \prod_{i=1}^{p} \Gamma[n_2/2 + bh - (i-1)/2]}{\prod_{i=1}^{p} \Gamma[n/2 + (a+b)h + k_i - (i-1)/2]}\, dh. \tag{2.7}$$

Noting that the integral on the R.H.S. of (2.7) is in the form of the H-function, the noncentral density of Y for test (1) can be put in a single general form for different sets of values of a and b as follows:

$$f(Y) = C(p, n, \mathbf{\Lambda})\alpha \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa\, \delta C_\kappa(\mathbf{M})}{k!}\, Y^{-1} H_{t,u}^{r,s}\left(Y \,\middle|\, \begin{matrix}(a_i, \alpha_i)\ i = 1,..., t \\ (b_i, \beta_i)\ i = 1,..., u\end{matrix}\right), \tag{2.8}$$

where $C(p, n, \mathbf{\Lambda})$ is as in (2.5) and the constants are as given in Table I.

(b) *Manova*

Let $\theta_1, ..., \theta_p$ be the characteristic roots of $\mathbf{S}_1(\mathbf{S}_1 + \mathbf{S}_2)^{-1}$ where $\mathbf{S}_1$ is a $p \times p$ matrix distributed as noncentral Wishart with n_1 d.f., $\mathbf{\Omega}$ is a matrix of noncentrality parameters and $\mathbf{S}_2$ has the Wishart distribution with n_2 d.f., the covariance matrix in each case being $\mathbf{\Sigma}$. The distribution of $\theta_1, ..., \theta_p$ is given by Constantine [3, Eq. (41)] using which $E(Y^h)$ can be obtained in the same manner as before and is given by

$$E(Y^h) = C_1(p, n, \mathbf{\Omega}) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa\, C_\kappa(\mathbf{\Omega})}{(n_1/2)_\kappa\, k!} \frac{\Gamma_p(n_1/2 + ah, \kappa)\, \Gamma_p(n_2/2 + bh)}{\Gamma_p[n/2 + (a+b)h, \kappa]}, \tag{2.9}$$

where $C_1(p, n, \mathbf{\Omega}) = \Gamma_p(n/2)\, [\Gamma_p(n_1/2)\, \Gamma_p(n_2/2)]^{-1}\, e^{-\mathrm{Tr}\,\mathbf{\Omega}}$.
Noting that (2.9) can be obtained from (2.6) by making the following substitution

$$(|\mathbf{\Lambda}|^{-n_1/2}, \mathbf{M}, (n/2)_\kappa) \to (e^{-\mathrm{Tr}\,\mathbf{\Omega}}, \mathbf{\Omega}, (n/2)_\kappa/(n_1/2)_\kappa), \tag{2.10}$$

we can write the density of Y for this model in the general form

$$f(Y) = C_1(p, n, \mathbf{\Omega})\alpha \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa\, \delta C_\kappa(\mathbf{\Omega})}{(n_1/2)_\kappa\, k!}\, Y^{-1} H_{t,u}^{r,s}\left(Y \,\middle|\, \begin{matrix}(a_i, \alpha_i)\ i = 1,..., t \\ (b_i, \beta_i)\ i = 1,..., u\end{matrix}\right), \tag{2.11}$$

where the constants α, δ, r, s, t, u, (a_1, α_i) and (b_i, β_i) are as in Table I.

(c) *Canonical correlation*

Let the columns of $\binom{\mathbf{X}_1}{\mathbf{X}_2}$ be independent normal $(p+q)$ variate $(p \leqslant q$, $p+q \leqslant n$, n is the sample size) with zero means and covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{12}' & \mathbf{\Sigma}_{22} \end{pmatrix}.$$

Let $\mathbf{R}^2 = \operatorname{diag}(r_1^2, \ldots, r_p^2)$ where r_i^2 are the char roots of

$$| \mathbf{X}_1\mathbf{X}_2'(\mathbf{X}_2\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{X}_1' - r^2\mathbf{X}_1\mathbf{X}_1' | = 0$$

and $\mathbf{P}^2 = \operatorname{diag}(\rho_1^2, \ldots, \rho_p^2)$ where ρ_i^2 are the char. roots of

$$| \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{12}' - \rho^2\mathbf{\Sigma}_{11} | = 0.$$

Using the density of $r_1^2, \ldots, r_p^2$ given by Constantine [3], we obtain $E(Y^h)$ in the form (noting that $r_i^2 = \theta_i$, $i = 1, \ldots, p$)

$$\begin{aligned} E(Y^h) = C_2(p, n, q, \mathbf{P}^2) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa (n/2)_\kappa C_\kappa(\mathbf{P}^2)}{(q/2)_\kappa k!} \\ \cdot \frac{\Gamma_p(q/2 + ah, \kappa)\, \Gamma_p(n_2/2 + bh)}{\Gamma_p[n/2 + (a+b)h, \kappa]}, \end{aligned} \tag{2.12}$$

where $C_2(p, n, q, \mathbf{P}^2) = \Gamma_p(n/2)[\Gamma_p(q/2)\, \Gamma_p[(n-2)/2]]^{-1} | \mathbf{I}_p - \mathbf{P}^2 |^{n/2}$ and $n_2 = n - q$. Noting that (2.12) can be obtained from (2.9) by making the following substitution,

$$[C_1(p, n, \mathbf{\Omega}), 1/(n_1/2)_\kappa, \mathbf{\Omega}, n_1] \rightarrow [C_2(p, n, q, \mathbf{P}^2), (n/2)_\kappa/(q/2)_\kappa, \mathbf{P}^2, q] \tag{2.13}$$

we can write the density of $Y = \prod_{i=1}^{p} (r_i^2)^a (1 - r_i^2)^b$ for this case in the general form

$$\begin{aligned} f(Y) = C_2(p, n, q, \mathbf{P}^2)\alpha \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa (n/2)_\kappa\, \delta C_\kappa(\mathbf{P}^2)}{(q/2)_\kappa k!} \\ \cdot H_{t,u}^{r,s}\left(Y \,\middle|\, \begin{matrix} (a_i, \alpha_i)\ i = 1, \ldots, t \\ (b_i, \beta_i)\ i = 1, \ldots, u \end{matrix}\right), \end{aligned} \tag{2.14}$$

where the constants $\alpha, \delta, r, s, t, u, (a_i, \alpha_i)$ and (b_i, β_i) are as in Table I in which n_1 is to be replaced by q throughout.

