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Calderón–Zygmund operators on product Hardy spaces ☆

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Abstract

Let T be a product Calderón–Zygmund singular integral introduced by Journé. Using an elegant rectangle atomic decomposition of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and Journé's geometric covering lemma, R. Fefferman proved the remarkable $H^p(\mathbb{R}^n \times \mathbb{R}^m) - L^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of T. In this paper we apply vector-valued singular integral, Calderón's identity, Littlewood–Paley theory and the almost orthogonality together with Fefferman's rectangle atomic decomposition and Journé's covering lemma to show that T is bounded on product $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $\max\{\frac{n}{n+\varepsilon}, \frac{m}{m+\varepsilon}\} < p \leqslant 1$ if and only if $T_1^*(1) = T_2^*(1) = 0$, where ε is the regularity exponent of the kernel of T.

Keywords: Calderón–Zygmund operators; Journé's class; Littlewood–Paley function; Product Hardy spaces

1. Introduction

The product Hardy space was first introduced by M.P. Malliavin and P. Malliavin [11] and Gundy and Stein [8]. Chang and R. Fefferman [3] provided the atomic decomposition of

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$H^p(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$. However, atomic decomposition of the product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ is more complicated than the classical $H^p(\mathbb{R}^n)$. Indeed it was conjectured that the product atomic Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ could be characterized by rectangle atoms (see definition below). This conjecture, however, was disproved by Carleson [2] based on a counterexample. This leads that the role of cubes in the classical atomic Hardy space $H^p(\mathbb{R}^n)$ was replaced by arbitrary open sets on $\mathbb{R}^n \times \mathbb{R}^m$ with finite measures. It was quite surprising that using the rectangle atomic decomposition of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and a geometric covering lemma due to Journé [10], R. Fefferman [5] proved the remarkable $H^p(\mathbb{R}^n \times \mathbb{R}^m) - L^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of product singular integrals introduced by Journé. Nevertheless, the $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of Journé's product singular integrals is still open. The purpose of the current article is to study this issue.

Let us recall the classical $H^p(\mathbb{R}^n)$ boundedness of singular integrals. We first begin with recalling the definition of a Calderón–Zygmund kernel.

Definition 1. A continuous complex-valued function $K(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$ is called a *Calderón–Zygmund kernel* if there exist constants $C > 0$ and a regularity exponent $\varepsilon \in (0, 1]$ such that

(i) $|K(x, y)| \leqslant C|x - y|^{-n}$,
(ii) $|K(x, y) - K(x', y)| \leqslant C|x - x'|^{\varepsilon}|x - y|^{-n-\varepsilon}$ if $|x - x'| \leqslant |x - y|/2$,
(iii) $|K(x, y) - K(x, y')| \leqslant C|y - y'|^{\varepsilon}|x - y|^{-n-\varepsilon}$ if $|y - y'| \leqslant |x - y|/2$.

The smallest such constant C is denoted by $|K|_{CZ}$.

We say that an operator T is a *Calderón–Zygmund operator* if the operator T is a continuous linear operator from $C_0^\infty(\mathbb{R}^n)$ into its dual associated with a Calderón–Zygmund kernel $K(x, y)$ given by

$$\langle Tf, g\rangle = \iint g(x)K(x, y)f(y)\,dy\,dx$$

for all test functions f and g with disjoint supports and T is bounded on $L^2(\mathbb{R}^n)$. If T is a Calderón–Zygmund operator associated with a kernel K, its Calderón–Zygmund operator norm is defined by $\|T\|_{CZ} = \|T\|_{L^2 \mapsto L^2} + |K|_{CZ}$.

Given $0 < p \leqslant 1$, let

$$C_{0,0}^\infty(\mathbb{R}^n) = \Big\{\psi \in C^\infty(\mathbb{R}^n)\colon\ \psi \text{ has a compact support and } \int_{\mathbb{R}^n} \psi(y)y^\alpha\,dy = 0 \text{ for } 0 \leqslant |\alpha| \leqslant N_{p,n}\Big\},$$

where $N_{p,n}$ is a large integer depending on p and n. Let $\psi \in C_{0,0}^\infty(\mathbb{R}^n)$ satisfy the condition

$$\int_0^\infty \big|\hat{\psi}(t\xi)\big|^2 \frac{dt}{t} = 1 \quad \text{for all } \xi \neq 0. \tag{1.1}$$

For $t>0$ and $x\in\mathbb{R}^n$, set $\psi_t(x)=t^{-n}\psi(x/t)$. The Littlewood–Paley square function of $f\in\mathcal{S}'(\mathbb{R}^n)$ is defined by

$$g(f)(x)=\left\{\int_0^\infty\left|\psi_t*f(x)\right|^2\frac{dt}{t}\right\}^{1/2}.$$

The classical Hardy space $H^p(\mathbb{R}^n)$ can be defined by

$$H^p(\mathbb{R}^n)=\left\{f\in\mathcal{S}'(\mathbb{R}^n):\ g(f)\in L^p(\mathbb{R}^n)\right\}$$

with $\|f\|_{H^p(\mathbb{R}^n)}:=\|g(f)\|_{L^p(\mathbb{R}^n)}$.

The criterion for the $H^p(\mathbb{R}^n)$ boundedness of Calderón–Zygmund operators is given as follows.

Theorem A. *Let T be a Calderón–Zygmund operator associated to a kernel with regularity exponent ε. Then T is bounded on $H^p(\mathbb{R}^n)$, $\frac{n}{n+\varepsilon}<p\leqslant 1$, if and only if $T^*(1)=0$.*

Here, $T^*(1)=0$ means that $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K(x,y)\psi(y)\,dy\,dx=0$ for all $\psi\in C^\infty_{0,0}(\mathbb{R}^n)$.

Remark 1. Theorem A still holds for any given p, $0<p\leqslant 1$, if one requires more regularity conditions on the kernel of T and high order cancellation conditions on T (see [7] for more details). One proof of Theorem A was shown in terms of atomic decomposition together with the maximal function characterization of $H^p(\mathbb{R}^n)$ (see [12, p. 115, Theorem 4]). Another proof was given by molecule decomposition of $H^p(\mathbb{R}^n)$ (see [7, p. 335, Theorem 7.18]).

These methods, however, cannot be carried out to the product Hardy space $H^p(\mathbb{R}^n\times\mathbb{R}^m)$. To see this, let us recall the definition and atomic decomposition of $H^p(\mathbb{R}^n\times\mathbb{R}^m)$. Let $n_1=n$, $n_2=m$, $\psi^i\in C^\infty_{0,0}(\mathbb{R}^{n_i})$ supported in the unit ball of $\mathbb{R}^{n_i}$, and ψ^i satisfy condition (1.1), $i=1,2$. For $t_i>0$ and $(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^m$, set $\psi^i_{t_i}(x_i)=t_i^{-n_i}\psi(x_i/t_i)$ and $\psi_{t_1t_2}(x_1,x_2)=\psi^1_{t_1}(x_1)\psi^2_{t_2}(x_2)$. The product Littlewood–Paley square function of $f\in\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^m)$ is defined by

$$g(f)(x_1,x_2)=\left\{\int_0^\infty\int_0^\infty\left|\psi_{t_1t_2}*f(x_1,x_2)\right|^2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right\}^{1/2}.$$

For $0<p\leqslant 1$, the product Hardy space $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ can be defined by

$$H^p(\mathbb{R}^n\times\mathbb{R}^m)=\left\{f\in\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^m):\ g(f)\in L^p(\mathbb{R}^n\times\mathbb{R}^m)\right\}$$

with $\|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)}:=\|g(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}$.

A function $a(x_1,x_2)$ defined in $\mathbb{R}^n\times\mathbb{R}^m$ is called an $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ *atom* if $a(x_1,x_2)$ is supported in an open set $\Omega\subset\mathbb{R}^n\times\mathbb{R}^m$ with finite measure and satisfies the following conditions:

(i) $\|a\|_2\leqslant|\Omega|^{1/2-1/p}$,

(ii) a can further be decomposed as $a(x_1, x_2) = \sum_{R\in\mathcal{M}(\Omega)} a_R(x_1, x_2)$, where a_R are supported on the double of $R = I \times J$ (I a dyadic cube in $\mathbb{R}^n$, J a dyadic cube in $\mathbb{R}^m$) and $\mathcal{M}(\Omega)$ is the collection of all maximal dyadic rectangles contained in Ω,

$$\left\{\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|_2^2\right\}^{1/2} \leqslant |\Omega|^{1/2-1/p},$$

(iii) $\int_{2I} a_R(x_1, x_2)x_1^\alpha\, dx_1 = 0$ for all $x_2 \in \mathbb{R}^m$, $0 \leqslant |\alpha| \leqslant N_{p,n}$,

$\int_{2J} a_R(x_1, x_2)x_2^\beta\, dx_2 = 0$ for all $x_1 \in \mathbb{R}^n$, $0 \leqslant |\beta| \leqslant N_{p,m}$,

where $N_{p,n}$ and $N_{p,m}$ are given in the definition of $C_{0,0}^\infty$.

Chang and R. Fefferman [3] provided the following atomic decomposition of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$.

Theorem B. *A distribution $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$ if and only if $f = \sum_j \lambda_j a_j$, where a_j are $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ atoms, $\sum_j |\lambda_j|^p < \infty$, and the series converges in the distribution sense. Moreover, $\|f\|_{H^p}^p$ is equivalent to* $\inf\{\sum_j |\lambda_j|^p$: *for all $f = \sum_j \lambda_j a_j$*}.

The fact that the support of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ atom is an open set prevents from applications of atomic decomposition of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$. However, it was quite surprising that R. Fefferman [5] proved the following remarkable result.

Theorem C. *Let $0 < p \leqslant 1$ and T be a bounded linear operator on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$. Suppose that there exist constants $C > 0$ and $\delta > 0$ such that, for any $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ rectangle atom a supported on R,*

$$\int_{\mathbb{R}^n\times\mathbb{R}^m\setminus\gamma R} \left|Ta(x_1, x_2)\right|^p dx_1\, dx_2 \leqslant C\gamma^{-\delta} \quad \text{for all } \gamma \geqslant 2, \tag{1.2}$$

where γR denotes the concentric γ-fold dilation of R. Then T is a bounded operator from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

Here a function $a(x_1, x_2)$ supported on a rectangle $R = I \times J$ (I a cube in $\mathbb{R}^n$, J a cube in $\mathbb{R}^m$) is called an $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ *rectangle atom* provided

(i) $\|a\|_2 \leqslant |R|^{1/2-1/p}$,

(ii) $\int_I a(x_1, x_2)x_1^\alpha\, dx_1 = 0$, $0 \leqslant |\alpha| \leqslant N_{n,p}$ for all $x_2 \in J$,

(iii) $\int_J a(x_1, x_2)x_2^\beta\, dx_2 = 0$, $0 \leqslant |\beta| \leqslant N_{m,p}$ for all $x_1 \in I$.

Definition 2. A singular integral operator T is said to be in *Journé's class* if

$$Tf(x_1,x_2)=\int_{\mathbb{R}^n\times\mathbb{R}^m} K(x_1,x_2,y_1,y_2)f(y_1,y_2)\,dy_1\,dy_2,$$

where the kernel $K(x_1,x_2,y_1,y_2)$ satisfies the following conditions. For each $x_1,y_1\in\mathbb{R}^n$, set $\widetilde{K}^1(x_1,y_1)$ to be the singular integral operator acting on functions on $\mathbb{R}^m$ with the kernel $\widetilde{K}^1(x_1,y_1)(x_2,y_2)=K(x_1,x_2,y_1,y_2)$, and similarly, $\widetilde{K}^2(x_2,y_2)(x_1,y_1)=K(x_1,x_2,y_1,y_2)$. There exist constants $C>0$ and $\varepsilon\in(0,1]$ such that

(A_1) T is bounded on $L^2(\mathbb{R}^{n+m})$,
(A_2) $\|\widetilde{K}^1(x_1,y_1)\|_{CZ}\leqslant C|x_1-y_1|^{-n}$,
$\|\widetilde{K}^1(x_1,y_1)-\widetilde{K}^1(x_1,y_1')\|_{CZ}\leqslant C|y_1-y_1'|^{\varepsilon}|x_1-y_1|^{-(n+\varepsilon)}$ for $|y_1-y_1'|\leqslant|x_1-y_1|/2$,
$\|\widetilde{K}^1(x_1,y_1)-\widetilde{K}^1(x_1',y_1)\|_{CZ}\leqslant C|x_1-x_1'|^{\varepsilon}|x_1-y_1|^{-(n+\varepsilon)}$ for $|x_1-x_1'|\leqslant|x_1-y_1|/2$,
(A_3) $\|\widetilde{K}^2(x_2,y_2)\|_{CZ}\leqslant C|x_2-y_2|^{-m}$,
$\|\widetilde{K}^2(x_2,y_2)-\widetilde{K}^2(x_2,y_2')\|_{CZ}\leqslant C|y_2-y_2'|^{\varepsilon}|x_2-y_2|^{-(m+\varepsilon)}$ for $|y_2-y_2'|\leqslant|x_2-y_2|/2$,
$\|\widetilde{K}^2(x_2,y_2)-\widetilde{K}^2(x_2',y_2)\|_{CZ}\leqslant C|x_2-x_2'|^{\varepsilon}|x_2-y_2|^{-(m+\varepsilon)}$ for $|x_2-x_2'|\leqslant|x_2-y_2|/2$.

