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The Log-effect for 2 by 2 hyperbolic systems

Tamotu Kinoshita [a,*], Michael Reissig [b]

[a] Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan
[b] Faculty for Mathematics and Computer Science, TU Bergakademie Freiberg, Prüferstr. 9, 09596 Freiberg, Germany

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ABSTRACT

In the present paper we are interested to extend the Log-effect from wave equations with time-dependent coefficients to 2 by 2 strictly hyperbolic systems $\partial_t U - A(t)\partial_x U = 0$. Besides the effects of oscillating entries of the matrix $A = A(t)$ and interactions between the entries of A we have to take into consideration the system character itself. We will prove by tools from phase space analysis results about H^∞ well- or ill-posedness. The precise loss of regularity is of interest. Finally, we discuss the cone of dependence property.

1. Introduction

Recent papers have dealt with the Log-effect for strictly hyperbolic wave models with time-dependent oscillating coefficients. Let us explain the results by the aid of two examples. Here and in the following we are interested in the oscillating behavior of coefficients near $t = 0$. For this purpose we may choose in the following the interval $[0, T]$ with a sufficiently small positive T.

Example 1.1. (See [5].) Let us consider the strictly hyperbolic Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \tag{1.1}$$

where the coefficient a satisfies the following conditions:

* Corresponding author.
E-mail address: kinosita@math.tsukuba.ac.jp (T. Kinoshita).

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- $a \in C[0,T] \cap C^2(0,T]$, where

$$\left|a^{(k)}(t)\right| \leqslant M_k \left(\frac{1}{t}\left(\log\frac{1}{t}\right)^{\gamma}\right)^k, \quad \gamma \geqslant 0,\ k = 1,2,\ \text{for all } t \in (0,T]. \tag{1.2}$$

Then the Cauchy problem is H^∞ well-posed (even C^∞ well-posed due to the finite propagation speed) with the (at most) loss of derivatives $\exp(C_1(\log\langle D_x\rangle)^\gamma)$, $C_1 > 0$, that is, the energy inequality

$$\left\|(u_x,u_t)(t,\cdot)\right\|_{H^s} \leqslant C\left\|\exp\bigl(C_1\bigl(\log\langle D_x\rangle\bigr)^\gamma\bigr)(u_x,u_t)(0,\cdot)\right\|_{H^s} \tag{1.3}$$

holds for all $s \in \mathbb{R}$. In the following we use C and C_1 as universal constants.

Example 1.2. (See [7].) Let us consider the strictly hyperbolic Cauchy problem

$$u_{tt} + b(t)u_{xt} - a(t)u_{xx} = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x). \tag{1.4}$$

Then there exist coefficients a and b satisfying

- $a, b \in C[0,T] \cap C^2(0,T]$, where both coefficients fulfill (1.2) with $\gamma > 0$, such that

$$\left\|(u_x,u_t)(t,\cdot)\right\|_{H^s} \leqslant C\left\|\exp\bigl(C_1\bigl(\log\langle D_x\rangle\bigr)^\beta\bigr)(u_x,u_t)(0,\cdot)\right\|_{H^s} \tag{1.5}$$

  holds for all $s \in \mathbb{R}$ with $\beta = \gamma + 1$, but the energy inequality (1.5) is in general not valid for $\beta < \gamma + 1$.

Remark 1.1. Let us compare the statements of both examples. Example 1.1 describes the influence of oscillations. If we choose $\gamma = 0$, $\gamma \in (0,1)$, $\gamma = 1$, $\gamma > 1$ in (1.2), that is, we assume *very slow, slow, fast or very fast oscillations*, respectively, then we obtain in (1.3) *no loss, at most an arbitrary small, a finite or an infinite loss of derivatives*. In Example 1.2 we feel *interactions of oscillations* in a and b. Although the oscillations in a and b are slow, the interaction implies *an infinite loss of derivatives*.

In the present paper we are interested to extend the Log-effect to 2 by 2 *strictly hyperbolic systems*. Besides the effects of *oscillating entries of the matrix* $A = A(t)$ and *interactions between the entries of* A we have to take into consideration *the system character itself*.

Let us consider on $[0,T] \times \mathbb{R}$ the 2 by 2 strictly hyperbolic Cauchy problem

$$\partial_t U - A(t)\partial_x U = 0, \qquad U(0,x) = U_0(x), \quad \text{where } A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}. \tag{1.6}$$

We assume the following conditions:

strict hyperbolicity there exists a positive constant δ such that

$$\text{(A1)} \quad \Delta(t) = \bigl(a(t) - d(t)\bigr)^2 + 4b(t)c(t) \geqslant \delta \quad \text{for } t \in [0,T];$$

regularity we assume

$$\text{(A2)} \quad A \in L^\infty(0,T) \cap C^2(0,T];$$

oscillating behavior we assume with a non-negative constant C the estimate

$$\text{(A3)}\quad \left\|A'(t)-\frac{1}{2}\operatorname{tr}A'(t)I\right\|^2+\left\|A''(t)-\frac{1}{2}\operatorname{tr}A''(t)I\right\|\leqslant C\left(\frac{1}{t}\left(\log\frac{1}{t}\right)^{\gamma}\right)^2,$$
$$\gamma\geqslant 0,\ \text{for } t\in(0,T].$$

To characterize *interactions and the system character* we assume two conditions. To formulate these conditions we introduce the function

$$\psi=\psi(t)=\frac{(c-b-i\sqrt{\Delta})((a-d)(b+c)'-(a-d)'(b+c))}{2\sqrt{\Delta}((b+c)^2+(a-d)^2)}$$
$$\left(\Re\psi=\Re\psi(t)=\frac{(c-b)((a-d)(b+c)'-(a-d)'(b+c))}{2\sqrt{\Delta}((b+c)^2+(a-d)^2)}\right). \tag{1.7}$$

Then we suppose with non-negative constants C

$$\text{(A4)}\quad \left|\int_t^T \Re\psi(s)\,ds\right|\leqslant C\left(\log\frac{1}{t}\right)^{\alpha},\quad \alpha\geqslant 0,\ \text{for } t\in(0,T];$$

$$\text{(A5)}\quad \int_t^T \left|\Re\psi(s)\right|ds\leqslant C\left(\log\frac{1}{t}\right)^{\beta},\quad \beta\geqslant 0,\ \text{for } t\in(0,T].$$

Remark 1.2. It is clear that $\alpha\leqslant\beta$. In the following we will present concrete examples. Example 1.6 shows that in some sense both conditions (A4) and (A5) are independent. Both have their meaning in explaining the structure of hyperbolic systems and interactions between the entries of A.

Example 1.3. Let us come back to Example 1.1. After transformation to a system of first order (1.6) one gets $\Re\psi\equiv 0$. Consequently, the conditions (A4) and (A5) are satisfied with $\alpha=\beta=0$ and $C=0$. The condition (A3) is satisfied with $\gamma\geqslant 0$.

Example 1.4. Let us come back to Example 1.2. After transformation to a system of first order (1.6) one can show that both conditions $(A4)$ and $(A5)$ are satisfied with $\alpha=\beta=\gamma+1$. The condition (A3) is satisfied with $\gamma>0$.

Example 1.5. Let us choose the following entries in $A(t)$:

$$a(t)=-\cos\bigl(\cos\omega(t)\bigr),\qquad d(t)=\cos\bigl(\cos\omega(t)\bigr),$$
$$b(t)=\sin\bigl(\cos\omega(t)\bigr)-\frac{1}{\sqrt{1+(2+\sin\omega(t))^2}},\qquad c(t)=\sin\bigl(\cos\omega(t)\bigr)+\frac{1}{\sqrt{1+(2+\sin\omega(t))^2}}.$$

Here we take $\omega(t)=(\log\frac{1}{t})^p$ with $p\geqslant 1$. The condition (A3) is satisfied with $\gamma=p-1$. Simple calculations yield

$$a(t)-d(t)=-2\cos\bigl(\cos\omega(t)\bigr),\qquad b(t)-c(t)=-\frac{2}{\sqrt{1+(2+\sin\omega(t))^2}},$$
$$b(t)+c(t)=2\sin\bigl(\cos\omega(t)\bigr),\qquad \Delta(t)=4\frac{(2+\sin\omega(t))^2}{1+(2+\sin\omega(t))^2}.$$

Hence, we get $\Re\psi(t) = -\frac{(\cos\omega(t))'}{2(2+\sin\omega(t))}$. Thus we have to consider

$$\left|\int_t^T -\frac{(\cos\omega(s))'}{2(2+\sin\omega(s))}\,ds\right| = \left|\int_t^T \frac{\omega'(s)}{2}\frac{\sin\omega(s)}{2+\sin\omega(s)}\,ds\right|.$$

Using the strict monotonicity of $\omega(s)$, the last term can be reduced to

$$\left|\int_{\omega(T)}^{\omega(t)} \frac{\sin x}{2(2+\sin x)}\,dx\right|.$$

Taking account of

$$\int_{-\pi}^{\pi} \frac{-\sin x}{2+\sin x}\,dx = \frac{4\sqrt{3}-6}{3}\pi \quad \text{and} \quad \omega(t) = \left(\log\frac{1}{t}\right)^p$$

gives us the estimate to below

$$\left|\int_{\omega(T)}^{\omega(t)} \frac{\sin x}{2+\sin x}\,dx\right| \geqslant C\left(\log\frac{1}{t}\right)^p.$$

We conclude immediately

$$\int_t^T \left|\Re\psi(s)\right| ds \leqslant C\left(\log\frac{1}{t}\right)^p.$$

Resume of this example: We understand from this example that both conditions (A4) and (A5) may have the same priority. The condition (A3) is satisfied for $\gamma = p-1$. The conditions (A1) and (A2) are satisfied, too.

Example 1.6. Let us choose the following entries in $A(t)$:

$$a(t) = -\cos\bigl(\omega(t)\bigr), \qquad d(t) = \cos\bigl(\omega(t)\bigr), \qquad b(t) = \sin\bigl(\omega(t)\bigr) + \frac{1}{\sqrt{2}}, \qquad c(t) = \sin\bigl(\omega(t)\bigr) - \frac{1}{\sqrt{2}}.$$

Here we choose $\omega(t) = (\log\frac{1}{t})^r(2-\cos((\log\frac{1}{t})^p))$ with $p>0$, $r \geqslant 0$ and $p+r \geqslant 1$. The condition (A3) is satisfied for $\gamma = p+r-1$. Simple calculations yield

$$a(t)-d(t) = -2\cos\bigl(\omega(t)\bigr), \qquad b(t)-c(t) = \sqrt{2}, \qquad b(t)+c(t) = 2\sin\bigl(\omega(t)\bigr), \qquad \Delta(t) = 2.$$

We have $\Re\psi(t) = \frac{\omega'(t)}{2}$. Thus we have to consider

$$\left|\int_t^T \Re\psi(s)\,ds\right| = \left|\int_t^T \frac{\omega'(s)}{2}\,ds\right| = \frac{1}{2}\bigl|\omega(T)-\omega(t)\bigr| \leqslant C\left(\log\frac{1}{t}\right)^r.$$

Now let us devote to estimate

$$\int_t^T |\Re\psi(s)|\,ds = \int_t^T \frac{|\omega'(s)|}{2}\,ds$$
$$= \int_t^T \frac{1}{2s}\left| r\left(\log\frac{1}{s}\right)^{r-1}\left(2-\cos\left(\log\frac{1}{s}\right)^p\right) + p\left(\log\frac{1}{s}\right)^{p+r-1}\sin\left(\log\frac{1}{s}\right)^p\right| ds.$$

For T small (this is sufficient) we define t_{2k}, t_{2k-1} by

$$\left(\log\frac{1}{t_{2k}}\right)^p = k\pi + \frac{3\pi}{4}, \qquad \left(\log\frac{1}{t_{2k-1}}\right)^p = k\pi + \frac{\pi}{4}.$$

It is clear that on $[t_{2k}, t_{2k-1}]$ the term with sin is the dominant one in the above sum. Thus we can estimate

$$\int_t^T |\Re\psi(s)|\,ds \geqslant C\sum_{k=1}^{N(t)} \int_{t_{2k}}^{t_{2k-1}} \frac{1}{s}\left(\log\frac{1}{s}\right)^{p+r-1} ds$$
$$= C\sum_{k=1}^{N(t)}\left(\left(k\pi+\frac{3\pi}{4}\right)^{1+\frac{r}{p}} - \left(k\pi+\frac{\pi}{4}\right)^{1+\frac{r}{p}}\right) \geqslant C\sum_{k=1}^{N(t)} k^{\frac{r}{p}} \geqslant CN(t)^{1+\frac{r}{p}}.$$

Using $N(t) \sim (\log\frac{1}{t})^p$ we obtain the lower bound $C(\log\frac{1}{t})^{p+r}$ for $\int_t^T |\Re\psi(s)|\,ds$. The same we conclude for the upper bound.

Resume of this example: We understand from this example that both conditions (A4) and (A5) may have different priorities. On the one hand (A4) is satisfied for $\alpha = r \geqslant 0$, on the other hand (A5) is satisfied with $\beta = p + r$, $p > 0$. The assumption (A3) is satisfied for $\gamma = p + r - 1$. The conditions (A1) and (A2) are satisfied, too.

Remark 1.3. The term

$$\Re\psi = \frac{(c-b)((a-d)(b+c)' - (a-d)'(b+c))}{2\sqrt{\Delta}((b+c)^2 + (a-d)^2)}$$

is regarded as a very important term in the study of *weakly* hyperbolic systems (see [1,2,11] and [13]). This paper explains its importance also in study of the Log-effect for *strictly* hyperbolic systems. It seems to be interesting that our considerations for the *strictly* hyperbolic case derive the term $\frac{(c-b)((a-d)(b+c)'-(a-d)'(b+c))}{2\sqrt{\Delta}((b+c)^2+(a-d)^2)}$ in a different way than in the *weakly* hyperbolic case.

