On reorienting graphs by pushing down maximal vertices—II

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Received 6 March 2002; received in revised form 28 April 2003; accepted 5 May 2003

Abstract

We provide a new proof of Propp’s Theorem that the set of orientations of a graph $G$ with a given flow difference can be made into a distributive lattice by allowing all vertices except a distinguished sink vertex to be pushed down. The method used allows us to determine the irreducible elements of this lattice and describe how the lattice changes if the sink varies. © 2003 Elsevier B.V. All rights reserved.

Keywords: Graph; Orientation; Pushing down; Flow difference; Lattice

0. Introduction

An orientation of a finite, undirected, simple, connected graph $G$ with $n > 1$ vertices assigns a direction to each of its edges so that one of its endpoints becomes the head of the edge and the other its tail. A vertex is maximal if it is the head of every edge incident with it. Pushing down a maximal vertex $v$ reverses the direction of all edges incident with $v$ so that they are now all directed away from $v$, and $v$ has become minimal. This operation was first studied by Mosesian in 1972 [3–5]. The main result of [7], the first paper of this series, which appeared in 1985, was that orientations that could be obtained from one another by pushing down could be characterized by having the same value for an invariant (the flow difference) defined on the cycles of a graph. In [8] Youngs and I showed that any integer valued function on the cycles of a graph that satisfied three natural properties of the flow difference could indeed be realized as the flow difference of an orientation. At around the same time Liu and Rival [2] using
a different approach investigated pushing down from the point of view of inverting monotone cuts.

All these papers considered pushing down any maximal vertex as legitimate, but already Mosesian had shown in 1972 that if one starts with an acyclic orientation and prohibits pushing down a single vertex \( s \) one eventually ends with an orientation in which \( s \) is the unique maximal vertex. In his unpublished 30 page preprint of 1994 Propp [9] considers the family of all orientations of a graph with a given flow difference. I shall call such a family a push down class, because these are precisely the orientations that can be obtained by pushing down from a given starting point. Propp proves that any push down class of acyclic orientations becomes a distributive lattice if one vertex, the sink, is not allowed to be pushed down. Most recently, Latapy and Magnien [1] have shown that every finite distributive lattice can be realized as an initial segment of one of these lattices. Figs. 1 and 2 show a small tree and three of the lattices obtained from it in this way (for a tree there is only one push down class). I omit a fourth lattice \( \mathcal{P}_c \).

After a preliminary section on notation and basic results, Section 2 of this paper gives an alternative proof of Propp’s theorem by assigning a non-negative integer vector \( \text{pd}_s(R) \) to each orientation \( R \) in a natural way and showing first that Propp’s ordering of the orientations is the natural product partial ordering of these vectors and then that the vectors form a sublattice of the product of \( n \) copies of the chain \( 1, \ldots, n \) (\( n \) is the number of vertices of \( G \)).

Section 3 analyses the structure of these graphs by determining their join irreducible elements. These fall into two classes: the orientations in the first class are join irreducible regardless of the choice of sink, while those in the second class change depending on the sink. The join irreducibles split into chains whose maximal and minimal elements are precisely the orientations in the first class.
The last section of the paper is devoted to describing how the lattices of a given push down class change if the sink is changed from $s$ to $t \neq s$. It turns out that the resulting lattices can be partitioned into sublattices $\mathcal{P}_i$ such that the order inside each sublattice remains unchanged while relations between elements in different lattices change drastically: if some element of one of these sublattices $\mathcal{P}_i$ lies below an element in a different sublattice $\mathcal{P}_j$ when $s$ is the sink, then no element of $\mathcal{P}_i$ lies below any element of $\mathcal{P}_j$ when $t$ is the sink.

1. Basic definitions and preliminary results

Throughout the paper we fix a finite simple connected graph $G$ with $n > 1$ vertices. We consider acyclic orientations $R$ on $G$, that is orientations for which there are no directed cycles.

It is easy to see that the operation of pushing down described in the introduction preserves this property. Indeed, it preserves the flow difference around any cycle, as will be shown directly. The flow difference plays an important role in some of the arguments that follow, so we digress to give its definition and state those of its properties proved in [7] that we shall make use of. We shall use the notation $x \prec_R y$ to say that $x$ and $y$ are adjacent and $y$ is the head of the edge $xy$ in $R$. If $x = v_1, \ldots, v_k = y$ is a path in $G$ and $R$ is an orientation of $G$, we define a forward edge of the path to be an edge $v_i \prec_R v_{i+1}$. The other edges of the path are backward. If necessary we signal the orientation in question by writing $R$-forward or $R$-backward. Note, however, that whether an edge is forward or backward depends both on $R$ and the implicit direction in which the path
is traversed. If there is a path from $x$ to $y$ with no $R$-backward edges we shall write $x \leq_R y$.

The flow difference of the path is the difference $f - b$ between the numbers $f$ of forward edges and $b$ of backward edges on the path. If a maximal vertex lies on a cycle (or on a path of which it is not one of the endpoints) the two adjacent edges point towards it, so one is forward and the other is backward. After it has been pushed down they point away from it, so they have just exchanged their directions and the flow difference is unchanged.

