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A one parameter family of locally quartically convergent zero-finding methods

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Abstract

A one parameter family of iteration functions for finding simple and multiple zeros of analytic functions is derived. The family includes, as a special case, Traub's quartic square root method and, as limiting cases, the Kiss method of order 4, the Halley and the Newton methods. All the methods of the family are locally quartically convergent for a simple or multiple zero with known multiplicity. The asymptotic error constants for the methods of the family are given. The decreasing ratio at infinity of iteration functions is discussed. The optimum parameter of the family for polynomials is given.

MSC: 65H05

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1. Introduction

Let $\phi(z)$ be an iteration function with a fixed point ζ. If there exists an integer $p>1$ and a nonzero constant C such that

$$\lim_{z\to\zeta}\frac{\phi(z)-\zeta}{(z-\zeta)^p}=C, \tag{1.1}$$

then the iteration function $\phi(z)$ is said to be of order p at ζ and the constant C is called the asymptotic error constant for $\phi(z)$. When $p=3$ the convergence is cubic. When $p=4$ the convergence is quartic.

Let $f(z)$ be an analytic function in a region R. A zero $\zeta\in R$ of $f(z)$ is of multiplicity $m\,(m\geqslant 1, m\in\mathbb{Z})$ if there exists an analytic function $g(z)$ in R such that $f(z)=(z-\zeta)^m g(z)$, $g(\zeta)\neq 0$. If $m=1$, ζ is said to be simple; if $m>1$, ζ is said to be multiple.

The Laguerre family

$$\hat{z}=z-\frac{\nu(f(z)/f'(z))}{1+\operatorname{sign}(\nu-m)\sqrt{((\nu-m)/m)[(\nu-1)-\nu(f(z)f''(z)/(f'(z))^2)]}}, \tag{1.2}$$

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where $v(\neq 0, m)$ is a real parameter, is a one parameter family of iteration functions for finding a simple or multiple zero of multiplicity $m(\geqslant 1)$ of an analytic function $f(z)$. This family is algebraically equivalent to the multiple zero counterpart of Hansen–Patrick family [6] (see, e.g. [12,14]). All the methods of the Laguerre family are locally cubically convergent for a simple or multiple zero with known multiplicity. When $f(z)$ is a polynomial of degree n, the method (1.2) with $v=n$ is the Laguerre method [1,2,11], whose initial behavior is superior, i.e., when starting from a point z for which $|z|$ is large, the next approximation $\hat{z}$ is close to one of zeros of $f(z)$ (see [6]).

The purposes of this paper are to give a one parameter family of quartically convergent zero finding methods for analytic functions and to propose the optimum parameter of this family for polynomials.

In Section 2, we derive a one parameter family of iteration functions for finding simple zeros of analytic functions. This family includes Traub's quartic square root method [15, p. 96] and, as limiting cases, the Kiss method [10] of order 4, the Halley, and the Newton methods.

In Section 3, we modify the family to be quartic convergence in the case of multiple zeros. The modified family includes, as limiting cases, Farmer–Loizou's method [3] of order 4, the multiple zero counterpart of Halley's method [6], and the Schröder method.

In Section 4, we prove that all the methods of the family are locally quartically convergent for a simple or multiple zero with known multiplicity.

In Section 5, we introduce the decreasing ratio at infinity and propose the optimum parameter of the family for polynomials.

In Section 6, we exhibit the local convergence and the initial behavior of some methods of the family defined in Section 3. Next we test the global convergence of the method with the optimum parameter and the Laguerre method by a numerical experiment on a large number of polynomials.

2. Derivation of the family for simple zeros

2.1. Notation

The sign function is defined by

$$\operatorname{sign}(x)=\begin{cases}1 & \text{if } x>0,\\ -1 & \text{if } x<0,\\ 0 & \text{if } x=0.\end{cases}$$

Let $w=r\mathrm{e}^{\mathrm{i}\theta}(r>0, -\pi<\theta\leqslant\pi)$ be a nonzero complex number. It is well known that the equation $z^2=w$ has exactly two solutions $\pm\sqrt{r}\mathrm{e}^{\mathrm{i}\theta/2}$. We define the square root of w by

$$\sqrt{w}=\sqrt{r}\mathrm{e}^{\mathrm{i}\theta/2}.$$

We remark that $\sqrt{w^2}=w$ if $\mathrm{Re}(w)>0$, and $\sqrt{w^2}=-w$ if $\mathrm{Re}(w)<0$.

For an analytic function $f(z)$, define

$$u(z)=u=\frac{f(z)}{f'(z)},\quad A_j(z)=A_j=\frac{f^{(j)}(z)}{j!f'(z)}\quad \text{for } j=2,3,4.$$

2.2. Derivation

Following the derivation of the Laguerre method found in [13], let $f(z)=\prod_{i=1}^{n}(z-\zeta_i)$ with $n\geqslant 4$. Define

$$s_j(z)=s_j=\frac{(-1)^{j-1}}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\left(\frac{f'(z)}{f(z)}\right)\quad \text{for } j=1,2,3.$$

By computing the derivative of $\log|f(z)|$, we have

$$s_1=\frac{f'(z)}{f(z)}=\sum_{i=1}^{n}\frac{1}{z-\zeta_i}=\frac{1}{u}. \tag{2.1}$$