The content of this paper is organized as follows:

In Section 2 we present the main results and apply them to the above Examples 1.5 and 1.6. In Section 3 we prove H^∞ well-posedness with an (at most) arbitrary small or finite loss of derivatives. Section 4 is devoted to the question if we have at least such a loss, in other words, if the loss of derivatives really appears. The example of Section 5 explains the complexity of hyperbolic systems and the difficulty to get general results for H^∞ well-posedness or ill-posedness. In Section 6 we discuss the question for the finite propagation speed of perturbations.

2. The main results and examples

We will prove the following results for

$$\partial_t U - A(t)\partial_x U = 0, \qquad U(0,x) = U_0(x). \tag{2.1}$$

Theorem 2.1. *Let us assume that $A = A(t)$ satisfies the assumptions* (A1), (A2), (A3) *with $\gamma \in [0,1]$ and* (A5) *with $\beta \in [0,1]$. Then the Cauchy problem is H^∞ well-posed with an* (*at most*) *loss of derivatives* $\exp(C_1(\log\langle D_x\rangle)^\kappa)$, *$\kappa = \max\{\gamma,\beta\}$. The following energy inequality holds*:

$$\big\|U(t,\cdot)\big\|_{H^s} \leqslant C\big\|\exp\big(C_1\big(\log\langle D_x\rangle\big)^\kappa\big)U(0,\cdot)\big\|_{H^s} \tag{2.2}$$

for all $s \in \mathbb{R}$.

Theorem 2.2. *Let us assume that $A = A(t)$ satisfies the assumptions* (A1), (A2), (A3) *with $\gamma \in [0,1]$ and* (A4) *with $\alpha = 0$. Then the Cauchy problem is H^∞ well-posed with an* (*at most*) *loss of derivatives* $\exp(C_1(\log\langle D_x\rangle)^\gamma)$. *The following energy inequality holds*:

$$\big\|U(t,\cdot)\big\|_{H^s} \leqslant C\big\|\exp\big(C_1\big(\log\langle D_x\rangle\big)^\gamma\big)U(0,\cdot)\big\|_{H^s} \tag{2.3}$$

for all $s \in \mathbb{R}$.

Theorem 2.3. *Let us assume that $A = A(t)$ satisfies the assumptions* (A1), (A2), (A3) *with $\gamma = 0$ and* (A4) *both sided with $\alpha \in (0,1]$, that is,*

$$C^{-1}\left(\log\frac{1}{t}\right)^\alpha \leqslant \left|\int_t^T \Re\psi(s)\,ds\right| \leqslant C\left(\log\frac{1}{t}\right)^\alpha \tag{2.4}$$

with a positive constant C. Moreover, we assume that there exists a function $\theta = \theta(t,\xi)$ satisfying

$$\Re\psi + \theta(t,\xi) \geqslant 0 \quad \text{for all } (t,\xi) \in (0,T] \times \big\{|\xi| \geqslant M\big\}, \tag{2.5}$$

and there exists a positive zero sequence $\{t_k\}_{k\geqslant 1}$ such that

$$\int_{t_{2k+1}}^{t_{2k-1}} \theta(s,\xi)\,ds = 0 \quad \text{for all } |\xi| \geqslant M, \tag{2.6}$$

$$\left|\int_t^{t_{2k-1}} \theta(s,\xi)\,ds\right| \leqslant C \quad \text{for all } t \in [t_{2k+1}, t_{2k-1}],\ t \geqslant \frac{N}{|\xi|}. \tag{2.7}$$

Here M and N are large constants. Under these conditions the Cauchy problem is H^∞ well-posed with an (*at least*) *loss of derivatives* $\exp(C_1(\log\langle D_x\rangle)^\alpha)$.

Remark 2.1. It is known that a finite loss of derivatives really appears for (1.1) if the coefficient $a = a(t)$ satisfies the log-Lipschitz condition or (1.2) with $\gamma = 1$ (see [3,6] and [10]). While a finite loss of derivatives in Theorem 2.3 is caused from the structure of systems. Actually, we can deal with coefficients consisting of $t^2\exp(t^{-2})\sin(\exp(t^{-2}))$ which satisfy a slightly better condition than log-Lipschitz and (1.2) with $\gamma = 0$.

Remark 2.2. The condition (A3) with $\gamma = 0$ implies the condition (A5) with $\beta = 1$. Thus we have from Theorem 2.1 *at most* a finite loss of derivatives. Theorem 2.3 explains conditions under which we have *at least* a loss of derivatives $\exp(C_1(\log\langle D_x\rangle)^\alpha)$.

Remark 2.3. The function θ from Theorem 2.3 is an important auxiliary function. It is used in the phase space analysis only in the interacting subzone. The main goal of this auxiliary function is to control the oscillating behavior of $\Re\psi$, to make $\Re\psi$ non-negative, that is to guarantee the inequality (2.5) without having an essential influence in the Riemann integral over time intervals $[t, T]$ (see assumptions (2.6) and (2.7)). At the beginning of the proof to Theorem 2.4 we explain one way how to find for Example 1.5 the function θ. Setting $p = 1$ in Example 1.5 this construction can be used as one example satisfying all assumptions from Theorem 2.3.

Theorem 2.4. *There exists a matrix $A = A(t)$ satisfying the assumptions* (A1), (A2) *and* (A3) *with $\gamma \in (0, 1)$ such that the Cauchy problem is H^∞ ill-posed.*

Remark 2.4. From Example 1.5 we conclude, that $\alpha > 0$ in (A4) cannot be used to prove a general H^∞ ill-posedness result. Example 1.6 tells us, that $\beta > 1$ in (A5) cannot be used to prove a general H^∞ ill-posedness result because $\alpha = p = 0$ can be chosen in (A4). The following theorem shows that there exist examples with $\gamma \in (0, 1)$ in (A3), with $\alpha \in (0, 1)$ in (A4) and with $\beta > 1$ in (A5) such that the Cauchy problem is H^∞ well-posed.

Theorem 2.5. *There exists a matrix $A = A(t)$ satisfying the assumptions* (A1), (A2), (A3) *with $\gamma \in (0, 1)$,* (A4) *with $\alpha < 1$ and* (A5) *with $\beta > 1$ such that the Cauchy problem is H^∞ well-posed.*

Remark 2.5. An assumption as (A4) was used in [8] in connection with the influence of oscillations on Levi conditions. Following the proposed strategy from [8] it would be interesting to understand the interplay between the assumptions (A4) with $\alpha < 1$ and (A5) with $\beta > 1$.

Remark 2.6. In the formulation of our main results we restricted to H^∞ well- or ill-posedness. In Section 6 we will study the property of finite propagation speed for systems (1.6). As a consequence we obtain even C^∞ well-posedness.

3. Proof of H^∞ well-posedness

3.1. Proof to Theorem 2.1

After partial Fourier transformation the Cauchy problem (2.1) is transferred to

$$\partial_t \hat{U} = A(t) i\xi \hat{U}, \qquad \hat{U}(0, \xi) = \hat{U}_0(\xi).$$

We divide the extended phase space $[0, T] \times \{|\xi| \geqslant M\}$ into the pseudo-differential zone $Z_{pd}(N, M)$ and the hyperbolic zone $Z_{hyp}(N, M)$. Both zones are defined by

$$Z_{pd}(N, M) := \bigl\{(t, \xi) \in [0, T] \times \bigl\{|\xi| \geqslant M\bigr\}:\ t|\xi| \leqslant N\bigl(\log|\xi|\bigr)^\kappa\bigr\},$$
$$Z_{hyp}(N, M) := \bigl\{(t, \xi) \in [0, T] \times \bigl\{|\xi| \geqslant M\bigr\}:\ t|\xi| \geqslant N\bigl(\log|\xi|\bigr)^\kappa\bigr\},$$

where $\kappa = \max\{\gamma, \beta\}$. The separating line $t_\xi = t(|\xi|)$ between both zones is defined by $t_\xi|\xi| = N(\log|\xi|)^\kappa$.

Considerations in $Z_{pd}(N,M)$. Here we define the micro-energy $E(t,\xi) := |\hat{U}(t,\xi)|^2$. By application of assumption (A2) we obtain after differentiation

$$E'(t,\xi) = 2\Re(\partial_t \hat{U}, \hat{U}) \leqslant C|\xi|E(t,\xi).$$

The Gronwall's inequality yields

$$\begin{cases} E(t,\xi) \leqslant E(0,\xi)\exp\big(C|\xi|t_\xi\big) = E(0,\xi)\exp\big(C_N\big(\log|\xi|\big)^\kappa\big), \\ \big|\hat{U}(t,\xi)\big| \leqslant \big|\hat{U}_0(\xi)\big|\exp\big(C_N\big(\log|\xi|\big)^\kappa\big) \quad \text{for all } (t,\xi) \in Z_{pd}(N,M), \end{cases} \tag{3.1}$$

respectively. Here and in the following C and C_N are used as universal constants.

Diagonalization procedure in $Z_{hyp}(N,M)$. Let us introduce the notations $\mu_\pm(t) := \frac{a(t)-d(t)\pm\sqrt{\Delta(t)}}{2}$. Then we define the first (non-singular) globally invertible diagonalizer $H = H(t)$ by

$$H(t) = (1+i)\begin{pmatrix} b(t) & \mu_-(t) \\ -\mu_-(t) & c(t) \end{pmatrix} + (1-i)\begin{pmatrix} \mu_+(t) & b(t) \\ c(t) & -\mu_+(t) \end{pmatrix}.$$

It holds

$$\det H(t) = 2\sqrt{\Delta(t)}\big(c(t) - b(t) + i\sqrt{\Delta(t)}\big).$$

By assumptions (A1) and (A2) both matrices H and H^{-1} belong to $L^\infty(0,T) \cap C^2(0,T]$. Thus, putting $\hat{U}(t,\xi) = H(t)V(t,\xi)$ we have

$$\partial_t V = H^{-1}(t)A(t)H(t)i\xi V - H^{-1}(t)H'(t)V, \qquad V(0,\xi) = H^{-1}(0)\hat{U}_0(\xi).$$

Since H is a diagonalizer and the eigenvalues of A are $\mu_\pm + d$ the above system simplifies to

$$\partial_t V - \begin{pmatrix} \mu_+ + d & 0 \\ 0 & \mu_- + d \end{pmatrix} i\xi V + H^{-1}(t)H'(t)V = 0.$$

We obtain for the entries $h_{lm} = h_{lm}(t)$, $1 \leqslant l,m \leqslant 2$, of the matrix $H^{-1}H'$ the following representations:

$$\begin{aligned} h_{11} &= \frac{(\det H)'}{2\det H} + \frac{(a-d)'(b+c) - (a-d)(b+c)'}{\det H}, \\ h_{22} &= \frac{(\det H)'}{2\det H} + \frac{(a-d)(b+c)' - (a-d)'(b+c)}{\det H}, \\ h_{21} &= \frac{\det H}{2\Delta}\left(\frac{\sqrt{\Delta}(b+c+i(d-a))}{\det H}\right)', \\ h_{12} &= \frac{\det H}{2\Delta}\left(\frac{\sqrt{\Delta}(i(d-a)-(b+c))}{\det H}\right)'. \end{aligned}$$

Now we are able to carry out the second step of diagonalization (but only in $Z_{hyp}(N,M)$) for the system

$$\partial_t V - \begin{pmatrix} \mu_+ + d & 0 \\ 0 & \mu_- + d \end{pmatrix} i\xi V + \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} V = 0.$$

Let us define the diagonalizer $K = K(t,\xi)$ by

$$K(t,\xi) := \begin{pmatrix} 1 & \frac{h_{12}(t)}{i\xi\sqrt{\Delta(t)}} \\ -\frac{h_{21}(t)}{i\xi\sqrt{\Delta(t)}} & 1 \end{pmatrix}.$$

The entries h_{12} and h_{21} depend on $a-d, b, c$ and their first derivatives. Due to assumption (A3) we may estimate

$$\left|\frac{h_{12}(t)}{\xi\sqrt{\Delta(t)}}\right| + \left|\frac{h_{21}(t)}{\xi\sqrt{\Delta(t)}}\right| \leqslant C\frac{1}{t|\xi|}\left(\log\frac{1}{t}\right)^{\gamma} \leqslant C\frac{1}{t_\xi|\xi|}\left(\log\frac{1}{t_\xi}\right)^{\gamma}$$
$$\leqslant \frac{C}{N}\left(\log|\xi|\right)^{\gamma-\kappa} \leqslant \frac{C}{N} \quad \text{for all } t\in[t_\xi,T],\ |\xi|\geqslant M.$$

A large N guarantees the invertibility of K in $Z_{hyp}(N,M)$. Setting $V(t,\xi) = K(t,\xi)W(t,\xi)$ in $Z_{hyp}(N,M)$ we arrive at the system

$$\partial_t W - K^{-1}\begin{pmatrix} \mu_+ + d & 0 \\ 0 & \mu_- + d \end{pmatrix} K i\xi W + K^{-1}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} KW + K^{-1}K'W = 0,$$
$$W(t_\xi,\xi) = K^{-1}(t_\xi,\xi)H^{-1}(t_\xi)\hat{U}(t_\xi,\xi).$$