3. Special Cases

(i) *Wilks' Λ criterion.* Taking $a = 0$ and $b = 1$ in (2.8) and using the relation between the H-function and the G-function, we find that the noncentral density of Wilks' $\Lambda = \prod_{i=1}^{p} (1 - \theta_i)$ is as obtained by Pillai, Al-Ani, Jouris in the three cases [6].

(ii) *Wilks–Lawley U-criterion.* If $a = 1$ and $b = 0$ in (2.8), we obtain the non-central density of Wilks–Lawley U-statistic, $U = \prod_{i=1}^{p} \theta_i$ for test (1), in the form

$$f(u) = C(p, n, \mathbf{\Lambda}) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa \, C_\kappa(\mathbf{M})}{k!} \Gamma_p(n_2/2) \cdot Y^{-1} H_{p,p}^{p,0}\left(Y \,\middle|\, \begin{matrix} (a_i, \alpha_i)\; i = 1,..., p \\ (b_i, \beta_i)\; i = 1,..., p \end{matrix}\right), \tag{3.1}$$

where $C(p, n, \mathbf{\Lambda})$ is as in (2.4), $(a_i, \alpha_i) = (n/2 + k_i - (i - 1)/2, 1)$, and $(b_i, \beta_i) = (n_1/2 + k_i - (i - 1)/2, 1)$ $i = 1,..., p$. Equation (3.1) can also be expressed in terms of the G-function. The density of U for the Manova and Canonical correlation cases can be written down using the substitution (2.10) and (2.13) respectively.

(iii) Taking $a = n_1/2$ and $b = n_2/2$ in (2.8) we obtain the noncentral density of the modified likelihood ratio criterion for testing $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$, i.e., of the statistic

$$\lambda = \prod_{i=1}^{p} \theta_i^{n_1/2}(1 - \theta_i)^{n_2/2} = | \mathbf{S}_1 |^{n_1/2} \, | \mathbf{S} |^{-n/2} \, | \mathbf{S}_2 |^{n_2/2}$$

where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, in the form

$$f(\lambda) = C(p, n, \mathbf{\Lambda}) \, \pi^{p(p-1)/4} \sum_{k=0}^{\infty} \frac{(n/2)_\kappa \, C_\kappa(\mathbf{M})}{k!} \lambda^{-1} H_{p,2p}^{2p,0}\left(\lambda \,\middle|\, \begin{matrix} (a_i, \alpha_i)\; i = 1,..., p \\ (b_i, \beta_i)\; i = 1,..., p \end{matrix}\right),$$

where

$$(a_i, \alpha_i) = (n/2 + k_i - (i - 1)/2, n/2),$$

and

$$(b_i, \beta_i) = \{(n_1/2 + k_i - (i - 1)/2, n_1/2), (n_2/2 - (i - 1)/2, n_2/2)\}, i = 1,..., p.$$

The densities in the other two cases can be written down using (2.10) and (2.13).

(iv) Taking $a = 1$ and $b = -1$ in (2.8) we obtain the noncentral density of the statistic $W = \prod_{i=1}^{p} \theta_i(1-\theta_i)^{-1} = | \mathbf{S}_1\mathbf{S}_2^{-1} |$ for test (1) in the form

$$f(Y) = C(p, n, \Lambda)\, \pi^{p(p-1)/2} \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa\, C_\kappa(\mathbf{M})}{k!\,\Gamma_p(n/2, \kappa)}\, Y^{-1} H_{p,p}^{p,p}\left(Y \,\middle|\, \begin{matrix} (a_i\,, 1)\; i = 1,\ldots, p \\ (b_i\,, 1)\; i = 1,\ldots, p \end{matrix}\right), \tag{3.2}$$

where $a_i = 1 - n_2/2 + (i-1)/2$ and $b_i = n_1/2 + k_i - (i-1)/2$. The density in (3.2) can be easily written down in terms of the G-function. The noncentral densities of W for the Manova and Canoncial correlation cases can be written down using (2.10) and (2.13).

4. Noncentral Distribution of Y in the Complex Case

The noncentral density of Y in the complex case can be obtained in a similar manner and is noted below.

(a′) The general form of the density of Y for test (1) can be written down from (2.8) by making the following substitutions.

$$\begin{aligned} &(\pi^d, n_1/2, n_2/2, n/2, (i-1)/2, \Gamma_p(\cdot), \Gamma_p(\cdot, \kappa), C_\kappa(\cdot), (\cdot)_\kappa) \\ &\to (\pi^{2d}, n_1\,, n_2\,, n, (i-1), \tilde{\Gamma}_p(\cdot), \tilde{\Gamma}_p(\cdot, \kappa), \tilde{C}_\kappa(\cdot), [\cdot]_\kappa), \end{aligned} \tag{4.1}$$

where $\tilde{\Gamma}_p(\cdot)$, $\tilde{\Gamma}_p(\cdot, \kappa)$, $\tilde{C}_\kappa(\cdot)$ and $[\cdot]_\kappa$ are as defined in James [4].

(b') For the Manova case the general form of the density of Y is obtained from (2.11) by making the substitutions as in (4.1).

(c') In the case of Canonical correlation also, the general form of the density of Y can be written down from (2.14) using (4.1).

5. Asymptotic Expansion of the Distribution of Y, $a = n_1/2$ and $b = n_2/2$

First we give some preliminaries.

(a) *Preliminaries.* For this case, putting $a = n_1/2$ and $b = n_2/2$ in (2.6) we have,

$$E(Y^h) = \frac{\Gamma_p(n/2)}{\Gamma_p(n_1/2)\,\Gamma_p(n_2/2)}\, | \Lambda |^{-n_1/2} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa\, \Gamma_p\left(\frac{n_1(1+h)}{2}, \kappa\right)}{k!\,\Gamma_p\left(\frac{n(1+h)}{2}, \kappa\right)} \cdot \Gamma_p\{n_2(1+h)/2\}\, C_\kappa(\mathbf{M}). \tag{5.1}$$

This can be easily written in the form

$$
\begin{aligned}
E(Y^h) = \{&\Gamma_p(n/2)\, \Gamma_p[n_1(1+h)/2] \\
&\cdot \Gamma_p[n_2(1+h)/2]/\Gamma_p[n(1+h)/2]\, \Gamma_p(n_1/2)\, \Gamma_p(n_2/2)\} \\
&\cdot |\, \mathbf{\Lambda}\, |^{-n_1/2} \cdot {}_2F_1(n/2, n_1(1+h)/2; n(1+h)/2, \mathbf{M}).
\end{aligned} \tag{5.2}
$$