We say that an orientation $S$ is accessible from $R$ if it is possible to get from $R$ to $S$ by pushing down a sequence of vertices $v_1, \ldots, v_k$. The sequence itself will be called a push down sequence from $R$ to $S$. The vertices need not be distinct, but each must be maximal in the orientation obtained by pushing down its predecessors.

The results from [7] are the following

**Theorem** (Pretzel [7, Theorem 1]). The orientation $S$ is accessible from $R$ if and only if the flow differences around cycles are the same in $S$ and $R$.

**Proposition** (Pretzel [7, Proposition 3]). If $S$ is accessible from $R$ then $R$ is accessible from $S$.

**Proposition** (Pretzel [7, Proposition 4]). If $S$ is accessible from $R$ and they both have the same unique maximal vertex $s$, then $S = R$.

An equivalence class of orientations under accessibility will be called a push down class. We shall fix an (arbitrarily chosen) push down class $\mathcal{P}$ of acyclic orientations of $G$.

In the following, we shall limit pushing down to vertices distinct from a fixed (but arbitrarily chosen) sink $s$ and we shall call a push down sequence not containing $s$ an $s$-sequence. If there is an $s$-sequence from $R$ to $S$, then we say $S$ is $s$-accessible from $R$ and write $S \preceq_s R$. In contrast to ordinary accessibility $s$-accessibility is antisymmetric as the following proposition shows.

**Proposition 1.** (a) If $v$ occurs in a push down sequence $(v_i)$ from $R$ to $S$ twice, $v = v_i = v_j$ with $i \neq j$, and $w$ is adjacent to $v$, then $w = v_k$ for some $k$ with $i < k < j$.

(b) If the sequence $(v_i)$ is an $s$-sequence and the distance $d(v, s) = d$, then $v$ occurs in $(v_i)$ at most $d$ times.

(c) $s$-accessibility is anti-symmetric.

**Proof.** (a) After $v = v_i$ is pushed down the head of the edge $vv$ is $w$. In order that $v$ can be pushed down again this edge must be reversed. That can only be achieved by pushing down $w$.

(b) This is proved by induction on $d(v, s)$ starting with the fact that $s$ is prohibited from occurring in the sequence. Suppose that the statement is true for vertices with $d(w, s) = d - 1$ and let $v = w_0, w_1, \ldots, w_d = s$ be a path of minimal length from $v$ to $s$. Then $s$ cannot be adjacent to $v_k$ for any $k$ with $i < k < j$.
Then \( d(w_1, s) = d - 1 \) and so by induction hypothesis \( w_1 \) occurs in \( v_1, \ldots, v_k \) at most \( d - 1 \) times. By part (a) \( w \) must occur in \( v_1, \ldots, v_k \) between any two occurrences of \( v \). Hence \( v \) cannot occur more than \( d \) times.

(c) If \( S \neq R \) were mutually \( s \)-accessible orientations it would be possible to go from \( R \) to \( S \) and back again arbitrarily often. That would produce \( s \)-sequences in which the first vertex \( v_1 \) occurred arbitrarily often, contradicting part (b). \( \square \)

Since \( s \)-accessibility is obviously transitive, part (c) of the proposition shows that the orientations of our given class form a partially ordered set under \( \leq_s \), which we shall denote by \((\mathcal{P}, \leq_s)\). We shall now show that \( \mathcal{P} \) has a unique minimal element and characterise it.

**Proposition 2.** There is a unique minimal orientation \( \hat{s} \in (\mathcal{P}, \leq_s) \). It is the only orientation in \( \mathcal{P} \) with \( s \) as its unique maximal vertex.

**Proof.** Let \( R \) be any orientation in \( \mathcal{P} \) and suppose that some vertex \( v \neq s \) is maximal in \( R \). Then \( v \) can be pushed down and thus \( R \) is not minimal.

The second statement is just [7, Proposition 4]. \( \square \)

2. Special push down vectors and the lattice of orientations

We now introduce the main tool used in our investigation of the partially ordered set \((\mathcal{P}, \leq_s)\). List the vertices of \( G \) as \( x_1, \ldots, x_n \). To any orientation \( R \in \mathcal{P} \) we define the \( s \)-vector \( \text{pd}_s(R) = (\text{pd}_s(R; x_1), \ldots, \text{pd}_s(R; x_n)) \) by letting \( \text{pd}_s(R; x_i) \) be the number of times \( x_i \) is pushed down in a push down sequence going from \( R \) to \( \hat{s} \). Since we are not permitted to push down \( s \) we know that \( \text{pd}_s(R; s) = 0 \), but we retain it in the vector, because it makes it easier to investigate change of sink later. It is not quite trivial that \( \text{pd}_s(R) \) is well defined, but that will be established in the next proposition.