Direct computations show that after two steps of diagonalization we deduce the system

$$\partial_t W - \begin{pmatrix} \mu_+ + d & 0 \\ 0 & \mu_- + d \end{pmatrix} i\xi W + \begin{pmatrix} h_{11} & 0 \\ 0 & h_{22} \end{pmatrix} W + J(t,\xi)W = 0,$$
$$W(t_\xi,\xi) = K^{-1}(t_\xi,\xi)H^{-1}(t_\xi)\hat{U}(t_\xi,\xi),$$

where the matrix $J = J(t,\xi)$ is equal to

$$J = \frac{1}{\det K}\begin{pmatrix} \frac{h_{12}h_{21}(h_{11}-h_{22})}{\xi^2\Delta} - \frac{2h_{12}h_{21}}{i\xi\sqrt{\Delta}} - \frac{h_{12}}{\xi^2\sqrt{\Delta}}(\frac{h_{21}}{\sqrt{\Delta}})' & -\frac{h_{12}(h_{22}-h_{11})}{i\xi\sqrt{\Delta}} + \frac{h_{12}^2h_{21}}{\xi^2\Delta} + \frac{1}{i\xi}(\frac{h_{12}}{\sqrt{\Delta}})' \\ -\frac{h_{21}(h_{22}-h_{11})}{i\xi\sqrt{\Delta}} + \frac{h_{12}h_{21}^2}{\xi^2\Delta} - \frac{1}{i\xi}(\frac{h_{21}}{\sqrt{\Delta}})' & \frac{h_{12}h_{21}(h_{22}-h_{11})}{\xi^2\Delta} + \frac{2h_{12}h_{21}}{i\xi\sqrt{\Delta}} - \frac{h_{21}}{\xi^2\sqrt{\Delta}}(\frac{h_{12}}{\sqrt{\Delta}})' \end{pmatrix}.$$

Denoting the entries of J by $J_{lm} = J_{lm}(t,\xi)$, $1 \leqslant l, m \leqslant 2$, they depend on $a-d, b$ and c, and due to assumption (A3) we obtain the estimate

$$\|J(t,\xi)\| \leqslant C\frac{1}{|\xi|t^2}\left(\log\frac{1}{t}\right)^{2\gamma} \quad \text{for } (t,\xi)\in Z_{hyp}(N,M).$$

Energy estimate in $Z_{hyp}(N,M)$. Let us recall the structure of h_{11} and h_{22}. Both consist of the term $\frac{(\det H)'}{2\det H}$, and the term

$$\psi = \psi(t) = \frac{(a-d)(b+c)' - (a-d)'(b+c)}{\det H} \quad \text{which coincides with (1.7).}$$

The influence of the imaginary part $\Im\psi$ is not important, but we are forced to control the real part $\Re\psi$. Introducing $W = W(t,\xi) = (\det H(t))^{-\frac{1}{2}} Z(t,\xi)$ the above Cauchy problem for W is transferred to

$$\partial_t Z - \begin{pmatrix} (\mu_+ + d)\xi + \Im\psi & 0 \\ 0 & (\mu_- + d)\xi - \Im\psi \end{pmatrix} iZ - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\Re\psi) Z + JZ = 0,$$
$$Z(t_\xi, \xi) = \sqrt{\det H(t_\xi)} K^{-1}(t_\xi, \xi) H^{-1}(t_\xi) \hat{U}(t_\xi, \xi).$$

In $Z_{hyp}(N, M)$ we define the micro-energy $E(t, \xi) := |Z(t, \xi)|^2$.

After differentiation with respect to t we conclude from the last Cauchy problem

$$E'(t, \xi) = 2\Re(\partial_t Z, Z) \leqslant C\left(\left|\Re\psi(t)\right| + \frac{1}{|\xi| t^2} \left(\log \frac{1}{t} \right)^{2\gamma} \right) E(t, \xi).$$

Now we control the influence of $|\Re\psi|$ by condition (A5). Together with (A3) and the definition of t_ξ it follows

$$\begin{aligned} E(t, \xi) &\leqslant E(t_\xi, \xi) \exp\left(C \int_{t_\xi}^{t} \left|\Re\psi(s)\right| ds + C \int_{t_\xi}^{t} \frac{1}{|\xi| s^2} \left(\log \frac{1}{s} \right)^{2\gamma} ds \right) \\ &\leqslant E(t_\xi, \xi) \exp\left(C \left(\log \frac{1}{t_\xi} \right)^{\beta} + C \left(\log \frac{1}{t_\xi} \right)^{\gamma} \right) \\ &\leqslant E(t_\xi, \xi) \exp\left(C_1 \left(\log |\xi| \right)^{\kappa} \right) \quad \text{for all } (t, \xi) \in Z_{hyp}(N, M). \end{aligned}$$

The backward transformation yields immediately

$$\left|\hat{U}(t, \xi)\right| \leqslant C \left|\hat{U}(t_\xi, \xi)\right| \exp\left(C_1 \left(\log |\xi| \right)^{\kappa} \right) \quad \text{for all } (t, \xi) \in Z_{hyp}(N, M). \tag{3.2}$$

Conclusion. From (3.1) and (3.2) we conclude

$$\left|\hat{U}(t, \xi)\right| \leqslant C \left|\hat{U}(0, \xi)\right| \exp\left(C_1 \left(\log |\xi| \right)^{\kappa} \right) \quad \text{for all } t \in [0, T], \ |\xi| \geqslant M.$$

This a priori estimate implies the statement of Theorem 2.1.

3.2. Proof to Theorem 2.2

The proof to Theorem 2.2 is only a slight modification of that one to Theorem 2.1. Both proofs coincide up to the second step of diagonalization (defining $t_\xi = t(|\xi|)$, $t_\xi |\xi| = N(\log |\xi|)^\gamma$)

$$\partial_t W - \begin{pmatrix} \mu_+ + d & 0 \\ 0 & \mu_- + d \end{pmatrix} i\xi W + \begin{pmatrix} h_{11} & 0 \\ 0 & h_{22} \end{pmatrix} W + J(t, \xi) W = 0,$$
$$W(t_\xi, \xi) = K^{-1}(t_\xi, \xi) H^{-1}(t_\xi) \hat{U}(t_\xi, \xi).$$

To derive the energy estimate in $Z_{hyp}(N, M)$ we introduce the transformation

$$\begin{aligned} W = W(t, \xi) &= \left(\det H(t) \right)^{-\frac{1}{2}} \begin{pmatrix} \exp(\int_{t_\xi}^{t} \Re\psi(s)\, ds) & 0 \\ 0 & \exp(-\int_{t_\xi}^{t} \Re\psi(s)\, ds) \end{pmatrix} Z(t, \xi) \\ &=: D(t, \xi) Z(t, \xi). \end{aligned}$$

Then we obtain the following Cauchy problem

$$\partial_t Z - \begin{pmatrix} (\mu_+ + d)\xi + \Im\psi & 0 \\ 0 & (\mu_- + d)\xi - \Im\psi \end{pmatrix} iZ + \tilde{J}Z = 0,$$
$$Z(t_\xi, \xi) = \sqrt{\det H(t_\xi)} K^{-1}(t_\xi, \xi) H^{-1}(t_\xi) \hat{U}(t_\xi, \xi).$$

Due to assumption (A4) with $\alpha = 0$ we conclude that the entries $\tilde{J}_{lm} = \tilde{J}_{lm}(t, \xi)$, $1 \leqslant l, m \leqslant 2$, of the matrix $\tilde{J} := D^{-1} J D$ fulfill the same estimates as those for $J = J(t, \xi)$, that is, the matrix $\tilde{J}$ satisfies

$$\|\tilde{J}(t, \xi)\| \leqslant C \frac{1}{|\xi| t^2} \left(\log \frac{1}{t}\right)^{2\gamma} \quad \text{for } (t, \xi) \in Z_{hyp}(N, M).$$

Defining the energy $E(t, \xi) := |Z(t, \xi)|^2$ we arrive as in the proof to Theorem 2.1 at

$$E(t, \xi) \leqslant E(t_\xi, \xi) \exp\left(C \int_{t_\xi}^{t} \frac{1}{|\xi| s^2} \left(\log \frac{1}{s}\right)^{2\gamma} ds\right) \leqslant E(t_\xi, \xi) \exp\left(C \left(\log \frac{1}{t_\xi}\right)^{\gamma}\right)$$
$$\leqslant E(t_\xi, \xi) \exp\left(C_1 \left(\log|\xi|\right)^{\gamma}\right) \quad \text{for all } (t, \xi) \in Z_{hyp}(N, M).$$

This implies (3.2), and together with (3.1) for $\kappa = \gamma$ it gives

$$\left|\hat{U}(t, \xi)\right| \leqslant C \left|\hat{U}(0, \xi)\right| \exp\left(C_1 \left(\log|\xi|\right)^{\gamma}\right) \quad \text{for all } t \in [0, T],\ |\xi| \geqslant M,$$

the estimate (2.3) of Theorem 2.2, respectively.

4. Does the loss of regularity really appear?

With Theorem 2.3 and its proof we present a general approach how to show for the Cauchy problem (2.1) which is H^∞ well-posed that a *loss of regularity really appears*. This loss is coming from interactions of oscillations.

4.1. The case of H^∞ well-posedness

4.1.1. Proof to Theorem 2.3

We divide the extended phase space $[0, T] \times \{|\xi| \geqslant M\}$ into the pseudo-differential zone and the hyperbolic zone. Both zones are defined by

$$Z_{pd}(N, M) := \left\{(t, \xi) \in [0, T] \times \left\{|\xi| \geqslant M\right\}:\ t|\xi| \leqslant N\right\},$$
$$Z_{hyp}(N, M) := \left\{(t, \xi) \in [0, T] \times \left\{|\xi| \geqslant M\right\}:\ t|\xi| \geqslant N\right\}.$$

The separating line $t_\xi = t(|\xi|)$ between both zones is defined by $t_\xi = N|\xi|^{-1}$.

Considerations in the pseudo-differential zone. As in the proof to Theorem 2.1 we obtain

$$\left|\hat{U}(t, \xi)\right| \leqslant C_N \left|\hat{U}_0(\xi)\right| \quad \text{for all } (t, \xi) \in Z_{pd}(N, M). \tag{4.1}$$

There is no loss of derivatives coming from this zone.

Diagonalization procedure and an auxiliary transformation in $Z_{hyp}(N,M)$. As in the proof to Theorem 2.1 we have

$$\partial_t Z - \begin{pmatrix} (\mu_+ + d)\xi + \Im\psi & 0 \\ 0 & (\mu_- + d)\xi - \Im\psi \end{pmatrix} iZ - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\Re\psi) Z + JZ = 0,$$
$$Z(t_\xi, \xi) = \sqrt{\det H(t_\xi)} K^{-1}(t_\xi, \xi) H^{-1}(t_\xi) \hat{U}(t_\xi, \xi).$$

Introducing in $Z_{hyp}(N,M)$

$$Z(t,\xi) =: \begin{pmatrix} \exp(-\int_{t_\xi}^t \theta(s,\xi)\,ds - M_3 \int_{t_\xi}^t \frac{1}{|\xi| s^2}\,ds) & 0 \\ 0 & \exp(\int_{t_\xi}^t \theta(s,\xi)\,ds + M_3 \int_{t_\xi}^t \frac{1}{|\xi| s^2}\,ds) \end{pmatrix} Y(t,\xi),$$

where the large constant M_3 will be chosen later, we get the Cauchy problem

$$\partial_t Y - \begin{pmatrix} (\mu_+ + d)\xi + \Im\psi & 0 \\ 0 & (\mu_- + d)\xi - \Im\psi \end{pmatrix} iY - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\Re\psi + \theta + \frac{M_3}{|\xi| t^2}\right) Y + \tilde{J} Y = 0,$$
$$Y(t_\xi, \xi) = Z(t_\xi, \xi).$$

Here we have to remark that due to the definition of $Z_{hyp}(N,M)$, (A3) with $\gamma = 0$ and assumptions (2.6) and (2.7) the following estimates hold:

$$C^{-1} \big|Z(t,\xi)\big| \leqslant \big|Y(t,\xi)\big| \leqslant C \big|Z(t,\xi)\big|, \qquad \big\|\tilde{J}(t,\xi)\big\| \leqslant M_4 \frac{1}{|\xi| t^2} \quad \text{for all } (t,\xi) \in Z_{hyp}(N,M).$$

Lyapunov functional versus energy functional. We define in $Z_{hyp}(N,M)$ the Lyapunov functional $\tilde{E} = \tilde{E}(t,\xi)$ and the energy functional $E = E(t,\xi)$ by

$$\tilde{E}(t,\xi) = \big|y_1(t,\xi)\big|^2 - \big|y_2(t,\xi)\big|^2, \qquad E(t,\xi) = \big|y_1(t,\xi)\big|^2 + \big|y_2(t,\xi)\big|^2, \qquad Y = (y_1, y_2)^T.$$

Differentiation of $\tilde{E}$ with respect to t gives

$$\begin{aligned} \partial_t \tilde{E}(t,\xi) &= 2\Re(\partial_t y_1, y_1) - 2\Re(\partial_t y_2, y_2) \\ &= 2\Re\left(\big((\mu_+ + d)\xi + \Im\psi\big) i y_1 + \left(\Re\psi + \theta + \frac{M_3}{|\xi| t^2}\right) y_1 - \tilde{J}_{11} y_1 - \tilde{J}_{12} y_2, y_1 \right) \\ &\quad - 2\Re\left(\big((\mu_+ + d)\xi - \Im\psi\big) i y_2 - \left(\Re\psi + \theta + \frac{M_3}{|\xi| t^2}\right) y_2 - \tilde{J}_{21} y_1 - \tilde{J}_{22} y_2, y_2 \right) \\ &\geqslant 2\left(\Re\psi + \theta + \frac{M_3}{|\xi| t^2}\right)\big(|y_1|^2 + |y_2|^2\big) - 6M_4 \frac{1}{|\xi| t^2}\big(|y_1|^2 + |y_2|^2\big). \end{aligned}$$

