Upper large deviations for the maximal flow in first-passage percolation

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Abstract

We consider the standard first-passage percolation in $\mathbb{Z}^d$ for $d \geq 2$ and we denote by $\phi_{n^{d-1}, h(n)}$ the maximal flow through the cylinder $]0, n]^{d-1} \times]0, h(n)]$ from its bottom to its top. Kesten proved a law of large numbers for the maximal flow in dimension 3: under some assumptions, $\phi_{n^{d-1}, h(n)}/n^{d-1}$ converges towards a constant ν. We look now at the probability that $\phi_{n^{d-1}, h(n)}/n^{d-1}$ is greater than $\nu + \varepsilon$ for some $\varepsilon > 0$, and we show under some assumptions that this probability decays exponentially fast with the volume $n^{d-1}h(n)$ of the cylinder. Moreover, we prove a large deviation principle for the sequence $(\phi_{n^{d-1}, h(n)}/n^{d-1}, n \in \mathbb{N})$.
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1. Definitions and main results

We will use generally the notation introduced in [6,7] but some changes will be made, for example to obtain independent objects. Let $d \geq 2$. We consider the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ having for vertices $\mathbb{Z}^d$ and for edges $\mathbb{E}^d$ the set of all the pairs of nearest neighbors for the standard L^1 norm. With each edge e in $\mathbb{E}^d$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. We suppose that the family $(t(e), e \in \mathbb{E}^d)$ is independent and identically distributed, with a common distribution function F. More formally, we take the product measure $\mathbb{P}$ on $\Omega = \prod_{e \in \mathbb{E}^d}[0, \infty[$, and we denote

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its expectation by $\mathbb{E}$. We interpret $t(e)$ as the capacity of the edge e; this means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time. For a given realization $(t(e), e \in \mathbb{E}^d)$ we denote by $\phi_{\vec{k},m}^- = \phi_B$ the maximal flow through the box

$$B(\vec{k}, m) = \prod_{i=1}^{d-1}]0, k_i] \times]0, m],$$

where $\vec{k} = (k_1, \ldots, k_{d-1}) \in \mathbb{Z}^{d-1}$, from its bottom

$$F_0 = \prod_{i=1}^{d-1}]0, k_i] \times \{0\}$$

to its top

$$F_m = \prod_{i=1}^{d-1}]0, k_i] \times \{m\}.$$

Let us define this quantity properly. We recall that $\mathbb{E}^d$ is the set of the edges of the graph. An edge $e \in \mathbb{E}^d$ can be written as $e = \langle x, y \rangle$, where $x, y \in \mathbb{Z}^d$ are the endpoints of e. The edges of $\mathbb{E}^d$ are unoriented; hence $\langle x, y \rangle = \langle y, x \rangle$. We will say that $e = \langle x, y \rangle$ is included in a subset A of $\mathbb{R}^d$ ($e \subset A$) if the segment joining x to y (except possibly its extremities) is included in A. Now we define $\widetilde{\mathbb{E}}^d$ as the set of all the oriented edges, i.e., an element $\widetilde{e}$ in $\widetilde{\mathbb{E}}^d$ is an ordered pair of vertices. We denote an element $\widetilde{e} \in \widetilde{\mathbb{E}}^d$ by $\langle\langle x, y \rangle\rangle$, where $x, y \in \mathbb{Z}^d$ are the endpoints of $\widetilde{e}$ and the edge is oriented from x towards y. We consider now the set $\mathcal{S}$ of all pairs of functions (g, o), with $g : \mathbb{E}^d \to \mathbb{R}^+$ and $o : \mathbb{E}^d \to \widetilde{\mathbb{E}}^d$ such that $o(\langle x, y \rangle) \in \{\langle\langle x, y \rangle\rangle, \langle\langle y, x \rangle\rangle\}$, satisfying:

- for each edge e in B we have

$$0 \leq g(e) \leq t(e),$$

- for each vertex v in $B \setminus F_m$ (remember that $F_0 \cap B = \emptyset$) we have

$$\sum_{e \in B : o(e) = \langle\langle v, \cdot \rangle\rangle} g(e) = \sum_{e \in B : o(e) = \langle\langle \cdot, v \rangle\rangle} g(e).$$

A couple $(g, o) \in \mathcal{S}$ is a possible stream in B: $g(e)$ is the amount of fluid that goes through the edge e, and $o(e)$ gives the direction in which the fluid goes through e. The first condition on (g, o) expresses only the fact that the amount of fluid that can go through an edge is bounded by its capacity. The second one is a balance equation: it means that there is no loss of fluid in the cylinder. With each possible stream we associate the corresponding flow

$$\text{flow}(g, o) = \sum_{u \in B \setminus F_m, v \in F_m : \langle u, v \rangle \in \mathbb{E}^d} g(\langle u, v \rangle) \mathbb{I}_{o(\langle u, v \rangle) = \langle\langle u, v \rangle\rangle} - g(\langle u, v \rangle) \mathbb{I}_{o(\langle u, v \rangle) = \langle\langle v, u \rangle\rangle}.$$

This is the amount of fluid that crosses the cylinder B if the fluid respects the stream (g, o). The maximal flow through the cylinder B from its bottom to its top is the supremum of this quantity over all possible choices of streams in $\mathcal{S}$:

$$\phi_B = \phi_{\vec{k},m}^- = \sup\{\text{flow}(g, o) : (g, o) \in \mathcal{S}\}.$$

If $\phi_B = \text{flow}(g, o)$ we say that the stream (g, o) realizes the flow ϕ_B.

Kesten proved in 1987 the following law of large numbers for the maximal flow in dimension 3 (see [7]):

Theorem 1. *We consider a cylinder $B((k, l), m)$ such that $\lim_{k \geq l \to \infty} m(k, l) = \infty$ in such a way that for some $\delta > 0$ we have*

$$\lim_{k \geq l \to \infty} \frac{\ln m(k, l)}{k^{1-\delta}} = 0.$$

There exists a positive p_0 with the following property: If F satisfies $F(0) < p_0$ and $\int_{[0,\infty[} e^{\theta x} dF(x)$ is finite for some positive θ, then there exists a constant $v(F) < \infty$ such that

$$\lim_{k,l \to \infty} \frac{\phi_{(k,l),m}}{kl} = v \quad \text{with probability one and in } L^1.$$

Actually, the constant $v(F)$ is defined as the limit of another object under weaker assumptions on F (see [7] and (1) in the next section), and we rely on this definition to state the following result. We are now interested in the deviations of the rescaled flow from its typical behavior. We will show two results for dimensions $d \geq 2$. The first one states the existence of a limit, and some of its properties.

Theorem 2. *We consider the maximal flow $\phi_{(n,\ldots,n),h(n)}$ through the cylinder $B((n, \ldots, n), h(n))$, where the function $h : \mathbb{N} \to \mathbb{N}$ satisfies*

$$\lim_{n \to \infty} \frac{h(n)}{\ln n} = \infty.$$

For every λ in $\mathbb{R}^+$, the limit

$$\psi(\lambda) = \lim_{n \to \infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi_{(n,\ldots,n),h(n)} \geq \lambda n^{d-1}\right]$$

exists and is independent of h. Moreover ψ is convex on $\mathbb{R}^+$, finite and continuous on the set $\{\lambda \mid F([\lambda, +\infty[) > 0\}$. If $\int_{[0,+\infty[} x dF(x)$ is finite, then ψ vanishes on $[0, v]$, where v is defined in (1). If $\int_{[0,+\infty[} e^{\theta x} dF(x)$ is finite for some positive θ, then ψ is positive on $]v, +\infty[$.

We say that a sequence $(X_n, n \in \mathbb{N})$ of random variables with values in $D \subset \mathbb{R}$ satisfies a large deviation principle with speed $v(n)$ and governed by the rate function $\mathcal{I}$ if and only if:

- for any closed subset $\mathcal{F} \subset D$, we have

$$\limsup_{n \to \infty} \frac{1}{v(n)} \ln \mathbb{P}[X_n \in \mathcal{F}] \leq -\inf_{\mathcal{F}} \mathcal{I},$$

- for any open subset $\mathcal{O} \subset D$, we have

$$\liminf_{n \to \infty} \frac{1}{v(n)} \ln \mathbb{P}[X_n \in \mathcal{O}] \geq -\inf_{\mathcal{O}} \mathcal{I}.$$

Now, with the help of the function ψ, we can state the following large deviation principle for the rescaled flow:

Theorem 3. *Let $h : \mathbb{N} \to \mathbb{N}$ be such that*

$$\lim_{n \to \infty} \frac{h(n)}{\ln n} = \infty.$$

If there exists a positive θ such that

$$\int_{[0,+\infty[} e^{\theta x} dF(x) < \infty,$$

then the sequence

$$\left(\frac{\phi_{(n,\ldots,n),h(n)}}{n^{d-1}} \right)_{n \in \mathbb{N}}$$

satisfies a large deviation principle, with speed $n^{d-1}h(n)$, and governed by the good rate function ψ.

Remark 1. When the capacity t of an edge is bounded, we do not need the condition

$$\lim_{n \to \infty} \frac{h(n)}{\ln n} = +\infty;$$

the results hold under the weaker condition

$$\lim_{n \to \infty} h(n) = +\infty.$$

Actually, the role of the condition $\lim_{n \to \infty} h(n)/\ln n = +\infty$ is not fully understood yet. For example, when t is equal in law to the absolute value of a Gaussian variable this condition can also be replaced by $\lim_{n \to +\infty} h(n) = +\infty$. Unfortunately we could not find satisfactory sufficient conditions (in particular on the moments of the law of t) to get rid of the condition $\lim_{n \to \infty} h(n)/\ln n = +\infty$.

A special aspect of the proof of Theorem 2 is the use of a discrete version of the model. Indeed, we are confronted with a combinatorial problem: we need to look at boundary conditions for streams to glue together streams in different cylinders, but when the capacity of an edge takes its values in $\mathbb{R}^+$ we cannot count the number of possible boundary conditions. Our strategy is to consider a discrete approximation of the capacity of the edges and the corresponding maximal flow. We work with these objects. To handle the boundary conditions, we use a technique introduced by Chow and Zhang [4]. We finally compare the real maximal flow to this approximation.

