Positive Dependence Properties of Elliptically Symmetric Distributions

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Let \( X_1, \ldots, X_p \) have p.d.f. \( g(x_1^2 + \cdots + x_p^2) \). It is shown that (a) \( X_1, \ldots, X_p \) are positively lower orthant dependent or positively upper orthant dependent if, and only if, \( X_1, \ldots, X_p \) are i.i.d. \( N(0, \sigma^2) \); and (b) the p.d.f. of \( |X_1|, \ldots, |X_p| \) is TP\(_2\) in pairs if, and only if, \( \ln g(u) \) is convex. Let \( X_1, X_2 \) have p.d.f. \( f(x_1, x_2) = |\Sigma|^{-1/2} g((x_1, x_2) \Sigma^{-1}(x_1, x_2)) \). Necessary and sufficient conditions are given for \( f(x_1, x_2) \) to be TP\(_2\) for fixed correlation \( \rho \). It is shown that if \( f \) is TP\(_2\), for all \( \rho > 0 \), then \( (X_1, X_2)^\top \sim N(0, \Sigma) \). Related positive dependence results and applications are also considered.

1. Introduction

Recently many new positive dependence results have been obtained for the multivariate normal distribution and other related distributions, with an excellent discussion of these results given by Tong [13]. In this paper we obtain new positive dependence results for both spherically and elliptically symmetric p.d.f.'s and related distributions.

The notation \( X_1, \ldots, X_p \) and \( X \) are used interchangeably depending on the context. The multivariate normal distribution is designated by \( N(\mu, \Sigma) \). To write a matrix in terms of its elements we write \( A = \{a_{ij}\} \) and \( a_{ij} = (A)_{ij} \). If \( A = \{a_{ij}\}, A \geq 0 \) denotes \( a_{ij} \geq 0 \); and \( A \) is p.d. denotes \( A \) is positive definite.

The random variables \( X_1, \ldots, X_p \) are said to be positively lower orthant dependent (PLOD) if \( P(\bigcap(X_i \leq x_i)) \geq \Pi P(X_i \leq x_i) \) and positively upper orthant dependent (PUOD) if \( P(\bigcap(X_i > x_i)) \geq \Pi P(X_i > x_i) \). In the case

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\( p = 2 \), \( \text{PLOD} \) and \( \text{PUOD} \) coincide, and are called (Lehmann [10]) positively quadrant dependent (PQD). Total positivity of order 2 in pairs (TP, in pairs) is a stronger condition which implies \( \text{PLOD} \) and \( \text{PUOD} \). For further discussion of these concepts and other related dependence concepts as well as their interrelationships, see Barlow and Proschan [2], Tong [13] and Block and Ting [3].

2. Spherically Symmetric Distributions

A random vector \( X \) is said to have an elliptically symmetric p.d.f. \( f(x) \) (with mean 0) if \( f(x) = |\Sigma|^{-1/2} g(x^T \Sigma^{-1} x) \) for some function \( g \) and \( \Sigma \) p.d. If \( \Sigma = I \), the p.d.f. is said to be spherically symmetric. (An extensive bibliography concerning general statistical results for elliptically symmetric distributions is given by Chmielewski [4].) Clearly, due to the invariance of \( \text{PLOD} \), \( \text{PUOD} \) and \( \text{TP, in pairs} \) under positive scale transformations on each variable separately, all results obtained for spherically symmetrical distributions apply to the case \( \Sigma \) being diagonal.

**Theorem 2.1.** Let \( X = (X_1, \ldots, X_p)' \) have joint p.d.f. \( f(x) = g(x'x) \), where \(-\infty < x < \infty \) and \( E(XX') \) is finite. Then there exists \( X_i, X_j, i \neq j \), that are PQD if, and only if, \( X \sim N(0, \sigma^2 I) \).