If we choose $M_3 \geqslant 3M_4$, then

$$\partial_t \tilde{E}(t,\xi) \geqslant 2(\Re\psi + \theta) E(t,\xi).$$

Using assumption (2.5) we can estimate to below in the last inequality the energy by the Lyapunov functional. Hence,

$$\partial_t \tilde{E}(t,\xi) \geqslant 2(\Re\psi + \theta) \tilde{E}(t,\xi).$$

By Gronwall's inequality we conclude

$$\tilde{E}(t,\xi) \geqslant \tilde{E}(t_\xi,\xi)\exp\left(2\int_{t_\xi}^{t}\big(\Re\psi(s)+\theta(s,\xi)\big)\,ds\right) \quad \text{for all } (t,\xi)\in Z_{hyp}(N,M).$$

Finally, by using condition (2.4) with $\alpha\in(0,1]$, (2.6), (2.7) and the definition of t_ξ it follows

$$\exp\left(2\int_{t_\xi}^{T}\big(\Re\psi(s)+\theta(s,\xi)\big)\,ds\right) \geqslant \exp\left(C\left(\log\frac{1}{t_\xi}\right)^{\alpha}\right) \geqslant \exp\big(C_1\big(\log|\xi|\big)^{\alpha}\big).$$

Consequently,

$$\tilde{E}(T,\xi) \geqslant \tilde{E}(t_\xi,\xi)\exp\big(C_1\big(\log|\xi|\big)^{\alpha}\big).$$

Conclusion. Let us choose with a sufficiently large Q the data $y_1(t_\xi,\xi)=\langle\xi\rangle^{-Q}$ and $y_2(t_\xi,\xi)\equiv 0$, thus $\tilde{E}(t_\xi,\xi)=E(t_\xi,\xi)=\langle\xi\rangle^{-2Q}$. Then from the estimate of the Lyapunov functional in $Z_{hyp}(N,M)$ we conclude

$$\begin{aligned} E(T,\xi) &\geqslant \tilde{E}(T,\xi) \geqslant \tilde{E}(t_\xi,\xi)\exp\big(C_1\big(\log|\xi|\big)^{\alpha}\big) = E(t_\xi,\xi)\exp\big(C_1\big(\log|\xi|\big)^{\alpha}\big) \\ &= \exp\big(C_1\big(\log|\xi|\big)^{\alpha}\big)\Big(\sqrt{\det H(t_\xi)}\big|K^{-1}(t_\xi,\xi)H^{-1}(t_\xi)\hat{U}(t_\xi,\xi)\big|\Big)^2. \end{aligned}$$

From (4.1) we get for the backward Cauchy problem in $Z_{pd}(N,M)$ the estimate $|\hat{U}_0(\xi)|\leqslant C_N|\hat{U}(t_\xi,\xi)|$. All together yields

$$E(T,\xi) \geqslant C_N\exp\big(C_1\big(\log|\xi|\big)^{\alpha}\big)\big|\hat{U}_0(\xi)\big|^2.$$

Finally, using

$$\begin{aligned} E(T,\xi) &= \big|Y(T,\xi)\big|^2 \\ &= \left|\begin{pmatrix} \exp(\int_{t_\xi}^{T}(\theta(s,\xi)+\frac{M_3}{|\xi|s^2})\,ds) & 0 \\ 0 & \exp(-\int_{t_\xi}^{T}(\theta(s,\xi)+\frac{M_3}{|\xi|s^2})\,ds) \end{pmatrix}\right. \\ &\qquad \left.\times\sqrt{\det H(T)}K^{-1}(T,\xi)H^{-1}(T)\hat{U}(T,\xi)\right|^2 \end{aligned}$$

brings with (2.4), (2.6) and (2.7) the desired estimate

$$\big|\hat{U}(T,\xi)\big| \geqslant C_N\exp\big(C_1\big(\log|\xi|\big)^{\alpha}\big)\big|\hat{U}_0(\xi)\big|.$$

Summarizing, the Cauchy problem is H^∞ well-posed with an (at least) loss of regularity $\exp(C_1(\log\langle D_x\rangle)^{\alpha})$.

4.2. The case of H^∞ ill-posedness

One can be satisfied with Example 1.2 from [7]. But our goal is to have an example with *interactions of oscillations and with system character* "far away from systems" appearing after transformation of second order equations. Let us come back to Example 1.5. We will show that $p \in (1,2]$ implies an example for $A = A(t)$, that the corresponding Cauchy problem (2.1) is H^∞ ill-posed. Taking $p \in (1,2]$, then (A3) is satisfied for $\gamma = p-1 \in (0,1]$.

4.2.1. Proof to Theorem 2.4

We choose the matrix A from Example 1.5 and get

$$\Re\psi(t) = \frac{\sin((\log\frac{1}{t})^p) d_t (\log\frac{1}{t})^p}{2(2+\sin((\log\frac{1}{t})^p))}.$$

With large constants N and M let us define for $\{|\xi| \geqslant M\}$ the functions t_ξ and $\tilde{t}_\xi$ by

$$|\xi| t_\xi = N \log\frac{1}{t_\xi}, \qquad |\xi|\tilde{t}_\xi = N\left(\log\frac{1}{\tilde{t}_\xi}\right)^2.$$

Then we divide the extended phase space $[0,T] \times \{|\xi| \geqslant M\}$ into

the pseudo-differential zone	$Z_{pd}(N,M) = \{(t,\xi)\colon\ t \leqslant t_\xi\}$,
the oscillations subzone	$Z_{osc}(N,M) = \{(t,\xi)\colon\ t_\xi \leqslant t \leqslant \tilde{t}_\xi\}$,
the interacting subzone	$Z_{intac}(N,M) = \{(t,\xi)\colon\ \tilde{t}_\xi \leqslant t \leqslant T\}$.

Both subzones form the hyperbolic zone $Z_{hyp}(N,M)$.

Some auxiliary functions and their properties. We introduce the sequence $\{t_k\}_{k\geqslant 1}$ with $t_k := \exp(-(k\pi)^{\frac{1}{p}})$. Then there exists a function $q = q(\xi)$ such that $t_{2q+1} < \tilde{t}_\xi \leqslant t_{2q-1}$.

We notice that $\Re\psi(t) \geqslant 0$ for $t \in [t_{2k}, t_{2k-1}]$ and $\Re\psi(t) \leqslant 0$ for $t \in [t_{2k+1}, t_{2k}]$. Now let us introduce the functions

$$\omega(t,\xi) := L\frac{(\log\frac{1}{t})^2}{|\xi| t^2} \quad \text{for } (t,\xi) \in (0,T] \times \{|\xi| \geqslant M\},$$

$$\theta(t,\xi) := -\frac{\|\Re\psi\|_{L^1(t_{2[(k+1)/2]+1}, t_{2[(k+1)/2]-1})}}{2\|\Re\psi\|_{L^1(t_{k+1},t_k)}}\Re\psi(t) - \frac{\|\omega(\cdot,\xi)\|_{L^1(t_{k+1},t_k)}}{\|\Re\psi\|_{L^1(t_{k+1},t_k)}}\left|\Re\psi(t)\right| + \omega(t,\xi) \quad \text{for } t \in [t_{k+1}, t_k].$$

The large constant L will be determined later.

Lemma 4.1. *It holds*

$$\int_{\tilde{t}_\xi}^{T} \omega(t,\xi)\,dt \leqslant \frac{L}{N} \ll 1$$

if N is chosen large enough in comparison with L.

Proof. The statement follows from

$$\int_{\tilde{t}_\xi}^{T} \omega(t,\xi)\,dt = L\int_{\tilde{t}_\xi}^{T} \frac{(\log\frac{1}{t})^2}{|\xi|t^2}\,dt \leqslant L\frac{(\log\frac{1}{\tilde{t}_\xi})^2}{|\xi|\tilde{t}_\xi} = \frac{L}{N} \ll 1. \qquad \square$$

Lemma 4.2. *We have for $t \in [t_{2k}, t_{2k-1}]$ the relation*

$$\Re\psi(t) + \theta(t,\xi) - \omega(t,\xi) = \left(\int_{t_{2k+1}}^{t_{2k-1}} \Re\psi(s)\,ds - 2\big\|\omega(\cdot,\xi)\big\|_{L^1(t_{2k},t_{2k-1})}\right)\frac{|\Re\psi(t)|}{2\|\Re\psi\|_{L^1(t_{2k},t_{2k-1})}}.$$

Proof. Choosing the above introduced definitions for ω and θ we have for $t \in [t_{2k}, t_{2k-1}]$

$$\Re\psi(t) + \theta(t,\xi) - \omega(t,\xi) = \left(1 - \frac{\|\Re\psi\|_{L^1(t_{2k+1},t_{2k-1})}}{2\|\Re\psi\|_{L^1(t_{2k},t_{2k-1})}}\right)\Re\psi(t) - \frac{\|\omega(\cdot,\xi)\|_{L^1(t_{2k},t_{2k-1})}}{\|\Re\psi\|_{L^1(t_{2k},t_{2k-1})}}\big|\Re\psi(t)\big|.$$

Taking account of

$$\int_{t_{2k+1}}^{t_{2k-1}} \Re\psi(s)\,ds = \|\Re\psi\|_{L^1(t_{2k},t_{2k-1})} - \|\Re\psi\|_{L^1(t_{2k+1},t_{2k})}$$

we conclude

$$\begin{aligned}
&\Re\psi(t) + \theta(t,\xi) - \omega(t,\xi) \\
&\quad = \frac{(\|\Re\psi\|_{L^1(t_{2k},t_{2k-1})} - \|\Re\psi\|_{L^1(t_{2k+1},t_{2k})})}{2\|\Re\psi\|_{L^1(t_{2k},t_{2k-1})}}\big|\Re\psi(t)\big| - \frac{\|\omega(\cdot,\xi)\|_{L^1(t_{2k},t_{2k-1})}}{\|\Re\psi\|_{L^1(t_{2k},t_{2k-1})}}\big|\Re\psi(t)\big| \\
&\quad = \left(\int_{t_{2k+1}}^{t_{2k-1}} \Re\psi(s)\,ds - 2\big\|\omega(\cdot,\xi)\big\|_{L^1(t_{2k},t_{2k-1})}\right)\frac{|\Re\psi(t)|}{2\|\Re\psi\|_{L^1(t_{2k},t_{2k-1})}}. \qquad \square
\end{aligned}$$

In the same way we are able to prove the following statement:

Lemma 4.3. *We have for $t \in [t_{2k+1}, t_{2k}]$ the relation*

$$\Re\psi(t) + \theta(t,\xi) - \omega(t,\xi) = \left(\int_{t_{2k+1}}^{t_{2k-1}} \Re\psi(s)\,ds - 2\big\|\omega(\cdot,\xi)\big\|_{L^1(t_{2k+1},t_{2k})}\right)\frac{|\Re\psi(t)|}{2\|\Re\psi\|_{L^1(t_{2k+1},t_{2k})}}.$$

Corollary 4.4. *We have for $t \in [t_{k+1}, t_k]$ the relation*

$$\Re\psi(t) + \theta(t,\xi) - \omega(t,\xi) = \left(\int_{t_{2[(k+1)/2]+1}}^{t_{2[(k+1)/2]-1}} \Re\psi(s)\,ds - 2\big\|\omega(\cdot,\xi)\big\|_{L^1(t_{k+1},t_k)}\right)\frac{|\Re\psi(t)|}{2\|\Re\psi\|_{L^1(t_{k+1},t_k)}}.$$

Proof. Setting $k = 2n$ or $k = 2n+1$ the statement follows from those ones of Lemmas 4.2 and 4.3. $\square$

Lemma 4.5. *We have the following relations*:

$$\int_{t_{2k+1}}^{t_{2k-1}} \Re\psi(s)\,ds \geqslant M_5 > 0, \qquad \int_{t_{2k+1}}^{t_{2k-1}} \left|\Re\psi(s)\right| ds \leqslant \pi \quad \textit{for all } k \geqslant 1.$$

Proof. It holds

$$\int_{t_{2k+1}}^{t_{2k-1}} \Re\psi(s)\,ds = \int_{t_{2k+1}}^{t_{2k-1}} \frac{\sin((\log\frac{1}{s})^p)d_s(\log\frac{1}{s})^p}{2(2+\sin((\log\frac{1}{s})^p))}\,ds = \int_{(2k-1)\pi}^{(2k+1)\pi} -\frac{\sin\theta}{2(2+\sin\theta)}\,d\theta$$

by using the definition of t_k. From the calculations in Example 1.5 the first statement follows. The second statement can be concluded from

$$\int_{t_{2k+1}}^{t_{2k-1}} \left|\Re\psi(s)\right| ds \leqslant -\int_{t_{2k+1}}^{t_{2k-1}} \frac{1}{2}d_s\left(\log\frac{1}{s}\right)^p ds = \pi. \qquad \square$$

Corollary 4.6. *In the interacting subzone* $Z_{intac}(N, M)$ *it holds* $\Re\psi(t) + \theta(t,\xi) - \omega(t,\xi) \geqslant 0$ *for* $p \geqslant 1$ *in the definition of the matrix A from Example* 1.5.