R. Fefferman [4] further proved that product singular integrals in Journé class satisfy the estimate (1.2), and hence such product singular integrals are bounded from $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ to $L^p(\mathbb{R}^{n+m})$.

Suppose that T is a singular integral in Journé's class. Then by a result in [4] T is bounded from $H^1(\mathbb{R}^n\times\mathbb{R}^m)$ to $L^1(\mathbb{R}^{n+m})$. Note that if $\varphi^1\in C^\infty_{0,0}(\mathbb{R}^n)$ and $\varphi^2\in C^\infty_{0,0}(\mathbb{R}^m)$ then $\varphi^1(y_1)\varphi^2(y_2)\in H^1(\mathbb{R}^n\times\mathbb{R}^m)$. Therefore, $T(\varphi^1\varphi^2)(x_1,x_2)\in L^1(\mathbb{R}^n\times\mathbb{R}^m)$. This implies that $T(\varphi^1\varphi^2)(x_1,x_2)$, as a function of x_1 is a integrable function on $\mathbb{R}^n$. Similarly, $T(\varphi^1\varphi^2)(x_1,x_2)$, as a function of x_2 is a integrable function on $\mathbb{R}^m$. Now we say that $T_1^*(1)=0$ if

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\times\mathbb{R}^m} K(x_1,x_2,y_1,y_2)\varphi^1(y_1)\varphi^2(y_2)\,dy_1\,dy_2\,dx_1=0$$

for all $\varphi^1\in C^\infty_{0,0}(\mathbb{R}^n)$, $\varphi^2\in C^\infty_{0,0}(\mathbb{R}^m)$, and $x_2\in\mathbb{R}^m$. Similarly, $T_2^*(1)=0$ if

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^n\times\mathbb{R}^m} K(x_1,x_2,y_1,y_2)\varphi^1(y_1)\varphi^2(y_2)\,dy_1\,dy_2\,dx_2=0$$

for all $\varphi^1\in C^\infty_{0,0}(\mathbb{R}^n)$, $\varphi^2\in C^\infty_{0,0}(\mathbb{R}^m)$, and $x_1\in\mathbb{R}^n$.

The main result of this paper is the following

Theorem 1. *Let T be a singular integral operator in Journé's class with regularity exponent ε. Then T is bounded on $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ for* $\max\{\frac{n}{n+\varepsilon},\frac{m}{m+\varepsilon}\}<p\leqslant 1$ *if and only if $T_1{}^*(1)=T_2{}^*(1)=0$.*

Remark 2. As in the classical $H^p(\mathbb{R}^n)$, Theorem 1 still holds for any given p, $0 < p \leqslant 1$, if the kernel of T satisfies more regularity conditions and T satisfies high order cancellation conditions. We leave these details to the reader.

The approach used in this paper is even new for the classical $H^p(\mathbb{R}^n)$. Therefore, we would like first to describe that how one can use this approach to prove the classical $H^p(\mathbb{R}^n)$ boundedness. This approach includes the following steps.

Step 1. Reduce the $H^p(\mathbb{R}^n)$ boundedness to $H^p(\mathbb{R}^n) - L^p_{\mathcal{H}_1}(\mathbb{R}^n)$ boundedness: we first introduce the Hilbert space $\mathcal{H}_1([0,\infty), \frac{dt}{t})$ by

$$\mathcal{H}_1\left([0,\infty), \frac{dt}{t}\right) = \left\{\{h_t\}_{t>0}\colon\ \big\|\{h_t\}\big\|_{\mathcal{H}_1([0,\infty), \frac{dt}{t})} = \left(\int_0^\infty |h_t|^2 \frac{dt}{t}\right)^{1/2} < \infty\right\}.$$

We will denote $\mathcal{H}_1([0,\infty), \frac{dt}{t})$ simply by $\mathcal{H}_1$. To reduce the H^p boundedness to $H^p - L^p_{\mathcal{H}_1}$ boundedness, by the Littlewood–Paley characterization of $H^p(\mathbb{R}^n)$, we write

$$\|Tf\|_{H^p} = \left\|\left(\int_0^\infty \big|\psi_t * Tf(x)\big|^2 \frac{dt}{t}\right)^{1/2}\right\|_p.$$

Set an $\mathcal{H}_1$-valued operator L which maps f into $\{T_t(f)\}_{t>0}$ by

$$T_t(f)(x) = \psi_t * Tf(x), \quad t > 0.$$

Therefore, the H^p boundedness of T is equivalent to the $H^p - L^p_{\mathcal{H}_1}$ boundedness of L.

Step 2. The almost orthogonal estimates and decomposition of T_t: this step is crucial. We first start with a function $f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By the classical Calderón identity,

$$f(x) = \int_0^\infty \psi_t * \psi_t * f(x) \frac{dt}{t}, \quad f \in L^2\big(\mathbb{R}^n\big),$$

where $\psi \in C^\infty_{0,0}$ satisfies condition (1.1). Since T is bounded on $L^2(\mathbb{R}^n)$, we rewrite

$$T_t(f)(x) = \psi_t * T\left(\int_0^\infty \psi_s * \psi_s * f(\cdot)\frac{ds}{s}\right)(x) = \int_0^\infty \psi_t * T\big(\psi_s * \psi_s * f(\cdot)\big)(x)\frac{ds}{s}.$$

Denote $T_t(x,y)$ to be the kernel of T_t. Then

$$T_t(x,y) = \int_0^\infty \iiint \psi_t(x-u)K(u,v)\psi_s(v-w)\psi_s(w-y)\,du\,dv\,dw\frac{ds}{s}.$$

The almost orthogonal estimate says that there exists a constant C such that

$$\left|\int \psi_t(x-z)\psi_s(z-y)\,dz\right| \leqslant C\left(\frac{t}{s}\wedge\frac{s}{t}\right)^{\varepsilon}\frac{(s\vee t)^{\varepsilon}}{((s\vee t)+|x-y|)^{n+\varepsilon}},$$

where $s\vee t=\max\{s,t\}$ and $s\wedge t=\min\{s,t\}$. Suppose that $K(x,y)$ is a Calderón–Zygmund kernel with regularity exponent ε. Then the following almost orthogonal estimates still hold: for $0<\varepsilon'<\varepsilon$,

$$\begin{aligned}&\left|\iint\big[\psi_t(x-u)-\psi_t(x-y)\big]K(u,v)\psi_s(v-y)\,du\,dv\right|\\&\quad\leqslant C|K|_{CZ}\left(\frac{s}{t}\right)^{\varepsilon'}\frac{t^{\varepsilon'}}{(t+|x-y|)^{n+\varepsilon'}}\end{aligned}\tag{1.3}$$

for $s\leqslant t$, and for $t\leqslant s$,

$$\begin{aligned}&\left|\iint\psi_t(x-u)K(u,v)\big[\psi_s(v-y)-\psi_s(x-y)\big]\,du\,dv\right|\\&\quad\leqslant C|K|_{CZ}\left(\frac{t}{s}\right)^{\varepsilon'}\frac{s^{\varepsilon'}}{(s+|x-y|)^{n+\varepsilon'}}.\end{aligned}\tag{1.4}$$

Suppose $T^*(1)=0$. These considerations lead to the following decomposition

$$\begin{aligned}T_t(x,y)&=\int_0^t\iiint\psi_t(x-u)K(u,v)\psi_s(v-w)\psi_s(w-y)\,du\,dv\,dw\frac{ds}{s}\\&\quad+\int_t^\infty\iiint\psi_t(x-u)K(u,v)\big[\psi_s(v-w)-\psi_s(x-w)\big]\psi_s(w-y)\,du\,dv\,dw\frac{ds}{s}\\&\quad+\psi_t*T(1)(x)\phi_t(x-y)\\&:=T_t^1(x,y)+T_t^2(x,y)+T_t^3(x,y),\end{aligned}$$

where $\phi_t(x-y)=\int_t^\infty\psi_s*\psi_s(x-y)\frac{ds}{s}$.

The almost orthogonal estimate (1.3) can be used to estimate the kernel of T_t^1 because $T^*(1)=0$. The estimate of the kernel of T_t^2 then follows immediately from (1.4). We remark that T_t^3 is an $\mathcal{H}_1$-valued para-product operator and the estimate of $T_t^3(x,y)$ then easily follows from the facts that $\psi_t\in H^1(\mathbb{R}^n)$ whose norm is bounded uniformly for $t>0$ and $T(1)\in BMO$. All these estimates together with the fact that the L^2 boundedness of T implies the $L^2-L^2_{\mathcal{H}_1}$ boundedness of L yield that the kernel of L satisfies the size condition (i) and the smoothness condition (iii) for variable y in Definition 1 with the norm replaced by $\mathcal{H}_1$-valued norm. Finally, the $H^p-L^p_{\mathcal{H}_1}$ boundedness of L then follows from the following:

Step 3. The $H^p-L^p_{\mathcal{H}_1}$ boundedness via atoms: suppose that L is an $L^2-L^2_{\mathcal{H}_1}$ bounded operator. A general result of the boundedness says that L extends to a bounded operator from

H^p into $L^p_{\mathcal{H}_1}$ if and only if $\|La\|_{L^p_{\mathcal{H}_1}}$ is uniformly bounded for any $H^p(\mathbb{R}^n)$ atom a. This is non-trivial, in the light of the Meyer–Bownik example [1]. Let L be a $L^2 - L^2_{\mathcal{H}_1}$ bounded operator and its kernel satisfies the condition (iii) of Definition 1 with the norm replaced by $\mathcal{H}_1$-valued norm. We then have the uniform boundedness of $\|La\|_{L^p_{\mathcal{H}_1}}$ for any $H^p(\mathbb{R}^n)$ atom a and hence L is bounded from H^p to $L^p_{\mathcal{H}_1}$ by the above general result of the boundedness.

In the next section, we will carry out these steps to the product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$.

2. The proof of Theorem 1

The necessary conditions of Theorem 1 follow from the classical results. To see this, let $f(x_1, x_2) \in H^p(\mathbb{R}^n \times \mathbb{R}^m) \cap L^2(\mathbb{R}^n \times \mathbb{R}^m)$ and $f^*(x_1, x_2)$ be the maximal function of f defined in [8]. By the maximal function characterization of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, $f^*(x_1, x_2) \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$ (see [8, Theorem 1]). Denote $f_1^*(x_1, x_2)$ by the maximal function of $f(x_1, x_2)$, as the function of variable x_1 when x_2 is fixed. Then $f_1^*(x_1, x_2) \leqslant Cf^*(x_1, x_2)$ for fixed x_2. This implies that $f_1^*(x_1, x_2) \in L^p(\mathbb{R}^n)$ and hence $f(x_1, x_2) \in H^p(\mathbb{R}^n)$ for fixed x_2. By a classical result on $H^p(\mathbb{R}^n)$, $\int f(x_1, x_2)\,dx_1 = 0$ for fixed x_2. Now, for $\varphi^1 \in C^\infty_{0,0}(\mathbb{R}^n)$ and $\varphi^2 \in C^\infty_{0,0}(\mathbb{R}^m)$, let $g(x_1, x_2) = \varphi^1(x_1)\varphi^2(x_2)$. It is easy to see that $\|g\|_{H^p(\mathbb{R}^n \times \mathbb{R}^m)} \leqslant C\|\varphi^1\|_{H^p(\mathbb{R}^n)}\|\varphi^2\|_{H^p(\mathbb{R}^m)}$ and hence $g \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$. Thus, by the $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness and $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of T, the above explanation yields $Tg(x_1, x_2)$, as a function of x_1, is in $H^p(\mathbb{R}^n)$. This implies that $\iiint K(x_1, x_2, y_1, y_2)\varphi^1(y_1)\varphi^2(y_2)\,dy_1\,dy_2\,dx_1 = \int Tg(x_1, x_2)\,dx_1 = 0$ for fixed x_2, which yields $T_1^*(1) = 0$. Similarly, $T_2^*(1) = 0$.

We now prove the sufficiency; that is, if $T_1^*(1) = T_2^*(1) = 0$, then T is bounded on $H^p(\mathbb{R}^n \times \mathbb{R}^m)$. As in the step 1 of Section 1, we define the Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = \left\{ \{h_{t,s}\}_{t,s>0}\colon\ \big\|\{h_{t,s}\}\big\|_{\mathcal{H}} = \left(\int_0^\infty \int_0^\infty |h_{t,s}|^2 \frac{dt}{t}\frac{ds}{s} \right)^{1/2} < \infty \right\}.$$

Set $T_{t,s}(f) = \psi_{t,s} * T(f)$. For $f \in L^2(\mathbb{R}^{n+m}) \cap H^p(\mathbb{R}^n \times \mathbb{R}^m)$, by the classical Calderón identity,

$$T_{t,s}(f)(x_1, x_2) = \psi_{t,s} * T\left(\int_0^\infty \int_0^\infty \psi_{t',s'} * \psi_{t',s'} * f(\cdot,\cdot)\frac{dt'}{t'}\frac{ds'}{s'} \right)(x_1, x_2). \tag{2.1}$$

By (2.1), $T_{t,s}(x_1, x_2, y_1, y_2)$, the kernel of $T_{t,s}$ is given by

$$T_{t,s}(x_1, x_2, y_1, y_2) = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n\times\mathbb{R}^m} \int_{\mathbb{R}^n\times\mathbb{R}^m} \psi_{t,s}(x_1 - u_1, x_2 - u_2)K(u_1, u_2, v_1, v_2)$$
$$\times\, \psi_{t',s'} * \psi_{t',s'}(v_1 - y_1, v_2 - y_2)\,du_1\,du_2\,dv_1\,dv_2 \frac{dt'}{t'}\frac{ds'}{s'}. \tag{2.2}$$

As mentioned in the step 1 for the classical $H^p(\mathbb{R}^n)$, by the Littlewood–Paley characterization of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, the $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of T is equivalent to the $H^p - L^p_{\mathcal{H}}(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of the $\mathcal{H}$-valued operator $\mathcal{L}$ which maps f into $\{T_{t,s}(f)\}_{t,s>0}$.