**Proposition 3.** Let \( R \) be an orientation in \( \mathcal{P} \) and let \( v \triangleleft_R w \) be adjacent vertices. Let \( (v_i) \) be a push down sequence of vertices from \( R \) to \( S \). Then \( v \) and \( w \) occur the same number of times in \( (v_i) \) if \( v \triangleleft_S w \) and otherwise \( w \) occurs once more than \( v \).

**Proof.** The vertices \( v \) and \( w \) must occur alternately in the sequence. Each time one of them occurs, the direction of \( vw \) is reversed. Thus if the edge ends with the same direction as it started they must be pushed down the same number of times. Since \( w \) is the head of \( vw \) it must be pushed down before \( v \) can be pushed down. If the direction of the edge is reversed then one vertex must occur more often than the other and that vertex can only be \( w \). \( \square \)

**Corollary 4.** (a) The number of times a vertex \( v \) occurs in any \( s \)-sequence from \( R \) to \( \hat{s} \) is constant \( \leq d(v, s) \).

(b) The orientation \( R \) is determined by its \( s \)-sequence \( \text{pd}_s(R) \).
Proof. (a) We prove this by induction on \(d(v, s)\). If \(d(v, s) = 0\), then \(v = s\) and the number in question is 0. Otherwise let \(w\) be adjacent to \(v\) with \(d(w, s) = d(v, s) - 1\). Then by induction hypothesis the number of times \(w\) occurs in such a sequence is a constant \(c \leq d(w, s)\). By the proposition the number of times \(v\) occurs is equal to \(c\) if the edge \(vw\) has the same direction in \(R\) and \(\hat{S}\). Otherwise it is equal to \(c + 1\) if \(v\) is the head of \(vw\) in \(R\) and equal to \(c - 1\) if \(w\) is its head in \(R\). Thus it is also constant. It is at most \(d(v, s)\) by Proposition 1(b).

(b) This now follows directly. \(R\) is obtained from \(\hat{S}\) by reversing precisely those edges \(xy\) for which \(p_d_s(R, x) \neq p_d_s(R, y)\). 

The set of \(s\)-vectors is given a natural (partial) order by setting \(p_d_s(S) \leq p_d_s(R)\) if \(p_d_s(S, x_i) \leq p_d_s(R, x_i)\) for all \(i\). It is fairly obvious that \(S \leq s\) \(R\) implies \(p_d_s(S) \leq p_d_s(R)\), but in fact the converse also holds, as we shall see. The key observation is contained in the next proposition.

Proposition 5. Let \(R \neq S\) be two orientations in \(P\). Then there exists an \(R\)-maximal vertex \(v\) with \(p_d_s(R, v) > p_d_s(S, v)\), or there exists an \(S\)-maximal vertex \(w\) with \(p_d_s(S, w) > p_d_s(R, w)\).

Proof. Since we know that the \(s\)-vectors determine the orientations and \(R \neq S\), it follows that there is a vertex \(u\) such that \(p_d_s(R, u) \neq p_d_s(S, u)\). We assume by symmetry that \(p_d_s(R, u) > p_d_s(S, u)\).

Let \(u = v_1, \ldots, v_k = v\) be an \(R\)-forward path from \(u\) to an \(R\)-maximal vertex \(v\). I claim that \(p_d_s(R, v_i) > p_d_s(S, v_i)\) for all \(i = 1, \ldots, k\). This is certainly true for \(v_1 = u\). Suppose then that it is true for \(v_{i-1}\). If the edge \(v_{i-1} \leq s v_i\), then

\[
p_d_s(R, v_i) - p_d_s(R, v_{i-1}) = p_d_s(S, v_i) - p_d_s(S, v_{i-1}).
\]

Otherwise

\[
p_d_s(R, v_i) \geq p_d_s(R, v_{i-1}) > p_d_s(S, v_{i-1}) \geq p_d_s(S, v_i).
\]

In either case it follows that \(p_d_s(R, v_i) > p_d_s(S, v_i)\).

An easy consequence of this proposition is our first main result.

Theorem 6. The orientation \(S\) is \(s\)-accessible from \(R\) if and only if \(p_d_s(S) \leq p_d_s(R)\).

Proof. Each time a vertex \(v\) is pushed down the value \(p_d_s(\cdot; v)\) is reduced by one, while all other entries in the \(s\)-vector remain unchanged. Hence if \(S\) is \(s\)-accessible from \(R\) we have \(p_d_s(S) \leq p_d_s(R)\).

To prove the converse we assume the condition holds and \(R \neq S\) (otherwise there is nothing to prove). There cannot be an \(S\)-maximal vertex \(w\) with \(p_d_s(S, w) > p_d_s(R, w)\) and so by the previous proposition there must be an \(R\)-maximal vertex \(v\) with \(p_d_s(R, v) > p_d_s(S, v)\). If we push down \(v\) to obtain an orientation \(R'\) then we still have \(p_d_s(R') \geq p_d_s(S)\). Inductively, it follows that \(S\) is \(s\)-accessible from \(R'\) and hence also from \(R\).
The theorem implies immediately that an orientation \( S \in \mathcal{P} \) is \( s \)-accessible from both \( R_1 \in \mathcal{P} \) and \( R_2 \in \mathcal{P} \) if and only if \( \text{pd}_s(S) \leq \text{pd}_s(R_1) \land \text{pd}_s(R_2) \), where the \( \land \) operation on integer vectors corresponds to taking the minimum value at each entry. We shall establish the lattice properties of \( \mathcal{P} \) by showing that \( \text{pd}_s(R_1) \land \text{pd}_s(R_2) \) is the \( s \)-vector of an orientation. This will follow easily from the characterization of \( s \)-vectors in the next theorem.

