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Equisingularity classes of birational projections of normal singularities to a plane

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Abstract

Given a birational normal extension $\mathcal{O}$ of a two-dimensional local regular ring $(R, \mathfrak{m})$, we describe all the equisingularity types of the complete $\mathfrak{m}$-primary ideals J in R whose blowing-up $X = Bl_J(R)$ has some point Q whose local ring $\mathcal{O}_{X,Q}$ is analytically isomorphic to $\mathcal{O}$.
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0. Introduction

A sandwiched surface singularity (X, Q) is a normal surface singularity that can be projected birationally to a non-singular surface. From a more algebraic point of view, the local ring $\mathcal{O}$ of any sandwiched singularity is a birational normal extension of a two-dimensional local regular ring R. Once a sandwiched surface singularity has been fixed, in this paper we address the problem of describing the equisingularity classes of all its birational projections to a plane.

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The problem of classifying the germs of sandwiched surface singularities was already posed by Spivakovsky. As he claims in [14] this problem has two parts: discrete and continuous. The continuous part is to some extent equivalent to the problem of the moduli of plane curve singularities, while the main result of this paper solves completely the combinatorial part.

Any birational projection from a sandwiched singularity to a plane is obtained by the morphism of blowing up a complete $\mathfrak{m}_O$-primary ideal in the local ring of a regular point O on the plane. Our goal is to give all the equisingularity types of these ideals. Namely, fixed a birational normal extension $\mathcal{O}$ of a local regular ring $(R, \mathfrak{m}_O)$, we describe the equisingularity type of any complete $\mathfrak{m}_O$-primary ideal $J \subset R$ such that its blowing-up $X = Bl_J(R)$ has some point Q whose local ring $\mathcal{O}_{X,Q}$ is analytically isomorphic to $\mathcal{O}$. In this case, we will say that the surface X contains the singularity $\mathcal{O}$ for short, making a slight abuse of language. This is done by describing the Enriques diagram of the cluster of base points of any such ideal J: such a diagram will be called an Enriques diagram for the singularity $\mathcal{O}$. Recall that an Enriques diagram is a tree together with a binary relation (proximity) representing the topological equivalence classes of clusters of points in the plane (see Section 1.3). Previous works by Spivakovsky [14] and Möhring [12] describe a type of the Enriques diagram that exists for any given sandwiched surface singularity (detailed in Section 2) and provide other types mostly in the case of cyclic quotients (see [12, 2.7]) and minimal singularities (see [12, 2.5]).

The organization of the paper is as follows. Section 1 is devoted to recalling some definitions concerning the language of infinitely near points, sandwiched surface singularities and graphs. Fundamental for our purpose will be the notion of the Enriques diagram, introduced in [2]. In Section 2, after some technical results, we introduce the concept of *contraction* for a sandwiched surface singularity $\mathcal{O}$. By a contraction we mean the resolution graph $\Gamma_{\mathcal{O}}$ of $\mathcal{O}$ (a sandwiched graph, as introduced in [14]) enriched by some proximities between their vertices, these proximities being compatible with the weights of the graph. Fixed a sandwiched graph, the problem of finding the whole list of possibilities for such proximities is the hard part of our work. This is achieved in Section 3, by proving that any contraction for $\Gamma_{\mathcal{O}}$ may be recovered from some contraction of the graph obtained from $\Gamma_{\mathcal{O}}$ by removing one end. This fact is the key result in order to describe a procedure to obtain all the contractions for $\mathcal{O}$. Finally, in Section 4, we explain how to complete contractions in order to obtain any Enriques diagram for $\mathcal{O}$.

1. Preliminaries

In this section, we fix notation and recall some of the facts concerning sandwiched surface singularities and base points of ideals which will be used throughout this paper, and we focus on our problem. A standard reference for most of the tools and techniques treated here is the book by Casas-Alvero [1].

1.1. Infinitely near points and complete ideals

Let $(R, \mathfrak{m}_O)$ be a regular local two-dimensional $\mathbb{C}$-algebra and $S = Spec(R)$. A *cluster* of points of S with origin O is a finite set K of points infinitely near or equal to O such that, for any $p \in K$, K contains all points to which p is infinitely near. A subset of a cluster K is a *subcluster* if it is a cluster (with origin some point of K). By assigning integral multiplicities $\nu = \{\nu_p\}$ to the points of K, we obtain a *weighted cluster* $\mathcal{K} = (K, \nu)$; the multiplicities ν are called the *virtual multiplicities* of $\mathcal{K}$. A point p is said to be *proximate* to another point q if p is infinitely near to q and lies on the strict transform of the exceptional divisor of blowing up q. We write $p \geqslant q$ if p

is infinitely near or equal to q, and $p \to q$ if p is proximate to q. The relation $\geqslant$ is an ordering of the infinitely near points, and it will be considered as their natural ordering. A point $p \in K$ is *free* if it is proximate to only one point, which is necessarily the immediate predecessor, and p is *satellite* if it is proximate to two points; otherwise, the point is necessarily the origin of the cluster. The number $\rho_p^{\mathcal{K}} = \nu_p - \sum_{q \to p} \nu_q$ is the *excess* at p of $\mathcal{K}$. *Consistent clusters* are those weighted clusters with non-negative excesses at all their points. We write $\mathcal{K}_+$ for the set of the *dicritical* points of $\mathcal{K}$, that is, the points with positive excess.

If $\mathcal{K} = (K, \nu)$ and $\mathcal{K}' = (K', \nu')$, define the *sum* $\mathcal{K} + \mathcal{K}'$ as the weighted cluster whose set of points is $K \cup K'$ and whose virtual multiplicities are $\nu_p + \nu'_p$ for $p \in K \cup K'$ [1, 8.4]. This operation is clearly associative and commutative, thus making the set of all weighted clusters with origin at O a semigroup. Consider the set $\mathbf{W}$ of all consistent clusters with origin at O with positive virtual multiplicities. Again, $\mathbf{W}$ equipped with the sum is clearly a semigroup. A weighted cluster $\mathcal{K} \in \mathbf{W}$ is called *irreducible* if it is so as an element of the semigroup $\mathbf{W}$, that is, $\mathcal{K}$ is not the sum of two elements of $\mathbf{W}$. To any point p, $p \geqslant O$, we associate the irreducible cluster $\mathcal{K}(p)$ in $\mathbf{W}$ which has virtual multiplicity one at p, which will be called the *irreducible cluster in $\mathbf{W}$ ending at p*.

Two *clusters K and K'* are called *similar* if there is a bijection (*similarity*) $\varphi : K \to K'$ so that both φ and φ^{-1} preserve ordering and proximity. Two *weighted clusters* $\mathcal{K} = (K, \nu)$ and $\mathcal{K}' = (K', \nu')$ are called *similar* if there is a similarity between K and K' preserving virtual multiplicities (see [1, 8.3]). An analytic isomorphism Φ defined in a neighborhood of O clearly induces a similarity between each cluster K with origin O and its image $\Phi(K)$ [1, 3.3]. Furthermore, if Φ is only a homeomorphism, then K and $\Phi(K)$ are still similar [1, 8.3.12].

If $\pi_K : S_K \to S$ is the composition of the blowing-ups of all points in K, write E_K for the exceptional divisor of π_K and $\{E_p\}_{p \in K}$ for its irreducible components. We denote by $\mathbf{A}_K = (E_p \cdot E_q)_{p,q \in K}$ the *intersection matrix* of E_K: if $p = q$, its coefficient is just the self-intersection of E_p, and equals $-r_p - 1$, where r_p is the number of points in K proximate to p; if $p \neq q$, $E_p \cdot E_q = 1$ in case $E_p \cap E_q \neq \emptyset$, and $E_p \cdot E_q = 0$ otherwise. It can be easily seen that $E_p \cap E_q \neq \emptyset$ if and only if p is maximal among the points of K proximate to q or vice-versa (cf. [1, 4.4.2]). Notice that $\mathbf{A}_K$ is an invariant of the similarity class of K.

If $\mathcal{K}$ is a weighted cluster, there is a well-established notion for a germ of a curve to go through $\mathcal{K}$ (which is a linear condition, see [1, 4.1]), and the equations of all curves going through $\mathcal{K}$ define a complete $\mathfrak{m}_O$-primary ideal $H_{\mathcal{K}}$ in R (see [1, 8.3]). Any complete $\mathfrak{m}_O$-primary ideal J in R has a *weighted cluster of base points*, denoted by $BP(J)$, which consists of the points shared by, and the multiplicities of, the curves defined by generic elements of J. Moreover, the maps $J \mapsto BP(J)$ and $\mathcal{K} \mapsto H_{\mathcal{K}}$ are reciprocal isomorphisms between the semigroup $\mathbf{I}_R$ of complete $\mathfrak{m}_O$-primary ideals in R (equipped with the product of ideals) and the semigroup $\mathbf{W}$ (see [1, 8.4.11] for details). If $p \geqslant O$, denote by $J(p)$ the ideal in $\mathbf{I}_R$ corresponding by the preceding isomorphism to the irreducible cluster $\mathcal{K}(p) \in \mathbf{W}$ ending at p, that is, $J(p) = H_{\mathcal{K}(p)}$.