Note that the $L^2(\mathbb{R}^{n+m})$ boundedness of T and the product Littlewood–Paley estimate [6] imply that $\mathcal{L}$ is bounded from $L^2(\mathbb{R}^{n+m})$ to $L^2_{\mathcal{H}}(\mathbb{R}^{n+m})$. Let ε be the regularity exponent satisfying (A_2) and (A_3). We will prove that $\{T_{t,s}(x_1,x_2,y_1,y_2)\}_{t,s>0}$ satisfies the following estimates:

(B_1) $\|\{T_{t,s}(x_1,x_2,y_1,y_2)\}\|_{\mathcal{H}} \leqslant C|x_1-y_1|^{-n}|x_2-y_2|^{-m}$,

(B_2) for $\varepsilon'<\varepsilon$,

(i) $\|\{T_{t,s}(x_1,x_2,y_1,y_2)-T_{ts}(x_1,x_2,y_1',y_2)\}\|_{\mathcal{H}} \leqslant C\frac{|y_1-y_1'|^{\varepsilon'}}{|x_1-y_1|^{n+\varepsilon'}}|x_2-y_2|^{-m}$ if $|y_1-y_1'|\leqslant |x_1-y_1|/2$,

(ii) $\|\{T_{t,s}(x_1,x_2,y_1,y_2)-T_{t,s}(x_1,x_2,y_1,y_2')\}\|_{\mathcal{H}} \leqslant C\frac{|y_2-y_2'|^{\varepsilon'}}{|x_2-y_2|^{m+\varepsilon'}}|x_1-y_1|^{-n}$ if $|y_2-y_2'|\leqslant |x_2-y_2|/2$,

(B_3) for $\varepsilon'<\varepsilon$,

$$\|\{[T_{t,s}(x_1,x_2,y_1,y_2)-T_{t,s}(x_1,x_2,y_1',y_2)]-[T_{t,s}(x_1,x_2,y_1,y_2')-T_{t,s}(x_1,x_2,y_1',y_2')]\}\|_{\mathcal{H}} \leqslant C\frac{|y_1-y_1'|^{\varepsilon'}}{|x_1-y_1|^{n+\varepsilon'}}\frac{|y_2-y_2'|^{\varepsilon'}}{|x_2-y_2|^{m+\varepsilon'}} \quad \text{if } |y_1-y_1'|\leqslant |x_1-y_1|/2,\ |y_2-y_2'|\leqslant |x_2-y_2|/2.$$

We would like to point out that the above estimates (B_1)–(B_3) show that $\mathcal{L}$ is an $\mathcal{H}$-valued singular integral operator. However, we will use the estimate (B_3) only. See Lemma 3 below for more regularities of $\mathcal{L}$ from L^2 to $L^2_{\mathcal{H}}$. To this end, according to the almost orthogonal estimates as we mentioned in the step 2 for the classical case, we decompose $T_{t,s}(x_1,x_2,y_1,y_2)$ as follows. Here and throughout, we denote $\int_{\mathbb{R}^n\times\mathbb{R}^m} du_1\,du_2$ simply by $\int du_1\,du_2$, and similarly for $\int dv_1\,dv_2$ and $\int dz_1\,dz_2$.

$$\begin{aligned}
&T_{t,s}(x_1,x_2,y_1,y_2)\\
&\quad=\int_0^t\int_0^s\iint \psi_t^1(x_1-u_1)\psi_s^2(x_2-u_2)K(u_1,u_2,v_1,v_2)\\
&\qquad\times \psi_{t'}^1*\psi_{t'}^1(v_1-y_1)\psi_{s'}^2*\psi_{s'}^2(v_2-y_2)\,du_1\,du_2\,dv_1\,dv_2\frac{dt'}{t'}\frac{ds'}{s'}\\
&\qquad+\int_0^t\int_s^\infty\iint\int_{\mathbb{R}^m} \psi_t^1(x_1-u_1)\psi_s^2(x_2-u_2)K(u_1,u_2,v_1,v_2)\\
&\qquad\times \psi_{t'}^1*\psi_{t'}^1(v_1-y_1)\big[\psi_{s'}^2(v_2-z_2)-\psi_{s'}^2(x_2-z_2)\big]\psi_{s'}^2(z_2-y_2)\,du_1\,du_2\,dv_1\,dv_2\,dz_2\frac{dt'}{t'}\frac{ds'}{s'}\\
&\qquad+\int_0^t\iint \psi_t^1(x_1-u_1)\psi_s^2(x_2-u_2)K(u_1,u_2,v_1,v_2)\\
&\qquad\times \psi_{t'}^1*\psi_{t'}^1(v_1-y_1)\,du_1\,du_2\,dv_1\,dv_2\frac{dt'}{t'}\phi_s^2(x_2-y_2)\\
&\qquad+\int_t^\infty\int_0^s\iint\int_{\mathbb{R}^n} \psi_t^1(x_1-u_1)\psi_s^2(x_2-u_2)K(u_1,u_2,v_1,v_2)\big[\psi_{t'}^1(v_1-z_1)-\psi_{t'}^1(x_1-z_1)\big]
\end{aligned}$$

$$\times \psi^1_{t'}(z_1-y_1)\psi^2_{s'}*\psi^2_{s'}(v_2-y_2)\,du_1\,du_2\,dv_1\,dv_2\,dz_1\frac{dt'}{t'}\frac{ds'}{s'}$$

$$+\int_0^s\iint \psi^1_t(x_1-u_1)\psi^2_s(x_2-u_2)K(u_1,u_2,v_1,v_2)$$

$$\times \psi^2_{s'}*\psi^2_{s'}(v_2-y_2)\,du_1\,du_2\,dv_1\,dv_2\frac{ds'}{s'}\phi^1_t(x_1-y_1)$$

$$+\int_t^\infty\int_s^\infty\iiint \psi^1_t(x_1-u_1)\psi^2_s(x_2-u_2)K(u_1,u_2,v_1,v_2)\big[\psi^1_{t'}(v_1-z_1)-\psi^1_{t'}(x_1-z_1)\big]$$

$$\times\big[\psi^2_{s'}(v_2-z_2)-\psi^2_{s'}(x_2-z_2)\big]\psi^1_{t'}(z_1-y_1)\psi^2_{s'}(z_2-y_2)\,du_1\,du_2\,dv_1\,dv_2\,dz_1\,dz_2\frac{dt'}{t'}\frac{ds'}{s'}$$

$$+\int_s^\infty\iint\int_{\mathbb{R}^m}\psi^1_t(x_1-u_1)\psi^2_s(x_2-u_2)K(u_1,u_2,v_1,v_2)$$

$$\times\big[\psi^2_{s'}(v_2-z_2)-\psi^2_{s'}(x_2-z_2)\big]\psi^2_{s'}(z_2-y_2)\,du_1\,du_2\,dv_1\,dv_2\,dz_2\frac{ds'}{s'}\phi^1_t(x_1-y_1)$$

$$+\int_t^\infty\iint\int_{\mathbb{R}^n}\psi^1_t(x_1-u_1)\psi^2_s(x_2-u_2)K(u_1,u_2,v_1,v_2)\big[\psi^1_{t'}(v_1-z_1)-\psi^1_{t'}(x_1-z_1)\big]$$

$$\times\psi^1_{t'}(z_1-y_1)\,du_1\,du_2\,dv_1\,dv_2\,dz_1\frac{dt'}{t'}\phi^2_s(x_2-y_2)$$

$$+\psi_{t,s}*T(1)(x_1,x_2)\phi_{t,s}(x_1-y_1,x_2-y_2)$$

$$:=\sum_{j=1}^9 T^j_{t,s}(x_1,x_2,y_1,y_2),$$

where $\phi^1_t=\int_t^\infty\psi^1_{t'}*\psi^1_{t'}(\cdot)\frac{dt'}{t'}$, $\phi^2_s=\int_s^\infty\psi^2_{s'}*\psi^2_{s'}(\cdot)\frac{ds'}{s'}$ and $\phi_{t,s}=\phi^1_t\phi^2_s$. By a result in [10], $T(1)\in BMO(\mathbb{R}^n\times\mathbb{R}^m)$ and hence $\psi_{t,s}*T(1)(x_1,x_2)$ makes sense because $\psi_{t,s}\in H^1(\mathbb{R}^n\times\mathbb{R}^m)$. It is also easy to see that $\phi_{t,s}$ satisfy the same size and smoothness conditions as $\psi_{t,s}$.

The estimates of (B_1)–(B_3) for $\{T_{t,s}(x_1,x_2,y_1,y_2)\}_{t,s>0}$ will follow easily by the following lemma.

Lemma 2. *For $1\leqslant j\leqslant 9$ and $t,s>0$, there exists a constant C such that*

(D_1) *for* $\varepsilon'<\varepsilon$, $|T^j_{t,s}(x_1,x_2,y_1,y_2)|\leqslant C\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}\frac{s^{\varepsilon'}}{(s+|x_2-y_2|)^{m+\varepsilon'}}$,

(D_2) *for* $\varepsilon''<\varepsilon'$,

(i) $|T^j_{t,s}(x_1,x_2,y_1,y_2)-T^j_{t,s}(x_1,x_2,y_1',y_2)|\leqslant C\big(\frac{|y_1-y_1'|}{t}\big)^{\varepsilon''}\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}\frac{s^{\varepsilon'}}{(s+|x_2-y_2|)^{m+\varepsilon'}}$ *if* $|y_1-y_1'|\leqslant t/2$,

(ii) $|T^j_{t,s}(x_1,x_2,y_1,y_2)-T^j_{t,s}(x_1,x_2,y_1,y_2')|\leqslant C\big(\frac{|y_2-y_2'|}{s}\big)^{\varepsilon''}\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}\frac{s^{\varepsilon'}}{(s+|x_2-y_2|)^{m+\varepsilon'}}$ *if* $|y_2-y_2'|\leqslant s/2$,

(D_3) *for* $\varepsilon'' < \varepsilon'$,
$|[T^j_{t,s}(x_1, x_2, y_1, y_2) - T^j_{t,s}(x_1, x_2, y'_1, y_2)] - [T^j_{t,s}(x_1, x_2, y_1, y'_2) - T^j_{t,s}(x_1, x_2, y'_1, y'_2)]|$
$\leqslant C\big(\frac{|y_1-y'_1|}{t}\big)^{\varepsilon''}\big(\frac{|y_2-y'_2|}{s}\big)^{\varepsilon''}\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}\frac{s^{\varepsilon'}}{(s+|x_2-y_2|)^{m+\varepsilon'}}$
if $|y_1 - y'_1| \leqslant t/2$ *and* $|y_2 - y'_2| \leqslant s/2$.

Before proving Lemma 2, we recall the orthogonal estimates on $\mathbb{R}^n$ (cf. [9, Lemma 4.3] for details). Let S be a Calderón–Zygmund operator with regularity exponent ε associated with a kernel $S(z,w)$ and satisfy $S^*(1) = 0$. Then, for $\psi \in C^\infty_{0,0}(\mathbb{R}^n)$, the following almost orthogonal estimates hold: for $\varepsilon'' < \varepsilon' < \varepsilon$,

$$\left|\iint \psi_t(x-z)S(z,w)\psi_s(w-u)\,dz\,dw\right| \leqslant C\|S\|_{CZ}\left(\frac{s}{t}\right)^{\varepsilon''}\frac{t^{\varepsilon'}}{(t+|x-u|)^{n+\varepsilon'}} \quad \text{for } s \leqslant t, \tag{2.3}$$

and

$$\left|\iint \psi_t(x-z)S(z,w)\big[\psi_s(w-u) - \psi_s(x-u)\big]\,dz\,dw\right| \leqslant C\|S\|_{CZ}\left(\frac{t}{s}\right)^{\varepsilon''}\frac{s^{\varepsilon'}}{(s+|x-u|)^{n+\varepsilon'}} \quad \text{for } t < s. \tag{2.4}$$

The estimate (2.3) and the size condition on ψ_s imply

$$\left|\iint \psi_t(x-z)S(z,w)\psi_s * \psi_s(w-y)\,dz\,dw\right| \leqslant C\|S\|_{CZ}\left(\frac{s}{t}\right)^{\varepsilon''}\frac{t^{\varepsilon'}}{(t+|x-y|)^{n+\varepsilon'}} \quad \text{for } s \leqslant t. \tag{2.5}$$

Similarly,

$$\left|\iiint \psi_t(x-z)S(z,w)\big[\psi_s(w-u) - \psi_s(x-u)\big]\psi_s(u-y)\,dz\,dw\,du\right| \leqslant C\|S\|_{CZ}\left(\frac{t}{s}\right)^{\varepsilon''}\frac{s^{\varepsilon'}}{(s+|x-y|)^{n+\varepsilon'}} \quad \text{for } t < s. \tag{2.6}$$

For $s \leqslant t$ and $\varepsilon''' < \varepsilon'' < \varepsilon'$, by the almost orthogonal estimate (2.3) and the smoothness condition on ψ_s,

$$\begin{aligned}
&\left|\iiint \psi_t(x-z)S(z,w)\psi_s(w-u)\big[\psi_s(u-y) - \psi_s\big(u-y'\big)\big]\,dz\,dw\,du\right| \\
&\quad\leqslant C\|S\|_{CZ}\left(\frac{s}{t}\right)^{\varepsilon''}\int_{\mathbb{R}^n}\frac{t^{\varepsilon'}}{(t+|x-u|)^{n+\varepsilon'}}\big|\psi_s(u-y) - \psi_s\big(u-y'\big)\big|\,du \\
&\quad\leqslant C\|S\|_{CZ}\left(\frac{s}{t}\right)^{\varepsilon''-\varepsilon'''}\left(\frac{|y-y'|}{t}\right)^{\varepsilon'''}\frac{t^{\varepsilon'}}{(t+|x-y|)^{n+\varepsilon'}} \quad \text{for } \big|y-y'\big| \leqslant t/2.
\end{aligned} \tag{2.7}$$

Similarly, for $t < s$, by the estimate (2.4) and the smoothness condition on ψ_s,

$$\left|\iiint \psi_t(x-z)S(z,w)\left[\psi_s(w-u)-\psi_s(x-u)\right]\left[\psi_s(u-y)-\psi_s\left(u-y'\right)\right]dz\,dw\,du\right|$$

$$\leqslant C\|S\|_{CZ}\left(\frac{t}{s}\right)^{\varepsilon''}\left(\frac{|y-y'|}{s}\right)^{\varepsilon''}\frac{s^{\varepsilon'}}{(s+|x-y|)^{n+\varepsilon'}}\quad \text{for } \left|y-y'\right|\leqslant t/2.$$

We now return to the proof of Lemma 2.