Twice differentiating (2.1), we find that

$$s_2 = -\left(\frac{f'(z)}{f(z)}\right)' = \sum_{i=1}^{n} \frac{1}{(z-\zeta_i)^2} = \frac{1}{u^2} - \frac{2A_2}{u}, \tag{2.2}$$

$$s_3 = \frac{1}{2}\left(\frac{f'(z)}{f(z)}\right)'' = \sum_{i=1}^{n} \frac{1}{(z-\zeta_i)^3} = \frac{1}{u^3} - \frac{3A_2}{u^2} + \frac{3A_3}{u}. \tag{2.3}$$

Let ζ_n be the zero we want to determine and let z be an approximation to ζ_n. Define

$$\alpha = \frac{1}{z-\zeta_n}, \quad \beta = \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{1}{z-\zeta_i},$$

$$\delta_i = \frac{1}{z-\zeta_i} - \beta \quad \text{for } i = 1, ..., n-1, \tag{2.4}$$

$$\delta^{(3)}(z) = \delta^{(3)} = \sum_{i=1}^{n-1} \delta_i^3.$$

Cubing δ_i, we obtain

$$\delta^{(3)} = \sum_{i=1}^{n-1} \frac{1}{(z-\zeta_i)^3} - 3\beta\sum_{i=1}^{n-1}\frac{1}{(z-\zeta_i)^2} + 3\beta^2\sum_{i=1}^{n-1}\frac{1}{z-\zeta_i} - (n-1)\beta^3. \tag{2.5}$$

Using (2.1)–(2.4), we have

$$\beta = \frac{s_1-\alpha}{n-1}, \quad \sum_{i=1}^{n-1}\frac{1}{(z-\zeta_i)^j} = s_j - \alpha^j \quad \text{for } j = 1, 2, 3.$$

Eliminating $\zeta_i (i = 1, \ldots, n-1)$ and β from (2.5), we get

$$\delta^{(3)} = -\frac{n(n+1)}{(n-1)^2}\alpha^3 + \frac{3(n+1)s_1}{(n-1)^2}\alpha^2 + \frac{3(n-1)s_2 - 6s_1^2}{(n-1)^2}\alpha + \frac{(n-1)^2 s_3 - 3(n-1)s_1 s_2 + 2s_1^3}{(n-1)^2}. \tag{2.6}$$

Lemma 2.1. *Suppose that all of $\zeta_i (i = 1, \ldots, n-1)$ are distant from ζ_n. Then*

(1) $\delta^{(3)} = O(1)$ *as* $z \to \zeta_n$.

(2) $\dfrac{(n-1)^2 s_3 - 3(n-1)s_1 s_2 + 2s_1^3}{(n-1)^2} \sim \dfrac{(n-2)(n-3)}{(n-1)^2(z-\zeta_n)^3}$ *as* $z \to \zeta_n$.

Proof. (1) Let $\eta = \min_{i=1,\ldots,n-1}\{|\zeta_n - \zeta_i|\}$. Let ε be a real with $0 < \varepsilon < \eta$. Let z be an approximation to ζ_n such that $|z-\zeta_n| < \varepsilon$. Then for $i = 1, \ldots, n-1$,

$$|z-\zeta_i| \geqslant |\zeta_n - \zeta_i| - |z-\zeta_n| > |\zeta_n-\zeta_i| - \varepsilon \geqslant \eta - \varepsilon.$$

Thus

$$|\beta| \leqslant \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{1}{|z-\zeta_i|} < \frac{1}{\eta-\varepsilon},$$

$$|\delta_i| \leqslant \frac{1}{|z-\zeta_i|} + |\beta| < \frac{2}{\eta-\varepsilon} \quad \text{for } i = 1, \ldots, n-1,$$

$$|\delta^{(3)}| \leqslant \sum_{i=1}^{n-1}|\delta_i^3| < \frac{8(n-1)}{(\eta-\varepsilon)^3}.$$

(2) By the assumption, we find that

$$\lim_{z\to\zeta_n} s_3(z)(z-\zeta_n)^3 = 1, \quad \lim_{z\to\zeta_n} s_1(z)s_2(z)(z-\zeta_n)^3 = 1,$$

$$\lim_{z\to\zeta_n} s_1(z)^3(z-\zeta_n)^3 = 1.$$

Therefore

$$\lim_{z\to\zeta_n} \frac{(n-1)^2 s_3 - 3(n-1)s_1 s_2 + 2s_1^3}{(n-1)^2} \frac{(n-1)^2(z-\zeta_n)^3}{(n-2)(n-3)} = 1. \qquad \square$$

By Lemma 2.1, when all other zeros are distant from ζ_n we can ignore $\delta^{(3)}$ in (2.6). Setting $\delta^{(3)} = 0$, we have the cubic equation for α

$$-n(n+1)\alpha^3 + 3(n+1)s_1\alpha^2 + (3(n-1)s_2 - 6s_1^2)\alpha + (n-1)^2 s_3 - 3(n-1)s_1 s_2 + 2s_1^3 = 0. \tag{2.7}$$