Proof. Let us choose without loss of generality $T = t_{2k_0-1}$. To a given ξ: $|\xi| \geqslant M$ there exists an index $q = q(\xi)$ such that $t_{2q+1} < \tilde{t}_\xi \leqslant t_{2q-1}$. For $(t,\xi) \in [t_{2q-1}, T] \times \{|\xi| \geqslant M\}$ the statement follows from those of Lemmas 4.1, 4.5 and Corollary 4.4. It remains to prove it for $(t,\xi) \in [\tilde{t}_\xi, t_{2q-1}] \times \{|\xi| \geqslant M\}$. First we estimate

$$\int_{t_{2q+1}}^{t_{2q-1}} \omega(s,\xi)\,ds \leqslant L\frac{(\log\frac{1}{t_{2q+1}})^2}{|\xi|t_{2q+1}}.$$

There exists a constant M_6 such that $t_{2q-1} \leqslant M_6 t_{2q+1}$, where M_6 is independent of q. This follows from

$$\frac{t_{2q-1}}{t_{2q+1}} = \exp\Bigl(\bigl((2q+1)\pi\bigr)^{\frac{1}{p}} - \bigl((2q-1)\pi\bigr)^{\frac{1}{p}}\Bigr) \tag{4.2}$$

and from

$$\bigl((2q+1)\pi\bigr)^{\frac{1}{p}} - \bigl((2q-1)\pi\bigr)^{\frac{1}{p}} \leqslant \frac{C}{q^{\frac{p-1}{p}}} \leqslant M_6 \quad \text{for } p \geqslant 1. \tag{4.3}$$

Consequently,

$$\frac{(\log\frac{1}{t_{2q+1}})^2}{|\xi|t_{2q+1}} \leqslant \frac{(\log\frac{M_6}{t_{2q-1}})^2 M_6}{|\xi|t_{2q-1}} \leqslant \frac{M_7(\log\frac{1}{t_{2q-1}})^2}{|\xi|t_{2q-1}} \leqslant \frac{M_7}{N}$$

for $q \geqslant k_0$ by taking into consideration the definition of $Z_{intac}(N, M)$. Hence, a sufficiently large N in the definition of the zones implies

$$\int_{\tilde t_\xi}^{t_{2q-1}} \omega(s,\xi)\,ds \leqslant \int_{t_{2q+1}}^{t_{2q-1}} \omega(s,\xi)\,ds \leqslant C_N \quad \text{with } C_N \to 0 \quad \text{if } N \to \infty. \tag{4.4}$$

Together with the first statement from Lemma 4.5 we obtain the desired statement. □

Finally, let us study properties of $\theta = \theta(t,\xi)$.

Lemma 4.7. *We have the following properties*:

$$\int_{t_{2k+1}}^{t_{2k-1}} \theta(s,\xi)\,ds = 0 \quad \textit{for } k_0 \leqslant k \leqslant q,$$

$$\left| \int_{t}^{t_{2q-1}} \theta(s,\xi)\,ds \right| \leqslant M_8 \quad \textit{for } (t,\xi) \in [\tilde t_\xi, t_{2q-1}] \times \big\{ |\xi| \geqslant M \big\},$$

where the constant M_8 is independent of (t,ξ).

Proof. We use the definition of θ in $[t_{2k+1}, t_{2k}]$, $[t_{2k}, t_{2k-1}]$, respectively, and get

$$\begin{aligned} \int_{t_{2k+1}}^{t_{2k-1}} \theta(s,\xi)\,ds = {} & \int_{t_{2k+1}}^{t_{2k}} \left(-\frac{\|\Re\psi\|_{L^1(t_{2k+1},t_{2k-1})}}{2\|\Re\psi\|_{L^1(t_{2k+1},t_{2k})}} \Re\psi(s) - \frac{\|\omega(\cdot,\xi)\|_{L^1(t_{2k+1},t_{2k})}}{\|\Re\psi\|_{L^1(t_{2k+1},t_{2k})}} \big|\Re\psi(s)\big| + \omega(s,\xi) \right) ds \\ & + \int_{t_{2k}}^{t_{2k-1}} \left(-\frac{\|\Re\psi\|_{L^1(t_{2k+1},t_{2k-1})}}{2\|\Re\psi\|_{L^1(t_{2k},t_{2k-1})}} \Re\psi(s) - \frac{\|\omega(\cdot,\xi)\|_{L^1(t_{2k},t_{2k-1})}}{\|\Re\psi\|_{L^1(t_{2k},t_{2k-1})}} \big|\Re\psi(s)\big| + \omega(s,\xi) \right) ds. \end{aligned}$$

Simple calculations give together with the positiveness of ω, with $\Re\psi \geqslant 0$ on $[t_{2k}, t_{2k-1}]$ and with $\Re\psi \leqslant 0$ on $[t_{2k+1}, t_{2k}]$

$$\begin{aligned} \int_{t_{2k+1}}^{t_{2k-1}} \theta(s,\xi)\,ds = {} & \frac{1}{2}\|\Re\psi\|_{L^1(t_{2k+1},t_{2k-1})} - \frac{1}{2}\|\Re\psi\|_{L^1(t_{2k+1},t_{2k-1})} - \big\|\omega(\cdot,\xi)\big\|_{L^1(t_{2k+1},t_{2k})} \\ & - \big\|\omega(\cdot,\xi)\big\|_{L^1(t_{2k},t_{2k-1})} + \big\|\omega(\cdot,\xi)\big\|_{L^1(t_{2k+1},t_{2k})} + \big\|\omega(\cdot,\xi)\big\|_{L^1(t_{2k},t_{2k-1})} = 0. \end{aligned}$$

By (4.4) and Lemma 4.5 we have

$$\begin{aligned} \left| \int_{t}^{t_{2q-1}} \theta(s,\xi)\,ds \right| & \leqslant \int_{t_{2q}}^{t_{2q-1}} \big|\theta(s,\xi)\big|\,ds + \int_{t_{2q+1}}^{t_{2q}} \big|\theta(s,\xi)\big|\,ds \\ & \leqslant \|\Re\psi\|_{L^1(t_{2q+1},t_{2q-1})} + \big\|\omega(\cdot,\xi)\big\|_{L^1(t_{2q+1},t_{2q})} + \big\|\omega(\cdot,\xi)\big\|_{L^1(t_{2q},t_{2q-1})} \\ & \leqslant \|\Re\psi\|_{L^1(t_{2q+1},t_{2q-1})} + 2\big\|\omega(\cdot,\xi)\big\|_{L^1(t_{2q+1},t_{2q-1})} \leqslant M_8. \qquad \square \end{aligned}$$

Considerations in the interacting subzone. We follow the proof to Theorem 2.3. After two steps of diagonalization we obtain the Cauchy problem

$$\partial_t Z - \begin{pmatrix} (\mu_+ + d)\xi + \Im\psi & 0 \\ 0 & (\mu_- + d)\xi - \Im\psi \end{pmatrix} iZ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\Re\psi)Z + JZ = 0,$$
$$(\tilde{t}_\xi, \xi) = \sqrt{\det H(\tilde{t}_\xi)} K^{-1}(\tilde{t}_\xi, \xi) H^{-1}(\tilde{t}_\xi) \hat{U}(\tilde{t}_\xi, \xi).$$

Introducing in $Z_{intac}(N, M)$

$$Z(t,\xi) =: \begin{pmatrix} \exp(-\int_{\tilde{t}_\xi}^t \theta(s,\xi)\,ds) & 0 \\ 0 & \exp(\int_{\tilde{t}_\xi}^t \theta(s,\xi)\,ds) \end{pmatrix} Y(t,\xi), \qquad Y = (y_1, y_2)^T,$$

we get the Cauchy problem

$$\partial_t Y - \begin{pmatrix} (\mu_+ + d)\xi + \Im\psi & 0 \\ 0 & (\mu_- + d)\xi - \Im\psi \end{pmatrix} iY - \begin{pmatrix} 1 & 0 \\ 0-1 & \end{pmatrix} (\Re\psi + \theta)Y + \tilde{J}Y = 0,$$
$$Y(\tilde{t}_\xi, \xi) = Z(\tilde{t}_\xi, \xi),$$

where the matrix $\tilde{J}$ is given by

$$\tilde{J} = \begin{pmatrix} J_{11} & \exp(2\int_{\tilde{t}_\xi}^t \theta(s,\xi)\,ds) J_{12} \\ \exp(-2\int_{\tilde{t}_\xi}^t \theta(s,\xi)\,ds) J_{21} & J_{22} \end{pmatrix}, \qquad J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}.$$

Here we have to remark that due to Lemma 4.7 we have $C^{-1}|Z(t,\xi)| \leqslant |Y(t,\xi)| \leqslant C|Z(t,\xi)|$ for all $(t,\xi) \in Z_{intac}(N, M)$. Moreover, we have

$$\|\tilde{J}(t,\xi)\| \leqslant M_9 \frac{(\log\frac{1}{t})^{2(p-1)}}{|\xi| t^2} \quad \text{for } (t,\xi) \in Z_{intac}(N, M).$$

We define the Lyapunov functional $\tilde{E}(t,\xi) = |y_1(t,\xi)|^2 - |y_2(t,\xi)|^2$. Then we conclude as in the proof to Theorem 2.3

$$\partial_t \tilde{E}(t,\xi) \geqslant 2\left(\Re\psi + \theta - M_9 \frac{(\log\frac{1}{t})^{2(p-1)}}{|\xi| t^2}\right)\left(\left|y_1(t,\xi)\right|^2 + \left|y_2(t,\xi)\right|^2\right)$$
$$\geqslant 2(\Re\psi + \theta - \omega)\left(\left|y_1(t,\xi)\right|^2 + \left|y_2(t,\xi)\right|^2\right)$$

if we choose $p \in (1,2]$ and the constant $L \geqslant M_9$ in the definition of ω. Now Corollary 4.6 is of importance. It allows to estimate to below the energy functional on the right-hand side by the Lyapunov functional, and the application of Gronwall's inequality implies

$$\tilde{E}(t,\xi) \geqslant \tilde{E}(\tilde{t}_\xi, \xi) \exp\left(2 \int_{\tilde{t}_\xi}^t \left(\Re\psi(s) + \theta(s,\xi) - \omega(s,\xi)\right) ds\right)$$

for all $(t,\xi) \in Z_{intac}(N, M)$. Applying systematically Lemmas 4.1, 4.7 and the computations from Example 1.5 brings

$$\exp\left(2\int_{\tilde{t}_\xi}^{t}\big(\Re\psi(s)+\theta(s,\xi)-\omega(s,\xi)\big)\,ds\right)\geqslant C\exp\big(C_1 q(\xi)\big).$$

From $\tilde{t}_\xi \sim t_{2q}$ it follows $q(\xi)\sim(\log\frac{1}{\tilde{t}_\xi})^p$. With the definition of $\tilde{t}_\xi$ we conclude

$$\tilde{E}(T,\xi)\geqslant\tilde{E}(\tilde{t}_\xi,\xi)\exp\big(C_1\big(\log\langle\xi\rangle\big)^p\big), \tag{4.5}$$

that is, *the desired estimate of the Lyapunov functional in* $Z_{intac}(N,M)$.

Considerations in oscillations subzone. We are interested in the backward Cauchy problem for $t\in[t_\xi,\tilde{t}_\xi]$

$$\partial_t Z-\begin{pmatrix}(\mu_++d)\xi+\Im\psi & 0\\ 0 & (\mu_-+d)\xi-\Im\psi\end{pmatrix}iZ-\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(\Re\psi)Z+JZ=0,$$
$$Z(\tilde{t}_\xi,\xi)=\sqrt{\det H(\tilde{t}_\xi)}K^{-1}(\tilde{t}_\xi,\xi)H^{-1}(\tilde{t}_\xi)\hat{U}(\tilde{t}_\xi,\xi).$$

Differentiation of the energy $E=E(t,\xi)=|z_1(t,\xi)|^2+|z_2(t,\xi)|^2$, $Z=(z_1,z_2)^T$, gives

$$\begin{aligned}\partial_t E(t,\xi)&\geqslant -C\big(\big|\Re\psi(t)\big|+\big\|J(t,\xi)\big\|\big)E(t,\xi)\\ &\geqslant -C\left(\frac{1}{t}\left(\log\frac{1}{t}\right)^{p-1}+\frac{1}{|\xi|t^2}\left(\log\frac{1}{t}\right)^{2(p-1)}\right)E(t,\xi).\end{aligned}$$

Gronwall's inequality yields

$$\begin{aligned}E(\tilde{t}_\xi,\xi)&\geqslant E(t_\xi,\xi)\exp\left(-C\int_{t_\xi}^{\tilde{t}_\xi}\left(\left(\frac{1}{s}\log\frac{1}{s}\right)^{p-1}+\frac{1}{|\xi|s^2}\left(\log\frac{1}{s}\right)^{2(p-1)}\right)ds\right)\\ &=E(t_\xi,\xi)\exp\left(-C_p\left(\log\frac{1}{s}\right)^p\Bigg|_{\tilde{t}_\xi}^{t_\xi}-C\int_{t_\xi}^{\tilde{t}_\xi}\frac{1}{|\xi|s^2}\left(\log\frac{1}{s}\right)^{2(p-1)}ds\right).\end{aligned}$$

On the one hand we use

$$\left(\log\frac{1}{t_\xi}\right)^p-\left(\log\frac{1}{\tilde{t}_\xi}\right)^p\leqslant C_p\left(\log\frac{1}{t_\xi}\right)^{p-1}\log\frac{\tilde{t}_\xi|\xi|}{t_\xi|\xi|}\leqslant C_p\left(\log\frac{1}{t_\xi}\right)^{p-1}\log\log\frac{1}{\tilde{t}_\xi}.$$

But $\frac{1}{\tilde{t}_\xi}\leqslant|\xi|$, $\frac{1}{t_\xi}\leqslant|\xi|$, so

$$\left(\log\frac{1}{t_\xi}\right)^p-\left(\log\frac{1}{\tilde{t}_\xi}\right)^p\leqslant C_p\big(\log|\xi|\big)^{p-1}\log\log|\xi|.$$

On the other hand

$$\int_{t_\xi}^{\tilde{t}_\xi} \frac{1}{|\xi|s^2}\left(\log\frac{1}{s}\right)^{2(p-1)} ds = -\frac{1}{|\xi|s}\left(\log\frac{1}{s}\right)^{2(p-1)}\Bigg|_{t_\xi}^{\tilde{t}_\xi} - 2(p-1)\int_{\tilde{t}_\xi}^{t_\xi} \frac{1}{|\xi|s^2}\left(\log\frac{1}{s}\right)^{2p-3} ds.$$

Hence, we conclude

$$\int_{t_\xi}^{\tilde{t}_\xi} \frac{1}{|\xi|s^2}\left(\log\frac{1}{s}\right)^{2(p-1)} ds \leqslant \frac{1}{|\xi|t_\xi}\left(\log\frac{1}{t_\xi}\right)^{2(p-1)} = C_N\left(\log\frac{1}{t_\xi}\right)^{2p-3}$$
$$\leqslant C_N\big(\log|\xi|\big)^{2p-3} \leqslant C_N\big(\log|\xi|\big)^{p-1}.$$

Summarizing we derived the following *energy inequality in $Z_{osc}(N,M)$*:

$$E(\tilde{t}_\xi,\xi) \geqslant E(t,\xi)\exp\Big(-C_1\big(\big(\log|\xi|\big)^{p-1}\log\log|\xi| + \big(\log|\xi|\big)^{p-1}\big)\Big),$$
$$E(t,\xi) \leqslant \exp\Big(C_1\big(\log|\xi|\big)^{p-1}\log\log|\xi|\Big)E(\tilde{t}_\xi,\xi) \quad \text{for all } (t,\xi)\in Z_{osc}(N,M), \tag{4.6}$$

respectively.