Proof of Lemma 2. The main idea is that the iteration method can be applied to reduce the product case to the classical case. To be precise, let us first prove that $T^1_{t,s}(x_1,x_2,y_1,y_2)$ satisfies the estimates (D_1)–(D_3). For fixed t, x_1 and y_1, set

$$K_2(u_2,v_2)=\int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\psi^1_t(x_1-u_1)\widetilde{K}^2(u_2,v_2)(u_1,v_1)\psi^1_{t'}*\psi^1_{t'}(v_1-y_1)\,du_1\,dv_1\frac{dt'}{t'}.$$

Note that when u_2 and v_2 are fixed, $\widetilde{K}^2(u_2,v_2)(u_1,v_1)=K(u_1,u_2,v_1,v_2)$ is a Calderón–Zygmund kernel on $\mathbb{R}^n\times\mathbb{R}^n$ with the norm $\|\widetilde{K}^2(u_2,v_2)\|_{CZ}\leqslant C|u_2-v_2|^{-m}$. By $T^*_1(1)=0$ and the almost orthogonal estimate (2.5) for the kernel $\widetilde{K}^2(u_2,v_2)(u_1,v_1)$ with fixed (u_2,v_2),

$$\left|K_2(u_2,v_2)\right|\leqslant C\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}|u_2-v_2|^{-m}. \tag{2.8}$$

Similarly, when u_2, u'_2 and v_2 are fixed, we have

$$K_2(u_2,v_2)-K_2\left(u'_2,v_2\right)=\int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\psi^1_t(x_1-u_1)\left[\widetilde{K}^2(u_2,v_2)(u_1,v_1)-\widetilde{K}^2\left(u'_2,v_2\right)(u_1,v_1)\right]$$

$$\times\,\psi^1_{t'}*\psi^1_{t'}(v_1-y_1)\,du_1\,dv_1\frac{dt'}{t'}.$$

Note again that $\widetilde{K}^2(u_2,v_2)(u_1,v_1)-\widetilde{K}^2(u'_2,v_2)(u_1,v_1)$ is a Calderón–Zygmund kernel on $\mathbb{R}^n\times\mathbb{R}^n$ with the norm $\|\widetilde{K}^2(u_2,v_2)-\widetilde{K}^2(u'_2,v_2)\|_{CZ}\leqslant C|u_2-u'_2|^{\varepsilon}|u_2-v_2|^{-m-\varepsilon}$ for $|u_2-u'_2|\leqslant\frac{1}{2}|u_2-v_2|$. The same argument as (2.8) gives

$$\left|K_2(u_2,v_2)-K_2\left(u'_2,v_2\right)\right|\leqslant C\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}\frac{|u_2-u'_2|^{\varepsilon}}{|u_2-v_2|^{m+\varepsilon}}\quad\text{for }\left|u_2-u'_2\right|\leqslant\frac{|u_2-v_2|}{2}.$$

A same process shows

$$\left|K_2(u_2,v_2)-K_2\left(u_2,v'_2\right)\right|\leqslant C\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}\frac{|v_2-v'_2|^{\varepsilon}}{|u_2-v_2|^{m+\varepsilon}}\quad\text{for }\left|v_2-v'_2\right|\leqslant\frac{|u_2-v_2|}{2}.$$

These imply that $K_2(u_2, v_2)$ is a Calderón–Zygmund kernel on $\mathbb{R}^m \times \mathbb{R}^m$ and

$$|K_2|_{CZ} \leqslant C \frac{t^{\varepsilon'}}{(t + |x_1 - y_1|)^{n+\varepsilon'}}. \tag{2.9}$$

Note that if S is an operator associated with the kernel $K_2(u_2, v_2)$, then the condition $T_2^*(1) = 0$ implies $S^*(1) = 0$. Therefore, first writing

$$T^1_{t,s}(x_1, x_2, y_1, y_2) = \int_0^s \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \psi_s^2(x_2 - u_2) K_2(u_2, v_2) \psi_{s'}^2 * \psi_{s'}^2(v_2 - y_2)\, du_2\, dv_2 \frac{ds'}{s'}$$

and then applying the orthogonal estimate (2.5) for $K_2(u_2, v_2)$ with the norm estimate in (2.9) imply

$$\left|T^1_{t,s}(x_1, x_2, y_1, y_2)\right| \leqslant C \frac{t^{\varepsilon'}}{(t + |x_1 - y_1|)^{n+\varepsilon'}} \frac{s^{\varepsilon'}}{(s + |x_2 - y_2|)^{n+\varepsilon'}}.$$

This shows that $T^1_{t,s}(x_1, x_2, y_1, y_2)$ satisfies (D_1).

To check (D_2) (i), we write

$$\begin{aligned} &T^1_{t,s}(x_1, x_2, y_1, y_2) - T^1_{t,s}\big(x_1, x_2, y_1', y_2\big) \\ &\quad = \int_0^s \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \psi_s^2(x_2 - u_2) K_{2,2}(u_2, v_2) \psi_{s'}^2 * \psi_{s'}^2(v_2 - y_2)\, du_2\, dv_2 \frac{ds'}{s'}, \end{aligned}$$

where for fixed t, x_1, y_1, y_1',

$$\begin{aligned} K_{2,2}(u_2, v_2) = &\int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi_t^1(x_1 - u_1) \widetilde{K}^2(u_2, v_2)(u_1, v_1) \psi_{t'}^1(v_1 - z_1) \\ &\times \big[\psi_{t'}^1(z_1 - y_1) - \psi_{t'}^1\big(z_1 - y_1'\big)\big]\, du_1\, dv_1\, dz_1 \frac{dt'}{t'}. \end{aligned}$$

By the estimate of (2.7) and the fact that $\|\widetilde{K}^2(u_2, v_2)\|_{CZ} \leqslant C|u_2 - v_2|^{-m}$,

$$\left|K_{2,2}(u_2, v_2)\right| \leqslant C\left(\frac{|y_1 - y_1'|}{t}\right)^{\varepsilon''} \frac{t^{\varepsilon'}}{(t + |x_1 - y_1|)^{n+\varepsilon'}} \frac{1}{|u_2 - v_2|^m} \quad \text{for } \left|y_1 - y_1'\right| \leqslant t/2.$$

A similar argument and $\|\widetilde{K}^2(u_2, v_2) - \widetilde{K}^2(u_2', v_2)\|_{CZ} \leqslant C|u_2 - u_2'|^{\varepsilon}|u_2 - v_2|^{-m-\varepsilon}$ yield that, for $|u_2 - u_2'| \leqslant \frac{1}{2}|u_2 - v_2|$ and $|y_1 - y_1'| \leqslant t/2$,

$$
\begin{aligned}
&\left|K_{2,2}(u_2,v_2)-K_{2,2}\big(u_2',v_2\big)\right| \\
&\quad=\left|\int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\psi_t^1(x_1-u_1)\big[\widetilde{K}^2(u_2,v_2)(u_1,v_1)-\widetilde{K}^2\big(u_2',v_2\big)(u_1,v_1)\big]\right. \\
&\qquad\left.\times\psi_{t'}^1(v_1-z_1)\big[\psi_{t'}^1(z_1-y_1)-\psi_{t'}^1\big(z_1-y_1'\big)\big]\,du_1\,dv_1\,dz_1\frac{dt'}{t'}\right| \\
&\quad\leqslant C\left(\frac{|y_1-y_1'|}{t}\right)^{\varepsilon''}\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}\frac{|u_2-u_2'|^{\varepsilon}}{|u_2-v_2|^{m+\varepsilon}}.
\end{aligned}
$$

Similarly, for $|v_2-v_2'|\leqslant\frac{1}{2}|u_2-v_2|$ and $|y_1-y_1'|\leqslant t/2$,

$$
\left|K_{2,2}(u_2,v_2)-K_{2,2}\big(u_2,v_2'\big)\right|\leqslant C\left(\frac{|y_1-y_1'|}{t}\right)^{\varepsilon''}\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}\frac{|v_2-v_2'|^{\varepsilon}}{|u_2-v_2|^{m+\varepsilon}}.
$$

Hence, $K_{2,2}(u_2,v_2)$ is a Calderón–Zygmund kernel and

$$
|K_{2,2}|_{CZ}\leqslant C\left(\frac{|y_1-y_1'|}{t}\right)^{\varepsilon''}\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}\quad\text{for }\left|y_1-y_1'\right|\leqslant t/2. \tag{2.10}
$$

Applying the estimate of (2.5) to $K_{2,2}(u_2,v_2)$ together with the estimate of (2.10) yields

$$
\begin{aligned}
&\left|\int_0^s\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\psi_s^2(x_2-u_2)K_{2,2}(u_2,v_2)\psi_{s'}^2*\psi_{s'}^2(v_2-y_2)\,du_2\,dv_2\frac{ds'}{s'}\right| \\
&\quad\leqslant C\left(\frac{|y_1-y_1'|}{t}\right)^{\varepsilon''}\frac{t^{\varepsilon'}}{(t+|x_1-y_1|)^{n+\varepsilon'}}\frac{s^{\varepsilon'}}{(s+|x_2-y_2|)^{m+\varepsilon'}}\quad\text{for }\left|y_1-y_1'\right|\leqslant t/2,
\end{aligned}
$$

and hence (D_2) (i) follows. The proof of (D_2) (ii) is the same. To prove (D_3) for $T_{t,s}^1(x_1,x_2,y_1,y_2)$, we write

$$
\begin{aligned}
&\big[T_{t,s}^1(x_1,x_2,y_1,y_2)-T_{t,s}^1\big(x_1,x_2,y_1',y_2\big)\big]-\big[T_{t,s}^1\big(x_1,x_2,y_1,y_2'\big)-T_{t,s}^1\big(x_1,x_2,y_1',y_2'\big)\big] \\
&\quad=\int_0^t\int_0^s\iiint\psi_t^1(x_1-u_1)\psi_s^2(x_2-u_2)\widetilde{K}^2(u_2,v_2)(u_1,v_1)\psi_{t'}^1(v_1-z_1) \\
&\qquad\times\big[\psi_{t'}^1(z_1-y_1)-\psi_{t'}^1\big(z_1-y_1'\big)\big]\psi_{s'}^2(v_2-z_2)\big[\psi_{s'}^2(z_2-y_2)-\psi_{s'}^2\big(z_2-y_2'\big)\big] \\
&\qquad\times du_1\,du_2\,dv_1\,dv_2\,dz_1\,dz_2\frac{ds'}{s'}\frac{dt'}{t'} \\
&\quad=\int_0^s\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\psi_s^2(x_2-u_2)K_{2,2}(u_2,v_2)\psi_{s'}^2(v_2-z_2)\big[\psi_{s'}^2(z_2-y_2)-\psi_{s'}^2\big(z_2-y_2'\big)\big] \\
&\qquad\times du_2\,dv_2\,dz_2\frac{ds'}{s'}.
\end{aligned}
$$

By the estimate (2.7) for the kernel $K_{2,2}(u_2, v_2)$, $T^1_{t,s}(x_1, x_2, y_1, y_2)$ satisfies (D_3). The proofs for $T^j_{t,s}$, $j = 2, 4, 6$, are similar provided replacing (2.3) and (2.5) by (2.4) and (2.6), so we leave details to the reader.