**Proof.** Sufficiency is trivial. To show necessity, note that spherical symmetry implies (e.g., Kelker [8]) that the covariance matrix of \( X \) is of the form \( \Sigma_x = \sigma^2 I \). Thus, \( \text{cov}(X_i, X_j) = 0 \), \( X_i \) and \( X_j \) are PQD and, hence (Lehmann [10]), \( X_i \) and \( X_j \) are independent. Because \( X_i \) and \( X_j \) marginally have a spherically symmetric distribution, it now follows (Kelker [8, Lemma 5]) that \( X_i \) and \( X_j \) each have a normal distribution. This then yields the result because within the class of elliptically symmetric distributions any marginals being normal imply the entire vector has a normal distribution (Kelker [8, Lemma 4]).

Hence, if we consider using the class of spherically symmetric distributions for modeling purposes, the underlying random variables cannot be positively dependent in the relatively weak \( \text{PLOD} \) or \( \text{PUOD} \) sense unless they are normal.

**Theorem 2.2** Let \( X = (X_1, \ldots, X_p)' \) have joint p.d.f. \( f(x) = g(x'x) \), \(-\infty < x < \infty \), where \( g(u) > 0 \) for all \( u \geq 0 \) and is twice differentiable. Define \( W_i = |X_i|, i = 1, \ldots, p \). Then \( h(w) \), the joint p.d.f. of \( W_1, \ldots, W_p \), is TP, in pairs if, and only if, \( \ln g(u) \) is convex for \( u \geq 0 \).

**Proof.** Note that \( h(w_1, \ldots, w_p) = 2^p g(w_1^2 + \cdots + w_p^2), \quad w_1 \geq 0, \ldots, w_p \geq 0. \)
By symmetry, it suffices to consider $h$ to be TP in $w_1, w_2$ with $d = w_1^2 + \cdots + w_p^2$ being fixed. This is then equivalent to

$$
2^{(2p+2)} w_1 w_2 \left[ g(w_1^2 + w_2^2 + d) g''(w_1^2 + w_2^2 + d) - (g'(w_1^2 + w_2^2 + d))^2 \right] \geq 0
$$
(2.1)

for all $w_1 \geq 0$, $w_2 \geq 0$ and all $d \geq 0$. The inequality (2.1) holds if, and only if

$$
g(u) g''(u) - (g'(u))^2 \geq 0
$$
(2.2)

for all $u \geq d$ and all $d \geq 0$, which is equivalent to $\ln g(u)$ being convex for $u \geq 0$.

Das Gupta et al. [5, Corollary 3.1] in considering certain positive dependence results for elliptically symmetric distributions show that

$$
\mathbb{P}(\mathcal{W}|(X_i \leq x_i)) \geq \mathbb{P}(\mathcal{W}|(X_i \leq x_i)),
$$

where $(\Psi)_{ij} = (\Sigma)_{ii}$, $i = 1, \ldots, p$, $(\Psi)_{ij} = \lambda_i \lambda_j \sigma_{ij}$, $i \neq j$, $0 \leq \lambda_i \leq 1$, $i = 1, \ldots, p$, $(\Sigma)_{ij} = \gamma_i \gamma_j \sigma_{ij}$, $i \neq j$, $0 \leq \gamma_i \leq 1$, $i = 1, \ldots, p$, and $\lambda_i - \gamma_i \geq 0$, $i = 1, \ldots, p$. This result and the PLOD implication of Theorem 2.2 prove the following corollary.

**Corollary 2.1.** Let $X_1, \ldots, X_p$ have a joint p.d.f. $|\Sigma|^{-1/2} g(x|\Sigma^{-1}x)$, where $g(u) > 0$ and twice differentiable. If $\ln g(u)$ is convex and $\Sigma \geq 0$, then $|X_1|, \ldots, |X_p|$ are PLOD.

A number of authors have previously obtained some PLOD and PUOD results for specific elliptically symmetric distributions, e.g., Šidák [11], Khatri [9], Šidák [12] and A.-Hameed and Sampson [1].