Considerations in the pseudo-differential zone. We consider the backward Cauchy problem for $t\in[0,t_\xi]$

$$\partial_t\hat{U} = A(t)i\xi\hat{U}, \qquad \hat{U}(t_\xi,\xi) = \frac{1}{\sqrt{\det H(t_\xi)}}H(t_\xi)K(t_\xi,\xi)Z(t_\xi,\xi).$$

As in the proof to Theorem 2.1 we get

$$E(t,\xi) \leqslant \exp\big(C_N\log|\xi|\big)E(t_\xi,\xi) \quad \text{for } (t,\xi)\in Z_{pd}(N,M). \tag{4.7}$$

Conclusion. Let us choose with a sufficiently large Q the data $y_1(\tilde{t}_\xi,\xi) = \langle\xi\rangle^{-Q}$, $y_2(\tilde{t}_\xi,\xi) = 0$, thus $\tilde{E}(\tilde{t}_\xi,\xi) = E(\tilde{t}_\xi,\xi) = \langle\xi\rangle^{-2Q}$. Choosing $p\in(1,2]$, then the energy estimate (4.7) gives an (at most) finite loss of derivatives. But, the estimates (4.5), (4.6) for the Lyapunov functional, the energy functional, respectively, imply

$$E(T,\xi) \geqslant \tilde{E}(T,\xi) \geqslant \tilde{E}(\tilde{t}_\xi,\xi)\exp\big(C_1\big(\log\langle\xi\rangle\big)^p\big) = E(\tilde{t}_\xi,\xi)\exp\big(C_1\big(\log\langle\xi\rangle\big)^p\big)$$
$$\geqslant E(t_\xi,\xi)\exp\big(C_1\big(\log\langle\xi\rangle\big)^p\big),$$

that is, an infinite loss of derivatives in $Z_{intac}(N,M)\cup Z_{osc}(N,M)$. So, we have H^∞ ill-posedness for $p\in(1,2]$. The matrix $A=A(t)$ from Example 1.5 satisfies assumptions (A1), (A2) and (A3) with $\gamma = p-1\in(0,1]$ if $p\in(1,2]$. This completes the proof of Theorem 2.4.

Remark 4.1. One can construct other examples, where γ in (A3) is independent of p in Example 1.5. With the matrix A from Example 1.5 let us choose

$$\tilde{A} = \left(2+\sin\left(\left(\log\frac{1}{t}\right)^{\gamma+1}\right)\right)A.$$

Then, $\Re\psi$ is independent of γ, and we can take any $\gamma\in(0,1]$ in (A3).

5. The complexity of hyperbolic systems

With Theorem 2.5 and its proof we explain the complexity of hyperbolic systems and the difficulty to find general results about H^∞ well-posedness or ill-posedness as we did in Theorems 2.1 to 2.4.

5.1. Proof to Theorem 2.5

Choice of the matrix. We shall choose the matrix $A(t)$ from Example 1.6 with

$$\omega(t)=\left(\log\frac{1}{t}\right)^r\left(2-\cos\left(\left(\log\frac{1}{t}\right)^p\right)\right),\quad r\in(0,1),\ p>1,\ \text{and } r+p\leqslant 2-r.$$

Then, we know that $A(t)$ satisfies (A3) with $\gamma=p+r-1\in(0,1)$, (A4) with $\alpha=r\in(0,1)$ and (A5) with $\beta=r+p>1$. Moreover, $\Re\psi(t)=\frac{\omega'(t)}{2}=:\phi_1(t)+\phi_2(t)$, where

$$\phi_1(t)=-\frac{r}{2t}\left(\log\frac{1}{t}\right)^{r-1}\left(2-\cos\left(\left(\log\frac{1}{t}\right)^p\right)\right),\qquad \phi_2(t)=-\frac{p}{2t}\left(\log\frac{1}{t}\right)^{r+p-1}\sin\left(\left(\log\frac{1}{t}\right)^p\right).$$

Properties of auxiliary functions. We define t_ξ and $\tilde{t}_\xi$ by $|\xi|t_\xi=N\log\frac{1}{t_\xi}$ and $|\xi|\tilde{t}_\xi=N\log\frac{1}{\tilde{t}_\xi}\times\exp(L(\log\frac{1}{\tilde{t}_\xi})^r)$, respectively, where the constant L will be determined later. We remark that $t_\xi<\tilde{t}_\xi$, since $f(t)=t/\log\frac{1}{t}$ is an increasing function for small $t>0$ and

$$\frac{f(\tilde{t}_\xi)}{f(t_\xi)}=\frac{\tilde{t}_\xi/\log\frac{1}{\tilde{t}_\xi}}{t_\xi/\log\frac{1}{t_\xi}}=\exp\left(L\left(\log\frac{1}{\tilde{t}_\xi}\right)^r\right)>1.$$

Now we introduce the sequence $\{t_k\}_{k\geqslant1}$ with $t_k:=\exp(-(k\pi)^{\frac{1}{p}})$, a function $q=q(\xi)$ such that $t_{2q+1}<\tilde{t}_\xi\leqslant t_{2q-1}$ and the hyperbolic subzones (cf. with the proof to Theorem 2.4)

$$Z_{intac}(M,N)=\{(t,\xi):\ t_{2q-1}\leqslant t\leqslant T\},\qquad Z_{osc}(M,N)=\{(t,\xi):\ t_\xi\leqslant t\leqslant t_{2q-1}\}.$$

Finally, let us define the function $\theta^*(t)$ by

$$\theta^*(t)=-\frac{\|\phi_2\|_{L^1(t_{2[(k+1)/2]+1},t_{2[(k+1)/2]-1})}}{2\|\phi_2\|_{L^1(t_{k+1},t_k)}}\phi_2(t)$$

for $t\in[t_{k+1},t_k]$. Taking account of $\int_{t_{2k+1}}^{t_{2k-1}}\phi_2(s)\,ds=\|\phi_2\|_{L^1(t_{2k},t_{2k-1})}-\|\phi_2\|_{L^1(t_{2k+1},t_{2k})}$ we have for $t\in[t_{2k},t_{2k-1}]$ the relation

$$\phi_2(t)+\theta^*(t)=\left(1-\frac{\|\phi_2\|_{L^1(t_{2k+1},t_{2k-1})}}{2\|\phi_2\|_{L^1(t_{2k},t_{2k-1})}}\right)\phi_2(t)=\left(\int_{t_{2k+1}}^{t_{2k-1}}\phi_2(s)\,ds\right)\frac{|\phi_2(t)|}{2\|\phi_2\|_{L^1(t_{2k},t_{2k-1})}}.$$

Similarly, we have for $t\in[t_{2k+1},t_{2k}]$ the relation

$$\phi_2(t)+\theta^*(t)=\left(\int_{t_{2k+1}}^{t_{2k-1}}\phi_2(s)\,ds\right)\frac{|\phi_2(t)|}{2\|\phi_2\|_{L^1(t_{2k+1},t_{2k})}}.$$

Consequently, we find for $t \in [t_{k+1}, t_k]$ the relation

$$\phi_2(t) + \theta^*(t) = \left(\int_{t_{2[(k+1)/2]+1}}^{t_{2[(k+1)/2]-1}} \phi_2(s)\,ds \right) \frac{|\phi_2(t)|}{2\|\phi_2\|_{L^1(t_{k+1},t_k)}}.$$

From this identity we get the following estimate:

Corollary 5.1. *It holds*

$$\int_{t_{2q-1}}^{t} \left|\phi_2(s) + \theta^*(s)\right| ds \leqslant C\left(\log \frac{1}{t_{2q-1}}\right)^r$$

for all $t \in [t_{2q-1}, T]$.

Proof. Let us choose without loss of generality $T = t_1$. Then we can estimate

$$\begin{aligned}
&\int_{t_{2q-1}}^{t} \left|\phi_2(s) + \theta^*(s)\right| ds \\
&\quad \leqslant \int_{t_{2q-1}}^{t_1} \left|\phi_2(s) + \theta^*(s)\right| ds \\
&\quad \leqslant \left| \int_{t_{2q-1}}^{t_1} \phi_2(s)\,ds \right| = \left| \int_{t_{2q-1}}^{t_1} \frac{p}{2s}\left(\log\frac{1}{s}\right)^{r+p-1} \sin\left(\left(\log\frac{1}{s}\right)^p\right) ds \right| \\
&\quad \leqslant \left| -\frac{1}{2}\left(\log\frac{1}{s}\right)^r \cos\left(\left(\log\frac{1}{s}\right)^p\right)\Big|_{t_{2q-1}}^{t_1} - \int_{t_{2q-1}}^{t_1} \frac{r}{2s}\left(\log\frac{1}{s}\right)^{r-1} \cos\left(\left(\log\frac{1}{s}\right)^p\right) ds \right| \\
&\quad \leqslant \left| -\frac{1}{2}\left(\log\frac{1}{s}\right)^r \cos\left(\left(\log\frac{1}{s}\right)^p\right)\Big|_{t_{2q-1}}^{t_1} + \frac{r}{2p}\left(\log\frac{1}{s}\right)^{r-p} \sin\left(\left(\log\frac{1}{s}\right)^p\right)\Big|_{t_{2q-1}}^{t_1} \right. \\
&\qquad \left. + \int_{t_{2q-1}}^{t_1} \frac{r(r-p)}{2ps}\left(\log\frac{1}{s}\right)^{r-p-1} \sin\left(\left(\log\frac{1}{s}\right)^p\right) ds \right| \\
&\quad \leqslant C\left(\log\frac{1}{t_{2q-1}}\right)^r.
\end{aligned}$$

This is the desired estimate. □

Lemma 5.2. *We have the following properties*:

$$\int_{t_{2k+1}}^{t_{2k-1}} \theta^*(s)\,ds = 0 \quad \textit{for } k \geqslant 1, \tag{5.1}$$

$$\int_{t_{2k+1}}^{t_{2k-1}} \left|\phi_2(s)\right| ds \leqslant C\left(\log\frac{1}{\tilde{t}_\xi}\right)^r \quad \textit{for } 1 \leqslant k \leqslant q, \tag{5.2}$$

$$\int_{t_\xi}^{t_{2q-1}} \left|\phi_2(s)\right| ds \leqslant C\log\frac{1}{t_\xi} \quad \textit{if } p > 1,\ 2r+p \leqslant 2, \tag{5.3}$$

$$\left|\int_{t_{2q-1}}^{t} \theta^*(s)\,ds\right| \leqslant C\left(\log\frac{1}{\tilde{t}_\xi}\right)^r \quad \textit{for } (t,\xi) \in [t_{2q-1}, T] \times \bigl\{|\xi| \geqslant M\bigr\}. \tag{5.4}$$

Proof. From the definition of $\theta^*(t)$ we compute

$$\begin{aligned}\int_{t_{2k+1}}^{t_{2k-1}} \theta^*(s)\,ds &= \int_{t_{2k}}^{t_{2k-1}} \theta^*(s)\,ds + \int_{t_{2k+1}}^{t_{2k}} \theta^*(s)\,ds \\ &= -\int_{t_{2k}}^{t_{2k-1}} \frac{\|\phi_2\|_{L^1(t_{2k+1},t_{2k-1})}}{2\|\phi_2\|_{L^1(t_{2k},t_{2k-1})}}\phi_2(s)\,ds - \int_{t_{2k+1}}^{t_{2k}} \frac{\|\phi_2\|_{L^1(t_{2k+1},t_{2k-1})}}{2\|\phi_2\|_{L^1(t_{2k+1},t_{2k})}}\phi_2(s)\,ds = 0.\end{aligned}$$