Since the proofs for $T^j_{t,s}(x_1, x_2, y_1, y_2)$, $j = 3, 5, 7, 8$, are similar, we estimate $T^3_{t,s}(x_1, x_2, y_1, y_2)$ only. For fixed x_2, set

$$K_1(u_1, v_1) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \psi^2_s(x_2 - u_2)\widetilde{K}^1(u_1, v_1)(u_2, v_2)\, du_2\, dv_2.$$

Note that for fixed (u_1, v_1), $\int_{\mathbb{R}^m} \widetilde{K}^1(u_1, v_1)(u_2, v_2)\, dv_2$, as a function of the variable u_2, is a *BMO* function and $\psi^2_s(x_2 - u_2)$ is a function in $H^1(\mathbb{R}^m)$ with $H^1(\mathbb{R}^m)$-norm uniformly bounded for all x_2 and s. Moreover,

$$\left\| \int_{\mathbb{R}^m} \widetilde{K}^1(u_1, v_1)(\cdot, v_2)\, dv_2 \right\|_{BMO(\mathbb{R}^m)} \leqslant C \left\| \widetilde{K}^1(u_1, v_1)\right\|_{CZ} \leqslant C|u_1 - v_1|^{-n},$$

which implies

$$\left|K_1(u_1, v_1)\right| \leqslant C|u_1 - v_1|^{-n}.$$

Similarly, for $|u_1 - u_1'| \leqslant \frac{1}{2}|u_1 - v_1|$, we have

$$\begin{aligned}\left\| \int_{\mathbb{R}^m} \left[\widetilde{K}^1(u_1, v_1)(\cdot, v_2) - \widetilde{K}^1\left(u_1', v_1\right)(\cdot, v_2)\right] dv_2 \right\|_{BMO(\mathbb{R}^m)} &\leqslant C \left\| \widetilde{K}^1(u_1, v_1) - \widetilde{K}^1\left(u_1', v_1\right)\right\|_{CZ} \\ &\leqslant C\left|u_1 - u_1'\right|^{\varepsilon}|u_1 - v_1|^{-n-\varepsilon},\end{aligned}$$

and hence

$$\begin{aligned}&\left|K_1(u_1, v_1) - K_1\left(u_1', v_1\right)\right| \\ &\quad = \left| \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \psi^2_s(x_2 - u_2)\left[\widetilde{K}^1(u_1, v_1)(u_2, v_2) - \widetilde{K}^1\left(u_1', v_1\right)(u_2, v_2)\right] du_2\, dv_2 \right| \\ &\quad \leqslant C\left|u_1 - u_1'\right|^{\varepsilon}|u_1 - v_1|^{-n-\varepsilon}.\end{aligned}$$

The estimate $|K_1(u_1, v_1) - K_1(u_1, v_1')|$ can be obtained by the same manner. Thus, $K_1(u_1, v_1)$ is a Calderón–Zygmund kernel and $|K_1|_{CZ} \leqslant C$. Note that

$$T^3_{t,s}(x_1, x_2, y_1, y_2) = \int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi^1_t(x_1 - u_1)K_1(u_1, v_1)\psi^1_{t'} * \psi^1_{t'}(v_1 - y_1)\, du_1\, dv_1 \frac{dt'}{t'}\phi^2_s(x_2 - y_2).$$

Applying the almost orthogonal estimate of (2.5) to $K_1(u_1, v_1)$ together with the size condition on ϕ^2_s leads to (D_1) for $T^3_{t,s}(x_1, x_2, y_1, y_2)$. The estimates of (D_2) and (D_3) can be proved by the same way.

Finally, note that $\int K(u_1,u_2,v_1,v_2)\,dv_1\,dv_2$, as a function of variables u_1 and u_2, belongs to $BMO(\mathbb{R}^n\times\mathbb{R}^m)$, and $\psi_t^1(x_1-u_1)\psi_s^2(x_2-u_2)$, as function of (u_1,u_2), is in $H^1(\mathbb{R}^n\times\mathbb{R}^m)$ with the bounded norm uniformly for all t,s and x_1,x_2. Thus, $\psi_{ts}*T(1)(x_1,x_2)$ is uniformly bounded for all t,s and x_1,x_2. Therefore, the estimates (D_1)–(D_3) for $T^9_{t,s}(x_1,x_2,y_1,y_2)$ are the same as those for $\phi_t^1(x_1-y_1)\phi_s^2(x_2-y_2)$, which can be immediately obtained. The proof of Lemma 2 is completed. $\square$

Now we demonstrate the regularity of the operator $T_{t,s}$ mapping from L^2 into $L^2_{\mathcal{H}}$.

Lemma 3. *Let $T_{t,s}$ be defined in* (2.2) *and ε be the regularity exponent of T. For $\varepsilon'<\varepsilon$,*

(i) *if $|y_1-x_I|\leqslant|x_1-x_I|/2$, then*

$$\left\|\left\{\int_{\mathbb{R}^m}\big[T_{t,s}(x_1,\cdot,y_1,y_2)-T_{t,s}(x_1,\cdot,x_I,y_2)\big]f(y_2)\,dy_2\right\}\right\|_{L^2_{\mathcal{H}}(\mathbb{R}^m)}\leqslant C\frac{|y_1-x_I|^{\varepsilon'}}{|x_1-x_I|^{n+\varepsilon'}}\|f\|_2;$$

(ii) *if $|y_2-y_J|\leqslant|x_2-y_J|/2$, then*

$$\left\|\left\{\int_{\mathbb{R}^n}\big[T_{t,s}(\cdot,x_2,y_1,y_2)-T_{t,s}(\cdot,x_2,y_1,y_J)\big]f(y_1)\,dy_1\right\}\right\|_{L^2_{\mathcal{H}}(\mathbb{R}^n)}\leqslant C\frac{|y_2-y_J|^{\varepsilon'}}{|x_2-y_J|^{n+\varepsilon'}}\|f\|_2.$$

Proof. The proofs of (i) and (ii) are the same, so we show the case (i) only. We will use $0<\varepsilon'''<\varepsilon''<\varepsilon'<\varepsilon$ through the proof. Note that

$$\left\|\left\{\int_{\mathbb{R}^m}\big[T_{t,s}(x_1,\cdot,y_1,y_2)-T_{t,s}(x_1,\cdot,x_I,y_2)\big]f(y_2)\,dy_2\right\}\right\|^2_{L^2_{\mathcal{H}}(\mathbb{R}^m)}$$
$$=\int_{\mathbb{R}^m}\left\|\left\{\int_{\mathbb{R}^m}\big[T_{t,s}(x_1,x_2,y_1,y_2)-T_{t,s}(x_1,x_2,x_I,y_2)\big]f(y_2)\,dy_2\right\}\right\|^2_{\mathcal{H}}dx_2.$$

We write

$$\int_{\mathbb{R}^m}\big[T_{t,s}(x_1,x_2,y_1,y_2)-T_{t,s}(x_1,x_2,x_I,y_2)\big]f(y_2)\,dy_2$$
$$=\int_{\mathbb{R}^m}\int_0^\infty\int_0^\infty\iint\psi_{t,s}(x_1-u_1,x_2-u_2)k(u_1,u_2,v_1,v_2)$$
$$\times\big[\psi^1_{t'}*\psi^1_{t'}(v_1-y_1)-\psi^1_{t'}*\psi^1_{t'}(v_1-x_I)\big]\psi^2_{s'}*\psi^2_{s'}(v_2-y_2)f(y_2)$$
$$\times du_1\,du_2\,dv_1\,dv_2\frac{dt'}{t'}\frac{ds'}{s'}\,dy_2$$
$$=\int_0^\infty\int_0^\infty\iint\psi_{t,s}(x_1-u_1,x_2-u_2)K(u_1,u_2,v_1,v_2)$$

$$\times\big[\psi^1_{t'}*\psi^1_{t'}(v_1-y_1)-\psi^1_{t'}*\psi^1_{t'}(v_1-x_I)\big]\psi^2_{s'}*\psi^2_{s'}*f(v_2)\,du_1\,du_2\,dv_1\,dv_2\frac{dt'}{t'}\frac{ds'}{s'}$$

$$=\int_0^\infty\iint\psi_{t,s}(x_1-u_1,x_2-u_2)K(u_1,u_2,v_1,v_2)$$

$$\times\big[\psi^1_{t'}*\psi^1_{t'}(v_1-y_1)-\psi^1_{t'}*\psi^1_{t'}(v_1-x_I)\big]f(v_2)\,du_1\,du_2\,dv_1\,dv_2\frac{dt'}{t'}$$

$$=\psi^2_s*\Bigg(\int_0^\infty\int\int_{\mathbb{R}^n}\psi^1_t(x_1-u_1)K(u_1,\cdot,v_1,v_2)$$

$$\times\big[\psi^1_{t'}*\psi^1_{t'}(v_1-y_1)-\psi^1_{t'}*\psi^1_{t'}(v_1-x_I)\big]f(v_2)\,du_1\,dv_1\,dv_2\frac{dt'}{t'}\Bigg)(x_2),$$

where we first write $\int\psi^2_{s'}*\psi^2_{s'}(v_2-y_2)f(y_2)\,dy_2=\psi^2_{s'}*\psi^2_{s'}*f(v_2)$ and then use the Calderón identity $\int_0^\infty\psi^2_{s'}*\psi^2_{s'}*f(v_2)\frac{ds'}{s'}=f(v_2)$.

The Littlewood–Paley estimate gives

$$\int_{\mathbb{R}^m}\bigg\|\bigg\{\int_{\mathbb{R}^m}\big[T_{t,s}(x_1,x_2,y_1,y_2)-T_{t,s}(x_1,x_2,x_I,y_2)\big]f(y_2)\,dy_2\bigg\}\bigg\|^2_{\mathcal{H}}dx_2$$

$$\leqslant C\int_0^\infty\int_{\mathbb{R}^m}\bigg|\int_0^\infty\int_{\mathbb{R}^n\times\mathbb{R}^m}\int_{\mathbb{R}^n}\psi^1_t(x_1-u_1)K(u_1,x_2,v_1,v_2)$$

$$\times\big[\psi^1_{t'}*\psi^1_{t'}(v_1-y_1)-\psi^1_{t'}*\psi^1_{t'}(v_1-x_I)\big]f(v_2)\,du_1\,dv_1\,dv_2\frac{dt'}{t'}\bigg|^2dx_2\frac{dt}{t}.\quad(2.11)$$

Dividing the integral with respect to t' into three parts, we obtain

$$\int_{\mathbb{R}^m}\bigg|\int_0^\infty\int\int_{\mathbb{R}^n}\psi^1_t(x_1-u_1)K(u_1,x_2,v_1,v_2)$$

$$\times\big[\psi^1_{t'}*\psi^1_{t'}(v_1-y_1)-\psi^1_{t'}*\psi^1_{t'}(v_1-x_I)\big]f(v_2)\,du_1\,dv_1\,dv_2\frac{dt'}{t'}\bigg|^2dx_2$$

$$\leqslant C\int_{\mathbb{R}^m}\bigg|\int_0^t\int\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\psi^1_t(x_1-u_1)K(u_1,x_2,v_1,v_2)\psi^1_{t'}(v_1-z_1)$$

$$\times\big[\psi^1_{t'}(z_1-y_1)-\psi^1_{t'}(z_1-x_I)\big]f(v_2)\,dz_1\,du_1\,dv_1\,dv_2\frac{dt'}{t'}\bigg|^2dx_2$$

$$+ C \int_{\mathbb{R}^m} \left| \int_t^\infty \int \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi_t^1(x_1 - u_1) K(u_1, x_2, v_1, v_2) \right.$$

$$\times \left[\psi_{t'}^1(v_1 - z_1) - \psi_{t'}^1(x_1 - z_I)\right]\left[\psi_{t'}^1(z_1 - y_1) - \psi_{t'}^1(z_1 - x_I)\right] f(v_2)$$

$$\left. \times dz_1 \, du_1 \, dv_1 \, dv_2 \frac{dt'}{t'} \right|^2 dx_2$$

$$+ C \int_{\mathbb{R}^m} \left| \int_t^\infty \int \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi_t^1(x_1 - u_1) K(u_1, x_2, v_1, v_2) \right.$$

$$\left. \times \psi_{t'}^1(x_1 - z_1)\left[\psi_{t'}^1(z_1 - y_1) - \psi_{t'}^1(z_1 - x_I)\right] f(v_2) \, dz_1 \, du_1 \, dv_1 \, dv_2 \frac{dt'}{t'} \right|^2 dx_2$$

$$:= E + F + G.$$

We first consider the item G and write

$$G = C \int_{\mathbb{R}^m} \left| \int_t^\infty \int \int_{\mathbb{R}^n} \psi_t^1(x_1 - u_1) K(u_1, x_2, v_1, v_2) \right.$$

$$\left. \times \left[\psi_{t'}^1 * \psi_{t'}^1(x_1 - y_1) - \psi_{t'}^1 * \psi_{t'}^1(x_1 - x_I)\right] f(v_2) \, du_1 \, dv_1 \, dv_2 \frac{dt'}{t'} \right|^2 dx_2$$

$$= C \left|\phi_t^1(x_1 - y_1) - \phi_t^1(x_1 - x_I)\right|^2$$

$$\times \int_{\mathbb{R}^m} \left| \int \int_{\mathbb{R}^n} \psi_t^1(x_1 - u_1) K(u_1, x_2, v_1, v_2) f(v_2) \, du_1 \, dv_1 \, dv_2 \right|^2 dx_2$$

$$= C \sup_{\|g\|_2 \leqslant 1} \left(\int_{\mathbb{R}^m} \int \int_{\mathbb{R}^n} \psi_t^1(x_1 - u_1) K(u_1, x_2, v_1, v_2) f(v_2) g(x_2) \, du_1 \, dv_1 \, dv_2 \, dx_2 \right)^2$$

$$\times \left|\phi_t^1(x_1 - y_1) - \phi_t^1(x_1 - x_I)\right|^2,$$

where $\phi_t^1(\cdot) = \int_t^\infty \psi_{t'}^1 * \psi_{t'}^1(\cdot) \frac{dt'}{t'}$.