Using (2.1)–(2.3), Eq. (2.7) becomes

$$n(n+1)\alpha^3 - \frac{3(n+1)}{u}\alpha^2 + \frac{3(-n+3+2(n-1)A_2 u)}{u^2}\alpha - \frac{(n-2)(n-3) - 3(n-1)(n-3)A_2 u + 3(n-1)^2 A_3 u^2}{u^3} = 0. \tag{2.8}$$

On the other hand, by Taylor's expansion we have

$$0 = f(z) - f'(z)(z-\zeta_n) + \frac{f''(z)}{2}(z-\zeta_n)^2 - \frac{f'''(z)}{6}(z-\zeta_n)^3 + \mathrm{O}((z-\zeta_n)^4).$$

Neglecting the term $(z-\zeta_n)^4$, we get

$$f(z) - \frac{f'(z)}{\alpha} + \frac{f''(z)}{2\alpha^2} - \frac{f'''(z)}{6\alpha^3} = 0,$$

and by multiplying $\alpha^3/f(z)$ we have

$$\alpha^3 - \frac{1}{u}\alpha^2 + \frac{A_2}{u}\alpha - \frac{A_3}{u} = 0. \tag{2.9}$$

Eliminating the term α^3 from (2.8) and (2.9), we get

$$\frac{(n+1)(n-3)}{u}\alpha^2 - \frac{3(n-3) + (n-2)(n-3)A_2 u}{u^2}\alpha - \frac{(n-2)(n-3) - 3(n-1)(n-3)A_2 u + (2n-1)(n-3)A_3 u^2}{u^3} = 0,$$

which yields the quadratic equation

$$(n+1)\alpha^2 - \left(\frac{3}{u} + (n-2)A_2\right)\alpha + \left(-\frac{n-2}{u^2} + \frac{3(n-1)}{u}A_2 - (2n-1)A_3\right) = 0. \tag{2.10}$$

Solving this equation for α, we obtain

$$\alpha = \frac{(3/u) + (n-2)A_2 \pm (1/u)\sqrt{R_n}}{2(n+1)},$$

where

$$R_n = (2n-1)^2 - 6n(2n-1)A_2 u + (n-2)^2 A_2^2 u^2 + 4(n+1)(2n-1)A_3 u^2.$$

Since $\alpha = 1/(z - \zeta_n)$, we have

$$\zeta_n = z - \frac{2(n+1)u}{3 + (n-2)A_2 u \pm \sqrt{R_n}}. \tag{2.11}$$

Let choose the sign in (2.11) to be maximize $|3 + (n-2)A_2 u \pm \sqrt{R_n}|$ as $z \to \zeta_n$. Since ζ_n is a simple zero of $f(z)$, we have

$$u(\zeta_n) = 0, \quad A_2(\zeta_n)u(\zeta_n) = 0, \quad A_3(\zeta_n)(u(\zeta_n))^2 = 0,$$

hence we obtain

$$R_n \to (2n-1)^2 \quad \text{as } z \to \zeta_n.$$

Because

$$|3 - (2n-1)| < |3 + (2n-1)|, \quad n \geqslant 1,$$

(2.11) becomes

$$\zeta_n = z - \frac{2(n+1)u}{3 + (n-2)A_2 u + \sqrt{R_n}}. \tag{2.12}$$

Let $n(\neq -1, \frac{1}{2})$ be any real. Let $f(z)$ be an analytic function and let ζ be a simple zero of $f(z)$. Let z be an approximation to ζ and put $\alpha = 1/(z - \zeta)$. Suppose ζ equals to the right hand of (2.12). Then α satisfies (2.10). By (2.9), α approximately satisfies (2.8). Hence a polynomial $f(z)$ and its degree n in (2.12) can be generalized to an analytic function and a real number, respectively.

Let $f(z)$ be an analytic function and let $v(\neq -1, \frac{1}{2})$ be a real parameter. Now we define a one parameter family of iteration functions by

$$\phi_v(z) = z - \frac{2(v+1)u}{3 + (v-2)A_2 u + \operatorname{sign}(2v-1)\sqrt{R_v}}, \tag{2.13}$$

where

$$R_v = (2v-1)^2 - 6v(2v-1)A_2 u + (v-2)^2 A_2^2 u^2 + 4(v+1)(2v-1)A_3 u^2. \tag{2.14}$$

2.3. *Special or limiting cases*

Example 1. When $v = 2$ in (2.13),

$$\phi_2(z) = z - \frac{2u}{1 + \sqrt{1 - 4A_2 u + 4A_3 u^2}}, \tag{2.15}$$

which is Traub's quartic square root method [15, p. 96].

Example 2. Rationalizing (2.13), we have

$$\phi_v(z) = z + \frac{u(3 + (v-2)A_2 u - \operatorname{sign}(2v-1)\sqrt{R_v})}{2(v-2) - 6(v-1)A_2 u + 2(2v-1)A_3 u^2}. \tag{2.16}$$

When $v = -1$ in (2.14), $R_{-1} = 9(1 - A_2 u)^2$. If $\operatorname{Re}(1 - A_2 u) > 0$, letting $v \to -1$ in (2.16), we obtain

$$\hat{z} = z - \frac{(1 - A_2 u)u}{1 - 2A_2 u + A_3 u^2},$$

which is the Kiss method [10] of order 4.

Example 3. When $v=\frac{1}{2}$ in (2.14), $R_{1/2}=(\frac{3}{2}A_2u)^2$.