**Theorem 7.** A sequence \( p \) of non-negative integers \( p = (p_x; x \in G) \) is the \( s \)-vector of a (necessarily unique) orientation in \( \mathcal{P} \) if and only if \( p_s = 0 \) and it satisfies 
\[ p_y \leq p_x \leq p_y + 1 \quad \text{for all pairs of adjacent vertices } x \prec y. \]

**Proof.** The necessity of the condition is a special case of Proposition 3 with \( S = \hat{s} \).

To prove sufficiency we define the orientation \( R \) by directing each edge \( x \prec y \) with \( p_x = p_y \) towards \( y \) and each edge for which \( p_x = p_y + 1 \) towards \( x \). We must show that \( R \in \mathcal{P} \) and \( \text{pd}_s(R) = p \).

Let \( x = v_1, \ldots, v_k = y \) be a path from \( x \) to \( y \) in \( G \) and suppose that \( f' \) of its \( \hat{s} \)-forward edges and \( b \) of its \( \hat{s} \)-backward edges are reversed in \( R \). Then by the definition of \( R \) it follows that \( p_x = p_y + f' - b \). So the flow difference along \( (v_i) \) in \( R \) differs from that in \( \hat{s} \) by \( p_y - p_x \). As this holds for any path, the number of forward edges around a cycle that are reversed is equal to the number of backward edges that are reversed. Hence the flow difference around any cycle is the same in \( R \) and in \( \hat{s} \), which shows that \( R \in \mathcal{P} \).

Since \( R \in \mathcal{P} \), \( \text{pd}_s(R) \) is defined. From Corollary 4(a) it follows that \( \text{pd}_s(R; x) = \text{pd}_s(R; x) - \text{pd}_s(R; s) = p_x - p_s = p_s \).

Propps’s Theorem now follows as a simple corollary of the next proposition.

**Proposition 8.** Let \( R_1 \in \mathcal{P} \) and \( R_2 \in \mathcal{P} \). Then there are (necessarily unique) orientations \( S, T \in \mathcal{P} \), such that \( \text{pd}_s(S) = \text{pd}_s(R_1) \land \text{pd}_s(R_2) \) and \( \text{pd}_s(T) = \text{pd}_s(R_1) \lor \text{pd}_s(R_2) \).

**Proof.** Let \( x \prec y \) be adjacent vertices. Since both \( \text{pd}_s(R_1; x) \geq \text{pd}_s(R_1; y) \) and \( \text{pd}_s(R_2; x) \geq \text{pd}_s(R_2; y) \) it follows that
\[ \min_i \{\text{pd}_s(R_i; x) : i = 1, 2\} \geq \min_i \{\text{pd}_s(R_i; y) : i = 1, 2\}. \]

Similarly as \( \text{pd}_s(R_1; x) \leq \text{pd}_s(R_1; y) + 1 \) and \( \text{pd}_s(R_2; x) \leq \text{pd}_s(R_2; y) + 1 \) it follows that
\[ \min_i \{\text{pd}_s(R_i; x) : i = 1, 2\} \leq \min_i \{\text{pd}_s(R_i; y) : i = 1, 2\} + 1. \]

Hence \( \text{pd}_s(R_1) \lor \text{pd}_s(R_2) \) satisfies the condition of Theorem 7.

A similar argument proves the statement for \( \text{pd}_s(R_1) \lor \text{pd}_s(R_2) \).  

**Corollary 9.** (a) The set \( \text{pd}_s(\mathcal{P}) \) of \( s \)-vectors of orientations in \( \mathcal{P} \) is a sublattice of \( \mathbb{Z}_n^m \) (where \( \mathbb{Z}_n \) denotes the totally ordered set \( 0 < 1 < \cdots < n-1 \) and \( n \) is the number of vertices of \( G \)).
The set of orientations in $\mathcal{P}$ forms a distributive lattice under $s$-accessibility.

**Proof.** (a) The entry $\text{pd}(R; v)$ of an $s$-vector is at most $d(v, s) < n$ so the vectors form a subset of $\mathbb{Z}^n$. From Corollary 9 it follows that the set is closed under intersections and unions. Hence our set is indeed a sublattice of $\mathbb{Z}^n$.

(b) Theorem 6 states that the partially ordered set $(\mathcal{P}, \leq_s)$ is order-isomorphic to $(\text{pd}_s(\mathcal{P}), \leq)$. That set is a sublattice of $\mathbb{Z}^n$ which is distributive. □

3. The structure of the lattice of orientations

Every distributive lattice $A$ is determined by the partially ordered set of its join irreducible elements (which are those elements that cannot be represented as unions of smaller elements). For reasons of symmetry we include the minimal element of $A$ among the irreducibles.