We now consider applications of Theorem 2.2 and Corollary 2.1. In studying these applications, it is convenient to work with the distribution of $U = R^2 = \sum_{i=1}^p X_i^2$. Denote by $g(x^t x)$ the joint p.d.f. of $X_1, \ldots, X_p$, so that the p.d.f. of $U$ is given by

$$
f_U(u) = cu^{p/2-1} g(u),
$$
(2.3)

where $c$ is a constant. Thus to demonstrate the log-convexity of $g(u)$ it is equivalent to showing that $u^{1-p/2} f_U(u)$ is log-convex.

**Example 2.1** (Gamma/Normal). Let $f_U(u) = \lambda^\alpha u^\alpha e^{\lambda u (F(\alpha))}$, $u \geq 0$, and $\lambda > 0$, $\alpha > 0$. Then direct calculation shows that $g(u)$ defined by (2.3) is log-convex if $\alpha \leq p/2$, while $f_U(u)$ has decreasing failure rate (DFR) if $0 < \alpha \leq 1$, and increasing failure rate (IFR) if $\alpha \geq 1$. (The latter two results can be found in Barlow and Proschan [2, p. 75].) Observe that $\alpha = p/2$ corresponds to $(X_1, \ldots, X_p)^t \sim N(0, \sigma^2 I)$. 

EXAMPLE 2.2 (Weibull). Let \( f_{\lambda}(u) = a \lambda (\lambda u)^{a-1} e^{-(\lambda u)^a}, \ u \geq 0 \) and \( \lambda > 0, \ a > 0 \). Then the corresponding \( g(u) \) as defined in (2.3) is log-convex if \( a > 1 \). Note that this is equivalent to \( f_{\lambda}(u) \) being DFR (see Barlow and Proschan [2, p. 73]).

EXAMPLE 2.3 (F-distribution/Multivariate t distribution). The random variables \( X_1, \ldots, X_p \) have a multivariate t distribution if the p.d.f. of \( X \) is given by

\[
f(x) = \frac{\Gamma((v+p)/2)}{(\pi v)^{p/2} \Gamma(v/2) |\Sigma|^{1/2}} \left( 1 + \frac{(x - \mu)^t \Sigma^{-1}(x - \mu)}{v} \right)^{-(v+p)/2}, \quad (2.4)
\]

\(-\infty < x < \infty, \ v \geq 2, \ p \geq 1 \) and \( \Sigma \) is p.d. For spherical symmetry around the origin, i.e., \( \mu = 0, \ \Sigma = I, \ g(u) = c(1 + u/v)^{-((v+p)/2)} \), so that \( g(u) \) in (2.3) can be shown to be log-convex for \( v \geq 2 \) and \( p \geq 1 \). In this case \( U = R^2 \sim cF_{p,v} \), where \( F \) denotes the F-distribution with \( p \) and \( v \) degrees of freedom and \( c \) is a constant.

COROLLARY 2.2. Let \( X = (X_1, \ldots, X_p)' \) have p.d.f. \( g(x'x), \ -\infty < x < \infty \), where \( g(u) > 0 \) and twice differentiable. Define \( W_i = |X_i|, \ i = 1, \ldots, p \), and denote the p.d.f. of \( W = (W_1, \ldots, W_p)' \) by \( f_{\mu}(w) \). Let \( U = X'X \) and denote the p.d.f. of \( U \) by \( f_{\lambda}(u) \). If \( \log f_{\lambda}(u) \) is convex, then \( f_{\mu}(w) \) is TP, in pairs.

The condition that \( f_{\lambda}(u) \) be log-convex implies that the distribution of \( U \) is DFR. However, there are DFR distributions whose corresponding p.d.f.'s are not log-convex (Barlow and Proschan [5, p. 79]). Furthermore, from Example 2.1 it is seen that the condition that \( g \) be log-convex is weaker than \( f_{\lambda}(u) \) being DFR.