We shall assume that

$$n_i = \tau_i n, \ (i = 1, 2), \qquad \text{where} \quad \tau_1 + \tau_2 = 1. \tag{5.3}$$

The asymptotic expansion of the distribution of Y will be derived in terms of n increasing and also in terms of $m = \rho n$ increasing where $0 < \rho < 1$ and is defined later, with τ_1 and τ_2 fixed. (See Anderson [1, p. 254]). The h-th moment of

$$W = [n^{(1/2)pn}/n_1^{(1/2)pn_1} n_2^{(1/2)pn_2}].\ Y \tag{5.4}$$

is given by

$$
\begin{aligned}
E(W^h) = n&^{(1/2)pnh} n_1^{-(1/2)pn_1h} n_2^{-(1/2)pn_2h} \\
&\cdot \{\Gamma_p(n/2)\, \Gamma_p[n_1(1+h)/2]/\Gamma_p[n(1+h)/2]\, \Gamma_p(n_1/2)\} \\
&\cdot \{\Gamma_p[n_2(1+h)/2]/\Gamma_p(n_2/2)\} \\
&\ |\, \mathbf{\Lambda}\, |^{-n_1/2} \cdot {}_2F_1(n/2, n_1(1+h)/2; n(1+h)/2, \mathbf{M}).
\end{aligned} \tag{5.5}
$$

We shall obtain the asymptotic expansion for (i) $-2 \log W$ in terms of n increasing and assuming $\mathbf{M}$ to be of the form $\mathbf{M} = (2/n)\, \mathbf{P}$ where $\mathbf{P}$ is a fixed matrix, and (ii) $-2\rho \log W$ in terms of $m = \rho n$ increasing instead of n and assuming $\mathbf{M} = (2/m)\, \mathbf{P}$ where $\mathbf{P}$ is a fixed matrix and the correction factor ρ is given by (see Anderson [1, p. 255])

$$m = \rho n = n - 2\alpha \qquad \text{where} \quad \alpha = (\tau_1^{-1} + \tau_2^{-1} - 1)(2p^2 + 3p - 1)/12(p+1). \tag{5.6}$$

We will need the following lemmas proved in [8].

LEMMA 5.1. *Let $C_\kappa(\mathbf{Z})$ be a zonal polynomial corresponding to the partition $\kappa = \{k_1, k_2, ..., k_p\}$ with $k_1 + k_2 + \cdots + k_p = k$ and $k_1 \geqslant k_2 \geqslant k_3 \cdots \geqslant k_p \geqslant 0$. Putting*

$$a_1(\kappa) = \sum_{i=1}^{p} k_i(k_i - i), \qquad a_2(\kappa) = \sum_{i=1}^{p} k_i(4k_i^2 - 6ik_i + 3i^2). \tag{5.7}$$

Then the following equalities hold:

$$\sum_{k=0}^{\infty} \sum_{\kappa} x^k C_\kappa(\mathbf{Z})\, a_1(\kappa)/k! = (x^2 \operatorname{Tr} \mathbf{Z}^2)\, e^{\operatorname{Tr}(x\mathbf{Z})}, \tag{5.8}$$

$$\sum_{k=1}^{\infty} \sum_{\kappa} x^k C_\kappa(\mathbf{Z})\, a_1(\kappa)/(k-1)! = (2x^2 \operatorname{Tr} \mathbf{Z}^2 + x^3 \operatorname{Tr} \mathbf{Z}^2 \operatorname{Tr} \mathbf{Z})\, e^{\operatorname{Tr}(x\mathbf{Z})}, \tag{5.9}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} x^k (a_1(\kappa))^2\, C_\kappa(\mathbf{Z})/k!$$
$$= \{x^4(\operatorname{Tr} \mathbf{Z}^2)^2 + 4x^3 \operatorname{Tr} \mathbf{Z}^3 + x^2 \operatorname{Tr} \mathbf{Z}^2 + x^2(\operatorname{Tr} \mathbf{Z})^2\}\, e^{\operatorname{Tr}(x\mathbf{Z})}, \tag{5.10}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} x^k C_\kappa(\mathbf{Z})\, a_2(\kappa)/k!$$
$$= \{4x^3 \operatorname{Tr} \mathbf{Z}^3 + 3x^2 \operatorname{Tr} \mathbf{Z}^2 + 3x^2(\operatorname{Tr} \mathbf{Z})^2 + x \operatorname{Tr} \mathbf{Z}\}\, e^{\operatorname{Tr}(x\mathbf{Z})}, \tag{5.11}$$

$$\sum_{k=1}^{\infty} \sum_{\kappa} C_\kappa(\mathbf{Z})/(k-1)! = (\operatorname{Tr} \mathbf{Z})\, e^{\operatorname{Tr} \mathbf{Z}}, \tag{5.12}$$

and

$$\sum_{k=2}^{\infty} \sum_{\kappa} C_\kappa(\mathbf{Z})/(k-2)! = (\operatorname{Tr} \mathbf{Z})^2\, e^{\operatorname{Tr} \mathbf{Z}}. \tag{5.13}$$

LEMMA 5.2. *With the notations of the lemma* 5.1, *for large* n,

$$(n/2)_\kappa = (n/2)^k\, [1 + n^{-1} a_1(\kappa) + (1/6n^2)\{k - a_2(\kappa) + 3(a_1(\kappa))^2\} + O(n^{-3})], \tag{5.14}$$

and

$$(n/2 + a)_\kappa = (n/2)^k\, [1 + (1/2n)\{4ak + 2a_1(\kappa)\} + (1/24n^2)\{4k + 48a^2k(k-1)$$
$$+ 48a(k-1)\, a_1(\kappa) - 4a_2(\kappa) + 12(a_1(\kappa))^2\} + O(n^{-3})]. \tag{5.15}$$

(*b*) *Derivation of Asymptotic Expansions.* We consider below asymptotic expansions of the distributions of (i) and (ii) above.