A couple of ideals J, J' in $\mathbf{I}_R$ are *equisingular* if $BP(J)$ and $BP(J')$ are similar [1, 8.3]. Notice that two equisingular complete ideals in $\mathbf{I}_R$ have equisingular (that is, topologically equivalent) generic germs and equal codimensions [1, 8.3.9].

1.2. Sandwiched surface singularities

The main references here are [3,14]. If $I \in \mathbf{I}_R$, we denote by $\pi_I : X = Bl_I(R) \to S$ the blowing-up of I. The surface X is not regular in general, and its singularities are sandwiched

singularities. Moreover, if K is the set of base points of I, we have a commutative diagram

$$
\begin{array}{ccc}
S_K & \xrightarrow{\;f\;} & X \\
 & \searrow{\scriptstyle \pi_K} & \downarrow{\scriptstyle \pi_I} \\
 & & S
\end{array}
\tag{1.1}
$$

where the morphism f, given by the universal property of the blowing-up, is the minimal res-
olution of the singularities of X [14, Remark 1.4]. Let $\mathcal{O}$ be any singularity of X; then we say
that I is an *ideal for* $\mathcal{O}$. It follows that the exceptional divisor $E_{\mathcal{O}}$ associated with the minimal
resolution of $\mathcal{O}$ is a connected subset of the exceptional divisor E_K. There is a bijection between
the set of irreducible components of $\pi_I^{-1}(O)$ and the set of dicritical points of $\mathcal{K} = BP(I)$ (see
[5,11]). This allows to write $\{L_p\}_{p \in \mathcal{K}_+}$ for the set of these components on X. Because of this,
we may think of $\mathcal{O}$ as a singularity obtained by contracting a connected curve (which will be
called $E_{\mathcal{O}}$) of E_K containing no component with self-intersection -1 (such a component E_p
is necessarily the exceptional divisor of the blowing-up of some maximal point of K and thus,
a dicritical point).

For any ideal $J = \prod_{p \in \mathcal{K}_+} J(p)^{\alpha(p)}$ with positive $\alpha(p)$, we have an analytic isomorphism $X \cong
Bl_J(R)$ (cf. [14, Corollary I.1.5]). Since we are interested in sandwiched singularities modulo
analytic isomorphism, the relevant information we need to retain about $\mathcal{K} = BP(I)$ is, on one
hand, its set of points K and, on the other, knowing which of the points of K are dicritical (the
rest being non-dicritical, of excess zero).

1.3. Enriques diagrams and dual graphs

We introduce the Enriques diagrams and the weighted dual graphs related to them. The En-
riques diagrams are combinatorial objects that enclose the topological information of the clusters
of infinitely near points in S, namely they represent the similarity classes of clusters.

A *tree* is a finite graph with a partial order relation $\leqslant$ between the vertices, without loops,
which has a single initial vertex, or *root*, and every other vertex has a unique immediate prede-
cessor. The vertex q is said to be a *successor* of p if p is the immediate predecessor of q. If p
has no successors then it is an *extremal vertex*. The set of vertices of a graph will be denoted by
the same letter as the graph itself. An *Enriques diagram D* ([2, Enriques IV.I], [1, Casas 3.9];
see also [6,8] for a combinatorial presentation) is a tree with a binary relation between vertices,
called *proximity* and denoted by $\rightarrow_D$, which satisfies:

1. Every vertex but the root is proximate to its immediate predecessor; the root is proximate to
   no vertex.
2. If $p \rightarrow_D q$, then $p > q$ and there is at most one other vertex in D proximate to both of them.
3. Any vertex is proximate to at most two other vertices. The vertices which are proximate
   to two points are called *satellite*, the other vertices, but the root, are called *free*. If q is the
   immediate predecessor of p, and $p \rightarrow_D q'$, then $q \rightarrow_D q'$.

If p is a vertex in D, we write $r_D(p)$ for the number of vertices in D proximate to p. A satellite
vertex is said to be *satellite of* the last free vertex that precedes it. In order to express graphically
the proximity relation, Enriques diagrams are drawn according to the following rules:

1. If q is a free successor of p then the edge going from p to q is smooth and curved and, if p is not the root, it has at p the same tangent as the edge joining p to its predecessor.
2. The sequence of edges connecting a maximal succession of vertices proximate to the same vertex p are shaped into a line segment, orthogonal to the edge joining p to the first vertex of the sequence.

If K is a cluster, there is an Enriques diagram D_K naturally associated with it by taking one vertex for each point of K and the proximity of the cluster as the proximity of D_K; conversely, for any Enriques diagram D there is some cluster K with origin O whose Enriques diagram D_K is D. If no confusion may arise, we will label the points in K and their corresponding vertices in D_K with the same symbol. A connected subtree of an Enriques diagram D is a *subdiagram* if it is an Enriques diagram with a root some vertex of D and whose proximity is the restriction of the proximity of D. Observe that K' is a subcluster of K if and only if the associated Enriques diagram $D_{K'}$ is a subdiagram of D_K. If D is the Enriques diagram associated with K and $p \in K$, we denote by $D(p)$ the Enriques diagram of the irreducible cluster $\mathcal{K}(p)$ ending at p. If p is extremal, $D(p)$ is called a *branch* of D.

By assigning to an Enriques diagram D a *marking map* $\rho : D \rightarrow \{+, 0\}$, we obtain a *marked Enriques diagram* $\mathcal{D} = (D, \rho)$. Any consistent cluster $\mathcal{K}$ induces a marking map $\rho : D_K \rightarrow \{+, 0\}$ by taking $\rho(p) = +$ if p corresponds to a dicritical point of $\mathcal{K}$ (in this case, p is called a *dicritical vertex*), and $\rho(p) = 0$ otherwise. A *marked subdiagram* $\mathcal{D}' = (D', \rho')$ of $\mathcal{D}$ is a marked Enriques diagram where D' is a subdiagram of D and ρ' is the restriction of ρ to D'. Observe that the extremal vertices of a marked Enriques diagram associated with some $\mathcal{K} \in \mathbf{W}$ are always dicritical.

If $\mathcal{O}$ is a sandwiched surface singularity, we say that $\mathcal{D}$ is an *Enriques diagram for* $\mathcal{O}$ if it is the marked Enriques diagram of $BP(I)$, for some ideal $I \in \mathbf{I}_R$ for $\mathcal{O}$. Under this framework, the goal of this paper is to describe all the Enriques diagrams for a given $\mathcal{O}$.

Incidence between the irreducible components of a divisor E on a surface is usually represented by means of the *weighted dual graph* of E. It is defined by taking a vertex for each component of E, and by joining two vertices by an edge if and only if the corresponding components of E meet; each vertex is weighted by taking minus the self-intersection of the corresponding component. If D is the Enriques diagram of a cluster K, the (*weighted*) *dual graph* of D, denoted by Γ_D, is the weighted dual graph of the exceptional divisor E_K (which has no loops). Since the information enclosed in the weighted dual graph is the same as that contained in the intersection matrix of K, this definition is consistent.

Remark 1.1. The similarity class of a cluster may be represented either by its Enriques diagram or by its weighted dual graph, since from the intersection matrix the ordering $\leqslant$ (of being infinitely near) and the proximity may be inferred. In fact, this is also true for rational surface singularities. From the intersection matrix $\mathbf{A}$ of a rational surface singularity, the fundamental cycle Z may be computed (see [10, Theorem 4.2]) and from it, the order of the blowing-ups performed to resolve the singularity: the negative entries of $\mathbf{A}Z$ correspond to the exceptional components having appeared in the last blowing-up (cf. Theorem 1.14 of [13]). It is worth noticing that the proximity of D cannot be recovered in general only from its dual graph without weights (see [1, 4.4]).

A *non-singular graph* is the weighted dual graph of some Enriques diagram (cf. [14]). The vertex in Γ_D corresponding to p in D will be denoted by p, written in the `typewriter` font.

By the (weighted) dual graph of a marked Enriques diagram $\mathcal{D} = (D, \rho)$ we mean the dual graph of D and it will also be denoted by Γ_D. The vertices of Γ_D corresponding to dicritical vertices (non-dicritical, respectively) of $\mathcal{D}$ will be called *dicritical* (*non-dicritical*, respectively), too. If I is a complete $\mathfrak{m}_O$-primary ideal in R, we will write $\mathcal{D}_I$ and Γ_I to mean the marked Enriques diagram and the weighted dual graph of its cluster of base points $BP(I)$, respectively.