(1) Suppose $\mathrm{Re}(A_2u)>0$. Letting $v\to\frac{1}{2}-0$ in (2.16), we get

$$\hat{z}=z-\frac{u}{1-A_2u},$$

which is the Halley method. Letting $v\to\frac{1}{2}+0$ in (2.16), we have

$$\hat{z}=z-u,$$

which is the Newton method.

(2) If $\mathrm{Re}(A_2u)<0$, then we get the Newton method and the Halley method by letting $v\to\frac{1}{2}-0$ and $v\to\frac{1}{2}+0$, respectively.

3. Modification for multiple zeros

3.1. Derivation

Let $f(z)$ be an analytic function and let ζ be an m-fold zero of $f(z)$. Let $v(\neq -1,\frac{1}{2})$ be a real number. Let $h(z)=\sqrt[m]{f(z)}$. Since ζ is a simple zero of $h(z)$, we can apply $\phi_v(z)$ to $h(z)$. Define

$$U=\frac{h(z)}{h'(z)},\quad B_j=\frac{h^{(j)}(z)}{j!h'(z)}\quad \text{for } j=2,3.$$

By differentiating $h(z)$ three times, we find that

$$U=mu,\quad B_2U=\frac{1-m}{2}+mA_2u,$$

$$B_3U^2=\tfrac{1}{6}(1-m)(1-2m)+m(1-m)A_2u+m^2A_3u^2.$$

Substituting U, B_2, and B_3 for u, A_2, and A_3, respectively, in (2.13), we have

$$\phi_{m,v}(z)=z-\frac{2m(v+1)u}{3+(v-2)(\frac{1}{2}(1-m)+mA_2u)+\mathrm{sign}(2v-1)\sqrt{R_{m,v}}}, \tag{3.1}$$

where

$$R_{m,v}=\tfrac{1}{12}(5mv-v+2m-4)(7mv+5v-2m-4)-3mv(3mv+v-2)A_2u$$
$$+\ (v-2)^2m^2A_2^2u^2+4(v+1)(2v-1)m^2A_3u^2. \tag{3.2}$$

It is easy to see that $\phi_v(z)$ in (2.13) coincides with $\phi_{1,v}(z)$ in (3.1).

3.2. Special or limiting cases

Example 4. When $v=2$ in (3.1),

$$\phi_{m,2}(z)=z-\frac{2mu}{1+\sqrt{\frac{1}{3}(4m^2-1)-4m^2A_2u+4m^2A_3u^2}},$$

which is the multiple zero counterpart of Traub's quartic method (2.15).

Example 5. Letting $v\to\infty$ in (3.1), we obtain

$$\hat{z}=z-\frac{2mu}{\frac{1}{2}(1-m)+mA_2u+\sqrt{\bar{R}_m}},$$

where

$$\begin{aligned}\bar{R}_m &= \lim_{v\to\infty} \frac{R_{m.v}}{v^2}\\ &= \frac{1}{12}(5m-1)(7m+5) - 3m(3m+1)A_2u + m^2A_2^2u^2 + 8m^2A_3u^2.\end{aligned}$$

Example 6. In a way similar to Example 2, if $\mathrm{Re}(1+m-2mA_2u)>0$, letting $v\to -1$ in (3.1), we get

$$\hat{z} = z - \frac{m(\frac{1}{2}(1+m)-mA_2u)u}{\frac{1}{6}(m+1)(2m+1)-m(m+1)A_2u+m^2A_3u^2},$$

which is Farmer–Loizou's method [3] of order 4.

Example 7. In a way similar to Example 3, letting $v\to\frac{1}{2}\pm 0$ in (3.1), we obtain the multiple zero counterpart of Halley's method

$$\hat{z} = z - \frac{u}{(m+1)/2m - A_2u},$$

or the Schröder method $\hat{z} = z - mu$.

4. Order of convergence

Let $f(z)=(z-\zeta)^m g(z)$ be an analytic function with $g(\zeta)\neq 0$, $m\geqslant 1$. In this section, we denote $g(\zeta)$, $g'(\zeta)$, $g''(\zeta)$, and $g'''(\zeta)$ by g_0, g_0', g_0'', and g_0''', respectively. In the asymptotic formulas we omit the qualifying phrase "as $z\to\zeta$".

Lemma 4.1. *The following asymptotic formulas are valid*:

$$\begin{aligned}A_2u &= \frac{m-1}{2m} + \frac{1}{m^2}\frac{g_0'}{g_0}(z-\zeta) + \left(-\frac{3(m+1)}{2m^3}\left(\frac{g_0'}{g_0}\right)^2 + \frac{3}{2m^2}\frac{g_0''}{g_0}\right)(z-\zeta)^2\\ &\quad + \left(\frac{2(m+1)^2}{m^4}\left(\frac{g_0'}{g_0}\right)^3 - \frac{3m+4}{m^3}\frac{g_0'g_0''}{g_0^2} + \frac{1}{m^2}\frac{g_0'''}{g_0}\right)(z-\zeta)^3 + \mathrm{O}((z-\zeta)^4),\end{aligned}$$