Noting that $q = q(\xi) \sim (\log\frac{1}{\tilde{t}_\xi})^p$ from $\tilde{t}_\xi \sim t_{2q}$ we get

$$\begin{aligned}\int_{t_{2k+1}}^{t_{2k-1}} \left|\phi_2(s)\right| ds &\leqslant \int_{t_{2k+1}}^{t_{2k-1}} \frac{p}{2s}\left(\log\frac{1}{s}\right)^{r+p-1} ds = C\left(\left(\log\frac{1}{t_{2k+1}}\right)^{r+p} - \left(\log\frac{1}{t_{2k-1}}\right)^{r+p}\right) \\ &= C\left(\bigl((2k+1)\pi\bigr)^{1+\frac{r}{p}} - \bigl((2k+1)\pi\bigr)^{1+\frac{r}{p}}\right) \leqslant Ck^{\frac{r}{p}} \leqslant Cq^{\frac{r}{p}} \leqslant C\left(\log\frac{1}{\tilde{t}_\xi}\right)^r.\end{aligned}$$

Using (4.2) and (4.3) we conclude for $p \geqslant 1$ the inequalities

$$t_{2q-1} \leqslant \bigl(\exp(M_6)\bigr)t_{2q+1} < \bigl(\exp(M_6)\bigr)\tilde{t}_\xi.$$

Hence, taking into consideration $2r + p - 1 \leqslant 1$ and $t_\xi < \tilde{t}_\xi$ we get

$$\begin{aligned}\log\frac{t_{2q-1}}{t_\xi}\left(\log\frac{1}{t_\xi}\right)^{r+p-1} &\leqslant C\left(1 + \log\frac{\tilde{t}_\xi}{t_\xi}\right)\left(\log\frac{1}{t_\xi}\right)^{r+p-1} \leqslant CL\left(\log\frac{1}{t_\xi}\right)^r\left(\log\frac{1}{t_\xi}\right)^{r+p-1} \\ &\leqslant CL\left(\log\frac{1}{t_\xi}\right)^{2r+p-1} \leqslant CL\log\frac{1}{t_\xi}.\end{aligned}$$

This implies (5.3). By (5.1) and (5.2) with $k = k(t)$ satisfying $t_{2k+1} < t \leqslant t_{2k-1}$ we have

$$\left|\int_{t_{2q-1}}^{t} \theta^*(s)\,ds\right| \leqslant C\int_{t_{2k+1}}^{t} \left|\phi_2(s)\right| ds \leqslant C\int_{t_{2k+1}}^{t_{2k-1}} \left|\phi_2(s)\right| ds \leqslant C\left(\log\frac{1}{\tilde{t}_\xi}\right)^r.$$

In this way we obtain all estimates (5.1) to (5.4). $\square$

Estimate in $Z_{intac}(M,N)$. Putting in $Z_{intac}(M,N)$

$$Z(t,\xi) =: \begin{pmatrix} \exp(-\int_{t_{2q-1}}^t \theta^*(s)\,ds) & 0 \\ 0 & \exp(\int_{t_{2q-1}}^t \theta^*(s)\,ds) \end{pmatrix} Y(t,\xi), \qquad Y = (y_1, y_2)^T,$$

we shall transform

$$\partial_t Z - \begin{pmatrix} (\mu_+ + d)\xi + \Im\psi & 0 \\ 0 & (\mu_- + d)\xi - \Im\psi \end{pmatrix} iZ - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\Re\psi) Z + JZ = 0$$

into

$$\partial_t Y - \begin{pmatrix} (\mu_+ + d)\xi + \Im\psi & 0 \\ 0 & (\mu_- + d)\xi - \Im\psi \end{pmatrix} iY - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\Re\psi + \theta^*) Y + \tilde{J} Y = 0,$$

where

$$\tilde{J} = \begin{pmatrix} J_{11} & \exp(2\int_{t_{2q-1}}^t \theta^*(s)\,ds) J_{12} \\ \exp(-2\int_{t_{2q-1}}^t \theta^*(s)\,ds) J_{21} & J_{22} \end{pmatrix}.$$

By (5.4) we have

$$\|\tilde{J}\| \leqslant C \frac{(\log\frac{1}{t})^{2(r+p-1)}}{|\xi| t^2} \exp\left(L\left(\log\frac{1}{\tilde{t}_\xi}\right)^r\right) \quad \text{for } (t,\xi) \in Z_{intac}(M,N).$$

We define in $Z_{intac}(M,N)$ the energy

$$E(t,\xi) = \left|y_1(t,\xi)\right|^2 + \left|y_2(t,\xi)\right|^2$$

and obtain with the definition of $\Re\psi$ the estimates

$$\begin{aligned} E'(t,\xi) = 2\Re(\partial_t Y, Y) &\leqslant 2\left(\left|\Re\psi(t) + \theta^*(t)\right| + C\frac{(\log\frac{1}{t})^{2(r+p-1)}}{|\xi| t^2} \exp\left(L\left(\log\frac{1}{\tilde{t}_\xi}\right)^r\right)\right) E(t,\xi) \\ &\leqslant 2\left(\left|\phi_2(t) + \theta^*(t)\right| + C\frac{(\log\frac{1}{t})^{r-1}}{t} + C\frac{(\log\frac{1}{t})^{2(r+p-1)}}{|\xi| t^2} \exp\left(L\left(\log\frac{1}{\tilde{t}_\xi}\right)^r\right)\right) E(t,\xi). \end{aligned}$$

Thus, by Corollary 5.1 the application of Gronwall's inequality yields

$$E(t,\xi) \leqslant E(t_{2q-1},\xi) \exp\left(C\left(\log\frac{1}{t_{2q-1}}\right)^r + C\int_{t_{2q-1}}^t \frac{(\log\frac{1}{s})^{2(r+p-1)}}{|\xi| s^2}\,ds \exp\left(L\left(\log\frac{1}{\tilde{t}_\xi}\right)^r\right)\right).$$

Since $r < 1$ and $r + p \leqslant 2 - r \leqslant 2$ by using $t_\xi < \tilde{t}_\xi < t_{2q-1}$ we have

$$E(t,\xi) \leqslant E(t_{2q-1},\xi)\exp\left(C\log\frac{1}{\tilde t_\xi} + C\int_{\tilde t_\xi}^{t}\frac{(\log\frac{1}{s})^2}{|\xi|s^2}\,ds\exp\left(L\left(\log\frac{1}{\tilde t_\xi}\right)^r\right)\right)$$

$$\leqslant E(t_{2q-1},\xi)\exp\left(C\log\frac{1}{\tilde t_\xi} + C\left(-\frac{(\log\frac{1}{s})^2}{|\xi|s}\bigg|_{\tilde t_\xi}^{t} - \int_{\tilde t_\xi}^{t}\frac{2\log\frac{1}{s}}{|\xi|s^2}\,ds\right)\exp\left(L\left(\log\frac{1}{\tilde t_\xi}\right)^r\right)\right)$$

$$\leqslant E(t_{2q-1},\xi)\exp\left(C\log\frac{1}{\tilde t_\xi}\right) \leqslant E(t_{2q-1},\xi)\exp\left(C\log\frac{1}{t_\xi}\right) \leqslant |\xi|^C E(t_{2q-1},\xi).$$

Estimate in $Z_{osc}(M,N)$. Using (5.3) we have the following estimate in $Z_{osc}(M,N)$:

$$E(t_{2q-1},\xi) \leqslant E(t_\xi,\xi)\exp\left(C\int_{t_\xi}^{t_{2q-1}}\left|\Re\psi(s)\right|ds + C\log\frac{1}{t_\xi}\right)$$

$$\leqslant E(t_\xi,\xi)\exp\left(C\left(\log\frac{1}{t_\xi}\right)^r + C\int_{t_\xi}^{t_{2q-1}}\left|\phi_2(s)\right|ds + C\log\frac{1}{t_\xi}\right)$$

$$\leqslant E(t_\xi,\xi)\exp\left(C_1\log\frac{1}{t_\xi}\right) \leqslant |\xi|^{C_1}E(t_\xi,\xi).$$

Combining the two estimates we get in the hyperbolic subzones

$$E(t,\xi) \leqslant |\xi|^{C_1}E(t_\xi,\xi) \quad \text{for all } \left\{(t,\xi) \in [t_\xi,T]\times\left\{|\xi| \geqslant M\right\}\right\}.$$

In the pseudo-differential zone we can repeat the estimate (3.1) with $\kappa = 1$. All together gives the desired H^∞ well-posedness with an *at most finite loss of regularity*.

6. About the C^∞ well-posedness

The goal of this section is to prove the following result:

Theorem 6.1. *Let us consider the strictly hyperbolic Cauchy problem for the* 2 *by* 2 *system*

$$\partial_t U - A(t)\partial_x U = 0, \qquad U(0,x) = U_0(x), \quad U_0 \in \mathcal{A}',$$

where the matrix $A = A(t) \in C[0,T] \cap C^2(0,T]$ satisfies the assumption (A1). *Then the solutions $U \in C^1([0,T_0],\mathcal{A}')$ which are valued in the space of analytic functionals possess the domain of dependence property in the following sense: If $U_0 \equiv 0$ in $B(\rho_0)$, then $U(t,\cdot) \equiv 0$ in $B(\rho^-(t))$ for all $t \in [0,T_0]$. Here $B(\rho)$ denotes the ball around the origin with radius ρ and $\rho^-(t) := \rho_0 - ct$, where c is a constant depending on the coefficient A and T_0 is sufficiently small such that $cT_0 < \rho_0$.*

The statements of Theorems 2.1, 2.2, 2.3 and 6.1 allow us to draw the following conclusion.

Corollary 6.2. *Let us assume additionally $A \in C[0,T]$. Then the Cauchy problems from Theorems* 2.1, 2.2 *and* 2.3 *are C^∞ well-posed.*

Now let us devote to the proof of Theorem 6.1.

Proof of Theorem 6.1. We know that without any new assumptions to $A'(t)$ the only well-posedness results we can expect for the above Cauchy problem are in spaces of analytic functions $\mathcal{A}$ or in spaces of analytic functionals $\mathcal{A}'$ with respect to x. For this reason our strategy is to follow the main steps of the proof of finite propagation speed for the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad a \in L^1(0,T),$$

from [4].

The key point for the proof to the above theorem seems to be the proof of a statement similar to Lemma 2 from [4] which reads as follows in the system case:

Lemma 6.3. *Let U belong to $C^1([0,T_0],\mathcal{A}')$ as a solution to the strictly hyperbolic Cauchy problem from Theorem* 6.1. *We define*

$$\omega(A,\mu) = \sup_{0\leqslant\tau\leqslant\mu} \int_0^{T_0-\tau} \left|A(t+\tau) - A(t)\right| dt, \quad 0 < \mu < T_0.$$

Then we have for all $|\zeta| \geqslant 1$, $\zeta = \xi + i\eta$, and for all $t \in [0,T_0]$ the estimate

$$\left|\hat{U}(t,\zeta)\right|^2 \leqslant M \exp\left(c_1\omega\left(A, \frac{T_0}{2|\zeta|}\right)|\zeta| + c_2 t|\eta|\right)\left|\hat{U}(0,\zeta)\right|^2$$

with suitable constants c_1, c_2 and M.

Proof. It is sufficient to prove the result for a sufficiently small $T_0 > 0$ because of the property $A \in C^2[T_0,T]$. In the case $A \in C^2[T_0,T]$ Theorem 6.10 from [12] gives the cone of dependence property. Let us suppose that $t \in [0,T_0]$.

Step 1: Regularization of the matrix $A = A(t)$. Let us denote by $a_- := \min_{t\in[0,T_0]} a(t)$, $a_+ := \max_{t\in[0,T_0]} a(t)$ and in the same way we may introduce $b_-, b_+, c_-, c_+, d_-, d_+$. Due to the smallness of T_0 and the continuity of the matrix on $[0,T_0]$ we are able to introduce instead of (A1) the condition

$$(\text{A1})' \quad \Delta(t_1,t_2,t_3,t_4) = \big(a(t_1) - d(t_2)\big)^2 + 4b(t_3)c(t_4) \geqslant \delta \quad \text{for all } t_1,t_2,t_3,t_4 \in [0,T_0]$$

and with an eventually smaller, but positive constant δ. Now let us define the matrix $\tilde{A}$ with entries $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$, where

$$\tilde{a}(t) = \begin{cases} a(t) & \text{if } 0 \leqslant t \leqslant T_0, \\ m_a & \text{if } t > T_0, \end{cases} \qquad m_a = \frac{1}{T_0}\int_0^{T_0} a(t)\,dt.$$

It is clear that there exists a time $t_0 \in [0,T_0]$ with $a(t_0) = m_a$. In the same way we define $\tilde{b}, \tilde{c}, \tilde{d}$.

Now we are in position to define the desired regularization of A. It is the matrix $B(t,\zeta)$ which is defined as follows:

$$B(t,\zeta) := \int_0^\infty \tilde{A}(t+\tau)\rho_\zeta(\tau)\,d\tau, \quad \rho_\zeta(t) = \begin{cases} -140\ t^3(t-\varepsilon)^3\varepsilon^{-7} & \text{if } t \in [0,\varepsilon], \\ 0 & \text{if } t \geqslant \varepsilon, \end{cases}$$

where $\varepsilon = T_0(2|\zeta|)^{-1}$. Among a lot of useful properties of the function ρ_ζ we only mention that it belongs to $C^2[0,\infty)$.

Step 2: Properties of $B(t,\zeta)$. Let us denote by $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ the entries of B. Then due to the definition we have for all $t \in [0, T_0]$

$$a_- \leqslant \hat{a}(t,\zeta) \leqslant a_+, \qquad b_- \leqslant \hat{b}(t,\zeta) \leqslant b_+, \qquad c_- \leqslant \hat{c}(t,\zeta) \leqslant c_+, \qquad d_- \leqslant \hat{d}(t,\zeta) \leqslant d_+.$$

Using the property $(\mathrm{A1})'$ we conclude the next statement:

Lemma 6.4. *The entries of the matrix B satisfy the condition*

$$(\mathrm{A1}) \quad \hat{\Delta}(t,\zeta) = \big(\hat{a}(t,\zeta) - \hat{d}(t,\zeta)\big)^2 + 4\hat{b}(t,\zeta)\hat{c}(t,\zeta) \geqslant \delta \quad \textit{for all } t \in [0, T_0]$$

and with a positive constant δ.