Remark 1.2. The weighted dual graph Γ_D can be constructed as follows: take one vertex in Γ_D for each vertex of D, and connect two vertices in Γ_D by an edge if and only if one of the corresponding vertices in D is maximal among the vertices in D proximate to the other. Moreover, if p is a vertex of Γ_D, its weight $\omega(\mathrm{p})$ is $r_D(p) + 1$ (cf. §4.4 of [1] for details). A vertex $\mathrm{p} \in \Gamma_D$ has weight $\omega(\mathrm{p}) = 1$ if and only if p is extremal in D.

A *chain* $ch_\Gamma(\mathrm{q}, \mathrm{p})$ of a graph Γ without loops is the subgraph composed of all vertices and edges between the vertices $\mathrm{q}, \mathrm{p} \in \Gamma$; it will be described by the ordered sequence of vertices between q and p, and $d_\Gamma(\mathrm{q}, \mathrm{p})$ will denote its length. Two vertices $\mathrm{q}, \mathrm{p} \in \Gamma$ are *adjacent* if $d_\Gamma(\mathrm{p}, \mathrm{q}) = 1$; a vertex is an *end* if it is adjacent to only one vertex. A *weighted subgraph* of a weighted graph Γ is a subgraph of Γ whose vertices have the same weights as Γ.

The following result describes the proximity relations between the vertices of a chain:

Lemma 1.3. *Let $q \leqslant p$ be two vertices of an Enriques diagram D, and consider the non-singular graph Γ of D.*

(a) *If $\mathrm{u} \in ch_\Gamma(\mathrm{q}, \mathrm{p})$, then $q \leqslant u$; if $u \neq p$, either $u \leqslant p$ or $p \leqslant u$. Moreover, all the vertices of $ch_\Gamma(\mathrm{q}, \mathrm{p})$ correspond to vertices in the same branch of D.*

(b) *Write $ch_\Gamma(\mathrm{q}, \mathrm{p}) = \{\mathrm{u}_0 = \mathrm{q}, \mathrm{u}_1, \ldots, \mathrm{u}_n, \mathrm{u}_{n+1} = \mathrm{p}\}$. There exists some $i_0 \in \{0, \ldots, n+1\}$ satisfying $u_{k+1} \rightarrow_D u_k$ for $k \in \{0, \ldots, i_0 - 1\}$, and $u_k \rightarrow_D u_{k+1}$ for $k \in \{i_0, \ldots, n\}$. Furthermore, if $j \geqslant i_0$, u_j is proximate to some $u_{\sigma(j)}$ with $\sigma(j) \leqslant i_0 - 1$.*

Proof. The first assertion of (a) is just Lemma 3.2 of [4]. Now, if $\mathrm{u}, \mathrm{v} \in ch_\Gamma(\mathrm{q}, \mathrm{p})$, either $\mathrm{u} \in ch_\Gamma(\mathrm{q}, \mathrm{v})$ or $\mathrm{v} \in ch_\Gamma(\mathrm{q}, \mathrm{u})$; in any case, either u is infinitely near to v or vice-versa, and hence u and v cannot belong to different branches of D. Now, we prove (b). First of all, note that for any $i \in \{0, \ldots, n\}$ either u_i is proximate to u_{i+1} or vice-versa (cf. 1.2). By (a), u_1 is necessarily infinitely near to u_0 and so, proximate to it. If each u_{i+1} is proximate to u_i, the first claim is obvious by taking $i_0 = n + 1$. Assume that there exists some $i \in \{1, \ldots, n\}$ such that u_i is proximate to u_{i+1}, and take i_0 to be minimal with this property. We claim that $u_{k+1} \rightarrow_D u_k$ for $k \in \{0, \ldots, i_0 - 1\}$, and $u_k \rightarrow_D u_{k+1}$ for $k \in \{i_0, \ldots, n\}$. To show this, assume that there exists some $j \geqslant i_0 + 1$ such that $u_{j+1} \rightarrow_D u_j$ and take j_0 to be minimal. Then, both u_{j_0-1} and u_{j_0+1} are proximate to u_{j_0} and, since they are adjacent to it, they are maximal among the vertices of D proximate to u_{j_0}. However, by (a) they are in the same branch of D, so they must be equal, which is impossible. Note that u_{i_0} is the maximal point in D among the vertices belonging to $ch_\Gamma(\mathrm{q}, \mathrm{p})$. By (a) we know that every u_j, $j \geqslant i_0$, is infinitely near to q. Write $u_{\sigma(j)}$ for the maximal vertex in D among the vertices belonging to $ch_\Gamma(\mathrm{q}, \mathrm{u}_{i_0-1})$ such that u_j is infinitely near to it. By (a) applied to $ch_\Gamma(\mathrm{u}_{\sigma(j)}, \mathrm{u}_j)$ and the maximality of $u_{\sigma(j)}$, necessarily $u_{\sigma(j)+1}$ is infinitely near to u_j and, because $u_{\sigma(j)+1}$ is proximate to $u_{\sigma(j)}$, so is u_j. This completes the proof. □

The *resolution graph* of a sandwiched singularity $\mathcal{O}$ is the weighted dual graph of the exceptional divisor of the minimal resolution of $\mathcal{O}$. These graphs are called *sandwiched graphs*

and they are characterized as the weighted subgraphs of some non-singular graph containing no vertices of weight 1 (see [14, Proposition II.1.11]; cf. forthcoming 1.4). In particular, the graph obtained from a sandwiched graph by removing an end is still a sandwiched graph.

Remark 1.4. If $\mathcal{D}$ is an Enriques diagram for $\mathcal{O}$ and Γ_D^0 is the weighted subgraph of Γ_D comprising only the non-dicritical vertices, then $\Gamma_{\mathcal{O}}$ equals one of the connected components of Γ_D^0, whose vertices (and their corresponding vertices in D) will be called *non-dicritical vertices relative* to $\mathcal{O}$.

2. Sandwiched singularities and their Enriques diagrams

In Remark 1.4 we have observed that if $\mathcal{D}$ is an Enriques diagram for a sandwiched surface singularity $\mathcal{O}$, then its dual graph Γ_D contains the resolution graph $\Gamma_{\mathcal{O}}$ as a weighted subgraph. Given any Enriques diagram D, the following proposition shows that this combinatorial condition is sufficient to infer a result of geometrical nature: suitable marking maps ρ can be chosen so that (D, ρ) becomes an Enriques diagram for $\mathcal{O}$.

Proposition 2.1. *Let $\mathcal{O}$ be a sandwiched surface singularity and let $\mathcal{D} = (D, \rho)$ be a marked Enriques diagram. Then, $\mathcal{D}$ is an Enriques diagram for $\mathcal{O}$ if and only if*

1. *The dual graph Γ_D contains $\Gamma_{\mathcal{O}}$ as a weighted subgraph.*
2. *$\rho(p) = 0$ if $\mathrm{p} \in \Gamma_{\mathcal{O}}$; and $\rho(p) = +$ if $\mathrm{p} \in \Gamma_D \setminus \Gamma_{\mathcal{O}}$ and it is adjacent to some vertex of $\Gamma_{\mathcal{O}}$.*

Proof. The "only if" part follows from 1.4, and, since $\mathcal{D} = (D, \rho)$ is an Enriques diagram for $\mathcal{O}$, the marking map ρ must satisfy the statement in order to assure that the non-dicritical vertices of D relative to $\mathcal{O}$ correspond exactly to the vertices of $\Gamma_{\mathcal{O}}$. For the "if" part, consider the reduced exceptional divisor $E_{\mathcal{O}}$ of the minimal resolution of $\mathcal{O}$. Using plumbing (see [10], [14, Remark I.1.10]), we can glue smooth rational curves in order to obtain a configuration E_{Γ_D} containing $E_{\mathcal{O}}$, having dual graph Γ_D and being contained on a smooth surface S'. Since Γ_D is a non-singular graph, by Castelnuovo's criterion E_{Γ_D} contracts to a non-singular point O [14, II.1.10] on a surface S. This contraction factors into a composition of point blowing-ups [9, Theorem 5.7], so it determines a cluster with origin at O and having Enriques diagram D [1, 4.4]. It is clear that S' is the surface obtained from S by blowing up all points in K. Consider a system of virtual multiplicities $\nu = \nu_K$ for K in such a way that all the points have positive excess except for those corresponding to the vertices of the subgraph $\Gamma_{\mathcal{O}} \subset \Gamma_D$, which have excess 0 (such a system exists by [1, 8.4.1]). Notice that no maximal point of K corresponds to a vertex of $\Gamma_{\mathcal{O}}$, since any vertex of the resolution graph $\Gamma_{\mathcal{O}}$ has weight strictly greater than 1. Hence $\mathcal{K} = (K, \nu)$ is a consistent cluster with positive virtual multiplicities. Let $I = H_{\mathcal{K}}$ and $X = Bl_I(R)$. The morphism $f : S' \to X$ given by the universal property of blowing up is the minimal resolution of the singularities of X [14, II.1.4] and its exceptional components correspond to those points of $\mathcal{K}$ having excess 0. Therefore, $E_{\mathcal{O}}$ is the exceptional divisor of f and so, the singularity on X given by its contraction is isomorphic to $\mathcal{O}$ [9, Theorem 3.13]. $\quad\square$

Given a sandwiched surface singularity $\mathcal{O}$, there is not a unique non-singular graph Γ containing $\Gamma_{\mathcal{O}}$ as a weighted subgraph (see Example 2.5). In fact, it is possible to construct infinitely many non-singular graphs Γ containing a given sandwiched graph $\Gamma_{\mathcal{O}}$, as Example 2.2 shows.