$$\begin{aligned}(A_2u)^2 &= \frac{(m-1)^2}{4m^2} + \frac{m-1}{m^3}\frac{g_0'}{g_0}(z-\zeta) + \left(\frac{5-3m^2}{2m^4}\left(\frac{g_0'}{g_0}\right)^2 + \frac{3(m-1)}{2m^3}\frac{g_0''}{g_0}\right)(z-\zeta)^2\\ &\quad + \left(\frac{(m+1)(2m^2-5)}{m^5}\left(\frac{g_0'}{g_0}\right)^3 + \frac{-3m^2-m+7}{m^4}\frac{g_0'g_0''}{g_0^2} + \frac{m-1}{m^3}\frac{g_0'''}{g_0}\right)(z-\zeta)^3 + \mathrm{O}((z-\zeta)^4),\end{aligned}$$

$$\begin{aligned}A_3u^2 &= \frac{(m-1)(m-2)}{6m^2} + \frac{m-1}{m^3}\frac{g_0'}{g_0}(z-\zeta) + \left(-\frac{(m-1)(3m+4)}{2m^4}\left(\frac{g_0'}{g_0}\right)^2 + \frac{3m-2}{2m^3}\frac{g_0''}{g_0}\right)(z-\zeta)^2\\ &\quad + \left(\frac{2(m^2-1)(3m+5)}{3m^5}\left(\frac{g_0'}{g_0}\right)^3 - \frac{3m^2+3m-4}{m^4}\frac{g_0'g_0''}{g_0^2} + \frac{3m-1}{3m^3}\frac{g_0'''}{g_0}\right)(z-\zeta)^3 + \mathrm{O}((z-\zeta)^4).\end{aligned}$$

Proof. It is simple so we omit the proof. □

Lemma 4.2. *Suppose $v \neq \frac{1}{2}$ and $m \geqslant 1$. For $R_{m,v}$ in* (3.2),

$$\sqrt{R_{m,v}} = |2v-1|\left(1 - \frac{3v}{(2v-1)m}\frac{g_0'}{g_0}(z-\zeta) + \frac{7v-2}{2(2v-1)}\left(\frac{m+1}{m^2}\left(\frac{g_0'}{g_0}\right)^2 - \frac{1}{m}\frac{g_0''}{g_0}\right)(z-\zeta)^2\right.$$
$$+\left(\frac{(m+1)(-20v^2m-25v^2+26mv+19v-8m-10)}{3(2v-1)^2m^3}\left(\frac{g_0'}{g_0}\right)^3\right.$$
$$\left.\left.+\frac{10v^2m+15v^2-13mv-15v+4m+6}{(2v-1)^2m^2}\frac{g_0'g_0''}{g_0^2} - \frac{5v-4}{3(2v-1)m}\frac{g_0'''}{g_0}\right)(z-\zeta)^3 + \mathrm{O}((z-\zeta)^4)\right).$$

Proof. By means of Lemma 4.1, we can prove it. □

Theorem 4.3. *Let $f(z) = (z-\zeta)^m g(z)$ with $g(\zeta) \neq 0$, $m \geqslant 1$. Then $\phi_{m,v}(z)$ in* (3.1) *with $v \neq -1, \frac{1}{2}$ converges locally quartically to ζ, i.e.,*

$$\lim_{z\to\zeta}\frac{\phi_{m,v}(z)-\zeta}{(z-\zeta)^4} = \frac{(m+1)(4mv-v-2m-4)}{6m^3(2v-1)}\left(\frac{g_0'}{g_0}\right)^3 - \frac{2mv+v-m-2}{2m^2(2v-1)}\frac{g_0'g_0''}{(g_0)^2} + \frac{1}{6m}\frac{g_0'''}{g_0}.$$

Proof. Using Lemma 4.2, we have

$$3 + (v-2)\left(\frac{1}{2}(1-m) + mA_2u\right) + \mathrm{sign}(2v-1)\sqrt{R_{m,v}}$$
$$= 2(v+1)\left[1 - \frac{1}{m}\frac{g_0'}{g_0}(z-\zeta) + \left(\frac{m+1}{m^2}\left(\frac{g_0'}{g_0}\right)^2 - \frac{1}{m}\frac{g_0''}{g_0}\right)(z-\zeta)^2\right.$$
$$+\left(-\frac{(m+1)(8mv+13v-4m-2)}{6(2v-1)m^2}\left(\frac{g_0'}{g_0}\right)^3\right.$$
$$\left.\left.+\frac{4mv+7v-2m-2}{2(2v-1)m^2}\frac{g_0'g_0''}{g_0^2} - \frac{1}{3m}\frac{g_0'''}{g_0}\right)(z-\zeta)^3 + \mathrm{O}((z-\zeta)^4)\right].$$

Therefore,

$$\phi_{m,v}(z) = z - (z-\zeta)\left[1 + \left(-\frac{(m+1)(4mv-v-2m-4)}{6(2v-1)m^3}\left(\frac{g_0'}{g_0}\right)^3\right.\right.$$
$$\left.\left.+\frac{2mv+v-m-2}{2(2v-1)m^2}\frac{g_0'g_0''}{g_0^2} - \frac{1}{6m}\frac{g_0'''}{g_0}\right)(z-\zeta)^3 + \mathrm{O}((z-\zeta)^4)\right].$$