2. Max-flow min-cut theorem

It is difficult to work with the expression of the maximal flow that we have seen in the previous part, this is why we will use the max-flow min-cut theorem to express the maximal flow differently. First we need some definitions. A path on a graph ($\mathbb{Z}^d$ for example) from v_0 to v_n is a sequence $(v_0, e_1, v_1, \ldots, e_n, v_n)$ of vertices $v_0, \ldots, v_n$ alternating with edges $e_1, \ldots, e_n$ such that v_{i-1} and v_i are neighbors in the graph, joined by the edge e_i, for i in $\{1, \ldots, n\}$. Two paths are said to be disjoint if they have no common edge. A set E of edges of $B(\vec{k}, m)$ is said to separate F_0 from F_m in $B(\vec{k}, m)$ if there is no path from F_0 to F_m in $B(\vec{k}, m) \smallsetminus E$. We call E an (F_0, F_m)-cut if E separates F_0 from F_m in $B(\vec{k}, m)$ and if no proper subset of E does. With each set E of edges we associate the variable

$$V(E) = \sum_{e \in E} t(e).$$

The max-flow min-cut theorem (see [2]) states that

$$\phi_B = \min\{V(E) \mid E \text{ is an } (F_0, F_m) - \text{ cut in } B\}.$$

Remark 2. In the special case where $t(e)$ belongs to $\{0, 1\}$, let us consider the graph obtained from the initial graph (not necessarily $\mathbb{Z}^d$) by removing all the edges e with $t(e) = 0$. Menger's theorem (see [2]) states that the minimal number of edges in $B(\tilde{k}, m)$ that have to be removed from this graph to disconnect F_0 from F_m is exactly the maximal number of disjoint paths that connect F_0 to F_m. By the max-flow min-cut theorem, it follows immediately that the maximal flow in the initial graph through B from F_0 to F_m is exactly the maximal number of disjoint open paths from F_0 to F_m, where a path is open if and only if the capacity of all its edges is 1. Such a set of ϕ_B disjoint open paths from F_0 to F_m corresponds obviously to a stream (g, o):

- $g(e) = \begin{cases} 1 & \text{if } e \text{ belongs to one of these paths} \\ 0 & \text{otherwise,} \end{cases}$
- $o(e) = \begin{cases} \langle\langle x, y \rangle\rangle & \text{if } e = \langle x, y \rangle \text{ is crossed from } x \text{ to } y \text{ by one of these paths} \\ \langle\langle y, x \rangle\rangle & \text{if } e = \langle x, y \rangle \text{ is crossed from } y \text{ to } x \text{ by one of these paths} \\ \hat{o}(e) & \text{otherwise,} \end{cases}$

where $\hat{o}$ is some determined orientation ($\hat{o}(\langle x, y \rangle) \in \{\langle\langle x, y \rangle\rangle, \langle\langle y, x \rangle\rangle\}$) which does not matter. The stream (g, o) realizes the maximal flow ϕ_B (whatever $\hat{o}$).

We come back to the general case. We will also need the definition of a cut over a hyperrectangle. Let $S = \prod_{i=1}^{d-1}]k_i, l_i]$ be a hyper-rectangle, with $k_i \leq l_i$, k_i, l_i in $\mathbb{Z}$. We say that a set E of edges in $S \times \mathbb{R}$ separates $-\infty$ from $+\infty$ over S if there exists no path in $(S \times \mathbb{R}) \setminus E$ from $S \times \{-N\}$ to $S \times \{+N\}$ for some $N > 0$. Similarly, we call E a cut over S if E separates $-\infty$ from $+\infty$ over S, but no proper subset of E does. Let $\partial^{in}(S \times \mathbb{R})$ be the inner vertex boundary of the cylinder $S \times \mathbb{R}$:

$$\partial^{in}(S \times \mathbb{R}) = \{ x \in S \times \mathbb{R} \mid \exists y \notin S \times \mathbb{R}, \langle x, y \rangle \in \mathbb{E}^d \}.$$

We define the corresponding set of edges

$$\mathbb{E}(\partial^{in}(S \times \mathbb{R})) = \{ \langle x, y \rangle \mid x, y \in \partial^{in}(S \times \mathbb{R}) \}.$$

We say that an edge e is vertical if $e = \langle x, x + (0, \ldots, 0, 1) \rangle$; e is said to be horizontal otherwise. We denote by $(*)$ the condition on E

$$(*) \quad E \cap \mathbb{E}(\partial^{in}(S \times \mathbb{R})) \subset \{ e \in \mathbb{E}^d \mid e \text{ is vertical, } e \subset \mathbb{R}^{d-1} \times [0, 1] \},$$

which means in a way to say that the boundary of E is fixed on the perimeter of the rectangle $S \times \{0\}$. We define the variable τ by

$$\tau(S) = \inf\{V(E) \mid E \text{ is a cut over } S \text{ and } E \text{ satisfies } (*)\}.$$

For simplicity, we denote by $\tau_{k^{d-1}}$ the variable $\tau(]0, k]^{d-1})$. If S_1, S_2 are two disjoint hyperrectangles having a common side (so $S_1 \cup S_2$ is a hyper-rectangle too), then we have

$$\tau(S_1 \cup S_2) \leq \tau(S_1) + \tau(S_2).$$

Indeed if E_1 (respectively E_2) is a cut over S_1 (respectively S_2) satisfying $(*)$ for S_1 (respectively S_2) then E_1 and E_2 are both pinned at the boundary between $S_1 \times \{0\}$ and $S_2 \times \{0\}$ because they both satisfy $(*)$, so they can be glued together and $E_1 \cup E_2$ separates $-\infty$ from $+\infty$ over $S_1 \cup S_2$. By a subadditive argument (see [1]), the following limit exists almost surely:

$$\nu(F) = \lim_{k \to \infty} \frac{\tau_{k^{d-1}}}{k^{d-1}}, \tag{1}$$

where we know that $\nu(F)$ is a constant almost surely thanks to Kolmogorov's 0–1 law. We will denote it by ν when no doubt about F is possible. This is the "ν" in Theorems 1 and 2.

3. Proof of Theorem 2

We take $h : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n \to \infty} h(n) = +\infty.$$

We will see during the proof where we need the stronger condition

$$\lim_{n \to \infty} \frac{h(n)}{\ln n} = +\infty.$$

We will need to describe how the fluid goes in and out of a cylinder in order to glue together two cylinders without losing any flow. The problem is that we need too much information to describe this precisely. The feature of the proof is to consider a discrete approximation of the capacity of the edges (see Section 3.1), to work with this discrete model (Sections 3.2–3.4) and then to compare it to the original one (Section 3.5). The method used to prove the existence of the limit was developed in [4]. We study then the properties of ψ as in [3].

3.1. Discrete version

Let $k \in \mathbb{N}$ (we will choose it later). We associate with $(t(e), e \in \mathbb{Z}^d)$ a new family of independent and identically distributed variables $(t^k(e), e \in \mathbb{Z}^d)$ by setting

$$\forall e \in \mathbb{E}^d \quad t^k(e) = \lfloor k t(e) \rfloor \times \frac{1}{k},$$

and we denote by ϕ^k the maximal flow corresponding to these new variables.

Let us consider for a brief moment the graph G obtained by replacing each edge e by p edges $\widetilde{e}_1, \ldots, \widetilde{e}_p$, where $p = k t^k(e)$. In this new graph the capacity of each edge is simply 1. The Remark 2 also holds for G: the maximal flow ϕ_B^G for G from F_0 to $F_{h(n)}$ in $B = B((n, \ldots, n), h(n))$ is exactly the maximal number of disjoint paths connecting F_0 to $F_{h(n)}$ in G. We have seen that we can associate with each such family of ϕ_B disjoint paths in B a stream $(\widetilde{g}, \widetilde{o})$ in G that realizes ϕ_B^G. Actually, we can always reduce to the case where $\widetilde{o}(\widetilde{e}_1) = \widetilde{o}(\widetilde{e}_2)$ if the edges $\widetilde{e}_1$ and $\widetilde{e}_2$ are replacing in G the same edge $\langle x, y \rangle \in \mathbb{E}^d$. Indeed, if for such edges $\widetilde{e}_1$ and $\widetilde{e}_2$ we have $\widetilde{g}(\widetilde{e}_1) = \widetilde{g}(\widetilde{e}_2) = 1$ and $\widetilde{o}(\widetilde{e}_1) \neq \widetilde{o}(\widetilde{e}_2)$, we know that there exists a path l_1 (respectively l_2) from F_0 to $F_{h(n)}$ going through $\widetilde{e}_1$ (respectively $\widetilde{e}_2$) and crossing this edge from x to y (respectively from y to x). We can create two new disjoint paths in G, l_a which is equal to l_1 from F_0 to x and to l_2 from x to $F_{h(n)}$, and l_b which is equal to l_2 from F_0 to y and to l_1 from y to $F_{h(n)}$, that can replace l_1 and l_2 in the set of ϕ_B disjoint paths (see Fig. 1). The corresponding stream $(\widetilde{g}', \widetilde{o}')$ is equal to $(\widetilde{g}, \widetilde{o})$ except in $\widetilde{e}_1$ and $\widetilde{e}_2$ where we have $\widetilde{g}'(\widetilde{e}_1) = \widetilde{g}'(\widetilde{e}_2) = 0$ and $\widetilde{o}'(\widetilde{e}_1) = \widetilde{o}'(\widetilde{e}_2) = \hat{o}(\langle x, y \rangle)$. Then we just have to deal with the case $\widetilde{g}(\widetilde{e}_1) = 1$ and $\widetilde{g}(\widetilde{e}_2) = 0$. The definition of $\hat{o}$ is arbitrary; we can change the orientation of $\hat{o}(\widetilde{e}_2)$ to have $\widetilde{o}(\widetilde{e}_2) = \hat{o}(\widetilde{e}_2) = \widetilde{o}(\widetilde{e}_1)$. We obtain thus a stream such that $\widetilde{o}(\widetilde{e}_1) = \widetilde{o}(\widetilde{e}_2)$ if the edges $\widetilde{e}_1$ and $\widetilde{e}_2$ are replacing in G the same edge $\langle x, y \rangle \in \mathbb{E}^d$. Moreover we can assume that each path of a family of ϕ_B disjoint open paths has only its first vertex in F_0 and its last vertex in $F_{h(n)}$; otherwise we can restrict the path to obtain such a path. Thanks to a good choice of $\hat{o}$, we can thus suppose that if $\widetilde{e} \subset B$ has one endpoint x in $F_{h(n)}$ (respectively F_0) and one endpoint y not in $F_{h(n)}$ (respectively F_0) then $\widetilde{o}(\widetilde{e}) = \langle \langle y, x \rangle \rangle$ (respectively $\widetilde{o}(\widetilde{e}) = \langle \langle x, y \rangle \rangle$), and if $\widetilde{e}$ has both endpoints in $F_{h(n)}$ then $\widetilde{g}(\widetilde{e}) = 0$.
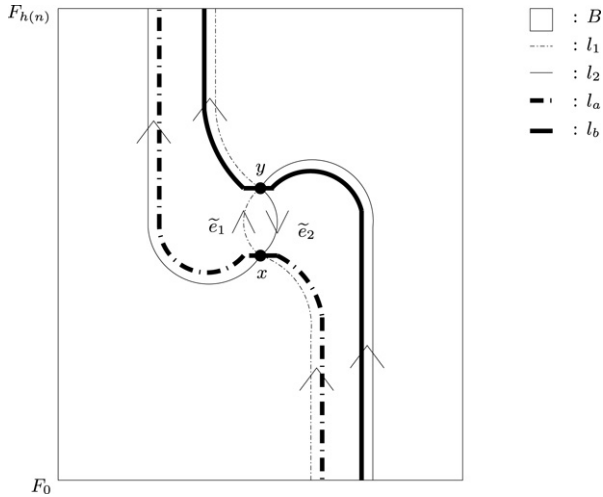
Fig. 1. Description of paths in G.

Coming back to the graph $\mathbb{Z}^d$, we remark that the maximal flow ϕ_B^k between F_0 and $F_{h(n)}$ in B is equal to ϕ_B^G/k, and it can be realized by the stream (g, o) defined as follows. Let e be an edge of $\mathbb{E}^d$. If there is no edge in G associated with e, we set $g(e) = 0$ and $o(e) = \hat{o}(e)$. Otherwise, we define

$$g(e) = \sum_{\widetilde{e} \sim e} \frac{\widetilde{g}(\widetilde{e})}{k}$$

where the sum is over the edges $\widetilde{e}$ that replace e in G, and $o(e) = \widetilde{o}(\widetilde{e})$ for some edge $\widetilde{e}$ associated with e (recall that if $\widetilde{e}_1 \sim e$ and $\widetilde{e}_2 \sim e$ then $\widetilde{o}(\widetilde{e}_1) = \widetilde{o}(\widetilde{e}_2)$). We will call such a stream, built from the graph G, a discrete stream. A discrete stream has three particular properties: g takes its values in $\mathbb{N}/k$, $o(\langle x, y \rangle) = \langle\langle x, y \rangle\rangle$ as soon as we have $x \in F_0$ and $y \in B$ ($y \notin F_0$) or $y \in F_{h(n)}$ and $x \in B \smallsetminus F_{h(n)}$, and $g(e) = 0$ if e has both endpoints in $F_{h(n)}$.