For fixed u_1 and v_1, set

$$\overline{K}(u_1, v_1) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} K(u_1, x_2, v_1, v_2) f(v_2) g(x_2) \, dv_2 \, dx_2.$$

Then the operator associated to the kernel $\overline{K}(u_1, v_1)$ is a Calderón–Zygmund operator with operator norm $C\|f\|_2\|g\|_2$. Since $\int_{\mathbb{R}^n} \overline{K}(u_1, v_1) \, dv_1$ is a BMO function for u_1,

$$\left| \int_{\mathbb{R}^n} \psi_t^1(x_1 - u_1) \int_{\mathbb{R}^n} \overline{K}(u_1, v_1) \, dv_1 \, du_1 \right| \leqslant C\|f\|_2\|g\|_2 \quad \text{uniformly for } x_1.$$

Hence, for $|y_1 - x_I| \leqslant t/2$,

$$G \leqslant C\left|\phi_t^1(x_1 - y_1) - \phi_t^1(x_1 - x_I)\right|^2 \|f\|_2^2 \leqslant C\left(\frac{|y_1 - x_I|}{t}\right)^{2\varepsilon} \frac{t^{2\varepsilon}}{(t + |x_1 - x_I|)^{2(n+\varepsilon)}} \|f\|_2^2. \tag{2.12}$$

To estimate E, we consider two cases $\{|x_1 - z_1| > 8t\}$ and $\{|x_1 - z_1| \leqslant 8t\}$. For the case $\{|x_1 - z_1| > 8t\}$, we will use the kernel estimates of K. To be precise, using the cancellation properties of $\psi_{t'}^1$ and duality, we get

$$E = C \int_{\mathbb{R}^m} \left| \int_0^t \int_{\mathbb{R}^n} \int \int_{\mathbb{R}^n} \psi_t^1(x_1 - u_1)\left[K(u_1, x_2, v_1, v_2) - K(u_1, x_2, z_1, v_2)\right] \times \psi_{t'}^1(v_1 - z_1)\left[\psi_{t'}^1(z_1 - y_1) - \psi_{t'}^1(z_1 - x_I)\right] f(v_2)\, du_1\, dv_1\, dv_2\, dz_1 \frac{dt'}{t'} \right|^2 dx_2$$

$$= C \sup_{\|h\|_2 \leqslant 1} \left(\int_{\mathbb{R}^m} h(x_2) \int_0^t \int_{\mathbb{R}^n} \int \int_{\mathbb{R}^n} \psi_t^1(x_1 - u_1)\left[K(u_1, x_2, v_1, v_2) - K(u_1, x_2, z_1, v_2)\right] \times \psi_{t'}^1(v_1 - z_1)\left[\psi_{t'}^1(z_1 - y_1) - \psi_{t'}^1(z_1 - x_I)\right] f(v_2)\, du_1\, dv_1\, dv_2\, dz_1 \frac{dt'}{t'}\, dx_2 \right)^2.$$

Note that the facts $|x_1 - z_1| > 8t$, $|x_1 - u_1| < t$ and $|v_1 - z_1| < t' \leqslant t$ easily imply $|v_1 - z_1| \leqslant |u_1 - v_1|/2$ and $|u_1 - v_1| > |x_1 - z_1|/2$. We apply (A_2) to obtain, for $|y_1 - x_I| \leqslant t/2$,

$$\begin{aligned} E^{1/2} &\leqslant C \sup_{\|h\|_2 \leqslant 1} \|h\|_2 \|f\|_2 \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v_1 - z_1|^\varepsilon}{|u_1 - v_1|^{n+\varepsilon}} \left|\psi_t^1(x_1 - u_1)\right| \\ &\quad \times \left|\psi_{t'}^1(v_1 - z_1)\right|\left|\psi_{t'}^1(z_1 - y_1) - \psi_{t'}^1(z_1 - x_I)\right| du_1\, dv_1\, dz_1 \frac{dt'}{t'} \\ &\leqslant C\|f\|_2 \int_0^t \int_{\mathbb{R}^n} \frac{t^\varepsilon}{(t + |x_1 - z_1|)^{n+\varepsilon}} \left(\frac{t'}{t}\right)^{\varepsilon' - \varepsilon''} \left(\frac{|y_1 - x_I|}{t}\right)^{\varepsilon''} \\ &\quad \times \left(\frac{(t')^\varepsilon}{(t' + |z_1 - y_1|)^{n+\varepsilon}} + \frac{(t')^\varepsilon}{(t' + |z_1 - x_I|)^{n+\varepsilon}}\right) dz_1 \frac{dt'}{t'} \\ &\leqslant C\left(\frac{|y_1 - x_I|}{t}\right)^{\varepsilon''} \frac{t^\varepsilon}{(t + |x_1 - x_I|)^{n+\varepsilon}} \|f\|_2. \end{aligned} \tag{2.13}$$

For the case $\{|x_1 - z_1| \leqslant 8t\}$, by the condition on the support of ψ_t^1, we write

$$E = C \int_{\mathbb{R}^m} \left| \int_0^t \int_{\mathbb{R}^n} \int \int_{|u_1 - z_1| \leqslant 9t} \psi_t^1(x_1 - u_1) K(u_1, x_2, v_1, v_2) \psi_{t'}^1(v_1 - z_1) \right.$$
$$\left. \times \left[\psi_{t'}^1(z_1 - y_1) - \psi_{t'}^1(z_1 - x_I)\right] f(v_2)\, du_1\, dv_1\, dv_2\, dz_1 \frac{dt'}{t'} \right|^2 dx_2.$$

Note that $E = 0$ if $|z_1 - y_1| > t$ and $|z_1 - x_I| > t$. It implies $|x_1 - x_I| \leqslant 10t$ provided $|x_1 - z_1| \leqslant 8t$ and $|y_1 - x_I| \leqslant t/2$. This fact will be used later.

Now let $\eta_0 \in C^\infty(\mathbb{R}^n)$ be 1 on the unit ball and 0 outside the ball $B(0, 2)$. Set $\eta_1 = 1 - \eta_0$. We use $T_1^* 1 = 0$ to obtain

$$E = C \int_{\mathbb{R}^m} \left| \int_0^t \int_{\mathbb{R}^n} \int \int_{|u_1 - z_1| \leqslant 9t} \left[\psi_t^1(x_1 - u_1) - \psi_t^1(x_1 - z_1)\right] K(u_1, x_2, v_1, v_2) \right.$$
$$\left. \times \psi_{t'}^1(v_1 - z_1)\left[\psi_{t'}^1(z_1 - y_1) - \psi_{t'}^1(z_1 - x_I)\right] f(v_2)\, du_1\, dv_1\, dv_2\, dz_1 \frac{dt'}{t'} \right|^2 dx_2$$
$$\leqslant C \int_{\mathbb{R}^m} \left| \int_0^t \int_{\mathbb{R}^n} \int \int_{|u_1 - z_1| \leqslant 9t} \eta_0\left(\frac{u_1 - z_1}{4t'}\right)\left[\psi_t^1(x_1 - u_1) - \psi_t^1(x_1 - z_1)\right] K(u_1, x_2, v_1, v_2) \right.$$
$$\left. \times \psi_{t'}^1(v_1 - z_1)\left[\psi_{t'}^1(z_1 - y_1) - \psi_{t'}^1(z_1 - x_I)\right] f(v_2)\, du_1\, dv_1\, dv_2\, dz_1 \frac{dt'}{t'} \right|^2 dx_2$$
$$+ C \int_{\mathbb{R}^m} \left| \int_0^t \int_{\mathbb{R}^n} \int \int_{|u_1 - z_1| \leqslant 9t} \eta_1\left(\frac{u_1 - z_1}{4t'}\right)\left[\psi_t^1(x_1 - u_1) - \psi_t^1(x_1 - z_1)\right] K(u_1, x_2, v_1, v_2) \right.$$
$$\left. \times \psi_{t'}^1(v_1 - z_1)\left[\psi_{t'}^1(z_1 - y_1) - \psi_{t'}^1(z_1 - x_I)\right] f(v_2)\, du_1\, dv_1\, dv_2\, dz_1 \frac{dt'}{t'} \right|^2 dx_2$$
$$:= E_1 + E_2.$$

By duality and the $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of T, for $|y_1 - x_I| \leqslant t/2$,

$$E_1^{1/2} = C \sup_{\|h\|_2 \leqslant 1} \left\langle h, \int_0^t \int_{\mathbb{R}^n} \int \int_{|u_1 - z_1| \leqslant 9t} f_{t', z_1}(u_1) K(u_1, \cdot, v_1, v_2) g_{t', z_1}(v_1) f(v_2) \right.$$
$$\left. \times du_1\, dv_1\, dv_2\, dz_1 \frac{dt'}{t'} \right\rangle$$
$$\leqslant C \sup_{\|h\|_2 \leqslant 1} \int_0^t \int_{\mathbb{R}^n} \|h\|_2 \|f_{t', z_1}\|_2 \|g_{t', z_1}\|_2 \|f\|_2\, dz_1 \frac{dt'}{t'}$$

$$\leqslant C\int_0^t \frac{t'}{t^{n+1}}\big(t'\big)^{n/2}\bigg(\frac{|y_1-x_I|}{t'}\bigg)^{\varepsilon'}\big(t'\big)^{-n/2}\frac{dt'}{t'}\|f\|_2$$

$$\leqslant C|y_1-x_I|^{\varepsilon'}t^{-n-\varepsilon'}\|f\|_2,$$

where $f_{t',z_1}(u_1)=\eta_0(\frac{u_1-z_1}{4t'})[\psi_t^1(x_1-u_1)-\psi_t^1(x_1-z_1)]$ and $g_{t',z_1}(v_1)=\psi_{t'}^1(v_1-z_1)\times[\psi_{t'}^1(z_1-y_1)-\psi_{t'}^1(z_1-x_I)]$.

To estimate E_2, we use the cancellation property of $\psi_{t'}^1$ and write

$$E_2=C\int_{\mathbb{R}^m}\bigg|\int_0^t\int_{\mathbb{R}^n}\int\int_{|u_1-z_1|\leqslant 9t}\eta_1\bigg(\frac{u_1-z_1}{4t'}\bigg)\big[\psi_t^1(x_1-u_1)-\psi_t^1(x_1-z_1)\big]$$
$$\times\big[K(u_1,x_2,v_1,v_2)-K(u_1,x_2,z_1,v_2)\big]\psi_{t'}^1(v_1-z_1)$$
$$\times\big[\psi_{t'}^1(z_1-y_1)-\psi_{t'}^1(z_1-x_I)\big]f(v_2)\,du_1\,dv_1\,dv_2\,dz_1\frac{dt'}{t'}\bigg|^2\,dx_2.$$

By duality again,

$$E_2^{1/2}=C\sup_{\|h\|_2\leqslant 1}\int_{\mathbb{R}^m}h(x_2)\int_0^t\int_{\mathbb{R}^n}\int\int_{|u_1-z_1|\leqslant 9t}\eta_1\bigg(\frac{u_1-z_1}{4t'}\bigg)\big[\psi_t^1(x_1-u_1)-\psi_t^1(x_1-z_1)\big]$$
$$\times\big[K(u_1,x_2,v_1,v_2)-K(u_1,x_2,z_1,v_2)\big]\psi_{t'}^1(v_1-z_1)$$
$$\times\big[\psi_{t'}^1(z_1-y_1)-\psi_{t'}^1(z_1-x_I)\big]f(v_2)\,du_1\,dv_1\,dv_2\,dz_1\frac{dt'}{t'}\,dx_2.$$

By the conditions on the supports of η_1 and $\psi_{t'}^1$, we have $|u_1-z_1|\geqslant 4t'$ and $|v_1-z_1|<t'$. This gives $|v_1-z_1|\leqslant|u_1-z_1|/2$. Applying (A_2) with the estimate

$$\big|\psi_t^1(x_1-u_1)-\psi_t^1(x_1-z_1)\big|\leqslant C\frac{|u_1-z_1|}{t^{n+1}},$$

we obtain, for $|y_1-x_I|\leqslant t/2$,

$$E_2^{1/2}\leqslant C\int_0^t\int_{\mathbb{R}^n}\int_{4t'\leqslant|u_1-z_1|\leqslant 9t}\int_{\mathbb{R}^n}\frac{|u_1-z_1|}{t^{n+1}}\frac{|v_1-z_1|^{\varepsilon}}{|u_1-z_1|^{n+\varepsilon}}\|f\|_2$$
$$\times\big|\psi_{t'}^1(v_1-z_1)\big|\big|\psi_{t'}^1(z_1-y_1)-\psi_{t'}^1(z_1-x_I)\big|\,dv_1\,du_1\,dz_1\frac{dt'}{t'}$$
$$\leqslant Ct^{-n}\|f\|_2\int_0^t\int_{\mathbb{R}^n}\int_{4t'\leqslant|u_1-z_1|\leqslant 9t}\bigg(\frac{|u_1-z_1|}{t}\bigg)^{\varepsilon}\frac{(t')^{\varepsilon}}{|u_1-z_1|^{n+\varepsilon}}\bigg(\frac{|y_1-x_I|}{t'}\bigg)^{\varepsilon''}$$

$$\times\left(\frac{(t')^{\varepsilon}}{(t'+|z_1-y_1|)^{n+\varepsilon}}+\frac{(t')^{\varepsilon}}{(t'+|z_1-x_I|)^{n+\varepsilon}}\right)du_1\,dz_1\frac{dt'}{t'}$$
$$\leqslant Ct^{-n-\varepsilon''}|y_1-x_I|^{\varepsilon''}\|f\|_2.$$