The result follows from this formula. □

Remark 4.1. As an immediate consequence of Theorem 4.3, the asymptotic error constant for (2.13) is

$$\frac{v-2}{2v-1}\left(\frac{g_0'}{g_0}\right)^3 - \frac{3(v-1)}{2(2v-1)}\frac{g_0'g_0''}{(g_0)^2} + \frac{1}{6}\frac{g_0'''}{g_0}.$$

In particular, when $v=2$, the asymptotic error constant for (2.15) is

$$-\frac{1}{2}\frac{g_0'g_0''}{(g_0)^2}+\frac{1}{6}\frac{g_0'''}{g_0}=-\frac{1}{12}\frac{f''(\zeta)f'''(\zeta)}{(f'(\zeta))^2}+\frac{1}{24}\frac{f^{(4)}(\zeta)}{f'(\zeta)}$$
$$=-A_2(\zeta)A_3(\zeta)+A_4(\zeta),$$

which is given by Traub [15, p. 96].

5. The optimum parameter

5.1. The decreasing ratio

Let $f(z)$ be a polynomial of order n. Let $\Phi_p(z)$ be the pth ($p>1$) order Newton iteration function [4,5], e.g., Newton ($p=2$), Halley ($p=3$), Kiss ($p=4$), etc. When $|z|$ is sufficiently large, $\Phi_p(z)$ satisfies

$$\Phi_p(z)-\zeta\approx\left(1-\frac{p-1}{n+p-2}\right)(z-\zeta), \tag{5.1}$$

where ζ is one of zeros of $f(z)$ (see [9]). Here the expression $x\approx y$ means that x/y nearly equals to unity. In this subsection we extend the behavior (5.1) to one-point iteration functions.

Let $\phi(z)$ be a one-point iteration function with a fixed point ζ. If there exists a constant D ($|D|<1$) such that

$$\lim_{z\to\infty}\frac{\phi(z)}{z}=D,$$

then D is called the decreasing ratio for $\phi(z)$ at infinity. In this case, since

$$\lim_{z\to\infty}\frac{\phi(z)-\zeta}{z-\zeta}=D,$$

we have

$$\phi(z)-\zeta\approx D(z-\zeta),$$

when $|z|$ is sufficiently large.

In particular, when $D=0$ and the absolute value of an initial approximation $|z_0|$ is sufficiently large, the next approximation $z_1=\phi(z_0)$ is expected to be close to one of fixed points.

The constant $1-(p-1)/(n+p-2)$ in (5.1) is the decreasing ratio for the pth order Newton method.

5.2. The decreasing ratio for the family: the polynomial case

Let $f(z)$ be a polynomial of degree n. It is easy to see that

$$\lim_{z\to\infty}\frac{u}{z}=\frac{1}{n},\quad \lim_{z\to\infty}A_2u=\frac{n-1}{2n},\quad \lim_{z\to\infty}A_3u^2=\frac{(n-1)(n-2)}{6n^2}. \tag{5.2}$$

For $v\neq-1,\frac{1}{2}$, let $\phi_{m,v}(z)$ be the iteration function defined in (3.1) and let $D_{n,m,v}=\lim_{z\to\infty}\phi_{m,v}(z)/z$. By using (5.2), we have

$$D_{n,m,v}=1-\frac{2m(v+1)}{3n+\frac{1}{2}(n-m)(v-2)+(1/2\sqrt{3})\mathrm{sign}(2v-1)\sqrt{R_{n,m,v}}},$$

where

$$R_{n,m,v}=(-5n^2+18nm+35m^2)v^2+(-16n^2-36nm+4m^2)v+16n^2-4m^2.$$

When $D_{n,m,v}=0$ and $|z_0|$ is large, the next approximation z_1 is expected to be close to one of zeros of $f(z)$. In the next subsection we give the necessary and sufficient condition for $D_{n,m,v}=0$.

5.3. Derivation of the optimum parameter for polynomials

Proposition 5.1. *Let* $v \neq -1, \frac{1}{2}$ *and* $0 < m < n$. *Then* $D_{n,m,v} = 0$ *if and only if*

$$v = \begin{cases} \dfrac{2(m-2n)}{n-5m} & \text{if } n \neq 5m, \\ \infty & \text{if } n = 5m. \end{cases} \tag{5.3}$$

Proof. The case $v < \infty$. $D_{n,m,v} = 0$ if and only if

$$\left(-\frac{1}{2}n + \frac{5}{2}m\right) v - 2n + m = \frac{1}{2\sqrt{3}} \operatorname{sign}(2v-1)\sqrt{R_{n,m,v}}.$$

Squaring the both sides, we have

$$12((-\tfrac{1}{2}n + \tfrac{5}{2}m)v - 2n + m)^2 = (-5n^2 + 18nm + 35m^2)v^2 + (-16n^2 - 36nm + 4m^2)v + 16n^2 - 4m^2.$$

That is,

$$(n-m)(v+1)(-2m + 4n - 5mv + nv) = 0. \tag{5.4}$$

The case $v = \infty$. Because

$$D_{n,m,\infty} = \lim_{v \to \infty} D_{n,m,v} = 1 - \frac{4m}{n - m + \sqrt{\frac{1}{3}(-n+5m)(5n+7m)}},$$

$D_{n,m,\infty} = 0$ if and only if

$$5m - n = \sqrt{\tfrac{1}{3}(-n+5m)(5n+7m)}.$$

Squaring both sides, we obtain,

$$(n-m)(5m-n) = 0. \tag{5.5}$$

The result follows from (5.4) and (5.5). □

The parameter v in (5.3) is called the optimum parameter.