Let λ be in $\mathbb{R}^+$. For a discrete stream (g, o) and $h \in \mathbb{Z}$ we define the truncated projection of g on the vertical edges that intersect the hyper-plane $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_d = h + 1/2\}$ by

$$\forall x \in \mathbb{Z}^{d-1} \cap]0, n]^{d-1} \qquad \pi_h^{\lambda, n}(g, x) = g(\langle (x, h), (x, h+1) \rangle) \wedge \left(\lfloor \lambda n^{d-1} \rfloor + 1 \right).$$

Thanks to the properties of discrete streams, we can state the following lemma:

Lemma 1 (*Junction of Two Boxes*). *Let* $B_1 =]0, n]^{d-1} \times]0, h(n)]$, $B_2 =]0, n]^{d-1} \times]h(n)$, $2h(n)]$. *If there exist a discrete stream* (g_1, o_1) *in* B_1 *and a discrete stream* (g_2, o_2) *in* B_2 *such that*

$$\mathrm{flow}(g_1, o_1) \geq \lambda n^{d-1} \quad and \quad \mathrm{flow}(g_2, o_2) \geq \lambda n^{d-1}$$

and

$$\forall x \in \mathbb{Z}^{d-1} \cap]0, n]^{d-1} \qquad \pi_{h(n)-1}^{\lambda, n}(g_1, x) = \pi_{h(n)}^{\lambda, n}(g_2, x),$$

then $\phi_{B_1 \cup B_2}^k \geq \lambda n^{d-1}$.

Proof. To prove this lemma, we consider two cases:

1. If for all $e = \langle (x, h(n) - 1), (x, h(n)) \rangle$ with $x \in \mathbb{Z}^{d-1} \cap]0, n]^{d-1}$ we have $g_1(e) \leq \lambda n^{d-1}$, then we can define the following discrete stream $(g_{\text{tot}}, o_{\text{tot}})$:

   - $g_{\text{tot}}(e) = \begin{cases} g_1(e) & \text{if } e \subset B_1 \\ g_2(e) & \text{if } e \subset B_2 \\ 0 & \text{otherwise,} \end{cases}$

   - $o_{\text{tot}}(e) = \begin{cases} o_1(e) & \text{if } e \subset B_1 \\ o_2(e) & \text{if } e \subset B_2 \\ \hat{o}(e) & \text{otherwise,} \end{cases}$

   where $\hat{o}$ is still some arbitrarily determined orientation. We can check that $(g_{\text{tot}}, o_{\text{tot}})$ is a discrete stream thanks to the properties of (g_1, o_1) and (g_2, o_2); in particular if $e_1 = \langle (x, h(n) - 1), (x, h(n)) \rangle$ and $e_2 = \langle (x, h(n)), (x, h(n) + 1) \rangle$ with $x \in]0, n]^{d-1}$, we have $g(e_1) = g(e_2)$, $g(e) = 0$ for all other edges $e = \langle (x, h(n)), \cdot \rangle$, $o(e_1) = \langle \langle (x, h(n) - 1), (x, h(n)) \rangle \rangle$ and $o(e_2) = \langle \langle (x, h(n)), (x, h(n) + 1) \rangle \rangle$, and hence the balance equation is satisfied. Moreover $\text{flow}(g_{\text{tot}}, o_{\text{tot}}) = \text{flow}(g_1, o_1) = \text{flow}(g_2, o_2)$ so $\phi_{B_1 \cup B_2}^k \geq \lambda n^{d-1}$.

2. Suppose there exists an edge $e = \langle (x, h(n) - 1), (x, h(n)) \rangle$ such that $g_1(e) > \lambda n^{d-1}$. The discrete stream (g_1, o_1) corresponds to $k \times \text{flow}(g_1, o_1)$ disjoint paths from F_0 to $F_{h(n)}$ in B_1 for the modified graph G. The inequality $g_1(e) > \lambda n^{d-1}$ implies that at least $q = \lceil \lambda n^{d-1} k \rceil$ of these paths, that we will denote by $l_1, \ldots, l_q$, go out of B_1 through e. The equality $\pi_{h(n)-1}^{\lambda, n}(g_1, x) = \pi_{h(n)}^{\lambda, n}(g_2, x)$ implies that $g_2(f) > \lambda n^{d-1}$ where $f = \langle (x, h(n)), (x, h(n)+1) \rangle$. By the same argument, we can find at least q disjoint paths $l'_1, \ldots, l'_q$ from $F_{h(n)}$ to $F_{2h(n)}$ in B_2 for G, all going in B_2 through the edge f. Now we can glue together these q paths $l_1, \ldots, l_q$ in B_1 with the q paths $l'_1, \ldots, l'_q$ in B_2 because e and f are adjacent. In this way we obtain q disjoint paths from F_0 to $F_{2h(n)}$ in $B_1 \cup B_2$ for G, and by considering the corresponding discrete flow $(g_{\text{tot}}, o_{\text{tot}})$ in the initial graph we obtain $\phi_{B_1 \cup B_2}^k \geq \text{flow}(g_{\text{tot}}, o_{\text{tot}}) = \lceil \lambda n^{d-1} k \rceil / k$. $\blacksquare$

We define the boundary conditions of the discrete stream (g, o) in the cylinder $B((n, \ldots, n), h(n))$ as

$$\Pi^{\lambda, n}(g) = \left(\Pi_1^{\lambda, n}(g), \Pi_2^{\lambda, n}(g) \right)$$
$$= \left(\left(\pi_0^{\lambda, n}(g, x), x \in \mathbb{Z}^{d-1} \cap]0, n]^{d-1} \right), \left(\pi_{h(n)-1}^{\lambda, n}(g, x), x \in \mathbb{Z}^{d-1} \cap]0, n]^{d-1} \right) \right).$$

The number $N_{\lambda, n}^k$ of possible boundary conditions for discrete streams satisfies

$$N_{\lambda, n}^k \leq \left(k \left(\lfloor \lambda n^{d-1} \rfloor + 1 \right) + 1 \right)^{2n^{d-1}}.$$

3.2. Existence of the limit for $\phi_{n^{d-1}, h(n)}^{k_n}$

In this section, we will prove the existence of the limit appearing in the Theorem 2 with ϕ^k instead of ϕ. We denote $\phi_{(n, \ldots, n), h(n)}$ by $\phi_{n^{d-1}, h(n)}$, and we define

$$\mu = \sup \{ \lambda \mid F([0, \lambda[) < 1 \}.$$

We will prove the following result:

Theorem 4. *For every pair $(h, (\chi_n, n \in \mathbb{N}))$ with $h : \mathbb{N} \to \mathbb{N}$ a function such that $\lim_{n \to +\infty} h(n) = +\infty$ and $(\chi_n, n \in \mathbb{N})$ a non-decreasing sequence of integers such that*

$\lim_{n \to +\infty} \chi_n = +\infty$, *satisfying*

$$\lim_{n \to +\infty} \frac{\chi_n \ln n}{h(n)} = 0, \tag{2}$$

for every λ in $\mathbb{R}^+ \smallsetminus \{\mu\}$ (or in $\mathbb{R}^+$ if μ is infinite), the limit

$$\widetilde{\psi}(\lambda, h, (\chi_n)) = \lim_{n \to +\infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi^{2\chi_n}_{n^{d-1}, h(n)} \geq \lambda n^{d-1} \right]$$

exists. Moreover $\widetilde{\psi}$ is independent of such a pair $(h, (\chi_n))$, i.e., if $(h, (\chi_n))$ and $(h', (\chi'_n))$ satisfy all the previous conditions, then $\widetilde{\psi}(\lambda, h, (\chi_n)) = \widetilde{\psi}(\lambda, h', (\chi'_n))$ for all λ in $\mathbb{R}^+ \smallsetminus \{\mu\}$ (or in $\mathbb{R}^+$). We will thus denote this limit by $\widetilde{\psi}(\lambda)$.

We now prove Theorem 4 by considering different cases.

- $\lambda > \mu$: Then

$$\forall k \in \mathbb{N}, \forall n \in \mathbb{N} \quad \mathbb{P}\left[\phi^k_{n^{d-1}, h(n)} \geq \lambda n^{d-1} \right] = 0,$$

  so for every sequence (χ_n) we have

$$\widetilde{\psi}(\lambda, h, (\chi_n)) = \lim_{n \to \infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi^{2\chi_n}_{n^{d-1}, h(n)} \geq \lambda n^{d-1} \right] = +\infty = \widetilde{\psi}(\lambda).$$

- $\lambda < \mu$: We take $N, n \in \mathbb{N}$ with $n \leq N$ and let $N = nm + r$ be the Euclidean algorithm. We consider two functions $h, \widetilde{h} : \mathbb{N} \to \mathbb{N}$, with $\lim_{n \to \infty} h(n) = \lim_{n \to \infty} \widetilde{h}(n) = +\infty$, and let $\widetilde{h}(N) = h(n)\widetilde{m} + \widetilde{r}$ be the Euclidean algorithm. We take $k \in \mathbb{N}$ which will be chosen later. We want to compare $\phi^k_{N^{d-1}, \widetilde{h}(N)}$ and $\phi^k_{n^{d-1}, h(n)}$.

  The idea is to divide $B((N, \ldots, N), \widetilde{h}(N))$ into m^{d-1} boxes which are disjoint translates of $B((n, \ldots, n), \widetilde{h}(N))$, then to cut again $B((n, \ldots, n), \widetilde{h}(N))$ into $\widetilde{m}$ disjoint translates of the elementary box $B((n, \ldots, n), h(n))$ and to use here the lemma of the junction (see Fig. 2).

  We define two quantities that will allow us to deal with the edges belonging to the part of $\phi^k_{N^{d-1}, \widetilde{h}(N)}$ that does not enter in any translate of $\phi^k_{n^{d-1}, h(n)}$. On one hand, by the definition of μ, $\lambda < \mu$ implies that $F([0, \lambda]) < 1$, so there exists a positive η such that

$$F([0, \lambda + \eta]) < 1, \qquad \text{i.e., } p(\eta) = \mathbb{P}[t(e) \geq \lambda + \eta] > 0.$$

It follows that there exists k_0 such that

$$\forall k \geq k_0 \quad \mathbb{P}\left[t^k(e) \geq \lambda + \frac{\eta}{2} \right] \geq p(\eta) > 0.$$

On the other hand, if we define

$$\gamma_k = \max\{p \in \mathbb{N} \mid \mathbb{P}[t(e) \geq pk] > 0\} \wedge \left(\lfloor \lambda k n^{d-1} \rfloor + 1 \right),$$

then we have

$$p_k = \mathbb{P}\left[t(e) \geq \gamma_k k \right] > 0.$$

Let $k \geq k_0$ in $\mathbb{N}$. For $i_1, \ldots, i_{d-1}$ in $\{0, \ldots, m\}$, we define

$$B_{i_1, \ldots, i_{d-1}} = \prod_{j=1}^{d-1}]i_j n, (i_j + 1)n] \times]0, \widetilde{h}(N)]$$
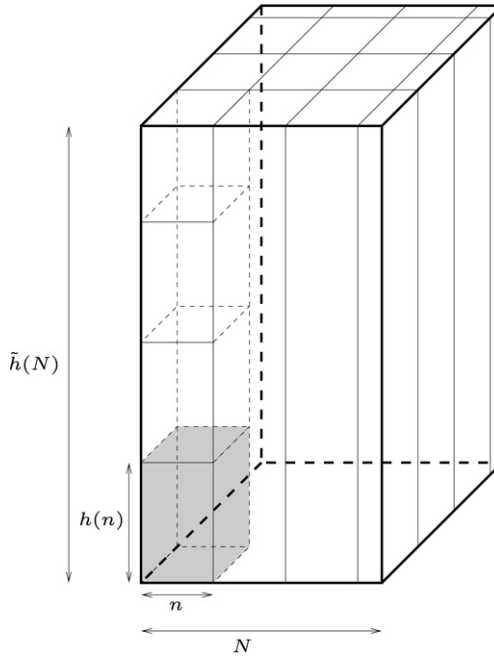
Fig. 2. Comparison between $\phi^k_{N^{d-1}, \widetilde{h}(N)}$ and $\phi^k_{n^{d-1}, h(n)}$.

and

$$B_{m^{d-1}+1} = B((N, \dots, N), \widetilde{h}(N)) \setminus \bigcup_{i_1, \dots, i_{d-1}=0}^{m-1} B_{i_1, \dots, i_{d-1}}.$$

Remark 3. It is easy (and very useful) to see that if $C_i \times]0, h]$, $i = 1, 2$, are two cylinders with disjoint bases $C_1, C_2 \subset \mathbb{R}^{d-1}$ having a common side and with maximal flows ϕ_i, $i = 1, 2$, the maximal flow through $(C_1 \cup C_2) \times]0, h]$ is at least $\phi_1 + \phi_2$.