(i) *Asymptotic expansion of the distribution of* -2 *log* W. Let $\phi(t)$ be the characteristic function of $-2 \log W$. Then from (5.2) we have

$$\phi(t) = E(e^{-2it \log W}) = E(W^{-2it}) = C_1(t)\, C_2(t)\, C_3(t)\, |\, \mathbf{\Lambda}\, |^{-n_1/2}, \tag{5.16}$$

where

$$C_1(t) = n^{-itpn} n_1^{itpn_1} n_2^{itpn_2}, \tag{5.17}$$

$$C_2(t) = \Gamma_p(n/2)\,\Gamma_p(n_1 g/2)\,\Gamma_p(n_2 g/2)\,[\Gamma_p(ng/2)\,\Gamma_p(n_1/2)\,\Gamma(n_2/2)]^{-1} \tag{5.18}$$

and

$$g = (1-2it),\ C_3(t) = {}_2F_1(n/2, n_1 g/2; ng/2, \mathbf{M}). \tag{5.19}$$

We shall use the following asymptotic formula for the gamma function as in Anderson [1, p. 204]

$$\log \Gamma(x+h) = \log \sqrt{2\pi} + (x+h-\tfrac{1}{2})\log x - x - \sum_{r=1}^{m} \frac{(-1)^r B_{r+1}(h)}{r(r+1)\,x^r} + O(|x|^{-m-1}), \tag{5.20}$$

which holds for large $|x|$ and fixed h. The Bernoulli polynomial $B_r(h)$ of degree r is given by $(te^{ht})/(e^t-1) = \sum_{r=0}^{\infty}(t^r/r!)\,B_r(h)$. Some of these which we shall need in the sequel, are listed below.

$$B_1(h) = h - 1/2, \qquad B_2(h) = h^2 - h + 1/6,$$
$$B_3(h) = h^3 - 3h^2/2 + h/2 \quad \text{and} \quad B_4(h) = h^4 - 2h^3 + h^2 - 1/30. \tag{5.21}$$

Applying the formula (5.20) to each gamma function in $C_2(t)$, we have

$$\log C_2(t) = it\,pn\log(n/2) - it\,pn_2\log(n_2/2) - it\,pn_1\log(n_1/2)$$
$$-f\log(g)/2 + (r/n)(g^{-1}-1) + (s/n^2)(1-g^{-2}) + O(n^{-3}), \tag{5.22}$$

where

$$f = p(p+1)/2, \qquad r = p(2p^2+3p-1)(\tau_1^{-1}+\tau_2^{-1}-1)/24 \tag{5.23}$$

and

$$s = p(p+1)(2-p^2-p)(\tau_1^{-2}+\tau_2^{-2}-1)/48.$$

It therefore follows that

$$C_1(t)\,C_2(t) = g^{-f/2}\exp[(r/n)(g^{-1}-1) + (s/n^2)(1-g^{-2}) + O(n^{-3})]$$
$$= g^{-f/2}[1 + (r/n)(g^{-1}-1) + n^{-2}\{s(1-g^{-2}) + (r^2/2)(g^{-1}-1)^2\} + O(n^{-3})]. \tag{5.24}$$

Let $\mathbf{M} = [I - \mathbf{\Lambda}^{-1}] = (2/n)\,\mathbf{P}$ where $\mathbf{P}$ is a fixed matrix.

Then

$$| \mathbf{\Lambda} |^{-n_1/2} = \left| \mathbf{I} - \frac{2}{n} \mathbf{P} \right|^{\tau_1 n/2}. \tag{5.25}$$

Now using the expansion

$$\log | \mathbf{I} - \frac{2}{n} \mathbf{P} | = -(2/n) \operatorname{Tr} \mathbf{P} - (2/n^2) \operatorname{Tr} (\mathbf{P})^2 - (8/3n^3) \operatorname{Tr} (\mathbf{P}^3) + O(n^{-4}), \tag{5.26}$$

we obtain

$$\begin{aligned} \left| \mathbf{I} - \frac{2}{n} \mathbf{P} \right|^{\tau_1 n/2} &= \exp \left[(\tau_1 n/2) \log \left| \mathbf{I} - \frac{2}{n} \mathbf{P} \right| \right] \\ &= e^{(\tau_1 n/2)[-(2/n \operatorname{Tr} \mathbf{P} - (2/n^2) \operatorname{Tr}(\mathbf{P}^2) - (8/3n^3) \operatorname{Tr}(\mathbf{P}^3) + O(n^{-4})]} \\ &= e^{-\tau_1 \operatorname{Tr} \mathbf{P}} [1 - n^{-1} A_1 - n^{-2} A_2 + O(n^{-3})], \end{aligned} \tag{5.27}$$

where

$$A_1 = \tau_1 \operatorname{Tr}(\mathbf{P}^2) \quad \text{and} \quad A_2 = (4/3) \tau_1 \operatorname{Tr}(\mathbf{P}^3) - \tau_1^2 (\operatorname{Tr} \mathbf{P}^2)^2/2. \tag{5.28}$$

Applying asymptotic formula (5.14) to $(n/2)\kappa$, $(n_1 g/2)\kappa$ and $(ng/2)\kappa$ we have after some algebraic simplication,

$$\begin{aligned} &(n/2)\kappa \, (n_1 g/2) \, \kappa / (ng/2)\kappa \\ &= (n\tau_1/2)^k \, [1 + n^{-1} a_1(\kappa) B(t) + (1/6n^2) \{((k - a_2(\kappa) + 3(a_1(\kappa))^2) \\ &\quad A(t) - g^{-2}(k - a_2(\kappa) - 3(a_1(\kappa))^2) - D(t)(a_1(\kappa))^2\} + O(n^{-3})], \end{aligned} \tag{5.29}$$

where

$$A(t) = 1 + (\tau_1 g)^{-2}, \qquad B(t) = 1 + (\tau_1^{-1} - 1)/g, \tag{5.30}$$

and

$$D(t) = 6[g^{-1} + (\tau_1 g^2)^{-1} - (\tau_1 g)^{-1}].$$

Using (5.29) and Lemma 5.1, we have on simplification,

$$\begin{aligned} C_3(t) &= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa \, (n_1 g/2)_\kappa \, C_\kappa \left(\frac{2}{n} \mathbf{P} \right)}{(ng/2)_\kappa \, k!} \\ &= e^{\tau_1 \operatorname{Tr} \mathbf{P}} [1 + (K/n) \, B(t) + (1/6n^2) \{ LA(t) - Mg^{-2} - ND(t) \} + O(n^{-3})], \end{aligned} \tag{5.31}$$

where

$$K = \tau_1^2 \operatorname{Tr} \mathbf{P}^2, \quad L = 8\tau_1^3 \operatorname{Tr} \mathbf{P}^3 + 3\tau_1^4(\operatorname{Tr} \mathbf{P}^2)^2,$$
$$M = -3\tau_1^4(\operatorname{Tr} \mathbf{P}^2)^2 - 16\tau_1^3 \operatorname{Tr} \mathbf{P}^3 - 6\tau_1^2\{\operatorname{Tr} \mathbf{P}^2 + (\operatorname{Tr} \mathbf{P})^2\}, \tag{5.32}$$

and

$$N = \tau_1^4(\operatorname{Tr} \mathbf{P}^2)^2 + 4\tau_1^3 \operatorname{Tr} \mathbf{P}^3 + \tau_1^2\{\operatorname{Tr} \mathbf{P}^2 + (\operatorname{Tr} \mathbf{P})^2\}.$$