Example 2.2. If $I \in \mathbf{I}_R$ is an ideal for $\mathcal{O}$ and $X = Bl_I(R)$, by choosing any non-singular point in the exceptional locus of X, and blowing up this point, we obtain a new surface X' containing $\mathcal{O}$, as well. This X' is the blowing-up of a complete $\mathfrak{m}_{\mathcal{O}}$-primary ideal $J_1 = II_1 \subset R$, where I_1 has codimension one in I (Theorem 3.5 of [3]), and the dual graph Γ_{J_1} contains a vertex more than Γ_I. In this way, an infinite chain of ideals in $\mathbf{I}_R$

$$\cdots \subset J_n \subset \cdots \subset J_1 \subset I \subset R$$

for $\mathcal{O}$ can be constructed, and each Γ_{J_n} contains $\Gamma_{\mathcal{O}}$ as a weighted subgraph. Moreover, for any n the Enriques diagram $\mathcal{D}_{J_{n-1}}$ is a marked subdiagram of $\mathcal{D}_{J_n}$.

Lemma 2.3. *Let $\mathcal{D}$ be an Enriques diagram for $\mathcal{O}$. Consider the set C of non-dicritical vertices of $\mathcal{D}$ relative to $\mathcal{O}$.*

(a) *There is a tree structure on C induced by the natural ordering $\leqslant$ of D.*
(b) *For any $p, q \in C$ define $p \to_C q$ if and only if $p \to_D q$. Then, $\to_C$ is a proximity, which turns C into an Enriques diagram.*

Proof. To exhibit the tree structure of C, we will prove that

1. There is a unique minimal element of C by $\leqslant$, which is taken as the root of C;
2. For any $p \in C$, its immediate predecessor in C is the maximal element of $\{q \in C : q < p\}$.

Suppose that p and q are two different minimal vertices in C, and write w for the maximal vertex in $D(p) \cap D(q)$ (this is the maximal vertex which both q and p are infinitely near or equal to). Then, as Γ_D contains no loops,

$$ch_{\Gamma_D}(q, p) = ch_{\Gamma_D}(q, w) \cup ch_{\Gamma_D}(w, p),$$

and, by the connectivity of Γ_C, we infer that $w \in C$, contradicting the minimality of q and p. We denote by O_C the minimal vertex of C, which is set as the root of C. On the other hand, if $p \in C$, $p \neq O_C$, the vertices in $D(p)$ are totally ordered by the natural ordering $\leqslant$ of D. Hence, there exists a unique immediate predecessor of p, which is the maximal element of $\{q \in C \mid q < p\}$, and this proves (a).

Now, to prove (b), we show that $\to_C$ defined as above is a proximity relation for C. Since the root is the minimal vertex of C, it is clear that it is proximate to no other vertex of C. If $p \neq O_C$, its immediate predecessor q_0 in C is the maximal element of $\{q \in C \mid q < p\}$; hence $q_0 < p$ and $q_0 \in C$. Then (b) of 1.3 says that p is proximate to some vertex w of $ch_{\Gamma_D}(q_0, p)$, and (a) of 1.3 says that w is infinitely near or equal to q_0. Since $ch_{\Gamma_D}(q_0, p) \subset \Gamma_C$, this leads to a contradiction. This proves the first condition of the proximity (see Section 1.3). The conditions 2 and 3 are clearly satisfied. □

An Enriques diagram C obtained as in Lemma 2.3 will be called *a contraction for $\mathcal{O}$* (or *for $\Gamma_{\mathcal{O}}$*) *associated with $\mathcal{D}$*. Reciprocally, we will also say that $\mathcal{D}$ is *associated with the contraction C*.

Remark 2.4. Notice that, by virtue of 2.1, any Enriques diagram (respectively, any contraction) for $\mathcal{O}$ is in fact an Enriques diagram (respectively, a contraction) for any sandwiched surface
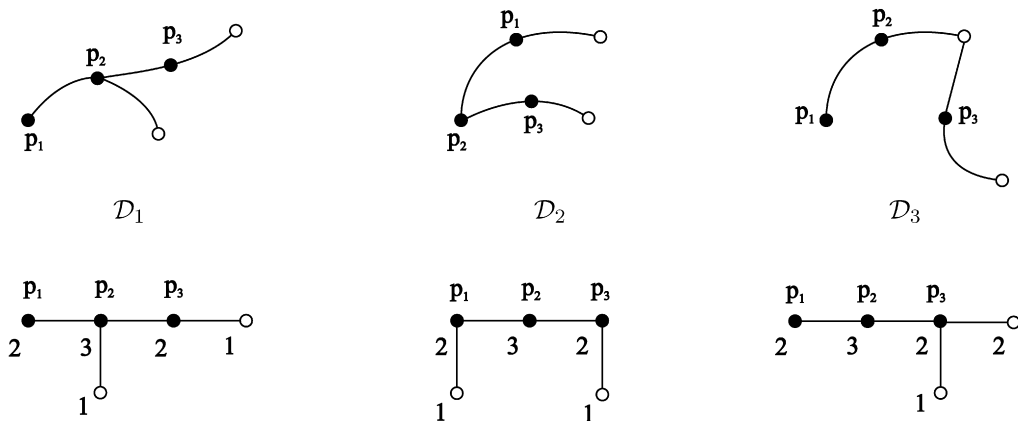
Fig. 1. Three different marked Enriques diagrams for the same singularity $\mathcal{O}$ and their corresponding dual graphs. Dicritical vertices are represented with the white-filled dots.

singularity whose resolution graph is $\Gamma_{\mathcal{O}}$. A contraction for $\mathcal{O}$ may also be regarded as an enrichment of the resolution graph $\Gamma_{\mathcal{O}}$ by some proximities between their vertices, these proximities being compatible with the weights of $\Gamma_{\mathcal{O}}$ in the sense of Lemma 2.7 below.

Example 2.5. Figure 1 provides three distinct Enriques diagrams for the same sandwiched singularity: they are not apparently related, namely one is not a subdiagram of the other, as was the case in Example 2.2. Notice that $\mathcal{D}_1$ and $\mathcal{D}_3$ give rise to the same contraction, which is the Enriques subdiagram of D_1 comprising the black dots.

Remark 2.6. If $\mathcal{D}$ is an Enriques diagram for $\mathcal{O}$ and C is the associated contraction, C is not, in general, an Enriques subdiagram of D (see Enriques diagram $\mathcal{D}_3$ of Fig. 1). In particular, if $I \in \mathbf{I}_R$ is an ideal for $\mathcal{O}$ with Enriques diagram $\mathcal{D}$, the set of points of $\mathcal{K} = BP(I)$ corresponding to the vertices of C does not constitute, in general, a subcluster of $\mathcal{K}$.

Lemma 2.7. *Let C be a contraction for $\Gamma_{\mathcal{O}}$ associated with an Enriques diagram $\mathcal{D}$. Then for any vertex $p \in C$,*

$$\omega_{\Gamma_C}(\mathrm{p}) \leqslant \omega_{\Gamma_D}(\mathrm{p}) = \omega_{\Gamma_{\mathcal{O}}}(\mathrm{p}), \tag{2.1}$$

and the inequality is strict at the extremal vertices of C. In particular, Γ_C is not a weighted subgraph of Γ_D.

Proof. The inequality comes from the definition of contraction, since the number of vertices proximate to p in C is less or equal than in D. If p is extremal in C, $\omega_{\Gamma_C}(\mathrm{p}) = 1$, while $\omega_{\Gamma_D}(\mathrm{p}) > 1$, since p is a non-dicritical vertex of $\mathcal{D}$ and hence necessarily non-extremal in D (see 1.2). The last assertion follows by considering the weights at the extremal vertices of C. □

In [14, Corollary II.1.14], Spivakovsky introduced a type of birational projection into a plane that could be achieved for any sandwiched singularity. Namely he showed that, once a sandwiched surface singularity $\mathcal{O}$ is fixed, an ideal $I \in \mathbf{I}_R$ can be chosen in such a way that:

(i) $\mathcal{O}$ is the only singularity of $X = Bl_I(R)$;
(ii) the strict transform (by the minimal resolution of X) of any exceptional component of $\pi_I^{-1}(O)$ is a curve of the first kind, that is, the strict transform by f (see diagram (1.1)) of any L_p with $p \in BP(I)_+$ has self-intersection equal to -1.