We deduce from this remark that if for every $i_1, \dots, i_{d-1}$ in $\{0, \dots, m-1\}$ we have $\phi^k_{B_{i_1, \dots, i_{d-1}}} \geq \lambda n^{d-1}$ and if all the vertical edges e in $B_{m^{d-1}+1}$ satisfy $t(e) \geq (\lambda + \eta)$, we have

$$\phi^k_{N^{d-1}, \widetilde{h}(N)} \geq \lambda N^{d-1}.$$

By independence we obtain

$$\mathbb{P}\left[\phi^k_{N^{d-1}, \widetilde{h}(N)} \geq \lambda N^{d-1}\right] \geq \prod_{i_1, \dots, i_{d-1}=0}^{m-1} \mathbb{P}\left[\phi^k_{B_{i_1, \dots, i_{d-1}}} \geq \lambda n^{d-1}\right] \times p(\eta)^{(d-1)N^{d-2}r\widetilde{h}(N)}$$

$$\geq \mathbb{P}\left[\phi^k_{n^{d-1}, \widetilde{h}(N)} \geq \lambda n^{d-1}\right]^{m^{d-1}} \times p(\eta)^{(d-1)N^{d-2}r\widetilde{h}(N)}.$$

We study next $\phi^k_{n^{d-1}, \widetilde{h}(N)}$. We define for j in $\{0, \dots, (\widetilde{m}-1)\}$

$$B'_j =]0, n]^{d-1} \times]jh(n), (j+1)h(n)]$$

and

$$B'_{\widetilde{m}} = B((n, \ldots, n), \widetilde{h}(N)) \smallsetminus \bigcup_{j=0}^{\widetilde{m}-1} B'_j.$$

The probability of the boundary conditions $\Pi \in \{0, 1/k, 2/k, \ldots, (\lfloor \lambda n^{d-1} \rfloor + 1)/k\}^{2n^{d-1}}$ in B is the probability that there exists a discrete stream (g, o) in B satisfying $\Pi^{\lambda, n}(g) = \Pi$. Remember that every discrete stream (g, o) must satisfy the balance equation, so once we know that such a discrete stream exists, flow(g, o) is just given by the projection of g on the vertical edges that intersect the hyper-plane $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_d = h(n) - 1/2\}$, so $\Pi^{\lambda, n}(g)$ contains enough information for us to know whether flow(g, o) is bigger than λn^{d-1} or not. We denote by $\Pi^k_{\lambda, n} = (\Pi^k_{\lambda, n, 1}, \Pi^k_{\lambda, n, 2})$ one of the boundary conditions of highest probability in $B((n, \ldots, n), h(n))$ which corresponds to a discrete stream (g, o) such that flow$(g, o) \geq \lambda n^{d-1}$ and we define $(\Pi^k_{\lambda, n})^* = (\Pi^k_{\lambda, n, 2}, \Pi^k_{\lambda, n, 1})$. The model is invariant under reflections in the coordinate hyperplanes or translates of these hyperplanes, so by symmetry we have $\mathbb{P}[\Pi^k_{\lambda, n}] = \mathbb{P}[(\Pi^k_{\lambda, n})^*]$. Using the lemma of junction (Lemma 1), we know that if:

- we can define a discrete stream in B'_0 with boundary conditions $\Pi^k_{\lambda, n}$,
- we can define a discrete stream in B'_1 with boundary conditions $(\Pi^k_{\lambda, n})^*$,
- we can define a discrete stream in B'_2 with boundary conditions $\Pi^k_{\lambda, n}$,
- $\cdots$,
- and all the vertical edges e in $B'_{\widetilde{m}}$ satisfy $t(e) \geq \gamma_k k$,

  then $\phi^k_{n^{d-1}, \widetilde{h}(N)} \geq \lambda n^{d-1}$.

Remark 4. It is not sufficient to impose here that all the vertical edges e in $B'_{\widetilde{m}}$ satisfy $t(e) \geq \lambda$, because the amount of fluid that goes out of $B'_{\widetilde{m}-1}$ at its top through one fixed edge f can exceed λ – we have no information about $\Pi^k_{\lambda, n}$ – and we cannot accept to lose fluid at the exit of f, unless it exceeds λn^{d-1}. This is why we introduced γ_k.

Now by independence we obtain

$$\mathbb{P}\left[\phi^k_{n^{d-1}, \widetilde{h}(N)} \geq \lambda n^{d-1}\right] \geq \mathbb{P}\left[\Pi^k_{\lambda, n}\right]^{\widetilde{m}} \times p_k^{n^{d-1}\widetilde{r}}, \tag{3}$$

whence

$$\mathbb{P}\left[\phi^k_{N^{d-1}, \widetilde{h}(N)} \geq \lambda N^{d-1}\right] \geq \mathbb{P}\left[\Pi^k_{\lambda, n}\right]^{m^{d-1}\widetilde{m}} \times p_k^{n^{d-1}\widetilde{r}m^{d-1}} p(\eta)^{(d-1)N^{d-2}r\widetilde{h}(N)}. \tag{4}$$

Let $\mathbf{\Pi}$ be the set of all the boundary conditions corresponding to a discrete stream (g, o) such that flow$(g, o) \geq \lambda n^{d-1}$. We have seen that a maximal flow ϕ^k is always realized by a discrete stream, so we have

$$\mathbb{P}\left[\phi^k_{n^{d-1}, h(n)} \geq \lambda n^{d-1}\right] \leq \mathbb{P}\left[\bigcup_{\Pi \in \mathbf{\Pi}} \Pi\right] \leq \sum_{\Pi \in \mathbf{\Pi}} \mathbb{P}[\Pi]$$
$$\leq N^k_{\lambda, n} \times \mathbb{P}\left[\Pi^k_{\lambda, n}\right],$$

where we recall that $N^k_{\lambda, n}$ is the number of possible boundary conditions for discrete streams.

To obtain later a result independent of k, we need to consider two sequences $(k_n, n \in \mathbb{N})$ and $(\widetilde{k}_n, n \in \mathbb{N})$ such that $\lim_{n \to \infty} k_n = \lim_{n \to \infty} \widetilde{k}_n = +\infty$. We want to get rid of $N^k_{\lambda,n}$. We recall that

$$N^{k_n}_{\lambda,n} \leq \left(k_n \left(\lfloor \lambda n^{d-1} \rfloor + 1 \right) + 1 \right)^{2n^{d-1}}.$$

Under the condition

$$\lim_{n \to \infty} \frac{\ln(k_n n)}{h(n)} = 0 \tag{5}$$

we have

$$\limsup_{n \to \infty} \frac{1}{n^{d-1} h(n)} \ln \mathbb{P} \left[\phi^{k_n}_{n^{d-1}, h(n)} \geq \lambda n^{d-1} \right] \leq \limsup_{n \to \infty} \frac{1}{n^{d-1} h(n)} \ln \mathbb{P} \left[\Pi^{k_n}_{\lambda,n} \right]. \tag{6}$$

Consider (4) again. This equation is satisfied for every $k \geq k_0$, so it is true for k_n with a fixed n not too small. We need to compare t^{k_n} with $t^{\widetilde{k}_N}$, but the relation is simple only if $\widetilde{k}_N$ is divisible by k_n. That is why from now on we will consider only sequences $(k_n, n \in \mathbb{N})$ and $(\widetilde{k}_n, n \in \mathbb{N})$ such that

$$\forall n \in \mathbb{N} \quad k_n = 2^{\chi_n} \quad \text{and} \quad \widetilde{k}_n = 2^{\widetilde{\chi}_n}$$

where $(\chi_n, n \in \mathbb{N})$ and $(\widetilde{\chi}_n, n \in \mathbb{N})$ are non-decreasing sequences of integers. Of course the condition $\lim_{n \to +\infty} k_n = \lim_{n \to +\infty} \widetilde{k}_n = +\infty$ implies $\lim_{n \to +\infty} \chi_n = \lim_{n \to +\infty} \widetilde{\chi}_n = +\infty$. In that case for large N we have $\widetilde{\chi}_N \geq \chi_n$, so $\widetilde{k}_N$ is divisible by k_n and then $t^{\widetilde{k}_N} \geq t^{k_n}$, whence

$$\phi^{\widetilde{k}_N}_{N^{d-1}, \widetilde{h}(N)} \geq \phi^{k_n}_{N^{d-1}, \widetilde{h}(N)}. \tag{7}$$

We use (4) with $k = k_n = 2^{\chi_n}$ and (7) to obtain for n and N large enough

$$\frac{1}{N^{d-1} \widetilde{h}(N)} \ln \mathbb{P} \left[\phi^{\widetilde{k}_N}_{N^{d-1}, \widetilde{h}(N)} \geq \lambda N^{d-1} \right] \geq \frac{1}{N^{d-1} \widetilde{h}(N)} \ln \mathbb{P} \left[\phi^{k_n}_{N^{d-1}, \widetilde{h}(N)} \geq \lambda N^{d-1} \right]$$

$$\geq \frac{m^{d-1} \widetilde{m}}{N^{d-1} \widetilde{h}(N)} \ln \mathbb{P} \left[\Pi^{k_n}_{\lambda,n} \right] + \frac{n^{d-1} m^{d-1} \widetilde{r}}{N^{d-1} \widetilde{h}(N)} \ln p_{k_n}$$

$$+ \frac{(d-1) N^{d-2} r \widetilde{h}(N)}{N^{d-1} \widetilde{h}(N)} \ln p(\eta).$$

We send first N to $+\infty$ and then n to $+\infty$; this gives us with the help of (6)

$$\liminf_{N \to \infty} \frac{1}{N^{d-1} \widetilde{h}(N)} \ln \mathbb{P} \left[\phi^{\widetilde{k}_N}_{N^{d-1}, \widetilde{h}(N)} \geq \lambda N^{d-1} \right]$$

$$\geq \limsup_{n \to \infty} \frac{1}{n^{d-1} h(n)} \ln \mathbb{P} \left[\Pi^{k_n}_{\lambda,n} \right]$$

$$\geq \limsup_{n \to \infty} \frac{1}{n^{d-1} h(n)} \ln \mathbb{P} \left[\phi^{k_n}_{n^{d-1}, h(n)} \geq \lambda n^{d-1} \right].$$

By considering the case $h = \widetilde{h}$ and $k_n = \widetilde{k}_n = 2^{\chi_n}$, under the condition (5) on h and (k_n) – i.e., the condition (2) on h and (χ_n) – we obtain the existence of the limit

$$\widetilde{\psi}(\lambda, h, (\chi_n)) = \lim_{n \to \infty} -\frac{1}{n^{d-1} h(n)} \ln \mathbb{P} \left[\phi^{2^{\chi_n}}_{n^{d-1}, h(n)} \geq \lambda n^{d-1} \right].$$

For general h, $\widetilde{h}$, χ and $\widetilde{\chi}$ we obtain that $\widetilde{\psi}(\lambda)$ is independent of the pair $(h, (\chi_n))$ satisfying (2), so Theorem 4 is proved.