From (5.16), (5.24), (5.27) and (5.31), we have

$$\phi(t) = g^{-f/2}[1 + n^{-1}\{\alpha_0 + g^{-1}\alpha_1\} + n^{-2}\{\alpha_2 + g^{-1}\alpha_3 + g^{-2}\alpha_4\} + O(n^{-3})], \tag{5.33}$$

where the coefficients α_i's are given by

$$\alpha_0 = K - A_1 - r, \qquad \alpha_1 = K(\tau_1^{-1} - 1) + r,$$
$$\alpha_2 = L/6 - A_2 - KA_1 + s + r^2/2 - Kr + A_1 r,$$
$$\alpha_3 = r(K - A_1) - r^2 + (\tau_1^{-1} - 1)(N - rK - A_1 K), \tag{5.34}$$

and

$$\alpha_4 = (L\tau_1^{-2} - M)/6 - N\tau_1^{-1} - s + r^2/2 + rK(\tau_1^{-1} - 1).$$

By inverting the characteristic function in (5.33), using the fact that $(g)^{-f/2}$ is the characteristic function of χ_f^2, a chi square variable with f degrees of freedom, we obtain the following asymptotic expansion for the distribution of $-2 \log W$.

$$P(-2 \log W \leqslant z) = P(\chi_f^2 \leqslant z) + n^{-1}\{\alpha_0 P(\chi_f^2 \leqslant z) + \alpha_1 P(\chi_{f+2}^2 \leqslant z)\} + n^{-2}\{\alpha_2 P(\chi_f^2 \leqslant z) + \alpha_3 P(\chi_{f+2}^2 \leqslant z) + \alpha_4 P(\chi_{f+4}^2 \leqslant z)\} + O(n^{-3}), \tag{5.35}$$

where $\alpha_{i's}$ are defined in (5.33).

(ii) *Asymptotic expansion of the distribution of -2ρ log W.* Here we shall derive the asymptotic expansion for $-2\rho \log W$ here ρ is given by (5.6). Put $m = \rho n$ and let m tend to infinity instead of n. From (5.2), the characteristic function $f(t)$ of $-2\rho \log W$ can be written as

$$f(t) = E(e^{-2\rho\, it \log W}) = C_4(t)\, C_5(t), \tag{5.36}$$

where $C_4(t)$ and $C_5(t)$ are given by

$$C_4(t) = \frac{n^{-pnit\rho}}{n^{-pn_1it\rho} n_2^{-pn_2it\rho}} \frac{\Gamma_p(n/2)\, \Gamma_p\left[\frac{n_1(1 - 2\rho it)}{2}\right] \Gamma_p\left[\frac{n_2(1 - 2it\rho)}{2}\right]}{\Gamma_p\left[\frac{n(1 - 2\rho it)}{2}\right] \Gamma_p(n_1/2)\, \Gamma_p(n_2/2)} \tag{5.37}$$

and

$$C_5(t) = |\,\mathbf{\Lambda}\,|^{-(\tau_1/2)(m+2\alpha)}\,{}_2F_1\left(\frac{m}{2}+\alpha,\ \frac{mg\tau_1}{2}+\alpha\tau_1\,;\ \frac{mg}{2}+\alpha,\ \mathbf{M}\right), \tag{5.38}$$

g and α being as defined in (5.19) and (5.6), respectively. Now the first factor $C_4(t)$ in (5.36) can be expanded asymptotically (See Anderson [1, p. 255]) as follows:

$$C_4(t) = g^{-f/2}[1 + (A/m^2)(g^{-2} - 1) + O(m^{-3})], \tag{5.39}$$

where

$$f = p(p+1)/2, \qquad A = [p(p+1)/48][(p-1)(p+2)(\tau_1^{-2} + \tau_2^{-2} - 1) - 6\gamma] \tag{5.40}$$

and

$$\gamma = (\tau_1^{-1} + \tau_2^{-1} - 1)^2(2p^2 + 3p - 1)^2/36(p+1)^2 = 4\alpha^2.$$

Now as stated before, let

$$\mathbf{I} - \mathbf{\Lambda}^{-1} = (2/m)\mathbf{P},$$

where $\mathbf{P}$ is a fixed matrix. We then have

$$C_5(t) = \left|\,\mathbf{I} - \frac{2}{m}\mathbf{P}\,\right|^{(\tau_1/2)(m+2\alpha)}{}_2F_1\left(\frac{m}{2}+\alpha,\ \frac{m\tau_1 g}{2}+\alpha\tau_1\,;\ \frac{mg}{2}+\alpha,\ \frac{2}{m}\mathbf{P}\right). \tag{5.41}$$

Using the asymptotic expansion (5.15) to $(m/2+\alpha)_\kappa$, $(m\tau_1 g/2 + \alpha\tau_1)_\kappa$ and $(mg/2+\alpha)_\kappa$, we have

$$\begin{aligned}(m/2 + \alpha)_\kappa\,&(m\tau_1 g/2 + \alpha\tau_1)_\kappa/(mg/2+\alpha)_\kappa\\ &= (m\tau_1/2)^k\,[1 + m^{-1}\{2\alpha k + \delta_1 a_1(\kappa)\} + m^{-2}\{k\delta_2 + k^2\delta_3 + \delta_4 a_1(\kappa)\\ &\quad + \delta_5 k a_1(\kappa) + \delta_6 a_2(\kappa) + \delta_7(a_1(\kappa))^2\} + O(m^{-3})],\end{aligned} \tag{5.42}$$

where

$$\begin{aligned}&\delta_1 = 1 + (\tau_1^{-1} - 1)\,g^{-1}, \qquad \delta_2 = -2\alpha^2 + [1 + (\tau_1^{-2} - 1)\,g^{-2}]/6,\\ &\delta_3 = 2\alpha^2,\\ &\delta_4 = -2\alpha + 2\alpha g^{-2}(1 - \tau_1^{-1}), \qquad \delta_5 = 2\alpha + 2\alpha(\tau_1^{-1} - 1)\,g^{-1},\\ &\delta_6 = \{(1 - \tau_1^{-2})\,g^{-2} - 1\}/6 \qquad \text{and} \qquad \delta_7 = \{1 + (\tau_1^{-1} - 1)\,g^{-1}\}^2/2.\end{aligned} \tag{5.43}$$