An ideal satisfying the above conditions (i) and (ii) (cf. [12, 2.3]) will be called an *S-ideal for* $\mathcal{O}$. A marked Enriques diagram associated with an S-ideal for $\mathcal{O}$ will be called an *S-Enriques diagram for* $\mathcal{O}$. The following result describes what the equisingularity classes of S-Enriques diagrams look like:

Lemma 2.8. *An ideal $I \in \mathbf{I}_R$ is an S-ideal if and only if the dicritical vertices of $\mathcal{D}_I$ are free and extremal.*

Proof. Write $\mathcal{D}$ for the Enriques diagram of $\mathcal{K} = BP(I)$. First of all, note that $X = Bl_I(R)$ has only one singularity $\mathcal{O}$ if and only if any non-dicritical vertex of $\Gamma_\mathcal{D}$ belongs to $\Gamma_\mathcal{O}$. Let $p \in \mathcal{K}_+$ and assume that there exists some $q \in K$ infinitely near to p. We may assume that q is an immediate successor of p. Then, $\omega_{\Gamma_\mathcal{D}}(\mathrm{p}) = r_\mathcal{D}(p) + 1 \geqslant 2$ against condition (ii). Therefore, p must be maximal in K. Now, assume that p is satellite, proximate to u_1 and u_2. Then, $\mathrm{p} \in ch_{\Gamma_\mathcal{D}}(\mathrm{u}_1, \mathrm{u}_2)$. Necessarily, u_1 and u_2 are not dicritical points of K and thus, $\mathrm{u}_1, \mathrm{u}_2 \in \Gamma_\mathcal{O}$. It follows that $\mathrm{p} \in \Gamma_\mathcal{O}$ against the assumption $p \in \mathcal{K}_+$.

   Conversely, if the dicritical vertices of $\mathcal{D}$ are free and extremal, the union of the non-dicritical vertices of $\Gamma_\mathcal{D}$ is connected and hence X has only one singularity. Moreover, as above, the self-intersection of the strict transform on S_K of any component L_p with $p \in \mathcal{K}_+$ is -1. □

   A contraction C associated with an S-Enriques diagram will be called an *S-contraction*. Contrary to what happened for general contractions (recall 2.6), an S-contraction C is a subdiagram of its associated S-Enriques diagram $\mathcal{D}$; furthermore, any S-contraction is associated with a unique S-Enriques diagram:

Proposition 2.9. *If $\mathcal{D}$ is an S-Enriques diagram for $\mathcal{O}$, then an S-contraction C associated with $\mathcal{D}$ satisfies*:

(a) *C is a subdiagram of D;*
(b) *D can be recovered from C by adding at any vertex $p \in C$ a number of $\omega_{\Gamma_\mathcal{O}}(\mathrm{p}) - \omega_{\Gamma_C}(\mathrm{p})$ free successors; $\mathcal{D} = (D, \rho)$ is recovered by defining the marking map ρ as $\rho(p) = 0$ if $p \in C$, and $\rho(p) = +$ otherwise.*

Proof. By virtue of 2.8, any dicritical point of D is free and extremal. Therefore, for any $p \in C$, there are exactly $\omega_{\Gamma_\mathcal{O}}(\mathrm{p}) - \omega_{\Gamma_C}(\mathrm{p})$ of these vertices in the first neighborhood of p. This gives both claims. □

   Next result is a sort of converse of Lemma 2.7:

Corollary 2.10. *Let C be an Enriques diagram and assume that the dual graph Γ_C equals $\Gamma_\mathcal{O}$ and satisfies $\omega_{\Gamma_C}(\mathrm{p}) \leqslant \omega_{\Gamma_\mathcal{O}}(\mathrm{p})$ at each vertex $p \in C$. Then C is an S-contraction for $\Gamma_\mathcal{O}$.*

Proof. To any $p \in C$ add exactly $\omega_{\Gamma_{\mathcal{O}}}(\mathrm{p}) - \omega_{\Gamma_C}(\mathrm{p})$ free successors to obtain from C a marked Enriques diagram $\mathcal{D}$ whose dicritical vertices are the extremal ones. By construction the dicritical vertices are also free, $\Gamma_{\mathcal{O}}$ is a weighted subgraph of Γ_D. Then invoking 2.1 and 2.8 we are done. □

Let us end this section by showing that the family of contractions for a sandwiched surface singularity equals the family of S-contractions:

Proposition 2.11. *Any contraction for $\Gamma_{\mathcal{O}}$ is an S-contraction.*

Proof. Let $\mathcal{D}$ be an Enriques diagram for $\mathcal{O}$ and let C be a contraction for $\Gamma_{\mathcal{O}}$ associated with $\mathcal{D}$. Define a new Enriques diagram D' from C by adding to each vertex $p \in C$ as many free successors as the number of vertices in $D \setminus C$ that are proximate to p. Taking $\rho(p) = 0$ if $p \in C$ and $\rho(p) = +$ otherwise, the resulting Enriques diagram $\mathcal{D}' = (D', \rho)$ for $\mathcal{O}$ is an S-Enriques diagram associated with C (see 2.1). □

3. Contractions for a sandwiched surface singularity

In this section we describe all the contractions for a given sandwiched surface singularity $\mathcal{O}$. Observe that, by virtue of 2.9 and 2.11, this is equivalent to listing all the equisingularity classes of the S-ideals for $\mathcal{O}$.

Suppose that the resolution graph $\Gamma_{\mathcal{O}}$ has n vertices and that v is an end of $\Gamma_{\mathcal{O}} = \Gamma_n$. The weighted graph obtained by removing v is again a sandwiched graph and will be denoted by Γ_{n-1} (see Section 1.3). We want to detail a procedure to obtain all the contractions for Γ_n from the contractions for Γ_{n-1}. Then, by induction on n, the whole list of contractions for a given sandwiched singularity will be inferred just from its resolution graph.

The first result of this section describes how the vertex v looks like in any contraction:

Lemma 3.1. *The vertex v (corresponding to the end v of $\Gamma_{\mathcal{O}}$) in any contraction C for $\mathcal{O}$ is either the root or free. Furthermore, if v is not the root of C, then either v is extremal or v has a unique successor, which is satellite of v.*

Proof. Assume that v is satellite, proximate to the vertices u_1 and u_2 in C, and suppose that u_2 is proximate to u_1. We will show that $\mathrm{v} \in ch_{\Gamma_C}(\mathrm{u}_1, \mathrm{u}_2)$, thus contradicting that v is an end, and proving the first claim. Consider the Enriques subdiagram $C(v)$ of C comprising all the points preceding or equal to v. In particular, u_1, u_2 are both in $C(v)$, and v is maximal among the points of $C(v)$ proximate to u_1, and also to u_2. Hence, as vertices of $\Gamma_{C(v)}$, v is adjacent to both u_1 and u_2 and so, $\mathrm{v} \in ch_{\Gamma_{C(v)}}(\mathrm{u}_1, \mathrm{u}_2)$. Now, the rest of vertices of $C \setminus C(v)$ all lie after some vertex of $C(v)$, giving rise to blowing-ups of extra points. The combinatorial effect of these blowing-ups is translated in the dual graph by the elementary modifications introduced in I.1.5 of [14], those of the first kind corresponding to the blowing-ups of free points while those of the second kind to the blowing-ups of satellite points. From their definition, it is immediate that these modifications respect the property of being in the chain determined by two vertices already in the graph.

For the second claim, let q be the only vertex in $\Gamma_{\mathcal{O}}$ to which v is adjacent, and assume that v is not the root of C. We distinguish two cases. The first one is when v is maximal among the points in C proximate to q. Since v is an end, there are no vertices in C proximate to v and v is an extremal vertex of C. The second case is when q is maximal among the points in C proximate

to v. Since v is an end, v must have a unique successor, say w, preceding q. Denote by u the immediate predecessor of v. Since v is free and v is adjacent only to q, v cannot be the last point in C proximate to u. Hence w must be proximate to u and so w is satellite of v. This completes the proof. □

The following result shows how to construct a contraction for Γ_n from a contraction for Γ_{n-1}. Moreover, any contraction for Γ_n can be obtained in this way.