Remark 5. Thanks to this independence, we can prove some properties of $\widetilde{\psi}$ by studying the behavior of the limit involved in Theorem 4 for specific choices of pairs $(h, (\chi_n))$.

Moreover, still in the case $\lambda < \mu$, we have immediately that for n sufficiently large

$$
\mathbb{P}\left[\phi^{k_n}_{n^{d-1}, h(n)} \geq \lambda n^{d-1}\right]
$$
$$
\geq \mathbb{P}\left[\text{all the vertical edges } e \text{ in } B((n, \ldots, n), h(n)) \text{ satisfy } t(e) \geq (\lambda + \eta)\right]
$$
$$
\geq p(\eta)^{n^{d-1} h(n)},
$$

and thus

$$
\widetilde{\psi}(\lambda) \leq -\ln p(\eta) < +\infty.
$$

If the capacity t of an edge is bounded by a constant M, we can simply define

$$
\forall x \in \mathbb{Z}^{d-1} \cap \,]0, n]^{d-1} \quad \pi_h^{\lambda, n}(g, x) = g\left(\langle (x, h), (x, h+1)\rangle\right)
$$

without truncating g because g is already bounded by M. Then the number of possible boundary conditions N_n^k satisfies

$$
N_n^k \leq (k(M+1))^{2n^{d-1}}
$$

so we can replace the hypothesis (5) by

$$
\lim_{n \to \infty} \frac{\ln k_n}{h(n)} = 0. \tag{8}
$$

Remark 6. We do not study the case $\lambda = \mu$ for the moment; it is more appropriate to study it with the continuity of $\widetilde{\psi}$.

3.3. Convexity of $\widetilde{\psi}$

Let $\lambda_1 \leq \lambda_2 < \mu$, and $\alpha \in\,]0, 1[$. We want to show that

$$
\widetilde{\psi}\left(\alpha\lambda_1 + (1-\alpha)\lambda_2\right) \leq \alpha\widetilde{\psi}(\lambda_1) + (1-\alpha)\widetilde{\psi}(\lambda_2). \tag{9}
$$

We know that $\widetilde{\psi}$ does not depend on the couple $(h, (\chi_n))$ satisfying (2), so we can take $h(n) = n$ to simplify the notation and we will take an adapted (χ_n). First we fix k in $\mathbb{N}$; we will make it vary later. We fix n, m in $\mathbb{N}$, and take $N = nm$. We set $u = \lfloor \alpha m^{d-1} \rfloor$. We keep the same notation as in the previous section for $B_{i_1, \ldots, i_{d-1}}, i_1, \ldots, i_{d-1}$ in $\{0, \ldots, m-1\}$. We use the lexicographic order to order $\{(i_1, \ldots, i_{d-1}), i_j \in \{0, \ldots, m-1\}, 1 \leq j \leq (d-1)\}$ and use this to rename these cylinders $(B_j, 1 \leq j \leq m^{d-1})$. On the event

$$
\left\{\forall j \in \{1, \ldots, u\}, \phi^k_{B_j} \geq \lambda_1\right\} \cap \left\{\forall j \in \{(u+1), \ldots, m^{d-1}\}, \phi^k_{B_j} \geq \lambda_2\right\}
$$

we have (see Remark 3)

$$
\phi^k_{N^{d-1}, N} \geq \left(u\lambda_1 n^{d-1} + (m^{d-1} - u)\lambda_2 n^{d-1}\right)
$$

$$\geq N^{d-1} \left(\frac{u}{m^{d-1}} \lambda_1 + \left(1 - \frac{u}{m^{d-1}} \right) \lambda_2 \right)$$

$$\geq N^{d-1} \left(\alpha \lambda_1 + (1 - \alpha) \lambda_2 \right)$$

because $\lambda_1 < \lambda_2$, so

$$\mathbb{P} \left[\phi^k_{N^{d-1},N} \geq N^{d-1} (\alpha \lambda_1 + (1 - \alpha) \lambda_2) \right] \geq \mathbb{P} \left[\phi^k_{n^{d-1},N} \geq \lambda_1 n^{d-1} \right]^{u}$$
$$\times \mathbb{P} \left[\phi^k_{n^{d-1},N} \geq \lambda_2 n^{d-1} \right]^{m^{d-1}-u}.$$

As in the previous section (see (3)), we have

$$\mathbb{P} \left[\phi^k_{n^{d-1},N} \geq \lambda_i n^{d-1} \right] \geq \mathbb{P} \left[\Pi^k_{\lambda_i,n} \right]^m \quad i = 1, 2$$

so

$$\frac{1}{N^d} \ln \mathbb{P}[\phi^k_{N^{d-1},N} \geq N^{d-1} (\alpha \lambda_1 + (1 - \alpha) \lambda_2)]$$
$$\geq \frac{m^{d-1}}{N^d} \left(\frac{u}{m^{d-1}} \ln \mathbb{P} \left[\Pi^k_{\lambda_1,n} \right] + \left(1 - \frac{u}{m^{d-1}} \right) \ln \mathbb{P} \left[\Pi^k_{\lambda_2,n} \right] \right).$$

We now make k vary, $k_n = 2^{\chi_n}$ with $(n, (\chi_n))$ satisfying the condition (2) (for example $\chi_n = \lfloor n^{1/2} \rfloor$), and we use the property $\phi^{2^{\chi_N}}_{N^{d-1},N} \geq \phi^{2^{\chi_n}}_{N^{d-1},N}$ for large N; we first send N to $+\infty$ and then n to $+\infty$. We proved in the previous section that

$$\limsup_{n \to \infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P} \left[\Pi^{2^{\chi_n}}_{\lambda,n} \right] = -\widetilde{\psi}(\lambda),$$

so we obtain (9).

3.4. Continuity of $\widetilde{\psi}$

We want to show that $\widetilde{\psi}$ is continuous on $[0, \mu]$ when μ is finite or on $[0, +\infty[$ when μ is infinite (remember that $\widetilde{\psi}$ is infinite on $]\mu, +\infty[$). The function $\widetilde{\psi}$ is convex and finite on $[0, \mu[$, so $\widetilde{\psi}$ is continuous on $]0, \mu[$. We assume then that $0 < \mu$: $\widetilde{\psi}(0) = 0$ and $\widetilde{\psi}$ is non-negative on $\mathbb{R}^+$, so $\widetilde{\psi}$ is right continuous at 0. We assume then that $0 < \mu < +\infty$ (there is nothing to prove otherwise). The only point which remains to study is the left continuity of $\widetilde{\psi}$ at μ. Remember that we did not define $\widetilde{\psi}$ at μ; we will do it now. We set

$$q_\mu = \mathbb{P}[t(e) = \mu].$$

Notice that q_μ can be null. We remark that

$$\mathbb{P} \left[\phi_{n^{d-1},h(n)} \geq \mu n^{d-1} \right]$$
$$= \mathbb{P} \left[\text{all the vertical edges } e \text{ in } B((n, \ldots, n), h(n)) \text{ satisfy } t(e) = \mu \right]$$
$$= q_\mu^{n^{d-1}h(n)},$$

so

$$\lim_{n \to \infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P} \left[\phi_{n^{d-1},h(n)} \geq \mu n^{d-1} \right] = - \ln q_\mu$$

is finite as soon as $q_\mu > 0$. Unfortunately, the existence of an atom for the law of $t(e)$ at μ does not imply the existence of an atom for the law of $t^k(e)$ at μ, so we can have $q_\mu > 0$ and

$$\lim_{n\to\infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \geq \mu n^{d-1}\right] = +\infty.$$

This is why we did not study $\widetilde{\psi}(\mu)$ previously. We define (for every pair $(h, (\chi_n))$ as in Theorem 4)

$$\widetilde{\psi}(\mu) = -\ln q_\mu,$$

which can eventually be infinite.

Now we want to check that $\widetilde{\psi}$ is left continuous at μ (for $q_\mu = 0$ we will show that $\lim \widetilde{\psi}(\lambda) = +\infty$ when $\lambda \leq \mu$ and $\lambda \to \mu$). The idea of the proof is simple: if the flow in a cylinder is high, it must be high in each horizontal section of this cylinder. We fix $\varepsilon > 0$, and we take $h(n), k_n \to_{n\to+\infty} +\infty, k_n = 2^{\chi_n}$, satisfying the condition (5). We define for i in $\{0, \ldots, (h(n) - 1)\}$

$$C_i = \,]0, n]^{d-1} \times]i, i+1]$$

and we denote by $t_1, \ldots, t_{n^{d-1}}$ the capacities of the n^{d-1} vertical edges in C_0. We have

$$\mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \geq (\mu - \varepsilon)n^{d-1}\right] \leq \mathbb{P}\left[\bigcap_{i=0}^{h(n)-1} \{\phi^{k_n}_{C_i} \geq (\mu - \varepsilon)n^{d-1}\}\right]$$

$$\leq \mathbb{P}\left[\phi^{k_n}_{n^{d-1},1} \geq (\mu - \varepsilon)n^{d-1}\right]^{h(n)},$$

and we know that

$$\phi^{k_n}_{n^{d-1},1} = \sum_{j=1}^{n^{d-1}} t_j^{k_n} \leq \sum_{j=1}^{n^{d-1}} t_j,$$

so we have

$$\mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \geq (\mu - \varepsilon)n^{d-1}\right] \leq \mathbb{P}\left[\sum_{j=1}^{n^{d-1}} (t_j - \mu) \geq -\varepsilon n^{d-1}\right]^{h(n)}.$$

For every positive ρ we obtain

$$\mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \geq (\mu - \varepsilon)n^{d-1}\right] \leq \mathrm{e}^{\rho\varepsilon n^{d-1}h(n)} \mathbb{E}[\mathrm{e}^{\rho(t-\mu)}]^{n^{d-1}h(n)}.$$

This expectation is well defined, because $(t - \mu) \leq 0$. Let $\eta > 0$. Since

$$\lim_{\rho\to+\infty} \mathbb{E}[\mathrm{e}^{\rho(t-\mu)}] = q_\mu,$$

then there exists ρ_0 such that

$$\forall \rho \geq \rho_0 \quad \mathbb{E}[\mathrm{e}^{\rho(t-\mu)}] \leq (q_\mu + \eta).$$

It follows that

$$\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi^{k_n}_{n^{d-1},h(n)} \geq (\mu - \varepsilon)n^{d-1}\right] \leq \rho_0\varepsilon + \ln(q_\mu + \eta),$$

so

$$\widetilde{\psi}(\mu - \varepsilon) \geq -\rho_0 \varepsilon - \ln(q_\mu + \eta),$$

whence

$$\lim_{\varepsilon \to 0} \widetilde{\psi}(\mu - \varepsilon) \geq -\ln(q_\mu + \eta).$$

This is true for every positive η, so

$$\lim_{\varepsilon \to 0} \widetilde{\psi}(\mu - \varepsilon) \geq \lim_{\eta \to 0} -\ln(q_\mu + \eta) = -\ln q_\mu = \widetilde{\psi}(\mu).$$

If $q_\mu = 0$, we have the desired equality. Otherwise, we remark that for every positive ε we have

$$\mathbb{P}\left[\phi_{n^{d-1},h(n)}^{k_n} \geq (\mu - \varepsilon)n^{d-1}\right]$$

$$\geq \mathbb{P}\left[\text{all the vertical edges } e \text{ in } B((n, \ldots, n), h(n)) \text{ satisfy } t^{k_n}(e) \geq (\mu - \varepsilon)\right]$$

$$\geq \mathbb{P}\left[t^{k_n}(e) \geq \mu - \varepsilon\right]^{n^{d-1}h(n)}.$$

Now for k_n sufficiently large we have

$$\mathbb{P}\left[t^{k_n}(e) \geq \mu - \varepsilon\right] \geq \mathbb{P}\left[t(e) \geq \mu - \frac{\varepsilon}{2}\right] \geq q_\mu,$$

and thus

$$\forall \varepsilon > 0 \quad \widetilde{\psi}(\mu - \varepsilon) \leq -\ln q_\mu = \widetilde{\psi}(\mu).$$

This ends the proof of the continuity of $\widetilde{\psi}$ on $[0, \mu]$ (or $[0, +\infty[$ if μ is infinite). We deduce immediately from this continuity that $\widetilde{\psi}$ is good.