Remark 5.1. By using (5.2), we have the decreasing ratio of the Laguerre family (1.2):

$$D_{n,m,v} = 1 - \frac{v/n}{1 + \operatorname{sign}(v-m)\sqrt{(v-m)(v-1)/m - v(n-1)(v-m)/nm}}.$$

Thus $D_{n,m,v} = 0$ if and only if $(n-m)(v-n) = 0$, i.e., $v = n$. Therefore, the Laguerre method is excellent initial behavior (see [6]). By approximating $f(z_0) \approx z_0^n$, $f'(z_0) \approx nz_0^{n-1}$, $f''(z_0) \approx n(n-1)z_0^{n-2}$, for large $|z_0|$, Igarashi [8] proved this fact in the case $m = 1$.

6. Numerical examples

6.1. The local convergence

In this subsection we demonstrate the local convergence of the methods of the presented family on the polynomial [2, p. 246; 3]

$$\begin{aligned} f(z) &= z^{10} - 20z^9 + 175z^8 - 882z^7 + 2835z^6 - 6072z^5 + 8777z^4 - 8458z^3 + 5204z^2 - 1848z + 288 \\ &= (z-1)^4(z-2)^3(z-3)^2(z-4). \end{aligned} \tag{6.1}$$

Table 1
$f(z) = (z-1)^4(z-2)^3(z-3)^2(z-4)$, $m = 1$, $z_0 = 4.1$

	$v = 10$	$v = 2$	$v \to -1$	$v \to \infty$	$v = -7.6$	Laguerre	Newton twice
z_1	3.9_326	3.9957	4.0014	3.9_366	4.0_471	4.0_324	4.0040
z_2	$3.9_{11}71$	3.9_787	4.0_911	$3.9_{13}36$	$4.0_{16}31$	$4.0_{11}43$	4.0_728
z_3	$3.9_{45}27$	$3.9_{29}89$	$4.0_{38}36$	$3.9_{52}21$	4.0_{60}	$4.0_{34}23$	$4.0_{28}71$

Table 2
$f(z) = (z-1)^4(z-2)^3(z-3)^2(z-4)$, $m = 4$, $z_0 = 1.1$

	$v = 10$	$v = 2$	$v \to -1$	$v \to \infty$	$v = 3.2$	Laguerre	Schröder twice
z_1	0.9_454	1.0_418	0.9_386	0.9_441	0.9_485	1.0_475	0.99981
z_2	$0.9_{17}85$	$1.0_{19}17$	$0.9_{15}57$	$0.9_{17}49$	$0.9_{20}46$	$1.0_{13}25$	$0.9_{14}83$

Table 3
$f(z) = (z-1)^4(z-2)^3(z-3)^2(z-4)$, $m = 1$, $z_0 = 1000$

	$v = 2$	$v \to \infty$	$v = -7.6$	Laguerre	Newton twice
	$D = 0.8485 + 0.08571i$	$D = 0.7955 + 0.2215i$	$D = 0$	$D = 0$	$D = 0.81$
z_1	$849 + 86i$	$796 + 221i$	4.99	4.99	810
z_2	$728 - 0.0_518i$	$682 + 0.0_418i$	4.045	4.056	657
z_3	$618 + 62i$	$543 - 151i$	4.0_539	4.0_446	532
z_4	$530 - 0.0_537i$	$466 - 0.0_414i$	$4.0_{21}29$	$4.0_{13}29$	432

Initial approximations are $z_0 = 4.1$ and 1.1 and the multiplicity is $m = 1$ and 4, respectively. We consider five parameters of the family: $v = \deg f = 10$, $v = 2$, $v \to -1$, $v \to \infty$, and v the optimum defined in (5.3). For comparison we take up the Laguerre method and the Schröder (Newton, for $m = 1$) method twice:

$$\hat{z} = z - m\,\frac{f(z)}{f'(z)} - m\,\frac{f(z - mf(z)/f'(z))}{f'(z - mf(z)/f'(z))}, \tag{6.2}$$

which is quartic convergence but is not one-point.

It is well known that the attainable number of correct significant digits of ζ of multiplicity m is about l/m decimal digits for l decimal digit computation (see, e.g., [16, p. 62]). In order to avoid round-off errors we carried out using g + + (gcc version egcs-2.90.29) with MPPACK [7] with 60 significant digits, in this and the next subsection.

We show numerical results in Tables 1 and 2. A subscripted digit in a number indicates the number of repetitions of this digit. e.g., $3.9_326 \equiv 3.99926$.

6.2. *The initial behavior*

In this subsection we demonstrate the initial behavior of three methods of the presented family on the polynomial (6.1). Parameters of the family are $v = 2$, $v \to \infty$, and the optimum. For comparison we take up the Laguerre method and the Schröder method twice. Initial approximations are $z_0 = 1000, -1000$ and the multiplicity is $m = 1, 4$, respectively. We show decreasing ratios D and numerical results in Tables 3 and 4.