Theorem 3.2. *Suppose Γ_{n-1} is obtained from a sandwiched graph Γ_n by removing an end v. Let u be the unique vertex in Γ_n to which v is adjacent, and let C' be a contraction for Γ_{n-1}. Define a new Enriques diagram by taking $C = C' \cup \{v\}$ and adding to C' extra proximities relating v according to one of the following rules:*

1. *If $\omega_{\Gamma_{C'}}(u) < \omega_{\Gamma_n}(u)$, set v in C as a free successor of u.*
2. *If $u = q_r \to_{C'} q_{r-1} \to_{C'} \cdots \to_{C'} q_1$ are free vertices in C' with $1 \leqslant r < \omega_{\Gamma_n}(v)$ and $q_1 \to_{C'} q_0$, then set $q_i \to_C v$ for all $i \in \{1, \ldots, r\}$, and either set $q_0 \to_C v$, provided q_0 is the root of C' and $r < \omega_{\Gamma_n}(v) - 1$ (with v becoming the root of C), or set $v \to_C q_0$, provided $\omega_{\Gamma_{n-1}}(q_0) < \omega_{\Gamma_n}(q_0)$.*

Then C is a contraction for Γ_n. Moreover, any contraction C for Γ_n can be constructed from some contraction C' for Γ_{n-1} as above.

Proof. Clearly C defined as above satisfies $\Gamma_C = \Gamma_n$ and $\omega_{\Gamma_C}(p) \leqslant \omega_{\Gamma_n}(p)$ at each vertex $p \in C$. Thus, invoking 2.10 and 2.11, the first claim follows. By virtue of 3.1, the vertex v of C corresponding to v is either free or the root of C. Since v is adjacent to u in Γ_n, there are only two possibilities for their corresponding vertices in C:

Case 1. v is maximal among the vertices in C proximate to u. Hence v cannot be the root of C, and by 3.1 v is free. Moreover, v is an extremal vertex of C: otherwise, 3.1 implies that v has a unique successor, which is satellite of v and thus proximate to u, contradicting the maximality of v among the vertices proximate to u. Therefore the set of vertices of $C' = C \setminus \{v\}$ is connected and has a tree structure. By considering the restriction of the proximity of C to C', C' becomes an Enriques diagram. Clearly the graphs $\Gamma_{C'}$ and Γ_{n-1} are equal (disregarding weights). Observe that $r_{C'}(u) = r_C(u) - 1$ and that $r_{C'}(q) = r_C(q) - 1$ if $q \in C$, $q \neq u$. Applying 1.2, we have

$$\omega_{\Gamma_{C'}}(q) = \omega_{\Gamma_C}(q) \leqslant \omega_{\Gamma_n}(q) = \omega_{\Gamma_{n-1}}(q),$$

for any $q \in C' \setminus \{u\}$ and

$$\omega_{\Gamma_{C'}}(u) = \omega_{\Gamma_C}(u) - 1 < \omega_{\Gamma_C}(u) \leqslant \omega_{\Gamma_n}(u) = \omega_{\Gamma_{n-1}}(u).$$

Thus, invoking 2.10 and 2.11, C' is a contraction for Γ_{n-1}. Finally, notice that the Enriques diagram C is obtained from C' by the procedure of the first rule of the statement, and we are done in this case.

Case 2. u is maximal among the vertices in C proximate to v. Let $p_1, \ldots, p_j = u$ be the vertices in C preceding or equal to u which are proximate to v. First of all, we define on the set of vertices of $C' = C \setminus \{v\}$ a tree structure. We distinguish two cases:

2.1. If v is the root of C, then take p_1 as the root of C', and for any $q \in C' \setminus \{p_1\}$ declare that p is the immediate predecessor of q in C' if and only if p is the immediate predecessor of q in C.

2.2. Otherwise, take the root of C as the root of C'; for any $q \in C' \setminus \{p_1\}$ declare that p is the immediate predecessor of q in C' if and only if p is the immediate predecessor of q in C; declare that the immediate predecessor of p_1 in C' is the immediate predecessor p_0 of v in C.

Restrict the proximity of C to C', namely, for any $q, q' \in C'$ set $q \to_{C'} q'$ if and only if $q \to_C q'$. Let us check that it satisfies the properties 1 to 3 of a proximity (see Section 1.3). In the first case (where v is the root of C) these properties are clearly satisfied. In the second case, the only condition that must be checked is property 1 for the vertex p_1, namely, that $p_1 \to_C p_0$. Since p_1 is a successor of v in C, by 3.1 we infer that p_1 is satellite of v. Thus p_1 is satellite in C: p_1 is proximate to its immediate predecessor in C, which is v, and to some point of C, say p; moreover, v must be proximate to p, as well. Since v is proximate to p_0, we infer that $p = p_0$ and $p_1 \to_C p_0$, as desired. Therefore C' is an Enriques diagram, whose dual graph $\Gamma_{C'}$ equals Γ_{n-1} disregarding weights. Observe that $r_{C'}(p_0) = r_C(p_0) - 1$ and $r_{C'}(q) = r_C(q)$ if $q \in D$, $q \neq p_0$. Applying 1.2, we have

$$\omega_{\Gamma_{C'}}(q) = \omega_{\Gamma_C}(q) \leqslant \omega_{\Gamma_n}(q) = \omega_{\Gamma_{n-1}}(q),$$

for any $q \in C' \setminus \{p_0\}$ and

$$\omega_{\Gamma_{C'}}(p_0) = \omega_{\Gamma_C}(p_0) - 1 < \omega_{\Gamma_C}(p_0) \leqslant \omega_{\Gamma_n}(p_0) = \omega_{\Gamma_{n-1}}(p_0).$$

Thus, invoking 2.10 and 2.11, C' is a contraction for Γ_{n-1}.

Finally, it remains to show that the Enriques diagram C may be obtained from C' by the procedure of the second rule of the statement. Indeed, according to the proximity defined in C', notice first that $\{u = p_j \to_{C'} \cdots \to_{C'} p_1\}$ is a chain of free vertices in C' preceding or equal to u, and that p_1 is the root of C' if and only if v is the root of C. On the other hand, recall that the proximity relations in C involving the vertex v are

$$p_1 \to_C v, \quad \ldots, \quad p_j \to_C v,$$

and the further proximity relation $v \to_C p_0$ must be added in case p_1 is not the root of C'. This is exactly what performs the operation of the second rule, and we are done. $\square$

The whole list of contractions for $\Gamma_{\mathcal{O}}$ are obtained by applying recursively the rules of 3.2. Let us just sketch the main steps of an implementation of this procedure. Each step of this procedure adds a new vertex keeping the proximities already defined. The idea is that the bigger the weights of $\Gamma_{\mathcal{O}}$ are, the more Enriques diagrams for $\mathcal{O}$ can be found.

Step 1. Choose any vertex of $\Gamma_{\mathcal{O}}$, say p_1, and take $\Gamma_1 = (p_1, \omega_{\Gamma_{\mathcal{O}}}(p_1))$ and $C_1 = \bullet_{p_1}$.

Step i. Assume that Γ_{i-1}, C_{i-1} have been obtained, where Γ_{i-1} is a subgraph of $\Gamma_{\mathcal{O}}$. Choose a vertex p_i adjacent to some $q \in \Gamma_{i-1}$. The graph Γ_i is obtained by adding p_i with weight $\omega_{\Gamma_{\mathcal{O}}}(p_i)$ to Γ_{i-1}, adjacent to q; the new Enriques diagram C_i is obtained from C_{i-1} by adding p_i according to one of the rules of 3.2:

1. If $\omega_{\Gamma_{i-1}}(\mathbf{q}) < \omega_{\Gamma_{\mathcal{O}}}(\mathbf{q})$, p_i can be added to C_{i-1} as a free successor of q, $p_i \to_{C_i} q$.
2. If $q = q_r \to_{C_{i-1}} q_{r-1} \to_{C_{i-1}} \cdots \to_{C_{i-1}} q_1$ are free vertices in C_{i-1} with $1 \leqslant r < \omega_{\Gamma_{\mathcal{O}}}(\mathbf{p}_i)$ and $q_1 \to_{C_{i-1}} q_0$, then set $q_j \to_{C_i} p_i$ for all $j \in \{1, \ldots, r\}$, and either set $q_0 \to_{C_i} p_i$, provided q_0 is the root of C_{i-1} and $r < \omega_{\Gamma_{\mathcal{O}}}(\mathbf{p}_i) - 1$ (with p_i becoming the root of C_i), or set $p_i \to_{C_i} q_0$, provided $\omega_{\Gamma_{i-1}}(\mathbf{q}_0) < \omega_{\Gamma_{\mathcal{O}}}(\mathbf{q}_0)$.

The procedure stops at step n, the number of vertices of $\Gamma_{\mathcal{O}}$. At this point, the obtained weighted graph Γ_n equals $\Gamma_{\mathcal{O}}$, and the Enriques diagram C_n is just a contraction for $\Gamma_{\mathcal{O}}$.

Remark 3.3. At any step of the procedure, there may be several choices to add a fixed new vertex (for example, we may apply either rule 1 or 2 to add the vertex p_i to C_{i-1}). In order to obtain the whole list of all the contractions for $\Gamma_{\mathcal{O}}$, all these possibilities must be performed. It might also happen that an Enriques diagram C_{i-1} to which the new vertex cannot be added is reached. This means that no Enriques diagram for $\mathcal{O}$ with the subset of proximities of C_{i-1} exists.