3. Elliptically Symmetric Distributions

Because of the complex nature of the \( p \)-variate elliptically symmetric density function, general TP, in pairs results appear to be analytically very difficult to obtain for these random variables and for their absolute values. There are, however, some quite interesting and insightful results for the bivariate case. We now show that under certain conditions, bivariate elliptically symmetric distributions can be TP,.

THEOREM 3.1. Let \((X, Y)' \) have joint p.d.f. \( f(x, y) = |\Sigma|^{-1/2} g((x, y)' \Sigma^{-1}(x, y)) \), \(-\infty < x < \infty, \ -\infty < y < \infty \), where \( g(u) > 0 \) for all \( u \geq 0 \) and is twice differentiable, and \( \Sigma = \{\sigma_{ij}\} \) is positive definite. Then \( f \) is TP, if, and only if, for each value of \( \xi > 0 \)
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(a) \( h''(\xi) > 0 \) and \( h''(\xi) \xi(-\rho - 1) - \rho h'(\xi) \geq 0 \) or
(b) \( h''(\xi) < 0 \) and \( h''(\xi) \xi(-\rho + 1) - \rho h'(\xi) \geq 0 \) or
(c) \( h''(\xi) = 0 \) and \( \rho h'(\xi) \leq 0 \),

where \( h(\xi) = \ln g(\xi) \) and \( \rho = \sigma_{12}/(\sigma_{11}\sigma_{22})^{1/2} \).

Proof. Observe that for fixed \( \Sigma \), showing \( f \) is TP is equivalent to showing \( g(u^2 - 2\rho uv + v^2) \) is TP in \( u, v \) for \(-\infty < u < \infty \), \(-\infty < v < \infty \) and fixed \( \rho \), where \( |\rho| < 1 \). This latter function is TP if, and only if,

\[
(g(\xi_{uv}) g''(\xi_{uv}) - g'(\xi_{uv})^2)(2u - 2\rho v)(2v - 2\rho u) - 2pg'(\xi_{uv}) g(\xi_{uv}) \geq 0 \tag{3.1}
\]

for \(-\infty < u, v < \infty \), and \( \rho \) fixed, where \( \xi_{uv} = u^2 - 2\rho uv + v^2 \).

Let \( u = r \cos \theta, v = r \sin \theta \) and \( t = \sin 2\theta \). Then (3.1) becomes

\[
(g(\xi_{rr}) g''(\xi_{rr}) - g'(\xi_{rr})^2) \xi(1 - \rho t)^{-1}(-2\rho + (1 + \rho^2) t) - \rho g'(\xi_{rr}) g(\xi_{rr}) \geq 0 \tag{3.2}
\]

for \( r \geq 0 \), \(-1 \leq t \leq 1 \) and \( \rho \) fixed, where \( \xi_{rt} = r^2(1 - \rho t) \).

Now let \( h(\xi) = \ln g(\xi) \) and further observe that (3.2) is equivalent to

\[
h''(\xi) \xi(1 - \rho t)^{-1}(-2\rho + (1 + \rho^2) t) - \rho h'(\xi) \geq 0, \tag{3.3}
\]

for \( t \geq 0 \), \(-1 \leq t \leq 1 \) and \( \rho \) fixed.

Define \( K_\rho(t) \equiv (1 - \rho t)^{-1}(-2\rho + (1 + \rho^2) t) \), so that \( K_\rho(-1) = -\rho - 1 \leq 0 \), \( K_\rho(1) = -\rho + 1 \geq 0 \) and \( K'_\rho(t) = (1 - \rho^2)(1 - \rho t)^{-2} > 0 \). Hence, \( K_\rho(t) \) ranges monotonically from \(-\rho - 1\) to \(-\rho + 1\) as \( t \) ranges from \(-1\) to \( 1\). The left hand side of (3.3) for a fixed value of \( \xi \) can be viewed as a linear function of \( K_\rho(t) \). The sign of the slope of this line is determined by the sign of \( h''(\xi) \). If \( h''(\xi) > 0 \), then the line has positive slope and (3.3) holds if, and only if, \( h''(\xi) K_\rho(-1) - \rho h'(\xi) \geq 0 \). Similarly, if \( h''(\xi) < 0 \) then (3.3) holds if, and only if, \( h''(\xi) K_\rho(1) - \rho h'(\xi) \geq 0 \). Finally, if \( h''(\xi) = 0 \), then (3.3) holds if, and only if, \(-\rho h'(\xi) \geq 0 \).