3.5. Existence of the limit for $\phi_{n^{d-1},h(n)}$

We come back to the existence of the limit involving ϕ in Theorem 2. We consider three cases.

- $\lambda > \mu$: Then

$$\forall n \in \mathbb{N} \quad \mathbb{P}\left[\phi_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right] = 0,$$

  so the limit involved in Theorem 2 exists and satisfies

$$\psi(\lambda) = \lim_{n \to \infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right] = +\infty = \widetilde{\psi}(\lambda).$$

- $\lambda = \mu$: As we saw by studying the continuity of $\widetilde{\psi}$, we have

$$\psi(\mu) = \lim_{n \to \infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi_{n^{d-1},h(n)} \geq \mu n^{d-1}\right] = -\ln q_\mu = \widetilde{\psi}(\mu)$$

  by definition of $\widetilde{\psi}(\mu)$.

- $\lambda < \mu$: We will compare $\phi_{n^{d-1},h(n)}^{k_n}$ with $\phi_{n^{d-1},h(n)}$. We fix $k \in \mathbb{N}$. We know that $t(e) \geq t^k(e)$ so $\phi_{n^{d-1},h(n)} \geq \phi_{n^{d-1},h(n)}^k$. For a set of edges E, we denote by $V^k(E)$ the quantity $\sum_{e \in E} t^k(e)$.

Thanks to the max-flow min-cut theorem we obtain

$$\phi_{n^{d-1},h(n)}^{k} = \min\{V^{k}(E) \mid E \text{ is an } (F_0, F_{h(n)})\text{-cut}\}.$$

Let E_0 be an $(F_0, F_{h(n)})$-cut realizing this minimum (it may depend on k). Then

$$\begin{aligned}
\phi_{n^{d-1},h(n)}^{k} &= V^{k}(E_0) \\
&= \sum_{e \in E_0} t^{k}(e) \\
&\geq \sum_{e \in E_0} t(e) - \frac{|E_0|}{k} \\
&\geq \min\{V(E) \mid E \text{ is an } (F_0, F_{h(n)})\text{-cut}\} - n^{d-1}\frac{h(n)}{k} \\
&\geq \phi_{n^{d-1},h(n)} - n^{d-1}\frac{h(n)}{k}.
\end{aligned}$$

We fix $\lambda \geq 0$, and we now make k vary. We take $k_n = 2^{\chi_n}$ (with (χ_n) a non-decreasing sequence of integers such that $\lim_{n \to +\infty} \chi_n = +\infty$) satisfying with h the condition (5). If the sequence (k_n) satisfies also the condition

$$\lim_{n \to \infty} \frac{h(n)}{k_n} = 0 \tag{10}$$

then we have for every $\lambda' < \lambda$ the existence of $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \quad \lambda - \frac{h(n)}{k_n} \geq \lambda'.$$

We deduce that under the condition (10) we have for all $n \geq n_0$

$$\mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}^{k_n}}{n^{d-1}} \geq \lambda\right] \leq \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \geq \lambda\right] \leq \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}^{k_n}}{n^{d-1}} \geq \lambda'\right].$$

We conclude thanks to the hypothesis (5) that

$$\widetilde{\psi}(\lambda) \geq \limsup_{n \to \infty}(\square) \geq \liminf_{n \to \infty}(\square) \geq \widetilde{\psi}(\lambda')$$

where

$$(\square) = -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right].$$

Sending λ' to λ, thanks to the continuity of $\widetilde{\psi}$ in $[0, \mu[$, we obtain the existence of the limit

$$\psi(\lambda, h) = \lim_{n \to \infty} -\frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\phi_{n^{d-1},h(n)} \geq \lambda n^{d-1}\right] = \widetilde{\psi}(\lambda).$$

Moreover we know that this limit is independent of h satisfying $\lim_{n \to +\infty} h(n) = +\infty$ and such that there exists a non-decreasing sequence of integers (χ_n), $\lim_{n \to +\infty} \chi_n = +\infty$ for which the pair $(h, (2^{\chi_n}))$ satisfies (5) and (10): we denote it by $\psi(\lambda)$. It is finally obvious that the existence

of such a sequence (χ_n) is equivalent to the condition

$$\lim_{n\to\infty} \frac{h(n)}{\ln n} = +\infty$$

(let $\chi_n = \lfloor 2\ln h(n)/\ln 2\rfloor$ for example). This ends the proof of the existence of the limit ψ in Theorem 2, and we have $\psi = \widetilde{\psi}$ so the properties proved for $\widetilde{\psi}$ still hold for ψ.

If the capacity t of an edge is bounded, we can replace the condition (5) by (8); in that case, as soon as

$$\lim_{n\to\infty} h(n) = +\infty$$

we can find a sequence $(k_n) = (2^{\chi_n})$ satisfying (8) and (10), so the limit exists.

3.6. The function ψ vanishes on $[0, \nu(F)]$

This could be proved easily thanks to Theorem 1 in dimension 3 and with the hypothesis on F required in Theorem 1, but we prefer to prove it directly in the general case without Theorem 1.

We suppose now that $\mathbb{E}[t]$ is finite. We suppose that $\nu > 0$ (otherwise there is nothing to prove), and we take $\lambda = \nu - \varepsilon$, with a positive ε. Remark 5 holds for ψ too: we know that ψ is independent of h satisfying $\lim_{n\to+\infty} h(n)/\ln n = +\infty$ so we can make a specific choice of function h and study the corresponding limit to show a general result on ψ. We take $h \to \infty$ such that

$$\lim_{n\to\infty} \frac{h(n)}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{h(n)}{\ln n} = +\infty. \tag{11}$$

We recall that

$$\tau_{n^{d-1}} = \tau(]0, n]^{d-1}) = \inf\{V(E) \mid E \text{ is a cut over}]0, n]^{d-1} \text{ and } E \text{ satisfies } (*)\},$$

where $(*)$ is defined at the end of the Section 2. We define for S a hyper-rectangle the variable

$$\tau(S, k) = \inf\{V(E) \mid E \text{ is a cut over } S, E \text{ satisfies } (*) \text{ and } E \subset S \times]-k, k]\},$$

and

$$\tau_{n^{d-1}, k} = \tau(]0, n]^{d-1}, k).$$

We define the set of edges F as

$$F = \{\langle x, y\rangle \mid x \in B, y \notin B \text{ and } \langle x, y\rangle \in \mathbb{R}^{d-1} \times [1, h(n)]\}.$$

This is the set of the edges through which some fluid could escape from B somewhere other than at its bottom or at its top. We denote by $|F|$ the cardinality of F, $|F| = 2(d-1)n^{d-2}h(n)$. We consider the larger cylinder

$$B' =]-1, n+1]^{d-1} \times]0, h(n)],$$

and we define

$$\tau'_{(n+2)^{d-1}, h(n)} = \tau\left(]-1, n+1]^{d-1}, h(n)\right).$$

We finally define the set of edges

$$F' = \{e \in B' \smallsetminus B \mid e \text{ is vertical}, e \in \mathbb{R}^{d-1} \times [0, 1]\}$$

of cardinality $|F'| = 2(d - 1)(n + 1)^{d-2}$ (see Fig. 3 in dimension 2). We remark that if E is an $(F_0, F_{h(n)})$-cut in $B((n, \ldots, n), h(n))$, the set of edges $E \cup F \cup F'$ contains a cut over $]-1, n + 1]^{d-1}$ satisfying the condition $(*)$ for $S =]-1, n + 1]^{d-1}$, so

$$\tau'_{(n+2)^{d-1}, h(n)} - \phi_{n^{d-1}, h(n)} \leq \sum_{e \in F} t(e) + \sum_{e \in F'} t(e).$$

We obtain for $M > \mathbb{E}[t]$

$$\mathbb{P}\left[\frac{\phi_{n^{d-1}, h(n)}}{n^{d-1}} \geq \lambda\right] \geq \mathbb{P}\left[\left\{\frac{\phi_{n^{d-1}, h(n)}}{n^{d-1}} \geq \lambda\right\} \cap \left\{\frac{\tau'_{(n+2)^{d-1}, h(n)} - \phi_{n^{d-1}, h(n)}}{|F| + |F'|} \leq M\right\}\right]$$

$$\geq \mathbb{P}\left[\left\{\frac{\tau'_{(n+2)^{d-1}, h(n)}}{n^{d-1}} \geq \lambda + M\frac{|F| + |F'|}{n^{d-1}}\right\}\right.$$

$$\left. \cap \left\{\frac{\tau'_{(n+2)^{d-1}, h(n)} - \phi_{n^{d-1}, h(n)}}{|F| + |F'|} \leq M\right\}\right].$$