Remark 3.4. Minimal singularities are rational surface singularities whose fundamental cycle is reduced. They are characterized as those sandwiched singularities that can be obtained by blowing up a complete ideal all whose base points are free (see [12, 2.5]; cf. [14]). As a consequence of our results, a sandwiched surface singularity $\mathcal{O}$ is minimal if and only if there exists a contraction for $\mathcal{O}$ that is obtained by applying the first rule at each step of the above procedure.

Example 3.5. Let $\mathcal{O}$ be a singularity whose resolution graph is shown at the bottom of Fig. 2. By applying the procedure just described, we obtain the whole list of contractions for $\mathcal{O}$. The S-Enriques diagrams shown in Fig. 2 are obtained by adding free successors to them as explained in (b) of 2.9.

4. Equisingularity classes of the ideals for a sandwiched singularity

In this section we address the problem of describing the equisingularity classes of the ideals for a given sandwiched surface singularity $\mathcal{O}$, that is, of describing all the possible Enriques diagrams for $\mathcal{O}$. The (finite) family of contractions for $\mathcal{O}$ was inferred from the resolution graph $\Gamma_{\mathcal{O}}$ of $\mathcal{O}$ by the procedure explained in the preceding section. It remains to find out all the different Enriques diagrams for $\mathcal{O}$ giving rise to the same contraction (an infinite family). Here we will show how to complete contractions in order to describe all the different Enriques diagrams for $\mathcal{O}$, thus solving completely the problem we are concerned with.

Given a contraction C for $\mathcal{O}$, our aim is to describe all the Enriques diagrams for $\mathcal{O}$ associated with C. Consider the marked Enriques diagram $\mathcal{C} = (C, \rho_C)$ with $\rho_C(p) = 0$ for any $p \in C$, and the number $\lambda_C = \sum_{p \in C}(\omega_{\Gamma_{\mathcal{O}}}(\mathbf{p}) - \omega_{\Gamma_C}(\mathbf{p}))$. By 2.7, $\lambda_C > 0$. Let us describe a procedure to add vertices to $\mathcal{C}$ in order to reach an Enriques diagram for $\mathcal{O}$. Write $\mathcal{C}_0 = (C_0, \rho_0) = \mathcal{C}$. For $1 \leqslant i \leqslant \lambda_C$, choose a vertex p_i in C such that $\omega_{\Gamma_{\mathcal{O}}}(\mathbf{p}_i) > \omega_{\Gamma_{C_{i-1}}}(\mathbf{p}_i)$ and then define inductively $\mathcal{C}_i = (C_i, \rho_i)$ by taking $C_i = C_{i-1} \cup \{q_i\}$ and $\rho_{i|C_{i-1}} = \rho_{i-1}$, where the new vertex q_i is set as a successor of p_i either

A. as a free successor of p_i, $q_i \to_{C_i} p_i$, and then set $\rho_i(q_i) = +$;

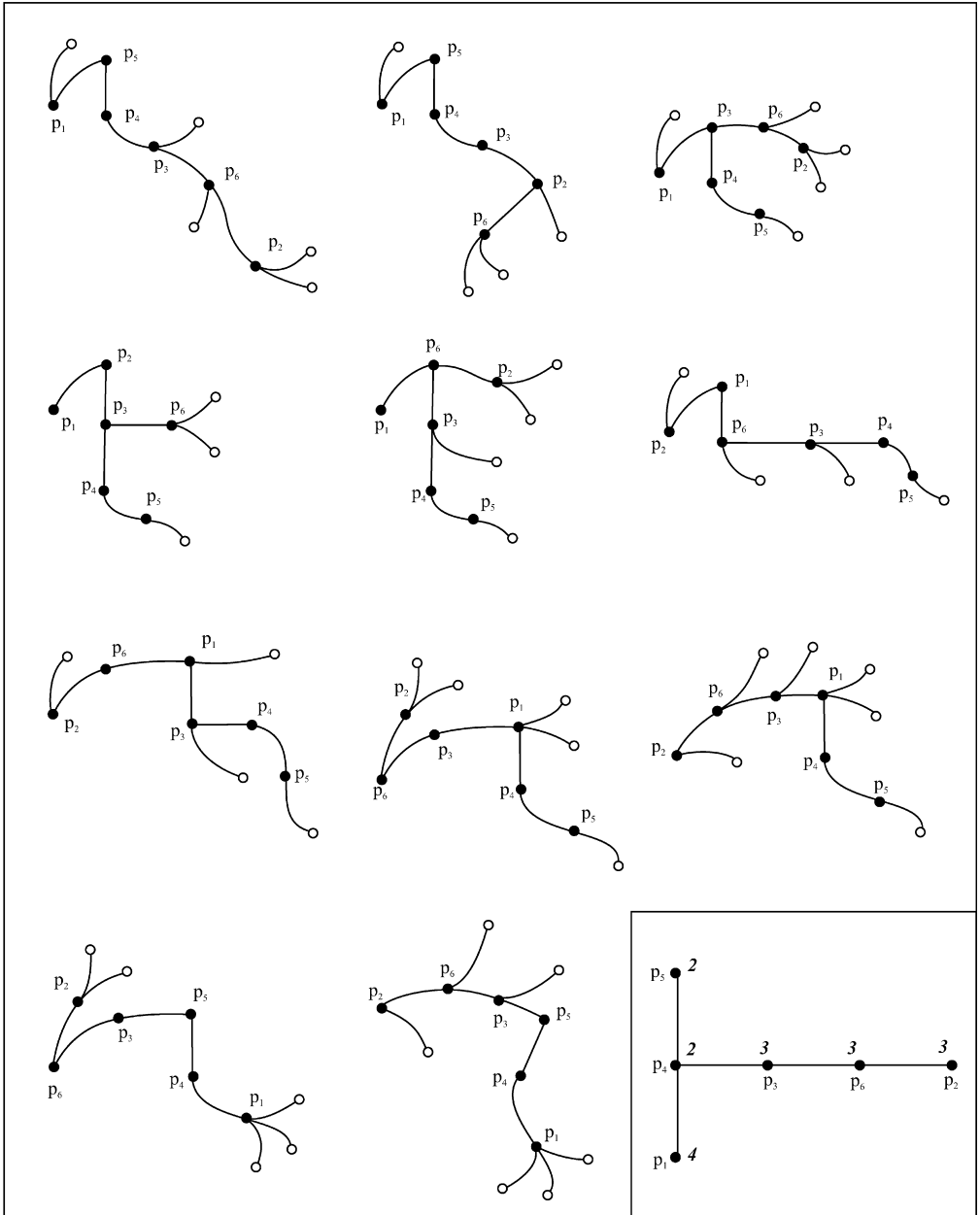or, if there is some free successor p_i' of p_i in C_{i-1},

Fig. 2. The complete list of S-Enriques diagrams for a singularity $\mathcal{O}$ with resolution graph $\Gamma_{\mathcal{O}}$ in Example 3.5. The white-filled dots represent the dicritical points, added to the contractions.

B. as a successor preceding p_i', namely $q_i \rightarrow_{C_i} p_i$ and $p_i' \rightarrow_{C_i} q_i$ are the only proximities relating q_i, and then set $\rho_i(q_i) = +$ in case $\rho_i(p_i') = 0$ (otherwise $\rho_i(q_i)$ can be chosen no matter 0 or +).

Notice that at step i the operation of type A may always be performed, independently of the existence of a free successor of p_i, which would offer the possibility to choose also an operation of type B. Observe that $\sum_{p \in C}(\omega_{\Gamma_{\mathcal{O}}}(\mathrm{p}) - \omega_{\Gamma_{C_i}}(\mathrm{p})) = \lambda_C - i$. Thus the procedure performs effectively the λ_C steps. Any of such marked Enriques diagram C_{λ_C}, obtained from C by the above procedure, will be called an *extension* of the contraction C. Clearly any extension of C is an Enriques diagram for $\mathcal{O}$ associated with C.

Remark 4.1. Notice that any extension of C all whose vertices have been added performing operation A at each step is an S-Enriques diagram for $\mathcal{O}$ (in fact, the unique S-Enriques diagram for $\mathcal{O}$ associated with C).

The set of all extensions of C forms a family of Enriques diagrams for $\mathcal{O}$ associated with C minimal in the following sense:

Theorem 4.2. *Any Enriques diagram $\mathcal{D}$ for a sandwiched singularity $\mathcal{O}$ contains, as a marked subdiagram, an extension of the contraction associated with $\mathcal{D}$.*

Conversely, if a marked Enriques diagram $\mathcal{D}$ contains, as a marked subdiagram, an extension $\mathcal{E}$ of some contraction C for $\mathcal{O}$ and satisfies that any vertex of $D \setminus E$ is proximate to no vertex of C, then $\mathcal{D}$ is an Enriques diagram for $\mathcal{O}$.