**Notation.** We denote by $\hat{t}$ the (unique) orientation in the push down class $\mathcal{P}$ with $t$ as its sole maximal vertex and by $\tilde{s}$ a (non-unique) orientation in $\mathcal{P}$ in which no vertices other than $t$ and $s$ are maximal (so type $\tilde{s}$ includes $\hat{t}$ and $\hat{s}$).

Similarly, we denote by $\hat{t}$ the (unique) orientation in the push down class $\mathcal{P}$ with $t$ as its sole minimal vertex and by $\tilde{s}$ a (non-unique) orientation in $\mathcal{P}$ in which no vertices other than $t$ and $s$ are minimal.

The orientations of type $\hat{s}$ are precisely the join irreducibles of $(\mathcal{P}, \leq_s)$, as the following proposition shows.

**Theorem 10.** An orientation $R \neq \tilde{s} \in \mathcal{P}$ is join irreducible under $\leq_s$ if and only if there is at most one $R$-maximal element among the vertices $t \neq s$ of $G$.

**Proof.** Suppose that $t$ is the only maximal element of $R$ that is different from $s$ and let $S$ be the orientation obtained from $R$ by pushing down $t$. Then $T \leq_s R$ implies $T \leq_s S$, and so $R$ cannot be a union of lower orientations.

Conversely, suppose that $R$ has at least two maximal elements $a$ and $b$ not equal to $s$ and let $A$ and $B$ be the orientations obtained from $R$ by pushing down these elements respectively. Then

$$\text{pd}_s(A; a) + 1 = \text{pd}_s(R; a) = \text{pd}_s(B; a),$$

$$\text{pd}_s(B; b) + 1 = \text{pd}_s(R; b) = \text{pd}_s(A; b),$$

and $\text{pd}_s(A; c) = \text{pd}_s(R; c) = \text{pd}_s(B; c)$ for all other vertices.

Therefore $R = A \cup B$ and $R$ is not join irreducible. □

The following proposition addresses the question, how many different orientations of type $\hat{s}$ there are in $\mathcal{P}$.
**Proposition 11.** Let \( R \) and \( S \) be two orientations in \( \mathcal{P} \) that have no maximal vertices except \( t \) and \( s \). If for any path \( t = v_1, \ldots, v_k = s \) the number of backward edges in \( R \) is the same as the number in \( S \), it follows that \( R = S \).

**Proof.** We first remark that since the flow difference around cycles is constant for all orientations in \( \mathcal{P} \) the condition holds for all paths from \( t \) to \( s \) if it holds for one. So we choose a path that is directed from \( t \) to \( s \) in \( \hat{s} \).

Now it follows from Proposition 3 that \( \text{pd}_s(R; t) \) is the number of backward edges in our path under \( R \). Hence \( \text{pd}_s(R; t) = \text{pd}_s(S; t) \). Since \( \text{pd}_s(R; s) = 0 = \text{pd}_s(S; s) \), it now follows from Proposition 5 that \( R = S \). \( \square \)

We now investigate how to determine which irreducibles lie below a given orientation \( R \). We start by determining the set of \( t \) such that \( \hat{t} \leq_s R \).

**Proposition 12.** Let \( R \in \mathcal{P} \). Then \( \hat{t} \leq_s R \) in \( \mathcal{P} \) if and only if \( s \leq_R t \) (that is, there is an \( R \)-forward path from \( s \) to \( t \)).

**Proof.** Suppose \( \hat{t} \leq_s R \) and let \( s = v_1, \ldots, v_k = t \) be a forward path from \( s \) to \( t \) in \( \hat{t} \).

We claim that \( v_1, \ldots, v_k \) is also forward in \( R \). By Proposition 3 the number of times the vertices of \( v_{i+1} \) is pushed down in an \( s \)-sequence from \( R \) to \( \hat{t} \) is at most equal to the number of times \( v_i \) is pushed down. But \( s \) is not pushed down at all in such a sequence. Hence none of the vertices of \( v_1, \ldots, v_k \) are pushed down, and the path has the same orientation in \( R \) as it has \( \hat{t} \).

Conversely suppose that \( s = v_1, \ldots, v_k = t \) is a forward path from \( s \) to \( t \) in \( R \). We repeatedly push down maximal vertices not lying on \( v_1, \ldots, v_k \) until there are no such vertices left. This leaves the directions of the edges of \( v_1, \ldots, v_k \) unchanged and so we end with an orientation in which the only possible maximal vertex is \( t \). Since \( s \) lies on \( v_1, \ldots, v_k \) it was not pushed down and thus \( \hat{t} \) is \( s \)-accessible from \( R \). \( \square \)

It is slightly more difficult to determine when \( \hat{t}s \leq_s R \), but the condition is analogous.