Moreover, we know that $B(\cdot,\zeta)$ is C^2 on $[0, T_0]$. As in Section 3.1 we are able to define the C^2 functions (in t) $\hat{\mu}_\pm(t,\zeta)$ and $\hat{H}(t,\zeta)$ on the interval $[0, T_0]$.

Step 3: Diagonalization of our starting system. Setting $A(t) = B(t,\zeta) + C(t,\zeta)$ we study instead of our starting system $\partial_t U - A(t)\partial_x U = 0$, $U(0,x) = U_0(x)$, the system

$$D_t V - B(t,\zeta)\zeta V - C(t,\zeta)\zeta V = 0, \qquad V(0,\zeta) = V_0(\zeta).$$

Introducing $V(t,\zeta) := \hat{H}(t,\zeta)W(t,\zeta)$ we obtain for W the Cauchy problem

$$D_t W - \hat{H}^{-1}(t,\zeta)B(t,\zeta)\hat{H}(t,\zeta)\zeta W + \hat{H}^{-1}(t,\zeta)D_t\hat{H}(t,\zeta)W - \hat{H}^{-1}(t,\zeta)C(t,\zeta)\hat{H}(t,\zeta)\zeta W = 0,$$
$$W(0,\zeta) = \hat{H}^{-1}(0,\zeta)V_0(\zeta).$$

This is equivalent to

$$\partial_t W - i\mathcal{D}(t,\zeta)\zeta W + \hat{H}^{-1}(t,\zeta)\partial_t\hat{H}(t,\zeta)W - i\hat{H}^{-1}(t,\zeta)C(t,\zeta)\hat{H}(t,\zeta)\zeta W = 0,$$
$$W(0,\zeta) = \hat{H}^{-1}(0,\zeta)V_0(\zeta),$$

where

$$\mathcal{D} = \begin{pmatrix} d_1(t,\zeta) & 0 \\ 0 & d_2(t,\zeta) \end{pmatrix} = \begin{pmatrix} \hat{\mu}_+(t,\zeta) + \hat{d}(t,\zeta) & 0 \\ 0 & \hat{\mu}_-(t,\zeta) + \hat{d}(t,\zeta) \end{pmatrix}$$

is real.

Step 4: Estimates of different terms. First we estimate the term $\Re(i\mathcal{D}\zeta W, W)$. Using that $\mathcal{D}$ is real we conclude with $\zeta = \xi + i\eta$

$$\Re(i\mathcal{D}\zeta W, W) = -\Re(\eta\mathcal{D}W, W) = -(\eta\mathcal{D}W, W),$$

and consequently,

$$\big|\Re(i\mathcal{D}\zeta W, W)\big| \leqslant c_2|\eta||W|^2.$$

Now let us devote to the term

$$\Re\big(\hat{H}^{-1}(t,\zeta)\partial_t\hat{H}(t,\zeta)W,W\big).$$

To estimate this term we have to estimate $|\partial_t\hat{a}(t,\zeta)|$, $|\partial_t\hat{b}(t,\zeta)|$, $|\partial_t\hat{c}(t,\zeta)|$ and $|\partial_t\hat{d}(t,\zeta)|$. For the first term we obtain for all $\tau\in[0,\varepsilon]$

$$\begin{aligned}\int_0^{T_0}\big|\partial_t\hat{a}(t,\zeta)\big|\,dt &= \int_0^{T_0}\left|\int_0^\infty\big(\tilde{a}(t)-\tilde{a}(t+\tau)\big)\rho_\zeta'(\tau)\,d\tau\right|dt\\ &\leqslant \int_0^{T_0-\tau}\big|a(t)-a(t+\tau)\big|\,dt\int_0^\infty\big|\rho_\zeta'(\tau)\big|\,d\tau+\int_{T_0-\tau}^{T_0}\big|a(t)-m_a\big|\,dt\int_0^\infty\big|\rho_\zeta'(\tau)\big|\,d\tau.\end{aligned}$$

Using corresponding estimates for the other terms we conclude for all $\tau\in[0,\varepsilon]$

$$\begin{aligned}&\int_0^{T_0}\big|\Re\big(\hat{H}^{-1}(t,\zeta)\partial_t\hat{H}(t,\zeta)W,W\big)\big|\,dt\\ &\quad\leqslant\left(\int_0^{T_0-\tau}\big|A(t)-A(t+\tau)\big|\,dt\int_0^\infty\big|\rho_\zeta'(\tau)\big|\,d\tau+\int_{T_0-\tau}^{T_0}\big|A(t)-A_0\big|\,dt\int_0^\infty\big|\rho_\zeta'(\tau)\big|\,d\tau\right)(W,W),\end{aligned}$$

where A_0 denotes the matrix $\binom{m_a\ m_b}{m_c\ m_d}$.

Finally, let us estimate

$$\Re\big(\hat{H}^{-1}(t,\zeta)C(t,\zeta)\hat{H}(t,\zeta)\zeta W,W\big).$$

It is important to understand the estimate for $C(t,\zeta)$ for all $t\in[0,T_0]$. We have

$$C(t,\zeta)=A(t)-\int_0^\infty\tilde{A}(t+\tau)\rho_\zeta(\tau)\,d\tau=\int_0^\infty\big(\tilde{A}(t)-\tilde{A}(t+\tau)\big)\rho_\zeta(\tau)\,d\tau.$$

As in the previous estimate we conclude for all $\tau\in[0,\varepsilon]$

$$\begin{aligned}&\int_0^{T_0}\big|\Re\big(\hat{H}^{-1}(t,\zeta)C(t,\zeta)\hat{H}(t,\zeta)\zeta W,W\big)\big|\,dt\\ &\quad\leqslant\left(\int_0^{T_0-\tau}\big|A(t)-A(t+\tau)\big|\,dt\int_0^\infty\rho_\zeta(\tau)\,d\tau+\int_{T_0-\tau}^{T_0}\big|A(t)-A_0\big|\,dt\int_0^\infty\rho_\zeta(\tau)\,d\tau\right)|\zeta|(W,W)\\ &\quad\leqslant\left(\int_0^{T_0-\tau}\big|A(t)-A(t+\tau)\big|\,dt+\int_{T_0-\tau}^{T_0}\big|A(t)-A_0\big|\,dt\right)|\zeta|(W,W).\end{aligned}$$

In this way we have estimates for all terms appearing in the energy estimate.

Step 5: Estimate of the energy. To derive the energy estimate we define

$$\omega(A,\mu)=\sup_{\tau\in[0,\mu]}\int_0^{T_0-\tau}\left|A(t+\tau)-A(t)\right|dt,\quad \mu\in(0,T_0).$$

Defining the energy $E(W)(t,\zeta)=(W(t,\zeta),W(t,\zeta))$ we conclude

$$\partial_t E(W)(t,\zeta)=2\Re(\partial_t W,W)=2\Re(i\mathcal{D}\zeta W,W)-2\Re\big(\hat{H}^{-1}(t,\zeta)\partial_t\hat{H}(t,\zeta)W,W\big)$$
$$+2\Re\big(\hat{H}^{-1}(t,\zeta)C(t,\zeta)\hat{H}(t,\zeta)\zeta W,W\big).$$

Using the above estimates from the previous step and a Gronwall argument implies immediately

$$E(W)(t,\zeta)\leqslant M\exp\left(c_1\omega\left(A,\frac{T_0}{2|\zeta|}\right)|\zeta|+c_2t|\eta|\right)E(W)(0,\zeta)$$

with suitable constants c_1,c_2 and M. The backward transformation leads to the desired result. □

We denote the ball around the origin with radius ρ by $B(\rho)$ and define $\rho^{\pm}(t):=\rho_0\pm\frac{c_2}{2}t>0$, where $2\rho_0>c_2T_0$ (cf. with Theorem 6.1). An immediate consequence of Lemma 6.3 is the following result:

Lemma 6.5. *Let us consider the strictly hyperbolic Cauchy problem for the* 2 *by* 2 *system*

$$\partial_t U-A(t)\partial_x U=0,\qquad U(0,x)=U_0(x).$$

If $U_0\in\mathcal{A}'$ with support in $B(\rho_0)$, then there exists a unique solution $U\in C([0,T_0],\mathcal{A}')$. For all $t\in[0,T_0]$ the solution has its support in $B(\rho^+(t))$.

Proof. Due to Paley–Wiener theorem for all $\varepsilon>0$ there exists a constant C_ε such that for all $|\zeta|\geqslant 1$ it holds

$$\left|\hat{U}_0(\zeta)\right|^2\leqslant C_\varepsilon\exp\big((2\rho_0+\varepsilon)|\eta|+\varepsilon|\zeta|\big).$$

The solution U belongs to $C^1([0,T_0],\mathcal{A}')$. By Lemma 6.3 we have

$$\left|\hat{U}(t,\zeta)\right|^2\leqslant M\exp\left(c_2t|\eta|+c_1\omega\left(A,\frac{T_0}{2|\zeta|}\right)|\zeta|\right)\left|\hat{U}_0(\zeta)\right|^2$$
$$\leqslant M_\varepsilon\exp\left(\big(2\rho^+(t)+\varepsilon\big)|\eta|+\left(c_1\omega\left(A,\frac{T_0}{2|\zeta|}\right)+\varepsilon\right)|\zeta|\right).$$

We remark that by Lebesgue's convergence theorem $\lim_{|\zeta|\to\infty}c_1\omega(A,\frac{T_0}{2|\zeta|})=0$. Moreover, the function ω is increasing in $|\zeta|^{-1}$. Therefore, there exists an increasing function $\varphi(p)$ satisfying $\varphi(0)=0$ such that $c_1\omega(A,\frac{T_0}{2|\zeta|})\leqslant\varphi(|\zeta|^{-1})$. Let $L_\varepsilon=\frac{1}{\varphi^{-1}(\varepsilon)}$, i.e., $\varepsilon=\varphi(L_\varepsilon^{-1})$. Then we get for $1\leqslant|\zeta|\leqslant L_\varepsilon$ the inequalities

$$\left|\hat{U}(t,\zeta)\right|^2\leqslant M_\varepsilon\exp\big(\varphi(1)L_\varepsilon\big)\exp\big(\big(2\rho^+(t)+\varepsilon\big)|\eta|+\varepsilon|\zeta|\big)\leqslant\tilde{M}_\varepsilon\exp\big(\big(2\rho^+(t)+\varepsilon\big)|\eta|+\varepsilon|\zeta|\big),$$

and for $|\zeta|\geqslant L_\varepsilon$ the inequality

$$\left|\hat{U}(t,\zeta)\right|^2 \leqslant M_\varepsilon \exp\bigl(\bigl(2\rho^+(t)+\varepsilon\bigr)|\eta|+2\varepsilon|\zeta|\bigr),$$

respectively. In conclusion, for all $\varepsilon > 0$ and for all $|\zeta| \geqslant 1$ we have

$$\left|\hat{U}(t,\zeta)\right|^2 \leqslant \tilde{M}_\varepsilon \exp\bigl(\bigl(2\rho^+(t)+\varepsilon\bigr)|\eta|+2\varepsilon|\zeta|\bigr).$$

Using again Paley–Wiener theorem we see that $U(t)$ belongs to $\mathcal{A}'(B(\rho^+(t)))$ what we wanted to prove. □

To complete the proof we recall a result from [9] about analytic functionals. If $U_0 \in \mathcal{A}'$, then

$$U_0 \equiv 0 \quad \text{in } \mathbb{R} \setminus B_0(R)$$

for a positive R, where $B_{x_0}(\rho)$ denotes the ball around x_0 with radius ρ. We find that

$$\operatorname{supp} U_0 \subset B_{\frac{R+\rho_0}{2}}\left(\frac{R-\rho_0}{2}\right) \cup B_{-\frac{R+\rho_0}{2}}\left(\frac{R-\rho_0}{2}\right).$$

Therefore, U_0 can be written as

$$U_0 = U_0^{(+)} + U_0^{(-)},$$

where

$$\operatorname{supp} U_0^{(\pm)} \subset B_{\pm\frac{R+\rho_0}{2}}\left(\frac{R-\rho_0}{2}\right).$$

By Lemma 6.5 it follows that the strictly hyperbolic Cauchy problem for the 2 by 2 system

$$\partial_t U^{(\pm)} - A(t)\partial_x U^{(\pm)} = 0, \qquad U^{(\pm)}(0,x) = U_0^{(\pm)}(x)$$

has a unique solution $U^{(\pm)} \in C([0,T_0],\mathcal{A}')$ with support in

$$B_{\pm\frac{R+\rho_0}{2}}\left(\frac{R-\rho_0}{2} + \frac{c_2}{2}t\right).$$

Since $U = U^{(+)} + U^{(-)}$ we conclude that $U \in C([0,T_0],\mathcal{A}')$ with

$$\operatorname{supp} U \subset B_{\frac{R+\rho_0}{2}}\left(\frac{R-\rho_0}{2} + \frac{c_2}{2}t\right) \cup B_{-\frac{R+\rho_0}{2}}\left(\frac{R-\rho_0}{2} + \frac{c_2}{2}t\right).$$

That is, $U(t,\cdot) \equiv 0$ in $B_0(\rho_0 - \frac{c_2}{2}t) = B(\rho^-(t))$. □

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