Thus, for $|x_1-z_1|\leqslant 8t$ and $|y_1-x_I|\leqslant t/2$, we have $E\leqslant Ct^{-n-\varepsilon''}|y_1-x_I|^{\varepsilon''}\|f\|_2$, which together with the fact $|x_1-x_I|\leqslant 10t$ as mentioned before implies

$$E\leqslant C\left(\frac{|y_1-x_I|}{t}\right)^{2\varepsilon''}\frac{t^{2\varepsilon''}}{(t+|x_1-x_I|)^{2(n+\varepsilon'')}}\|f\|_2^2. \tag{2.14}$$

The estimate of F is the same as the estimate of E. It follows from (2.12)–(2.14) that, for $|y_1-x_I|\leqslant t/2$,

$$\int\limits_{\mathbb{R}^m}\Bigg|\int\limits_0^{\infty}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n\times\mathbb{R}^m}\psi_t^1(x_1-u_1)k(u_1,x_2,v_1,v_2)$$
$$\times\big[\psi_{t'}^1*\psi_{t'}^1(v_1-y_1)-\psi_{t'}^1*\psi_{t'}^1(v_1-x_I)\big]f(v_2)\,du_1\,dv_1\,dv_2\frac{dt'}{t'}\Bigg|^2 dx_2$$
$$\leqslant C\left(\frac{|y_1-x_I|}{t}\right)^{2\varepsilon''}\frac{t^{2\varepsilon''}}{(t+|x_1-x_I|)^{2(n+\varepsilon'')}}\|f\|_2^2. \tag{2.15}$$

Inserting (2.15) into (2.11), we obtain the desired result (i) of Lemma 3. Hence, the proof of Lemma 3 is completed. □

To finish the proof of Theorem 1, as mentioned in step 3 of Section 1, we show the following general result.

Proposition 4. *Let $\mathcal{L}$ be a bounded operator from $L^2(\mathbb{R}^{n+m})$ to $L^2_{\mathcal{H}}(\mathbb{R}^{n+m})$. Then, for $0<p\leqslant 1$, $\mathcal{L}$ extends to be a bounded operator from $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ to $L^p_{\mathcal{H}}(\mathbb{R}^{n+m})$ if and only if $\|\mathcal{L}(a)\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n+m})}\leqslant C$ for all $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atoms a, where the constant C is independent of a.*

Proof. We only need to show the sufficiency. This follows from a special atomic decomposition. To be precise, for $f\in H^p(\mathbb{R}^n\times\mathbb{R}^m)$, Chang and R. Fefferman [3] gave an atomic decomposition $f=\sum_j\lambda_j a_j$, but the series converges only in the sense of distributions. In general, to estimate the $L^p_{\mathcal{H}}(\mathbb{R}^{n+m})$ norm of $\mathcal{L}(f)$, one cannot get $\mathcal{L}(f)=\sum_j\lambda_j\mathcal{L}(a_j)$. However, we prove this to be true if $f\in H^p(\mathbb{R}^n\times\mathbb{R}^m)\cap L^2(\mathbb{R}^{n+m})$. Indeed, for $f\in H^p(\mathbb{R}^n\times\mathbb{R}^m)\cap L^2(\mathbb{R}^{n+m})$, we will provide an atomic decomposition of f such that $f(x_1,x_2)=\sum_j\lambda_j a_j(x_1,x_2)$, where a_j are $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atoms and $\sum_j|\lambda_j|^p\leqslant C\|f\|^p_{H^p(\mathbb{R}^n\times\mathbb{R}^m)}$. The crucial point is that the series converges in both $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ and $L^2(\mathbb{R}^{n+m})$. Assuming this atomic decomposition for the moment, since $\mathcal{L}$ is bounded from $L^2(\mathbb{R}^{n+m})$ to $L^2_{\mathcal{H}}(\mathbb{R}^{n+m})$ and the series in atomic decomposition of f converges in $L^2(\mathbb{R}^{n+m})$, thus $\mathcal{L}(f)(x_1,x_2)=\sum_j\lambda_j\mathcal{L}(a_j)(x_1,x_2)$. Moreover, this series also converges in $L^2(\mathbb{R}^{n+m})$ and hence a subsequence (written in the same indices) converges almost everywhere. Therefore,

$$\begin{aligned}\big\|\mathcal{L}(f)\big\|^p_{L^p_{\mathcal{H}}(\mathbb{R}^{n+m})} &\leqslant \sum_j |\lambda_j|^p \big\|\big\{\mathcal{L}(a_j)\big\}_{t,s>0}\big\|^p_{L^p_{\mathcal{H}}(\mathbb{R}^{n+m})} \\ &\leqslant C\sum_j |\lambda_j|^p \leqslant C\|f\|^p_{H^p(\mathbb{R}^n\times\mathbb{R}^m)}.\end{aligned}$$

Since $H^p(\mathbb{R}^n \times \mathbb{R}^m) \cap L^2(\mathbb{R}^{n+m})$ is dense in $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, so $\mathcal{L}$ can be extended to a bounded operator from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p_{\mathcal{H}}(\mathbb{R}^{n+m})$.

To prove the atomic decomposition above, we recall the proof of atomic decomposition of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ given by Chang and R. Fefferman [3]. Given $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m) \cap L^2(\mathbb{R}^{n+m})$, set $\Omega_k = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m\colon\ S(f)(x_1, x_2) > 2^k\}$ where $S(f)$ is the double S-function defined in [3, p. 456], and set $B_k = \{\text{dyadic rectangle } R = I \times J\colon\ |R \cap \Omega_k| > \frac{1}{2}|R| \text{ and } |R \cap \Omega_{k+1}| \leqslant \frac{1}{2}|R|\}$ where I and J are cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. By the classical Calderón identity on $L^2(\mathbb{R}^{n+m})$,

$$\begin{aligned} f(x_1,x_2) &= \int_0^\infty\!\!\int_0^\infty \psi_{t,s} * \psi_{t,s} * f(x_1,x_2)\frac{dt}{t}\frac{ds}{s} \\ &= \sum_{k\in\mathbb{Z}}\sum_{R\in B_k}\int_{\widehat{R}} \psi_{t,s}(x_1-y_1, x_2-y_2)\psi_{t,s} * f(y_1,y_2)\,dy_1\,dy_2\frac{dt}{t}\frac{ds}{s}, \end{aligned} \tag{2.16}$$

where $\widehat{R} = \{(x_1, t, x_2, s)\colon\ R = I \times J,\ x_1 \in I,\ x_2 \in J,\ \frac{\ell(I)}{2} \leqslant t < \ell(I),\ \frac{\ell(J)}{2} \leqslant s < \ell(J)\}$ is the tent of R.

Chang and R. Fefferman [3] proved that (2.16) provided an atomic decomposition of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and the series converges in the sense of distribution. We would like to point out that the series (2.16) converges in $L^2(\mathbb{R}^{n+m})$ as well. To see this, let $g \in L^2(\mathbb{R}^{n+m})$ with $\|g\|_2 = 1$. By the duality argument,

$$\begin{aligned} &\bigg\|\sum_{k\in\mathbb{Z}}\sum_{R\in B_k}\int_{\widehat{R}} \psi_{t,s}(\cdot-y_1,\cdot-y_2)\psi_{t,s} * f(y_1,y_2)\,dy_1\,dy_2\frac{dt}{t}\frac{ds}{s}\bigg\|_2 \\ &\quad= \sup_{\|g\|_2\leqslant 1}\bigg|\bigg\langle \sum_{k\in\mathbb{Z}}\sum_{R\in B_k}\int_{\widehat{R}} \psi_{t,s}(\cdot-y_1,\cdot-y_2)\psi_{t,s} * f(y_1,y_2)\,dy_1\,dy_2\frac{dt}{t}\frac{ds}{s}, g\bigg\rangle\bigg| \\ &\quad= \sup_{\|g\|_2\leqslant 1}\bigg|\sum_{k\in\mathbb{Z}}\sum_{R\in B_k}\int_{\widehat{R}} \widetilde{\psi}_{t,s} * g(y_1,y_2)\psi_{t,s} * f(y_1,y_2)\,dy_1\,dy_2\frac{dt}{t}\frac{ds}{s}\bigg|, \end{aligned}$$

where $\widetilde{\psi}_{t,s}(x_1, x_2) = \psi_{t,s}(-x_1, -x_2)$. By Schwarz's inequality and the L^2 boundedness of the Littlewood–Paley square function,

$$\bigg\|\sum_{k\in\mathbb{Z}}\sum_{R\in B_k}\int_{\widehat{R}} \psi_{t,s}(\cdot-y_1,\cdot-y_2)\psi_{t,s} * f(y_1,y_2)\,dy_1\,dy_2\frac{dt}{t}\frac{ds}{s}\bigg\|_2 \leqslant C\|f\|_2,$$

which implies that the series (2.16) converges in $L^2(\mathbb{R}^{n+m})$. Hence the proof is completed. □

We now return to the proof of Theorem 1.

Proof of Theorem 1. Given $\max\{\frac{n}{n+\varepsilon}, \frac{m}{m+\varepsilon}\} < p \leqslant 1$, we may choose an ε''' such that $0 < \varepsilon''' < \varepsilon$ and $\max\{\frac{n}{n+\varepsilon'''}, \frac{m}{m+\varepsilon'''}\} < p$. Through the proof, we set $0 < \varepsilon''' < \varepsilon'' < \varepsilon' < \varepsilon$.

Since $\mathcal{L}$ which maps f to $\{T_{t,s}(f)\}_{t,s>0}$ is bounded from $L^2(\mathbb{R}^{n+m})$ to $L^2_{\mathcal{H}}(\mathbb{R}^{n+m})$, to show Theorem 1, by Proposition 4, we only need to prove

$$\big\|\big\{T_{t,s}(a)\big\}_{t,s>0}\big\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n+m})} \leqslant C \quad \text{for all } H^p\big(\mathbb{R}^n \times \mathbb{R}^m\big) \text{ atoms } a,$$

where the constant C is independent of a.

To do this, we follow R. Fefferman's idea [5]. Suppose that a is an $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ atom supported on an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ with finite measure. Furthermore, a can be decomposed as $a = \sum_{R\in\mathcal{M}(\Omega)} a_R$, where $\mathcal{M}(\Omega)$ is the collection of all maximal dyadic subrectangles contained in Ω, each a_R is supported on $2R = 2I \times 2J$, the double of $R = I \times J$, $\int_{2I} a_R(x_1, x_2)\,dx_1 = 0$ for all $x_2 \in 2J$, and $\int_{2J} a_R(x_1, x_2)\,dx_2 = 0$ for all $x_1 \in 2I$. Here the higher order moments vanishing of a_R are not needed because we only consider $\max\{\frac{n}{n+\varepsilon}, \frac{m}{m+\varepsilon}\} < p \leqslant 1$. Moreover, $\|a\|_2 \leqslant |\Omega|^{\frac{1}{2}-\frac{1}{p}}$ and $\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|_2^2 \leqslant |\Omega|^{1-\frac{2}{p}}$. Let $\widetilde{\Omega} = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m : M_s(\chi_\Omega)(x_1, x_2) > 4^{-n-m}n^{-n/2}m^{-m/2}\}$, where M_s is the strong maximal function defined by

$$M_s(f)(x_1, x_2) = \sup_{(x_1,x_2)\in P} \frac{1}{|P|} \int_P \big|f(y_1, y_2)\big|\,dy_1\,dy_2,$$

where the supremum is taken over all rectangles P (a product of a cube in $\mathbb{R}^n$ with a cube in $\mathbb{R}^m$) containing (x_1, x_2). It follows from the strong maximal theorem that $|\widetilde{\Omega}| \leqslant C|\Omega|$.

We now estimate $\|\{T_{t,s}(a)\}_{t,s>0}\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n+m})}$ as follows. Write $\widetilde{\widetilde{\Omega}} = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m: M_s(\chi_{\widetilde{\Omega}})(x_1, x_2) > 4^{-n-m}n^{-n/2}m^{-m/2}\}$ and similarly for $\widetilde{\widetilde{\widetilde{\Omega}}}$. Then

$$\begin{aligned}&\int \big\|\big\{T_{t,s}(a)\big\}(x_1, x_2)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2 \\ &\quad = \int_{\widetilde{\widetilde{\widetilde{\Omega}}}} \big\|\big\{T_{t,s}(a)\big\}(x_1, x_2)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2 + \int_{(\widetilde{\widetilde{\widetilde{\Omega}}})^c} \big\|\big\{T_{t,s}(a)\big\}(x_1, x_2)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2.\end{aligned}$$

By Hölder's inequality, the $L^2 - L^2_{\mathcal{H}}$ boundedness of $\mathcal{L}$, and the size condition of a,

$$\begin{aligned}\int_{\widetilde{\widetilde{\widetilde{\Omega}}}} \big\|\big\{T_{t,s}(a)\big\}(x_1, x_2)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2 &\leqslant \bigg(\int_{\widetilde{\widetilde{\widetilde{\Omega}}}} \big\|\big\{T_{t,s}(a)\big\}(x_1, x_2)\big\|^2_{\mathcal{H}}\,dx_1\,dx_2\bigg)^{\frac{p}{2}} |\widetilde{\widetilde{\widetilde{\Omega}}}|^{1-\frac{p}{2}} \\ &\leqslant C\|a\|_2^p|\Omega|^{1-\frac{p}{2}} \leqslant C.\end{aligned}$$

Therefore it remains to deal with

$$\int_{(\widetilde{\widetilde{\widetilde{\Omega}}})^c} \big\|\big\{T_{t,s}(a)\big\}(x_1, x_2)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2 \leqslant \sum_{R\in\mathcal{M}(\Omega)} \int_{(\widetilde{\widetilde{\widetilde{\Omega}}})^c} \big\|\big\{T_{t,s}(a_R)\big\}(x_1, x_2)\big\|^p_{\mathcal{H}}\,dx_1\,dx_2,$$

where we use the inequality $(\alpha + \beta)^p \leqslant \alpha^p + \beta^p$ for $p \leqslant 1$.