Proof. For the first assertion, we need to find a marked subdiagram $\mathcal{E}$ of $\mathcal{D}$ which is an extension of C. Take $F = \{p \in D: p \text{ is proximate to some } q \in C\}$, and define $\mathcal{E} = (E, \rho_E)$, where $E = C \cup F$ and ρ_E is the restriction of ρ_D to E. Notice that E is a connected subtree of D since, if p is proximate to some $q \in C$, then any vertex in $D(p)$ infinitely near to q is also proximate to q. Hence, E together with the proximities inherited from the proximity of D is an Enriques subdiagram of D. Furthermore, $\mathcal{E}$ is a marked subdiagram of $\mathcal{D}$.

Moreover, by 1.2, the cardinality of F equals $\lambda := \lambda_C$. Denote the vertices of F by $\{p_1, \ldots, p_\lambda\}$ so that p_i is not infinitely near to p_j if $j > i$. Write $\mathcal{E}_\lambda := \mathcal{E}$ and for $1 \geqslant i < \lambda$, define, recursively $\mathcal{E}_i$ as the marked Enriques diagram obtained from $\mathcal{E}_{i+1}$ by deleting p_i (and keeping the restricted proximity and marking map; the successors of p_i become successors of the immediate predecessor of p_i). Notice that the $\mathcal{E}_i$ are the marked Enriques diagrams generated by the procedure detailed above to reach $\mathcal{E}$, proving that $\mathcal{E}$ is an extension of C, as wanted.

For the converse, let $\mathcal{E} = (E, \rho_E)$ be an extension of a contraction C for $\mathcal{O}$. Thus, by 2.1, $\Gamma_E \supseteq \Gamma_{\mathcal{O}}$ as weighted graphs, $\rho_E(p) = 0$ for any $\mathrm{p} \in \Gamma_{\mathcal{O}}$ and $\rho_E(p) = +$ for any $\mathrm{p} \in \Gamma_E \setminus \Gamma_{\mathcal{O}}$ being adjacent to some vertex of $\Gamma_{\mathcal{O}}$. If $\mathcal{D} = (D, \rho_D)$ contains $\mathcal{E}$ as a marked subdiagram and there any vertex of $D \setminus E$ is proximate to no vertex of $C \subset E$, then $\Gamma_D \supseteq \Gamma_{\mathcal{O}}$ as weighted graphs, and ρ_D satisfies the marking map hypothesis of 2.1, 2: $\rho_D(p) = 0$ for any $\mathrm{p} \in \Gamma_{\mathcal{O}}$ and $\rho_D(p) = +$ for any $\mathrm{p} \in \Gamma_D \setminus \Gamma_{\mathcal{O}}$ adjacent to some vertex of $\Gamma_{\mathcal{O}}$. Hence, applying 2.1 to $\mathcal{D}$ we are done. $\square$

We have already pointed out that sandwiched singularities are normal birational extensions of the regular ring R. If $R \subset \mathcal{O}$ is such an extension, there exists a complete ideal $I \subset R$ such that $\mathcal{O} = R[I/a]_{N_Q}$, where N_Q is a height two maximal ideal in $R[I/a]$ containing $\mathfrak{m}_R$ (the maximal
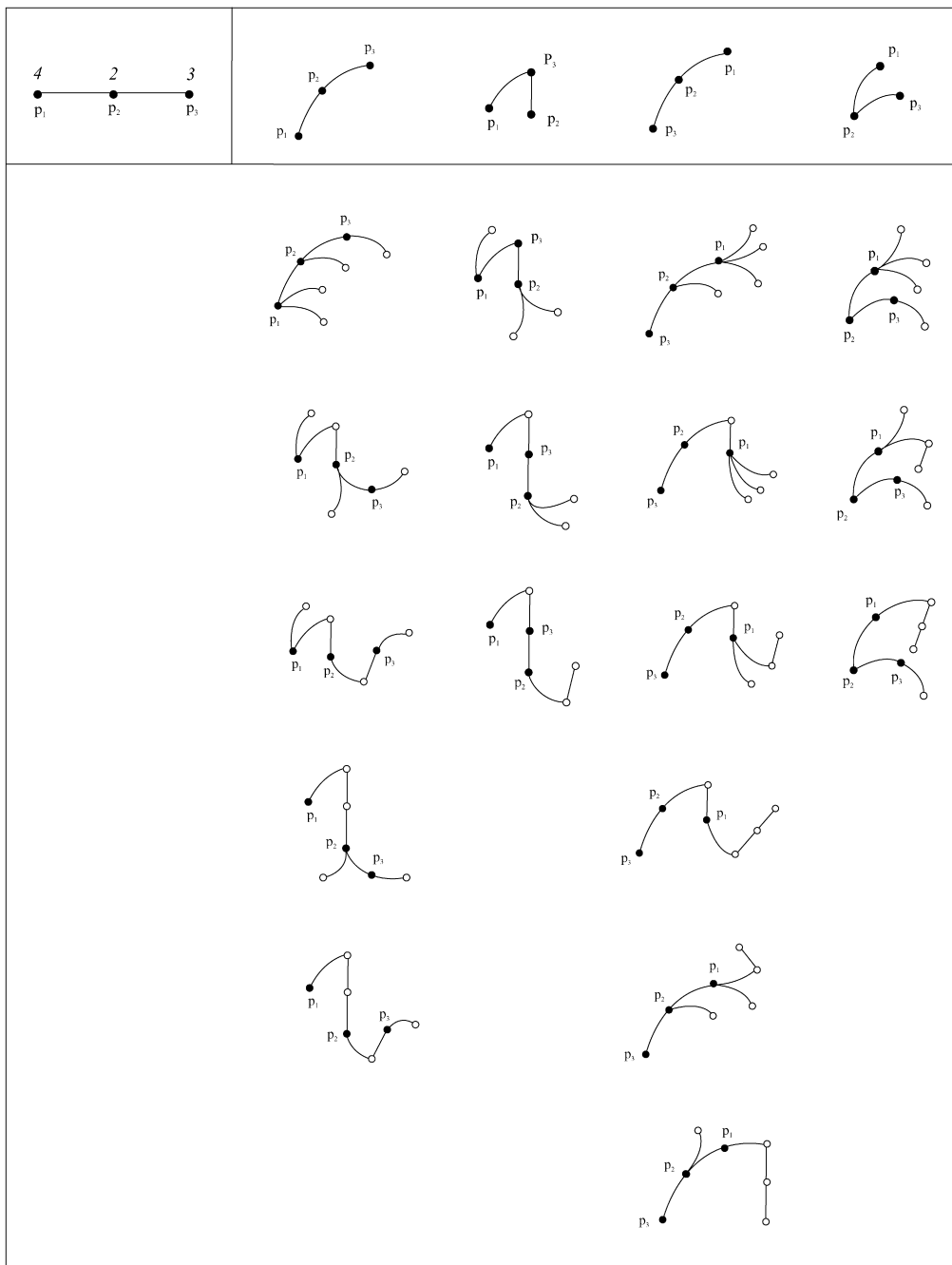
Fig. 3. A complete list of extensions for a sandwiched singularity whose resolution graph is drawn in the top left corner. At the top of the figure the contractions of $\mathcal{O}$ are shown. The white-filled dots represent dicritical vertices.

ideal of R), and a is a generic element of I (see [7]). R is said to be *maximally regular* in $\mathcal{O}$ if there is no other regular ring R' such that

$$R \subsetneq R' \subset \mathcal{O}.$$

Write $\mathcal{D}_I$ for the marked Enriques diagram of the base points of I. Let $\mathcal{E}$ and C be the extension and the contraction for $\mathcal{O}$ associated with $\mathcal{D}_I$. Then, by virtue of 4.2, $\mathcal{D}_I$ can be thought as being constructed from $\mathcal{E}$ by adding new vertices which are infinitely near to some dicritical vertex of $\mathcal{E}$ and not proximate to any vertex of C, or preceding the root of C (notice that in any case, the proximities of $\mathcal{E}$, and hence also the proximities of C, are preserved). Moreover, R is maximally regular in $\mathcal{O}$ if and only if the root of $\mathcal{D}_I$ equals the root of $\mathcal{E}$, i.e. no vertices have been added to $\mathcal{E}$ preceding the root.

Example 4.3. Let $\mathcal{O}$ be a sandwiched singularity whose resolution graph is shown in the top left corner of Fig. 3. The contractions for $\mathcal{O}$ are shown at the top of the figure, and below each one of them, a complete list of the associated extensions is drawn. Any Enriques diagram for $\mathcal{O}$ contains one of these extensions as a marked subdiagram.

Acknowledgments

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