Alperin–McKay Implies Brauer’s Problem 21

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There are several outstanding open problems in modular representation theory, including several on Brauer’s famous list [B]. One such conjecture is the Alperin–McKay conjecture (e.g., [A1], [F]): Whenever B is a p-block of a finite group G and B has defect group D, then the number $k_0(B)$ of irreducible characters $\chi$ in B with $\left[G : D\right]_p \| \chi(1)$ is the same as the corresponding number $k_0(b)$ for the unique p-block b of $N_G(D)$ with $b^G = B$. Another is R. Brauer’s Problem 21 (see [B], [F]): Let B be a p-block with defect group D and containing $k(B) = k$ irreducible characters. Is $|D|$ bounded in terms of $k$?

We prove here that a positive answer to Brauer’s question follows if the Alperin–McKay conjecture is correct, by virtue of Zelmanov’s solution of the restricted Burnside problem (see [Z1], [Z2]), and the positive solution of Brauer’s problem in the case of $p$-solvable G by the first author [K]. Our main result is:

THEOREM. Let B be a p-block with defect group D of a finite group G, b be the p-block of $N_G(D)$ with $b^G = B$. If $k_0(b) = k_0(B)$, then $|D|$ is bounded in terms of $k(B)$ alone. In particular, if the Alperin–McKay conjecture is true, then Brauer’s Problem 21 has a positive answer.

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Proof. By Corollary 7 of [Ro] (for example), there is a unique $p$-block $\overline{b}$ of $N_G(D) = N_G(D)/\Phi(D)$ with defect group $D/\Phi(D)$ whose irreducible characters “belong” to $b$. Evidently, $k(\overline{b}) = k_0(\overline{b}) \leq k_0(b) = k_0(B) \leq k(B)$. We may, and do, suppose that $|D| > 1$.

By results of W. F. Reynolds [Re], there is a $p$-closed group $H$ and a $p$-block $\overline{b}$ of $H$ with $k(\overline{b}) = k(b)$, and having defect group $(e \text{ Syl}_p(H))$ isomorphic to $\overline{D} = D/\Phi(D)$. By [K], $|\overline{D}|$ is bounded in terms of $k(\overline{b})$, so also of $k(B)$. Hence both $p$ and the number of generators of $D$ are bounded in terms of $k = k(b)$. By Zel'manov’s solution of the restricted Burnside problem (see [Z1], [Z2]), it only remains to prove that the exponent of $D$ is bounded in terms of $k$.

By Brauer’s well-known formula, $k(B) = \sum_x \sum_{b} l(b)$, where $x$ runs over elements of $D$ (up to $G$-conjugacy) and $b$ runs through $p$-blocks of $C_G(x)$ with $b^G = B$. For each $x \in D$ there is at least one such $b$, and $l(b) \in \mathbb{N}$ for this $b$. Let $D$ have exponent $p^e$. Then evidently $k(B) \geq e + 1$, so $p^e < p^k$, and since $p$ is already bounded in terms of $k$, we have bounded the exponent of $D$ in terms of $k$, as required to complete the proof.

Remarks. Although the Alperin–McKay conjecture remains open, there has been recent progress in reducing it to a question about simple groups. E. C. Dade has announced (see [D1, D2]) that he can reduce a conjecture which, partly thanks to the work in [KR], may be viewed as a simultaneous generalization of Alperin’s weight conjecture [A2] and the Alperin–McKay conjecture, to a question about simple groups. The equality predicted by the Alperin–McKay conjecture has been shown to hold in several special cases (e.g., [A1], [BM], [MO]).

REFERENCES