We remark that $\tau'_{(n+2)^{d-1}, h(n)}$ is equal in law to $\tau_{(n+2)^{d-1}, h(n)}$, so

$$\mathbb{P}\left[\frac{\phi_{n^{d-1}, h(n)}}{n^{d-1}} \geq \lambda\right] \geq 1 - \left(\mathbb{P}\left[\frac{\tau_{(n+2)^{d-1}, h(n)}}{n^{d-1}} < \lambda + M\frac{|F| + |F'|}{n^{d-1}}\right]\right.$$

$$\left. + \mathbb{P}\left[\frac{\tau'_{(n+2)^{d-1}, h(n)} - \phi_{n^{d-1}, h(n)}}{|F| + |F'|} > M\right]\right)$$

$$\geq 1 - \left(\mathbb{P}\left[\frac{\tau_{(n+2)^{d-1}}}{n^{d-1}} < \nu - \frac{\varepsilon}{2}\right]\right.$$

$$\left. + \mathbb{P}\left[\frac{1}{|F| + |F'|}\sum_{e \in F \cup F'} t(e) \geq M\right]\right)$$

for n sufficiently large, thanks to (11) and the fact that $\tau_{(n+2)^{d-1}, h(n)} \geq \tau_{(n+2)^{d-1}}$. We know that $M > \mathbb{E}[t]$ and $\lim_{n \to \infty} (\tau_{(n+2)^{d-1}}/n^{d-1}) = \nu$ almost surely, so

$$\lim_{n \to \infty} \mathbb{P}\left[\frac{\phi_{n^{d-1}, h(n)}}{n^{d-1}} \geq \lambda\right] = 1,$$

which leads to

$$\psi(\lambda) = 0.$$

To conclude that $\psi(\nu) = 0$ we need only to check that ψ is left continuous at ν, i.e., to be sure that $\nu \leq \mu$. Suppose that $\nu > \mu$; then $\mathbb{P}[t \geq \nu] = 0$, so $\mathbb{E}[t] < \nu$, and we can find a positive ε such that $\mathbb{E}[t] < \nu - \varepsilon$. Now if we denote by $(\widetilde{t}_i, i = 1, \ldots, n^{d-1})$ the capacities of the vertical edges in $]0, n]^{d-1} \times]0, 1]$, we have

$$\mathbb{P}\left[\frac{\tau_{n^{d-1}}}{n^{d-1}} \geq \nu - \varepsilon\right] \leq \mathbb{P}\left[\frac{\sum_{i=1}^{n^{d-1}} \widetilde{t}_i}{n^{d-1}} \geq \nu - \varepsilon\right] \xrightarrow[n \to \infty]{} 0.$$
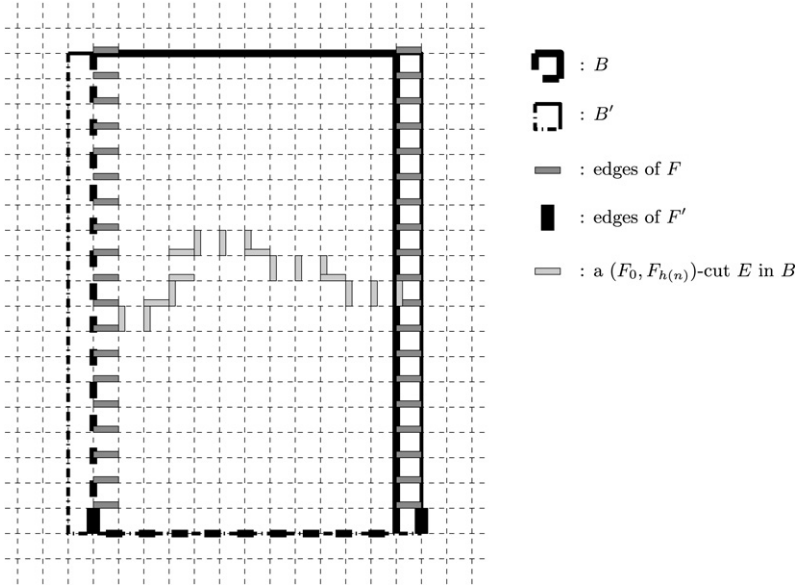
Fig. 3. Comparison between ϕ and τ in dimension 2.

This is absurd because $(\tau_{n^{d-1}}/n^{d-1})$ converges toward ν almost surely. We conclude that $\nu \leq \mu$ and that $\psi(\nu) = 0$.

3.7. The function ψ is positive on $]\nu(F), +\infty[$

We suppose that there exists a positive θ such that $\int_{[0,+\infty[} e^{\theta x} dF(x)$ is finite. The proof is based on the Cramér theorem in $\mathbb{R}$.

Let $\lambda = \nu + \varepsilon$, for a positive ε. We fix $k, N \in \mathbb{N}$; we will choose them later. We define

$$u = \left\lfloor \frac{h(N)}{2k} \right\rfloor.$$

Just as in the study of the continuity of ψ, by cutting $B((N, \ldots, N), h(N))$ into horizontal sections of height $2k$, we have

$$\mathbb{P}\left[\phi_{N^{d-1}, h(N)} \geq (\nu + \varepsilon)N^{d-1}\right] \leq \mathbb{P}\left[\phi_{N^{d-1}, 2k} \geq (\nu + \varepsilon)N^{d-1}\right]^u$$
$$\leq \mathbb{P}\left[\phi_{\mathcal{B}(k)} \geq (\nu + \varepsilon)N^{d-1}\right]^u,$$

where $\mathcal{B}(k) = N^{d-1} \times] - k, k]$, because $\phi_{N^{d-1}, 2k}$ and $\phi_{\mathcal{B}(k)}$ are equal in law.

Now $\mathbb{E}[\tau(S, k)]$ is subadditive in the sense that for disjoint hyper-rectangles S_1 and S_2 having a common side, we have

$$\tau(S_1 \cup S_2, k) \leq \tau(S_1, k) + \tau(S_2, k).$$

Moreover $\mathbb{E}[\tau(S, k)]$ is non-negative and finite (because $\mathbb{E}[t] < \infty$), so by a classical subadditive argument we have the existence of

$$\nu_k = \lim_{n \to \infty} \frac{\mathbb{E}[\tau_{n^{d-1}, k}]}{n^{d-1}}$$

and we know that

$$\nu_k = \inf_n \frac{\mathbb{E}[\tau_{n^{d-1},k}]}{n^{d-1}}.$$

The sequence $(\nu_k, k \in \mathbb{N})$ is non-increasing in k and non-negative, so it converges; we denote by $\widetilde{\nu}$ its limit: $\widetilde{\nu} = \lim_{k \to \infty} \nu_k = \inf_k \nu_k$. By the same subadditive argument, we have

$$\lim_{n \to \infty} \frac{\mathbb{E}[\tau_{n^{d-1}}]}{n^{d-1}} = \nu = \inf_n \frac{\mathbb{E}[\tau_{n^{d-1}}]}{n^{d-1}}.$$

We obtain

$$\widetilde{\nu} = \inf_k \inf_n \frac{\mathbb{E}[\tau_{n^{d-1},k}]}{n^{d-1}} = \inf_n \inf_k \frac{\mathbb{E}[\tau_{n^{d-1},k}]}{n^{d-1}} = \nu,$$

and thus we can choose k_0 such that $\nu_{k_0} \leq \nu + \varepsilon/4$. Then we choose n_0 such that

$$\frac{\mathbb{E}[\tau_{n_0^{d-1},k_0}]}{n_0^{d-1}} < \nu_{k_0} + \frac{\varepsilon}{2},$$

and we fix $N = n_0 m$, with $m \in \mathbb{N}$. We have

$$\phi_{\mathcal{B}(k_0)} \leq \tau_{N^{d-1},k_0} \leq \sum_{i_1,\dots,i_{d-1}=0}^{m-1} \tau\left(\prod_{j=1}^{d-1}]i_j n_0, (i_j+1)n_0], k_0\right).$$

The variables $(\tau(\prod_{j=1}^{d-1}]i_j n_0, (i_j+1)n_0], k_0), 0 \leq i_1, \dots, i_{d-1} \leq m-1)$ are independent and identically distributed, with the same law as $\tau_{n_0^{d-1},k_0}$. Their common expectation is

$$\mathbb{E}[\tau_{n_0^{d-1},k_0}] \leq \left(\nu_{k_0} + \frac{\varepsilon}{2}\right) n_0^{d-1}.$$

Moreover for some positive θ we know that $\mathbb{E}[e^{\theta t}]$ is finite so

$$\mathbb{E}\left[e^{\theta \tau_{n_0^{d-1},k_0}}\right] \leq \mathbb{E}\left[e^{\theta \sum_{i=1}^{n_0^{d-1}} \widetilde{t}_i}\right] \leq \mathbb{E}\left[e^{\theta t}\right]^{n_0^{d-1}} < \infty,$$

where $(\widetilde{t}_i, 1 \leq i \leq n_0^{d-1})$ are still the capacities of the vertical edges in $]0, n_0]^{d-1} \times]0, 1]$. We can thus apply the Cramér theorem in $\mathbb{R}$ (see [5]), which states the existence of a negative constant $c(n_0, k_0, \varepsilon)$ such that

$$\lim_{m \to \infty} \frac{1}{m^{d-1}} \ln \mathbb{P}\left[\frac{1}{m^{d-1}} \sum_{i_1,\dots,i_{d-1}=0}^{m-1} \frac{\tau\left(\prod_{j=1}^{d-1}]i_j n_0, (i_j+1)n_0], k_0\right)}{n_0^{d-1}} \geq \nu_{k_0} + \frac{3\varepsilon}{4}\right]$$

$$= c(n_0, k_0, \varepsilon).$$

It follows that for $u = \lfloor h(N)/(2k_0) \rfloor$ we have

$$\frac{1}{N^{d-1}h(N)} \ln \mathbb{P}\left[\phi_{N^{d-1},h(N)} \geq (\nu + \varepsilon)N^{d-1}\right]$$

$$\le \frac{u}{N^{d-1}h(N)} \ln \mathbb{P}\left[\phi_{\mathcal{B}(k_0)} \ge \left(\nu_{k_0} + \frac{3\varepsilon}{4}\right) N^{d-1}\right]$$

$$\le \frac{um^{d-1}}{N^{d-1}h(N)} \frac{1}{m^{d-1}}$$

$$\times \ln \mathbb{P}\left[\frac{1}{m^{d-1}} \sum_{i_1,\ldots,i_{d-1}=0}^{m-1} \frac{\tau\left(\prod_{j=1}^{d-1}]i_j n_0, (i_j+1)n_0], k_0\right)}{n_0^{d-1}} \ge \nu_{k_0} + \frac{3\varepsilon}{4}\right]$$

$$\xrightarrow[m\to\infty]{} \frac{c(n_0, k_0, \varepsilon)}{2k_0 n_0^{d-1}} < 0,$$

so $\psi(\lambda) > 0$. This ends the proof of Theorem 2.

Remark 7. The existence of a positive θ satisfying $\mathbb{E}[e^{\theta t}] < \infty$ is probably not a necessary condition for having the positivity of the function ψ on $]\nu, +\infty[$. However, a condition on the moments of t is necessary. Indeed, if the tail of the distribution of t is too big, the probability of having a vertical path of edges with big capacities (bigger than λn^{d-1}) is large; thus the probability of having $\phi_{n^{d-1}, h(n)} \ge \lambda n^{d-1}$ cannot decay exponentially fast in $n^{d-1}h(n)$.

4. Proof of Theorem 3

This is an adaptation of the proof of a large deviation principle in [3]. We take h such that $h(n)/\ln n \to \infty$ (we can do this again without loss of generality because ψ is independent of h) and we suppose that there exists a positive θ such that $\mathbb{E}[e^{\theta t}]$ is finite. We define

$$\beta = \inf\{v \mid \mathbb{P}[t(e) \le v] > 0\}.$$

We remark that $\phi_{(n,\ldots,n),h(n)}/n^{d-1}$ takes its values in $[\beta, +\infty[$. We have to prove that:

- for any closed subset $\mathcal{F} \subset [\beta, +\infty[$, we have

$$\limsup_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{(n,\ldots,n),h(n)}}{n^{d-1}} \in \mathcal{F}\right] \le -\inf_{\mathcal{F}} \psi,$$

- for any open subset $\mathcal{O} \subset [\beta, +\infty[$, we have

$$\liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{(n,\ldots,n),h(n)}}{n^{d-1}} \in \mathcal{O}\right] \ge -\inf_{\mathcal{O}} \psi.$$

By definition of β, for all positive η, we have

$$s_\beta(\eta) = \mathbb{P}[t(e) \le \beta + \eta] > 0.$$

4.1. Upper bound

Let $\mathcal{F}$ be a closed subset of $[\beta, +\infty[$, and $a = \inf \mathcal{F}$. Clearly

$$\mathbb{P}\left[\frac{\phi_{n^{d-1}, h(n)}}{n^{d-1}} \in \mathcal{F}\right] \le \mathbb{P}\left[\frac{\phi_{n^{d-1}, h(n)}}{n^{d-1}} \ge a\right],$$

so

$$\limsup_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P} \left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in \mathcal{F} \right] \le -\psi(a) = -\inf_{\mathcal{F}} \psi$$

because ψ is non-decreasing on $\mathbb{R}^+$.