From (5.42) and Lemma 5.1., it then easily follows that,

$$
\begin{aligned}
{}_2F_1&\left(\frac{m}{2}+\alpha, \frac{m\tau_1 g}{2}+\alpha\tau_1 ; \frac{mg}{2}+\alpha, \frac{2}{m}\mathbf{P}\right) \\
&= e^{\tau_1 \operatorname{Tr}\mathbf{P}}[1+m^{-1}(2a\alpha+b\delta_1) \\
&\qquad + m^{-2}(a\delta_2 + a(a+1)\,\delta_3 + b\delta_4 + c\delta_5 + d\delta_6 + e\delta_7) + O(m^{-3})],
\end{aligned}
\tag{5.44}
$$

where the constants a, b, c, d and e are given by

$$a = \tau_1 \operatorname{Tr}\mathbf{P},\ b = \tau_1^2 \operatorname{Tr}\mathbf{P}^2,\ c = 2\tau_1^2 \operatorname{Tr}\mathbf{P}^2 + \tau_1^3 \operatorname{Tr}\mathbf{P}^2 \operatorname{Tr}\mathbf{P},$$

$$d = 4\tau_1^3 \operatorname{Tr}\mathbf{P}^3 + 3\tau_1^2 \operatorname{Tr}\mathbf{P}^2 + 3\tau_1^2(\operatorname{Tr}\mathbf{P})^2 + \tau_1 \operatorname{Tr}\mathbf{P} \tag{5.45}$$

and $\quad e = \tau_1^4(\operatorname{Tr}\mathbf{P}^2)^2 + 4\tau_1^3 \operatorname{Tr}\mathbf{P}^3 + \tau_1^2[\operatorname{Tr}\mathbf{P}^2 + (\operatorname{Tr}\mathbf{P})^2].$

Also

$$\left|\mathbf{I}-\frac{2}{m}\mathbf{P}\right|^{(\tau_1 m/2)+\alpha\tau_1} = \left|\mathbf{I}-\frac{2}{m}\mathbf{P}\right|^{\tau_1 m/2}\left|\mathbf{I}-\frac{2}{m}\mathbf{P}\right|^{\alpha\tau_1} \tag{5.46}$$

and using (5.26) and (5.27) to the factor on the right hand side of (5.46), it can be easily checked that

$$\left|\mathbf{I}-\frac{2}{m}\mathbf{P}\right|^{(\tau_1 m/2)+\alpha\tau_1} = e^{-\tau_1 \operatorname{Tr}\mathbf{P}}[1 - m^{-1}\delta_8 + m^{-2}\delta_9 + O(m^{-3})], \tag{5.47}$$

where

$$\delta_8 = \tau_1\{\operatorname{Tr}\mathbf{P}^2 + 2\alpha \operatorname{Tr}\mathbf{P}\},$$

and

$$\delta_9 = \delta_8^2/2 - \tau_1\{4 \operatorname{Tr}\mathbf{P}^3/3 + 2\alpha \operatorname{Tr}\mathbf{P}^2\}.$$

We therefore have

$$
\begin{aligned}
C_5(t) = {}&[1 + m^{-1}(2a\alpha + b\delta_1 - \delta_8) + m^{-2}(a\delta_2 + a(a+1)\,\delta_3 + b\delta_4 + c\delta_5 \\
&+ d\delta_6 + e\delta_7 + \delta_9 - 2a\alpha\delta_8 - b\delta_1\delta_8) + O(m^{-3})]
\end{aligned}
\tag{5.48}
$$

and finally from (5.36), (5.39) and (5.48) we have

$$f(t) = g^{-f/2}[1 + m^{-1}\{\alpha_0 + \alpha_1 g^{-1}\} + m^{-2}\{\beta_0 + \beta_1 g^{-1} + \beta_2 g^{-2}\} + O(m^{-3})], \tag{5.49}$$

where

$$\alpha_0 = 2a\alpha + b - \delta_8, \qquad \alpha_1 = b(\tau_1^{-1} - 1),$$

$$\beta_0 = -A + 2\alpha^2 a^2 + (a - d)/6 - 2\alpha(b - c + a\delta_8) + e/2 + \delta_9 - b\delta_8, \tag{5.50}$$

$$\beta_1 = (\tau_1^{-1} - 1)(2c\alpha + e - b\delta_8)$$

and

$$\beta_2 = A + (\tau_1^{-2} - 1)(a - d)/6 + 2b\alpha(1 - \tau_1^{-1}) + e(\tau_1^{-1} - 1)^2/2.$$

Inverting the characteristic function in (5.49), we have the asymptotic expansion of the distribution of $-2\rho \log W$ in the form,

$$\begin{aligned} P(-2\rho \log W \leqslant z) &\\ = P_r(\chi_f^2 \leqslant z) &+ m^{-1}\{\alpha_0 P(\chi_f^2 \leqslant z) \\ &+ \alpha_1 P(\chi_{f+2}^2 \leqslant z)\} + m^{-2}\{\beta_0 P(\chi_f^2 \leqslant z) + \beta_1 P(\chi_{f+2}^2 \leqslant z) \\ &+ \beta_2 P(\chi_{f+4}^2 \leqslant z)\} + O(m^{-3}) \end{aligned} \tag{5.51}$$

6. Asymptotic Expansion of the Distribution of Y, $a = 1$ and $b = 0$

For this case putting $a = 1$ and $b = 0$ in (2.6) we can easily see that

$$E(Y^h) = \frac{\Gamma_p(n/2)\,\Gamma_p(n_1/2 + h)}{\Gamma_p(n/2 + h)\,\Gamma_p(n_1/2)} \mid \mathbf{\Lambda} \mid^{-n_1/2} {}_2F_1(n/2, n_1/2 + h; n/2 + h, \mathbf{M}). \tag{6.1}$$