The methods with $v = 2$ and $v \to \infty$ and the Schröder method twice linearly converge when $|z_i - \zeta|$ is large. On the other hand, when $|z_0|$ is sufficiently large, the next approximations z_1 by the methods of decreasing 0, i.e., the method with the optimum parameter and the Laguerre method, are close to one of zeros. The local convergence of the method with optimum parameter is better than that of the Laguerre method.

Table 4
$f(z) = (z-1)^4(z-2)^3(z-3)^2(z-4)$, $m = 4$, $z_0 = -1000$

	$v = 2$	$v \to \infty$	$v = 3.2$	Laguerre	Schröder twice
	$D = 0.2857 + 0.2474i$	$D = 0.2768$	$D = 0$	$D = 0$	$D = 0.36$
z_1	$-284 - 248i$	-275	0.78	0.78	-359
z_2	$-141 - 0.0_3 70i$	-75	0.99988	0.99956	-128
z_3	$-39 + 35i$	-19	$0.9_{16}82$	$0.9_{11}50$	-45

Table 5
Number of successful convergence (A) and average number of iteration times (B) for 10 000 polynomials

Degree	Optimum parameter		Laguerre method	
	A	B	A	B
10	9973	4.30	9990	4.26
20	9746	4.92	9772	4.84
30	9483	5.31	9485	5.19
40	9266	5.62	9256	5.46
50	9034	5.89	8985	5.67
60	8845	6.05	8791	5.83
70	8642	6.24	8600	6.00
80	8502	6.37	8462	6.12
90	8382	6.56	8335	6.24

6.3. A comparison of the global convergence

In this subsection we test the global convergence of two methods of the decreasing ratio 0. We experiment numerically on polynomials with complex coefficients with real and imaginary parts obtained random from a uniform distribution over $(-1, 1)$. Ten thousand polynomials each of degree 10, 20, 30, ..., 90 are tested. All of the initial approximations are $1000 + 1000i$ and all zeros are assumed to be simple. We decide that the iteration successfully converges if $|f(z_i)| < 10^{-6}$ for some $i \leqslant 50$. We show in Table 5 the number of successful convergence and the average number of iteration times in this case. Numerical computation are carried out using gcc version 3.3.2. on Pentium III, approximately 16 digits.

Numerical experiments show that the global convergence of two methods of decreasing ratio 0 are similarly excellent. More precisely, the percentage of successful convergence is more than 90% for polynomials of order less than 50.

7. Conclusions

All the methods of our family except for limiting cases converge locally quartically to a simple and multiple zero with known multiplicity of analytic functions.

For polynomials the initial behavior and the global convergence of the method with the optimum parameter and the Laguerre method are similarly excellent. However, the local convergence of the former method is better than that of the latter method.

References

[1] E. Bodewig, Sur la méthods de Laguerre pour l'approximation des racines de certaines équations algébriques et sur la critique d'Hermite, Akad. Wetensh. Proc. (Amst.) 49 (1946) 911–921.

[2] E. Durand, Solutions Numériques des Equations Algébriques, Tome I, Masson et Cie, Paris, 1960.

[3] M.R. Farmer, G. Loizou, An algorithm for the total, or partial, factorization of a polynomial, Math. Proc. Cambridge Philos. Soc. 82 (1977) 427–437.

[4] W.F. Ford, J.A. Pennline, Accelerated convergence in Newton's method, SIAM Rev. 38 (1996) 658–659.

[5] J. Gerlach, Accelerated convergence in Newton's method, SIAM Rev. 36 (1994) 272–276.
[6] E. Hansen, M. Patrick, A family of root finding methods, Numer. Math. 27 (1977) 257–269.
[7] H. Hirayama, Multiple Precision Package, ⟨http://phase.hpcc.jp/phase/mppack/⟩.
[8] M. Igarashi, On the global behavior of one parameter families of iteration methods for algebraic equations, Trans. Japan Soc. Indus. Appl. Math. 2 (1992) 69–78 (in Japanese).
[9] M. Igarashi, H. Nagasaka, The relationship between the iteration times and the convergence order for Newton–Raphson-like methods, J. Inform. Process. 32 (1991) 1349–1354 (in Japanese).
[10] I. Kiss, Über eine Verallgemeinerung des Newtonschen Näherungsverfahrens, Z. angew. Math. Mech. 34 (1954) 68–69.
[11] E.N. Laguerre, Sur une méthode pour obtenir par approximation les racines d'une équation algébraique qui a toutes ses racines réelles, Œuvre de Laguerre 1 (1898) 87–103 (The original publication: Nouvelles Ann. de Math. 2e série, t. XIX 1880.).
[12] N. Osada, Asymptotic error constants of cubically convergent zero finding methods, J. Comput. Appl. Math., to appear.
[13] B. Parlett, Laguerre's method applied to the matrix eigenvalue problem, Math. Comp. 18 (1964) 464–485.
[14] L.D. Petković, M.S. Petković, D. Živković, Hansen–Patrick's family is of Laguerre's type, Novi Sad J. Math. 33 (2003) 109–115.
[15] J.F. Traub, Iterative Methods for the Solution of Equations, second ed., Chelsea, New York, 1982.
[16] J.H. Wilkinson, Rounding Errors in Algebraic Processes, Her Majesty's Stationary Office, London, 1963.