**Theorem 13.** Let \( P = (v_1) \) be a path from \( t \) to \( s \) and let \( R \) be an orientation in \( \mathcal{P} \). Then there exists an orientation \( T \) of type \( \hat{t}s \) in which \( P \) has the same number of backward edges as it has in \( R \), or equivalently \( \text{pd}_s(T; t) = \text{pd}_s(R; t) \). Furthermore an orientation \( S \) of type \( \hat{t}s \) is \( s \)-accessible from \( R \) if and only if the number of \( S \)-backward edges in \( P \) is less than or equal to the number of \( R \)-backward edges in \( P \), or equivalently \( \text{pd}_s(S; t) \leq \text{pd}_s(R; t) \).

**Proof.** Again we note that the if either of the conditions on backward edges holds for one path \( P \) from \( t \) to \( s \) that condition will hold for all such paths. If we choose \( P \) to be an \( \hat{t}s \)-forward path, then the number of its \( Q \)-backward edges is \( \text{pd}_s(Q; t) \) for any orientation \( Q \). So we can concentrate on the versions of statements using \( \text{pd}_s(\cdot; t) \).

Starting with \( R \), repeatedly push down any maximal elements other than \( t \) and \( s \). When the process ends (which it must) we will have an orientation of type \( \hat{t}s \). Since
Suppose now that $S$ of type $\hat{s}$ is $s$-accessible from $R$. Then $\text{pd}_s(S) \leq \text{pd}_s(R)$ and so in particular $\text{pd}_s(S; t) \leq \text{pd}_s(R; t)$.

Conversely, suppose the condition is satisfied. We may suppose that $R \neq S$. Then there is no $S$-maximal vertex $v$ with $\text{pd}_s(S; v) > \text{pd}_s(R; v)$. So by Proposition 5 there is an $R$-maximal vertex $v$ with $\text{pd}_s(R; v) > \text{pd}_s(S; v)$. We can therefore push down $v$ to find an orientation $R' \leq_s R$ still satisfying the condition. Inductively it follows that $S \leq_s R' \leq_s R$. \hfill $\square$

**Corollary 14.** The orientations with no maximal vertices except $t$ and $s$ form a chain $\hat{s}=J_0, \ldots, J_d=\hat{t}$ in $(\mathcal{P}, \leq_s)$ of length at most $d(s, t)$. The orientation $J_i$ has $\text{pd}_s(J_i; t)=i$.

**Proof.** The first statement follows directly from the theorem. To prove the second we choose $P$ to be an $\hat{s}$-forward path from $t$ to $s$. Then by Theorem 7, $\text{pd}_s(R; t)$ is the number of $R$-backward edges on $P$. Starting with $J_i$ we can reduce $\text{pd}_s(J_i; t)$ by one by pushing down $t$. We then apply the process of the theorem to access an orientation of type $\hat{s}$ without reducing $\text{pd}_s(\cdot; t)$ further. That leads us to $J_{i-1}$. Thus $\text{pd}_s(J_{i-1}; t) = \text{pd}_s(J_i; t) - 1$. As $\text{pd}_s(\hat{s}; t) = 0$, the result follows. \hfill $\square$

By duality corresponding results hold for orientations of type $\hat{t}$ and $\tilde{s}$. They are the meet irreducibles of $(\mathcal{P}, \leq_s)$, they form a chain with $\hat{s}$ at the top and $\hat{t}$ at the bottom.

We shall also exploit the following result which links elements of the two types.

**Theorem 15.** The chains $\hat{s}=J_0 \leq_s J_1 \leq_s \cdots \leq_s J_d=\hat{t}$ of join irreducibles with no maximal elements other than $t$ and $s$, and $\tilde{t}=M_0 \leq_s M_1 \leq_s \cdots \leq_s M_d=\tilde{s}$ of meet irreducibles with no minimal elements except $t$ and $s$ have the same length. Furthermore $M_i$ is the minimal meet irreducible below $J_i$, and $J_i$ is the maximal join irreducible below $M_i$. Finally, the flow difference along any path from $t$ to $s$ is the same in $J_i$ and $M_i$ and $\text{pd}_s(J_i; t) = \text{pd}_s(M_i; t)$.

**Proof.** To each $M_i$ the maximal $J_k$ below $M_i$ has the same number of backward edges as $M_i$ on any path $t=v_1, \ldots, v_k=s$. Therefore the number of $M_i$ is at most equal to the number of $J_k$.

A dual argument shows that the minimal $M_i$ above $J_k$ also has the same number of backward edges as $J_k$ on any path $t=v_1, \ldots, v_k=s$. Hence the two chains have the same length. The other statements are special cases of Theorem 13. \hfill $\square$

### 4. General push down vectors and change of sink

In this final section we consider the relation between the lattices obtained by fixing different sinks $s$ and $t$. To that end we generalize our concept of an $s$-vector. Let $\hat{R}$ be an orientation and suppose we push down a sequence of vectors $v_1, \ldots, v_k$ to obtain an orientation $S$. List the vertices of $G$ as $x_1, \ldots, x_n$ and define the push down vector
pd(R, S) = (pd(R, S; x_1), ..., pd(R, S; x_n)) by letting pd(R, S; x_i) be the number of times x_i is pushed down in v_1, ..., v_k. Notice that we make no conditions on the sequence v_1, ..., v_k, in particular no vertex is ineligible for pushing down. Of course, pd(R, S) is not uniquely determined and we shall consider that problem first.