We assume (5.3) and obtain the asymptotic expansion of L_1 where

$$L_1 = \sqrt{n} \log(Y/\tau_1{}^p) \tag{6.2}$$

in terms of n increasing with τ_1 and τ_2 fixed assuming that $\mathbf{M} = [\mathbf{I} - \mathbf{\Lambda}^{-1}] = (2/n)\ \mathbf{P}$ where $\mathbf{P}$ is a fixed matrix. Let $\chi(t)$ be the characteristic function of L_1. Then

$$\chi(t) = E(e^{itL_1}) = C_6(t)\, C_7(t), \tag{6.3}$$

where

$$C_6(t) = (1/\tau_1)^{it\sqrt{n}p}\, \Gamma_p(n/2)\, \Gamma_p(n_1/2 + it\sqrt{n})[\Gamma_p(n/2 + it\sqrt{n})\, \Gamma_p(n_1/2)]^{-1} \tag{6.4}$$

and

$$C_7(t) = |\,\boldsymbol{\Lambda}\,|^{-n_1/2}\,{}_2F_1(n/2, n_1/2 + it\sqrt{n}; n/2 + it\sqrt{n}, \mathbf{M}). \tag{6.5}$$

Using the formula (5.18) to each gamma function on the right hand side of (6.4), we have

$$C_6(t) = e^{-pT_1t^2}[1 - n^{-1/2}\{fT_1(it) + 2pT_2(it)^3/3\} + n^{-1}\{(fT_2 + f^2T_1^2/2)(it)^2 + 2p(T_3 + fT_1T_2)(it)^4/3 + 2p^2T_2^2(it)^6/9\} + O(n^{-3/2})], \tag{6.6}$$

where

$$T_1 = (\tau_1^{-1} - 1), \quad T_2 = \tau_1^{-2} - 1, \quad T_3 = \tau_1^{-3} - 1 \quad \text{and} \quad f = p(p+1)/2. \tag{6.7}$$

Now using Lemma 5.2 to $(n/2)\kappa$, $(n/2 + it\sqrt{n})\kappa$ and $(n_1/2 + it\sqrt{n})\kappa$ we have

$$\begin{aligned}(n/2)_\kappa(n_1/2 + it\sqrt{n})_\kappa/(n/2 + it\sqrt{n})_\kappa &= (n_1/2)^k[1 + n^{-1/2}2\,it\,kT_1 + n^{-1}\\ &\quad\cdot\{\tau_1^{-1}a_1(\kappa) + 2(it)^2(k^2T_1^2 - kT_2)\} + O(n^{-3/2})].\end{aligned} \tag{6.8}$$

As before let $\mathbf{M} = (2/n)\mathbf{P}$ where $\mathbf{P}$ is a fixed matrix. Then from (6.8) and Lemma 5.1, we have after a little simplification,

$$\begin{aligned}&{}_2F_1(n/2, n_1/2 + it\sqrt{n}; n/2 + it\sqrt{n}, (2/n)\,\mathbf{P})\\ &= e^{\tau_1 \mathrm{Tr}\mathbf{P}}[1 - n^{-1/2}(it)\,A_1 + n^{-1}\{(it)^2\,A_2 + q\} + O(n^{-3/2})],\end{aligned} \tag{6.9}$$

where

$$A_1 = -2T_1\,\mathrm{Tr}(\tau_1\mathbf{P}), \qquad q = \tau_1\,\mathrm{Tr}\,\mathbf{P}^2$$

and

$$A_2 = 2\{T_1^2[(\mathrm{Tr}\,\tau_1\mathbf{P})^2 + \mathrm{Tr}(\tau_1\mathbf{P})] - T_2(\tau_1\,\mathrm{Tr}\,\mathbf{P})\}. \tag{6.10}$$

Also from (5.27) we have

$$|\,\boldsymbol{\Lambda}\,|^{-n_1/2} = |\,\mathbf{I} - \frac{2}{n}\mathbf{P}\,|^{\tau_1 n/2} = e^{-\tau_1 \mathrm{Tr}\mathbf{P}}[1 - n^{-1}q + O(n^{-3/2})] \tag{6.11}$$

and thus

$$C_7(t) = [1 - n^{-1/2}(it)\,A_1 + n^{-1}(it)^2\,A_2 + O(n^{-3/2})]. \tag{6.12}$$

From (6.3), (6.6) and (6.12) we, obtain the following asymptotic expansion for $\chi(t)$.

$$\chi(t/\sqrt{2pT_1}) = e^{-t^2/2}[1 - n^{-1/2}D_1 + n^{-1}D_2 + O(n^{-3/2})], \tag{6.13}$$

where the coefficients D_1 and D_2 are given by

$$D_1 = (2pT_1)^{-1/2}\,[(it)(A_1 + fT_1) + (3T_1)^{-1}\,T_2(it)^3]$$

and

$$\begin{aligned} D_2 = {} & (2pT_1)^{-3}\,4p^2[(it)^2(fT_2 + f^2T_1^2/2 + fT_1A_1 + A_2)\,T_1^2 \\ & + (it)^4\,T_1(T_3 + fT_1T_2 + A_1T_2)/3 + (it)^6\,T_2^2/18]. \end{aligned} \tag{6.14}$$

By inverting the characteristic function (6.13) we have the following asymptotic expansion for $L_1{}^* = \sqrt{n/2pT_1}\log(Y/\tau_1{}^p)$ up to the order of $n^{-3/2}$,

$$\begin{aligned} P(L_1{}^* \leqslant x) = {} & \Phi(x) + (2pT_1n)^{-1/2}\{(fT_1 + A_1)\,\Phi^1(x) + (3T_1)^{-1}\,T_2\Phi^3(x)\} \\ & + n^{-1}(2pT_1)^{-3}\,4p^2\{(fT_2 + f^2T_1^2/2 + fA_1T_1 + A_2)\,T_1^2\Phi^2(x) \\ & + T_1(T_3 + fT_1T_2 + A_1T_2)\,\Phi^4(x)/3 + T_2^2\Phi^6(x)/18\} + O(n^{-3/2}), \end{aligned} \tag{6.15}$$

where $\Phi^r(x)$ denotes the r-th derivative of the standard normal distribution $\Phi(x)$.