For each $R = I \times J \in \mathcal{M}(\Omega)$, we set a larger rectangle $\widetilde{R} = \widetilde{I} \times J$ such that $\widetilde{I}$ is the largest dyadic cube containing I and $\widetilde{I} \times J \subset \widetilde{\Omega}$. Similarly, $\widetilde{\widetilde{R}} = \widetilde{I} \times \widetilde{J}$ where $\widetilde{J}$ is the largest dyadic cube containing J and $\widetilde{I} \times \widetilde{J} \subset \widetilde{\widetilde{\Omega}}$. Let $\mathcal{M}_1(\Omega)$ denote the collection of all dyadic subrectangles $R \subset \Omega$, $R = I \times J$ that are maximal in the x_1 direction. It is clear that $R \in \mathcal{M}(\Omega)$ implies $R \in \mathcal{M}_2(\Omega)$ and $\widetilde{R} \in \mathcal{M}_1(\widetilde{\Omega})$. Define $\mathcal{M}_2(\Omega)$ similarly. Also note that $4\sqrt{n}\widetilde{I} \times 4\sqrt{m}\widetilde{J} \subset \widetilde{\widetilde{\Omega}}$. Then

$$\begin{aligned}
&\int\limits_{(\widetilde{\widetilde{\Omega}})^c} \big\|\big\{T_{t,s}(a_R)\big\}(x_1,x_2)\big\|_{\mathcal{H}}^p\,dx_1\,dx_2 \\
&\quad\leqslant \int\limits_{(4\sqrt{n}\widetilde{I})^c\times\mathbb{R}^m} \big\|\big\{T_{t,s}(a_R)\big\}(x_1,x_2)\big\|_{\mathcal{H}}^p\,dx_1\,dx_2 + \int\limits_{\mathbb{R}^n\times(4\sqrt{m}\widetilde{J})^c} \big\|\big\{T_{t,s}(a_R)\big\}(x_1,x_2)\big\|_{\mathcal{H}}^p\,dx_1\,dx_2 \\
&\quad:= U(R) + V(R).
\end{aligned}$$

We define $\gamma_1(R) = \gamma_1(R,\Omega) = \frac{\ell(\widetilde{I})}{\ell(I)}$ and $\gamma_2(\widetilde{R}) = \gamma_2(\widetilde{R},\widetilde{\Omega}) = \frac{\ell(\widetilde{J})}{\ell(J)}$, where $\ell(I)$ denotes the side length of I. To estimate $U(R)$, we write

$$\begin{aligned}
U(R) &= \int\limits_{(4\sqrt{n}\widetilde{I})^c\times 4\sqrt{m}J} \big\|\big\{T_{t,s}(a_R)\big\}(x_1,x_2)\big\|_{\mathcal{H}}^p\,dx_1\,dx_2 \\
&\quad + \int\limits_{(4\sqrt{n}\widetilde{I})^c\times(4\sqrt{m}J)^c} \big\|\big\{T_{t,s}(a_R)\big\}(x_1,x_2)\big\|_{\mathcal{H}}^p\,dx_1\,dx_2 \\
&:= U_1(R) + U_2(R).
\end{aligned}$$

By Hölder's inequality and Minkowski's inequality,

$$U_1(R) \leqslant C|J|^{1-\frac{p}{2}} \int\limits_{(4\sqrt{n}\widetilde{I})^c} \Bigg(\int\limits_{\mathbb{R}^m} \big\|\big\{T_{t,s}(a_R)\big\}(x_1,x_2)\big\|_{\mathcal{H}}^2\,dx_2\Bigg)^{\frac{p}{2}}\,dx_1. \tag{2.17}$$

The cancellation condition of a_R yields

$$\begin{aligned}
T_{t,s}(a_R)(x_1,x_2) &= \int T_{t,s}(x_1,x_2,y_1,y_2)a_R(y_1,y_2)\,dy_1\,dy_2 \\
&= \int \big[T_{t,s}(x_1,x_2,y_1,y_2) - T_{t,s}(x_1,x_2,x_I,y_2)\big]a_R(y_1,y_2)\,dy_1\,dy_2,
\end{aligned}$$

where x_I denotes the center of I. Now we apply Schwarz's inequality to get

$$\begin{aligned}
&\big\|\big\{T_{t,s}(a_R)\big\}(x_1,x_2)\big\|_{\mathcal{H}}^2 \\
&\quad\leqslant C|I| \int\limits_{2I} \Bigg\|\Bigg\{\int\limits_{2J} \big[T_{t,s}(x_1,x_2,y_1,y_2) - T_{t,s}(x_1,x_2,x_I,y_2)\big]a_R(y_1,y_2)\,dy_2\Bigg\}\Bigg\|_{\mathcal{H}}^2\,dy_1.
\end{aligned}$$

This estimate and Lemma 3 imply that, for $x_1 \in (4\sqrt{n}\widetilde{I})^c$ and $y_1 \in 2I$,

$$\begin{aligned}
&\int_{\mathbb{R}^m} \big\| \big\{ T_{t,s}(a_R) \big\}(x_1, x_2) \big\|_{\mathcal{H}}^2 \, dx_2 \\
&\quad \leqslant C|I| \int_{2I} \int_{\mathbb{R}^m} \bigg\| \bigg\{ \int_{2J} \big[T_{t,s}(x_1, x_2, y_1, y_2) - T_{t,s}(x_1, x_2, x_I, y_2) \big] a_R(y_1, y_2) \, dy_2 \bigg\} \bigg\|_{\mathcal{H}}^2 \, dx_2 \, dy_1 \\
&\quad \leqslant C|I| \bigg(\frac{\ell(I)^{\varepsilon'}}{|x_1 - x_I|^{n+\varepsilon'}} \bigg)^2 \|a_R\|_2^2.
\end{aligned}$$

Inserting the estimate above into (2.17) shows

$$\begin{aligned}
U_1(R) &\leqslant C|J|^{1-\frac{p}{2}} |I|^{\frac{p}{2}} \|a_R\|_2^p \int_{(4\sqrt{n}\widetilde{I})^c} \bigg(\frac{\ell(I)^{\varepsilon'}}{|x_1 - x_I|^{n+\varepsilon'}} \bigg)^p dx_1 \\
&\leqslant C|J|^{1-\frac{p}{2}} |I|^{\frac{p}{2}} \|a_R\|_2^p \, \ell(I)^{\varepsilon' p} \ell(\widetilde{I})^{n-(n+\varepsilon')p} \\
&= C\big(\gamma_1(R)\big)^{n-(n+\varepsilon')p} |R|^{1-\frac{p}{2}} \|a_R\|_2^p.
\end{aligned} \tag{2.18}$$

To estimate $U_2(R)$, we use the cancellation conditions of a_R to write

$$\begin{aligned}
&T_{t,s}(a_R)(x_1, x_2) \\
&\quad = \int T_{t,s}(x_1, x_2, y_1, y_2) a_R(y_1, y_2) \, dy_1 \, dy_2 \\
&\quad = \int \big[T_{t,s}(x_1, x_2, y_1, y_2) - T_{t,s}(x_1, x_2, x_I, y_2) - T_{t,s}(x_1, x_2, y_1, x_J) + T_{t,s}(x_1, x_2, x_I, x_J) \big] \\
&\qquad \times a_R(y_1, y_2) \, dy_1 \, dy_2,
\end{aligned}$$

where x_J is the center of J. For $x_1 \in (4\sqrt{n}\,\widetilde{I})^c$, $x_2 \in (4\sqrt{m}J)^c$, $y_1 \in 2I$, and $y_2 \in 2J$, we have $|y_1 - x_I| \leqslant \frac{1}{2}|x_1 - x_I|$ and $|y_2 - x_J| \leqslant \frac{1}{2}|x_2 - x_J|$. Thus, the estimate (B_3) gives

$$\big\| \big\{ T_{t,s}(a_R) \big\}(x_1, x_2) \big\|_{\mathcal{H}} \leqslant C \bigg(\int |R| \bigg(\frac{|y_1 - x_I|^{\varepsilon'}}{|x_1 - x_I|^{n+\varepsilon'}} \frac{|y_2 - x_J|^{\varepsilon'}}{|x_2 - x_J|^{m+\varepsilon'}} \big| a_R(y_1, y_2) \big| \bigg)^2 dy_1 \, dy_2 \bigg)^{1/2}.$$

Hence,

$$\begin{aligned}
U_2(R) &\leqslant C \int_{(4\sqrt{n}\widetilde{I})^c \times (4\sqrt{m}J)^c} |R|^{\frac{p}{2}} \frac{\ell(I)^{\varepsilon' p}}{|x_1 - x_I|^{(n+\varepsilon')p}} \frac{\ell(J)^{\varepsilon' p}}{|x_2 - x_J|^{(m+\varepsilon')p}} \|a_R\|_2^p \, dx_1 \, dx_2 \\
&\leqslant C|R|^{\frac{p}{2}} \ell(I)^{\varepsilon' p} \ell(\widetilde{I})^{n-(n+\varepsilon')p} \ell(J)^{\varepsilon' p} \ell(J)^{m-(m+\varepsilon')p} \|a_R\|_2^p \\
&\leqslant C\big(\gamma_1(R)\big)^{n-(n+\varepsilon')p} |R|^{1-\frac{p}{2}} \|a_R\|_2^p.
\end{aligned} \tag{2.19}$$

Both estimates (2.18) and (2.19) give

$$U(R) \leqslant C\big(\gamma_1(R)\big)^{n-(n+\varepsilon')p}|R|^{1-\frac{p}{2}}\|a_R\|_2^p.$$

The estimate for $V(R)$, though slightly different from $U(R)$, can be handled in much the same manner so that

$$V(R) \leqslant C\big(\gamma_2(\widetilde{R})\big)^{m-(m+\varepsilon')p}|R|^{1-\frac{p}{2}}\|a_R\|_2^p.$$

Summing over R gives

$$\begin{aligned}
&\sum_{R\in\mathcal{M}(\Omega)} \int_{(\widetilde{\widetilde{\Omega}})^c} \big\|\big\{T_{t,s}(a_R)\big\}(x_1,x_2)\big\|_{\mathcal{H}}^p\,dx_1\,dx_2 \\
&\quad\leqslant C\sum_{R\in\mathcal{M}(\Omega)}\big(\gamma_1(R)\big)^{n-(n+\varepsilon')p}|R|^{1-\frac{p}{2}}\|a_R\|_2^p + C\sum_{R\in\mathcal{M}(\Omega)}\big(\gamma_2(\widetilde{R})\big)^{m-(m+\varepsilon')p}|R|^{1-\frac{p}{2}}\|a_R\|_2^p \\
&\quad\leqslant C\bigg\{\bigg(\sum_{R\in\mathcal{M}_2(\Omega)}|R|\big(\gamma_1(R)\big)^{-\delta_1}\bigg)^{1-\frac{p}{2}} + \bigg(\sum_{\widetilde{R}\in\mathcal{M}_1(\widetilde{\Omega})}|\widetilde{R}|\big(\gamma_2(\widetilde{R})\big)^{-\delta_2}\bigg)^{1-\frac{p}{2}}\bigg\} \\
&\qquad\times\bigg(\sum_{R\in\mathcal{M}(\Omega)}\|a_R\|_2^2\bigg)^{\frac{p}{2}},
\end{aligned}$$

where $\delta_1 = \frac{2[n-(n+\varepsilon')p]}{p-2} > 0$ and $\delta_2 = \frac{2[m-(m+\varepsilon')p]}{p-2} > 0$.

To estimate the last part above, we use the following

Journé's lemma. *$\sum_{R\in\mathcal{M}_2(\Omega)}|R|(\gamma_1(R))^{-\delta} \leqslant C_\delta|\Omega|$ and $\sum_{R\in\mathcal{M}_1(\Omega)}|R|(\gamma_2(R))^{-\delta} \leqslant C_\delta|\Omega|$ for any $\delta > 0$, where C_δ is a constant depending on δ only.*

Journé's lemma and the size condition of a_R imply

$$\sum_{R\in\mathcal{M}(\Omega)} \int_{(\widetilde{\widetilde{\Omega}})^c} \big\|\big\{T_{t,s}(a_R)\big\}(x_1,x_2)\big\|_{\mathcal{H}}^p\,dx_1\,dx_2 \leqslant C|\Omega|^{1-\frac{p}{2}}|\Omega|^{\frac{p}{2}-1} \leqslant C.$$

This is the desired result, and hence the proof of Theorem 1 is completed. □

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