Proposition 16. (a) Given R, the push down vector pd(R, S) determines the orientation S.

(b) If p = (p_1, ..., p_n) and q = (q_1, ..., q_n) are two push down vectors both representing the transition from R to S, then q_i - p_i = q_1 - p_1 for all i = 1, ..., n.

(c) The vector 1 all of whose entries are 1 is a valid push down vector and represents the transition from R to itself.

Proof. (a) The argument of Proposition 3 adapts immediately to show that adjacent vertices are pushed down equally often if the edge connecting them has the same direction in S as its has in R. Otherwise the head of the edge in R is pushed down once more than its tail. Hence the direction of any edge in S can be read from pd(R, S).

(b) For x_i adjacent to x_j, the argument of part (a) shows that q_i - q_j = p_i - p_j. As the graph G is assumed to be connected it follows that q_i - q_j = p_i - p_j for any i and j. The claim now follows.

(c) It is shown in [6, Proposition 4] that if L is a linear extension of the orientation R then starting with the orientation R the vertices of G can be pushed down in descending order according to L. By part (a) the result of this sequence of push downs must be R. The corresponding push down vector is 1. 

In the light of this result we shall regard push down vectors as determined up to an additive constant and allow addition and subtraction of arbitrary multiples of 1, even if that produces negative entries. That gives general push down vectors pleasant additive properties.

Proposition 17. Push down vectors satisfy

pd(S, R) = -pd(R, S) and pd(R, S) + pd(S, T) = pd(R, T).

When we are dealing with (P, \leq_s) we shall normalize pd(R, S) to pd_s(R, S) in which the s-entry is 0. Then pd_s(R, S) = pd_s(R) - pd_s(S) and it follows directly from Theorem 6 that S is accessible from R if and only if pd_s(R, S) is non-negative.

Proposition 18. The orientation S is s-accessible from R if and only if the s-entry of pd(R, S) is minimal (in other words, pd(R, S; t) \geq pd(R, S; s) for all vertices of t \in G).

This allows us to establish a relation between the orders determined by s-accessibility and t-accessibility.

Theorem 19. Let R and S be two orientations in P.
(a) \( \text{pd}_i(\tilde{r}; t) = \text{pd}_j(\tilde{s}; s) \leq d(s, t) \), where \( d(s, t) \) is the distance between the vertices \( s \) and \( t \). Let us denote this common value by \( d \).

(b) \( \text{pd}_i(R; t) + \text{pd}_j(R; s) = d \).

(c) If \( \text{pd}_i(R; t) = \text{pd}_j(S; s) \) then \( S \) is \( s \)-accessible from \( R \) if and only if it is \( t \)-accessible from \( R \).

(d) If \( \text{pd}_i(R; t) > \text{pd}_j(S; s) \), then \( R \) is not \( s \)-accessible from \( S \) and \( S \) is not \( t \)-accessible from \( R \). \( S \) is \( s \)-accessible from \( R \) if and only if \( \text{pd}_i(R) \leq \text{pd}_j(S) \). \( R \) is \( t \)-accessible from \( S \) if and only if \( \text{pd}_i(R) \leq \text{pd}_j(S) + (\text{pd}_i(R; t) - \text{pd}_j(S; t)) \).

**Proof.** (a) By Proposition 17 we have \( \text{pd}(\hat{s}, \hat{t}) = -\text{pd}(\hat{t}, \hat{s}) \). If we normalize \( \text{pd}(\hat{t}, \hat{s}) \) to \( \text{pd}(\hat{t}, \hat{s}) = \text{pd}_i(\hat{t}) \) by making its \( s \) entry equal to 0, then the \( t \) entry of \( \text{pd}(\hat{s}, \hat{t}) \) is \( -\text{pd}_i(\hat{t}; t) \). Therefore \( \text{pd}_i(\hat{s}) = -\text{pd}_i(\hat{t}) + \text{pd}_i(\hat{t}; t) \).

(b) Again by Proposition 17

\[
\text{pd}_i(R) = \text{pd}_j(R) - \text{pd}_j(\tilde{t}) - (\text{pd}_j(R; t) - d) \leq \text{pd}_j(R; t) + d. \tag{*}
\]

Evaluating at \( s \) we obtain

\[
\text{pd}_i(R; s) = 0 - 0 - \text{pd}_i(R; t) + d. \tag{**}
\]

(c) It follows directly from part (b) that if \( \text{pd}_i(R; t) = \text{pd}_j(S; s) \) then \( \text{pd}_i(R) - \text{pd}_j(R) = \text{pd}_j(S) - \text{pd}_j(S) \). Hence \( \text{pd}_i(R) \leq \text{pd}_j(S) \) if and only if \( \text{pd}_i(R) \leq \text{pd}_j(S) \).