7. Asymptotic Expansion of the Distribution of Y, $a = 0$ and $b = 1$

For this case, we have from (2.6)

$$E(Y^h) = \frac{\Gamma_p(n/2)\,\Gamma_p\left(\frac{n_2}{2} + h\right)}{\Gamma_p\left(\frac{n}{2} + h\right)\Gamma_p(n_2/2)}\,|\,\mathbf{\Lambda}\,|^{-n_1/2}\,{}_2F_1\left(\frac{n}{2}, \frac{n_1}{2}; \frac{n}{2} + h, \mathbf{M}\right). \tag{7.1}$$

We shall obtain the asymptotic expansion for L_2 where

$$L_2 = \sqrt{n}\log(Y/\tau_2{}^p) \tag{7.2}$$

under the same condition as in Section 6. Let $H(t)$ be the characteristic function of L_2. Then

$$H(t) = E(e^{itL_2}) = C_8(t)\,C_9(t), \tag{7.3}$$

where

$$C_8(t) = (\tau_2)^{-it\sqrt{n}p}\,\Gamma_p(n/2)\,\Gamma_p\left(\frac{n_2}{2} + it\sqrt{n}\right)\left[\Gamma_p\left(\frac{n}{2} + it\sqrt{n}\right)\Gamma_p(n_2/2)\right]^{-1} \tag{7.4}$$

and

$$C_9(t) = |\,\mathbf{\Lambda}\,|^{-n_1/2}\,{}_2F_1\left(n/2, n_1/2; \frac{n}{2} + it\sqrt{n}, \mathbf{M}\right). \tag{7.5}$$

Using the formula (5.17) to each gamma function on the right-hand side of (7.4) we get

$$C_8(t) = e^{-pR_1t^2}[1 - n^{-1/2}\{fR_1(it) + 2pR_2(it)^3/3\} + n^{-1}\{(fR_2 + f^2R_1^2/2)(it)^2 + 2p(R_3 + fR_1R_2)(it)^4/3 + 2p^2R_2^2(it)^6/9\} + O(n^{-3/2})], \tag{7.6}$$

where the coefficients R_1, R_2, R_3 and R_4 are given by

$$R_1 = \tau_2^{-1} - 1, \quad R_2 = \tau_2^{-2} - 1, \quad R_3 = \tau_2^{-3} - 1 \quad \text{and} \quad f = p(p+1)/2. \tag{7.7}$$

Using Lemma 5.1 and 5.2, we have, proceeding as in Section 6,

$$\begin{aligned}{}_2F_1&\left(n/2, n_1/2; \frac{n}{2} + it\sqrt{n}, \frac{2}{n}\mathbf{P}\right)\\ &= e^{\tau_1 \operatorname{Tr} \mathbf{P}}[1 - n^{-1/2}(it)\,B_1 + n^{-1}\{B_2 + (it)^2\,B_3\} + O(n^{-3/2})],\end{aligned} \tag{7.8}$$

where $B_1 = 2(\operatorname{Tr} \tau_1\mathbf{P})$, $B_2 = \tau_1 \operatorname{Tr} \mathbf{P}^2$ and $B_3 = B_1(4 + B_1)/2$.
Using (7.8) and (6.11), we can write $C_9(t)$ as

$$C_9(t) = [1 - n^{-1/2}\,(it)B_1 + n^{-1}\,(it)^2B_3 + O(n^{-3/2})] \tag{7.9}$$

and thus we have the following asymptotic expansion for $H(t)$.

$$H(t/\sqrt{2R_1p}) = e^{-t^2/2}[1 - n^{-1/2}\beta_1 + n^{-1}\beta_2 + O(n^{-3/2})], \tag{7.10}$$

where

$$\beta_1 = 2p(2R_1\,p)^{-3/2}\,[(it)\,R_1(fR_1 + B_1) + (it)^3\,R_2/3]$$

and

$$\begin{aligned}\beta_2 = 4p^2(2R_1\,p)^{-3}\,&[(it)^2R_1^2(fR_2 + f^2R_1^2/2 + fR_1B_1 + B_3)\\ &+ (it)^4\,R_1(R_3 + qR_1R_2 + B_1R_2)/3 + R_2^2(it)^6/18].\end{aligned}$$

By inverting the characteristic function we have the following asymptotic expansion for $L_2^* = L_2/\sqrt{2pR_1}$ up to the order of $n^{-3/2}$

$$\begin{aligned}P(L_2^* \leqslant x) = \Phi(x) &+ n^{-1/2}(2pR_1)^{-3/2}\,2p[R_1(fR_1 + B_1)\,\Phi^1(x) + (R_2/3)\,\Phi^3(x)]\\ &+ n^{-1}(2pR_1)^{-3}\,4p^2[(fR_2 + f^2R_1^2/2 + B_3 + fR_1B_1)\,R_1^2\Phi^2(x)\\ &+ R_1(R_3 + fR_1R_2 + B_1R_2)\,\Phi^4(x)/3 + R_2^2\Phi^6(x)/18] + O(n^{-3/2}),\end{aligned} \tag{7.11}$$

where $\Phi^r(x)$ is as defined earlier.

The special case for $a = 1$, $b = -1$ has been already considered by Sugiura [7].

Acknowledgments

The authors wish to thank the Editor and the referees for their valuable suggestions which led to the inclusion of the latter sections on asymptotic distributions and for weeding out some computational errors from those sections.

References

[1] Anderson, T. W. (1958). *An Introduction to Multivariate Statistical Analysis*. Wiley, New York.

[2] Braaksma, B. L. J. (1964). Asymptotic expansions and analytic continuations for a class of Barnes-integrals. *Composite Math.* **15** 239–341.

[3] Constantine, A. G. (1963). Some non-central distribution problems in multivariate analysis. *Ann. Math. Statist.* **34** 1270–1285.

[4] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475–501.

[5] Khatri, C. G. (1967). Some distribution problems connected with the characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$. *Ann. Math. Statist.* **38** 944–948.

[6] Pillai, K. C. S., Al-Ani, S., and Jouris, G. M. (1969). On the distributions of the ratios of the roots of a covariance matrix and Wilks' criterion for tests of three hypotheses. *Ann. Math. Statist.* **40** 2033–2040.

[7] Sugiura, N. (1969). Asymptotic expansions of $| \mathbf{S}_1\mathbf{S}_2^{-1} |$ and $| \mathbf{S}_2(\mathbf{S}_1 + \mathbf{S}_2)^{-1} |$ for testing the equality of two covariance matrices. Mimeo. Series No. 653. Institute of Statistics, University of North Carolina.

[8] Sugiura, N. and Fujikoshi, Y. (1969). Asymptotic expansions of the non-null distributions of the likelihood ratio criteria for multivariate linear hypothesis and independence. *Ann. Math. Statist.* **40** 942–952.