4.2. Lower bound

We shall prove the following local lower bound:

$$\forall \alpha \in [\beta, +\infty[, \forall \varepsilon > 0$$

$$\liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P} \left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in]\alpha - \varepsilon, \alpha + \varepsilon[\right] \ge -\psi(\alpha). \tag{12}$$

If (12) holds, we have the desired lower bound. Indeed, if $\mathcal{O}$ is an open subset of $[\beta, +\infty[$, for every α in $\mathcal{O}$ there exists a positive ε such that $]\alpha - \varepsilon, \alpha + \varepsilon[\subset \mathcal{O}$, whence

$$\liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P} \left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in \mathcal{O} \right]$$

$$\ge \liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P} \left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in]\alpha - \varepsilon, \alpha + \varepsilon[\right]$$

$$\ge -\psi(\alpha).$$

By taking the supremum over α in $\mathcal{O}$, we obtain

$$\liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P} \left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in O \right] \ge -\inf_{\mathcal{O}} \psi.$$

To prove (12), we have to consider again different cases.

- $\alpha \ge \nu$: When $\psi(\alpha) = +\infty$, the result is obvious. For a finite $\psi(\alpha)$ we have $\psi(\alpha + \varepsilon) > \psi(\alpha)$ because the function ψ is convex on $[\nu, +\infty[$, $\psi(\nu) = 0$ and ψ is positive on $]\nu, +\infty[$ so ψ is increasing on $[\nu, +\infty[$ (or infinite). Now

$$\mathbb{P} \left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in]\alpha - \varepsilon, \alpha + \varepsilon[\right] \ge \mathbb{P} \left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \ge \alpha \right] - \mathbb{P} \left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \ge \alpha + \varepsilon \right],$$

so

$$\liminf_{n\to\infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P} \left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in]\alpha - \varepsilon, \alpha + \varepsilon[\right] \ge -\psi(\alpha).$$

- $\beta \le \alpha < \nu$: In our cylinder $B = B((n, \dots, n), h(n))$ we will isolate a smaller cylinder of adequate proportions in which we will impose that the rescaled flow is around its typical value ν, and we will control the amount of fluid that can circulate outside it (see Fig. 4). For that purpose, we consider a function h' such that

$$h' : \mathbb{N} \to \mathbb{N}, \quad h' \le h, \quad \lim_{n\to\infty} \frac{h'(n)}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{h'(n)}{\ln n} = +\infty$$
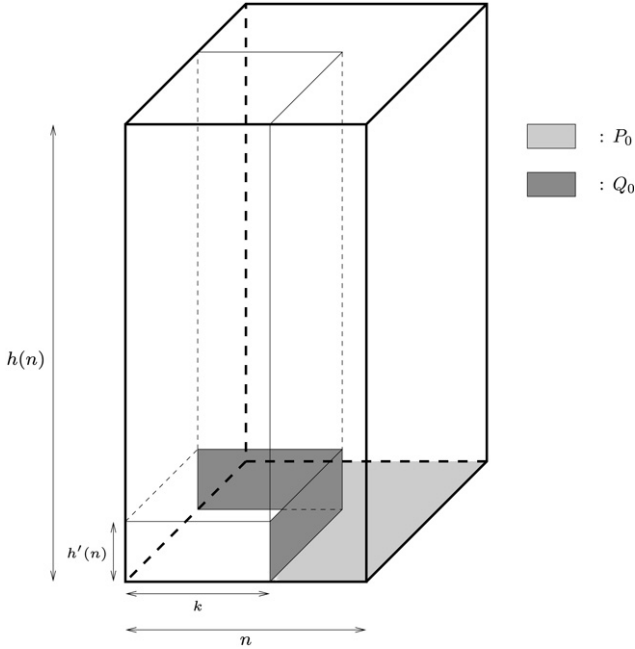
(then $\lim_{n\to\infty} h'(n) = +\infty$).

Fig. 4. Control of the flow.

We define the constants

$$v = \left(\frac{\alpha - \beta}{v - \beta} \right)^{\frac{1}{d-1}}, \quad k = \lfloor vn \rfloor, \quad 0 < \eta \leq \frac{\varepsilon}{4},$$

the set B' and the corresponding event A

$$B' = B((k, \ldots, k), h(n)), \qquad A = \{\phi_{B'} \geq (v - \eta)k^{d-1}\}.$$

For $i \in \mathbb{N}$, $0 \leq i \leq (\lfloor h(n)/h'(n) \rfloor - 1)$, we finally define the sets B_i, P_i and Q_i and the corresponding events A_i, E_i and F_i, and the global events E and F as follows:

$$B_i = B' \cap \left(\mathbb{R}^{d-1} \times]ih'(n), (i+1)h'(n)] \right),$$

$$P_i = (B \smallsetminus B') \cap \left(\mathbb{Z}^{d-1} \times \left\{ \frac{1}{2} + ih'(n) \right\} \right),$$

$$Q_i = \bigcup_{j=1}^{d-1} \left([0, k]^{j-1} \times \left\{ k + \frac{1}{2} \right\} \times [0, k]^{d-1-j} \times]ih'(n), (i+1)h'(n)] \right),$$

$$A_i = \{\phi_{B_i} \leq (v + \eta)k^{d-1}\},$$

$$E_i = \{\text{all the } (n^{d-1} - k^{d-1}) \text{ vertical edges } e \text{ of } B \smallsetminus B' \text{ that intersect } P_i$$
$$\text{satisfy } t(e) \leq \beta + \eta\},$$

$$E = \bigcap_i E_i,$$

$$F_i = \{\text{all the } (d-1)h'(n)k^{d-2} \text{ horizontal edges } e \text{ that intersect } Q_i \text{ satisfy } t(e) \leq \beta + \eta\},$$

$$F = \bigcap_i F_i.$$

Fix $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \quad (d-1)(\beta + \eta)\frac{h'(n)}{n} \leq \frac{\varepsilon}{8} \quad \text{and} \quad \left| \frac{\lfloor vn \rfloor^{d-1}}{n^{d-1}} - v^{d-1} \right| \leq \frac{\varepsilon}{8}.$$

On one hand, on the event A, we have for $n \geq n_0$

$$\begin{aligned}
\phi_B &\geq n^{d-1}(v-\eta)\frac{\lfloor vn \rfloor^{d-1}}{n^{d-1}} + \beta n^{d-1}\left(1 - \frac{\lfloor vn \rfloor^{d-1}}{n^{d-1}}\right) \\
&\geq n^{d-1}\left(vv^{d-1} + \beta(1 - v^{d-1}) - 2\frac{\varepsilon}{8} - \frac{\varepsilon}{4}\right) \\
&> n^{d-1}(\alpha - \varepsilon).
\end{aligned}$$

Here the term $\beta n^{d-1}(1 - \lfloor vn \rfloor^{d-1}/n^{d-1})$ is the minimal amount of fluid that crosses $B \setminus B'$ from its bottom to its top because the capacity of an edge cannot be smaller than β, by definition of β. On the other hand, if for some i in $\{0, \ldots, (\lfloor \frac{h(n)}{h'(n)} \rfloor - 1)\}$ the event $A_i \cap E_i \cap F_i$ occurs then we have

$$\forall n \geq n_0$$

$$\begin{aligned}
\phi_B &\leq n^{d-1}\left((v+\eta)\frac{\lfloor vn \rfloor^{d-1}}{n^{d-1}} + (\beta + \eta)\left(1 - \frac{\lfloor vn \rfloor^{d-1}}{n^{d-1}} + (d-1)\frac{\lfloor vn \rfloor h'(n)}{n^{d-1}}\right)\right) \\
&\leq n^{d-1}\left(\alpha + 2\frac{\varepsilon}{8} + 2\frac{\varepsilon}{4} + \frac{\varepsilon}{8}\right) \\
&< n^{d-1}(\alpha + \varepsilon).
\end{aligned}$$

We obtain then that

$$\forall n \geq n_0 \quad \mathbb{P}\left[\frac{1}{n^{d-1}}\phi_B \in]\alpha - \varepsilon, \alpha + \varepsilon[\right] \geq \mathbb{P}\left[A \cap \left(\bigcup_i A_i \cap E_i \cap F_i\right)\right]$$

$$\geq \mathbb{P}[E] \times \mathbb{P}[F] \times \mathbb{P}\left[A \cap \left(\bigcup_i A_i\right)\right].$$

Now we know that

$$\mathbb{P}[E] = s_\beta(\eta)^{(n^{d-1}-k^{d-1})\lfloor \frac{h(n)}{h'(n)} \rfloor}$$

and

$$\mathbb{P}[F] = s_\beta(\eta)^{(d-1)k^{d-2}\lfloor \frac{h(n)}{h'(n)} \rfloor h'(n)},$$

so

$$\lim_{n \to \infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}[E] = \lim_{n \to \infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}[F] = 0.$$

Moreover we have

$$\mathbb{P}[A \cap (\cup_i A_i)] \geq \mathbb{P}[A] - \mathbb{P}\left[\cap_i A_i^c\right]$$

$$\geq \mathbb{P}[A] - \mathbb{P}[A_0^c]^{\lfloor \frac{h(n)}{h'(n)} \rfloor}$$

$$\geq \mathbb{P}\left[\frac{\phi_{k^{d-1},h(n)}}{k^{d-1}} \geq \nu - \eta \right] - \mathbb{P}\left[\frac{\phi_{k^{d-1},h'(n)}}{k^{d-1}} \geq \nu + \eta \right]^{\lfloor \frac{h(n)}{h'(n)} \rfloor},$$

which leads, thanks to our previous study concerning ψ, to

$$\lim_{n \to \infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}[A \cap (\cup_i A_i)] = 0.$$

We conclude that

$$\lim_{n \to \infty} \frac{1}{n^{d-1}h(n)} \ln \mathbb{P}\left[\frac{\phi_{n^{d-1},h(n)}}{n^{d-1}} \in]\alpha - \varepsilon, \alpha + \varepsilon[\right] \geq 0 = -\psi(\alpha).$$

This ends the proof of the lower bound.

Acknowledgments

References

[1] M.A. Ackoglu, U. Krengel, Ergodic theorems for superadditive processes, Journal für die Reine und Angewandte Mathematik 323 (1981) 53–67.

[2] Béla Bollobás, Graph Theory, in: Graduate Texts in Mathematics, vol. 63, Springer-Verlag, New York, 1979, An introductory course.

[3] Raphaël Cerf, The Wulff crystal in Ising and percolation models, in: École d'Été de Probabilités de Saint Flour, in: Lecture Notes in Mathematics, vol. 1878, Springer-Verlag, 2006.

[4] Yunshyong Chow, Yu Zhang, Large deviations in first passage percolation, The Annals of Applied Probability 13 (4) (2003) 1601–1614.

[5] Richard Durrett, Probability: Theory and Examples, second edn, Duxbury Press, Belmont, CA, 1996.

[6] Harry Kesten, Aspects of first passage percolation, in: École d'Été de Probabilités de Saint Flour XIV, in: Lecture Notes in Mathematics, vol. 1180, Springer-Verlag, 1984.

[7] Harry Kesten, Surfaces with minimal weights and maximal flows: A higher dimensional version of first-passage percolation, Illinois Journal of Mathematics 31 (1) (1987) 99–166.