(d) That \( R \) is not \( s \)-accessible from \( S \) follows from Theorem 6. Using Eq. (**)) we find

\[
\text{pd}_i(S; s) = -\text{pd}_j(S; t) + d > -\text{pd}_j(R; s) + d = \text{pd}_j(R; t).
\]

So \( S \) is not \( t \)-accessible from \( R \). The condition for \( S \) to be \( s \)-accessible from \( R \) is just a restatement of the condition of Theorem 6. Using Eq. (*) and eliminating common terms we transform \( \text{pd}_i(R) \leq \text{pd}_j(S) \), the corresponding condition for \( R \) to be \( t \)-accessible from \( S \), into

\[
\text{pd}_i(R) - \text{pd}_i(R; t) \leq \text{pd}_j(S) - \text{pd}_j(S; t). \quad \Box
\]

The theorem gives a partial answer to the question when two distributive lattices of the same size can be represented as the lattice of the same push down class of a graph \( G \) with respect to two different points \( s \) and \( t \).

**Corollary 20.** Let \( s \) and \( t \) be two vertices of \( G \) and let \( d = \text{pd}_i(\tilde{r}; t) = \text{pd}_j(\tilde{s}; s) \). Then \( \mathcal{P} \) can be partitioned into subsets \( \mathcal{P}_0, \ldots, \mathcal{P}_d \) with the following properties for \( 0 \leq i < j \leq d \).

1. The subsets \( \mathcal{P}_i \) are sublattices with identical orderings in \( (\mathcal{P}, \leq_s) \) and \( (\mathcal{P}, \leq_t) \).
2. Elements of \( \mathcal{P}_i \) do not lie above elements of \( \mathcal{P}_j \) in \( (\mathcal{P}, \leq_s) \) and elements of \( \mathcal{P}_j \) do not lie above elements of \( \mathcal{P}_i \) in \( (\mathcal{P}, \leq_t) \).
3. The minimal element of \( \mathcal{P}_0 \) is \( \tilde{s} \); its maximal element is \( \tilde{t} \). The minimal element of \( \mathcal{P}_d \) is \( \tilde{r} \) its maximal element is \( \tilde{s} \). In general, the minimal element of \( \mathcal{P}_i \) is \( \tilde{i} \),
Table 1
Push down vectors for $P_x$, $P_a$, and $P_b$

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$xabc$</td>
<td>0111</td>
<td>0011</td>
<td>0101</td>
</tr>
<tr>
<td>$abc$</td>
<td>0110</td>
<td>0010</td>
<td>0100</td>
</tr>
<tr>
<td>$ab$</td>
<td>0010</td>
<td>1022</td>
<td>1202</td>
</tr>
<tr>
<td>$ac$</td>
<td>0101</td>
<td>0001</td>
<td>0001</td>
</tr>
<tr>
<td>$a$</td>
<td>0100</td>
<td>0000</td>
<td>1201</td>
</tr>
<tr>
<td>$b$</td>
<td>0010</td>
<td>1021</td>
<td>0000</td>
</tr>
<tr>
<td>$c$</td>
<td>0001</td>
<td>1012</td>
<td>1102</td>
</tr>
<tr>
<td>$x$</td>
<td>0000</td>
<td>1011</td>
<td>1101</td>
</tr>
</tbody>
</table>

and its maximal element is $\tilde{t}s$. In these two sets the flow difference along any path from $t$ to $s$ (or equivalently the value $pd_\ell(\cdot,t)$) is the same; indeed this holds for all orientations in $P_i$.

Example. Table 1 gives the $x$-, $a$- and $b$-vectors of the various orientations of the graph $G$ of Fig. 1. The lattice $P_x$ is illustrated in Fig. 1; the lattices $P_a$ and $P_b$ are illustrated in Fig. 2.

If we take $s = a$ and $t = x$ in the theorem, then $P_0 = \{a, ab, ac, abc\}$ and $P_1 = \{x, b, c, bc\}$. On the other hand, if we take $s = a$ and $t = b$ then we get $P_0 = \{a, ac\}$, $P_1 = \{ab, abc, x, c\}$ and $P_2 = \{b, bc\}$. Note that $\hat{abc} = \hat{x}$, $\hat{bc} = \hat{a}$, $\hat{ac} = \hat{b}$ and $\hat{c} = \hat{ab}$.

Proof. The subset $P_i$ is defined as $P_i = \{T \in P : pd_\ell(T; t) = i\}$. Then statement (a) follows directly from part (c) of the theorem. Similarly statement (b) follows directly from part (d).

To prove statement (c) note that the minimal element of $P_i$ must be obtained from any element without pushing down $t$ or $s$. Therefore, the flow difference in any path from $t$ to $s$ remains unchanged in all of $P_i$. Also since all elements except $t$ and $s$ can be pushed down, the minimal element must be of the form $\hat{t}s$, and so by duality the maximal element must be $\tilde{t}s$. The statements about the first and last class are just special cases of this general result.

References