Note that if \( f(x, y) \) is TP, then \( \rho \geq 0 \). This and Theorem 3.1 yield Remark 3.1 which provides a simple method for demonstrating that an elliptically symmetric bivariate density is not TP.

Remark 3.1. Let \((X, Y)' \) have joint p.d.f. \( f(x, y) = |\Sigma|^{-1/2}g((x, y) \Sigma^{-1}(x, y)) \), \(-\infty < x < \infty \), \(-\infty < y < \infty \), where \( g \) is positive and twice differentiable and \( \Sigma = |\sigma_{ij}| \) is positive definite. If \( f \) is TP, then either \( \sigma_{12} = 0 \), or \( g'(\xi) \leq 0 \) for all \( \xi \geq 0 \) and \( \sigma_{12} > 0 \).

In Theorem 3.1 and Remark 3.1, \( \rho \) was viewed as fixed with \( |\rho| < 1 \). However, a further consideration would be to characterize those bivariate elliptically symmetric densities which are TP for all \( \rho \), \( 0 < \rho < 1 \).
THEOREM 3.2. Let \((X, Y)'\) have joint p.d.f.\( f(x, y) = | \Sigma |^{-1/2} g((x, y) \Sigma^{-1}(x, y)'), -\infty < x < \infty, -\infty < y < \infty, \) where \(g\) is positive and twice differentiable and \(\Sigma = \{\sigma_{ij}\}\) is positive definite. If \(f\) is TP\(_2\) for all \(\rho, 0 < \rho < 1,\) then \((X, Y)' \sim N(O, \Sigma)\).

Proof. Let \(\xi\) be a point such that \(h''(\xi) > 0,\) where \(h(\xi) = \ln g(\xi).\) Then by Theorem 3.1

\[
h''(\xi) \xi(-\rho - 1) - \rho h'(\xi) \geq 0 \quad (3.4)
\]

for any \(\rho\) for which \(f\) is TP\(_2\). For (3.4) to hold for \(0 < \rho < 1,\) it is equivalent that

\[
h''(\xi) (1 + 1/\rho) + h'(\xi) \leq 0 \quad (3.5)
\]

for \(0 < \rho < 1.\) Since \(h''(\xi) > 0,\) (3.5) cannot hold for all \(\rho.\) A similar argument shows \(h''(\xi) < 0\) cannot hold and, hence, by Theorem 3.1, \(h''(\xi) = 0\) for all \(\xi \geq 0,\) so that the result now follows.

As may be seen from Theorem 3.2 the difficulty in requiring \(f\) to be TP\(_2\) for all \(\rho, 0 < \rho < 1,\) arises as \(\rho \to 0^+.\) The following example shows that if we bound \(\rho\) away from zero, then such families do exist.

Example 3.1. Let \(f(x, y) = | \Sigma |^{-1/2} g((x, y) \Sigma^{-1}(x, y)'),\) where \(g(\xi) = c(\alpha) e^{-\xi^\alpha},\) where \(\alpha > 0\) and \(c(\alpha)\) is a constant depending on \(\alpha\) (see DeSimoni [6] and Goodman and Kotz [7]). Then \(h(\xi) = \ln g(\xi) = \ln c(\alpha) - \xi^\alpha,\) so that

\(h''(\xi) = -\alpha(\alpha - 1) \xi^{\alpha - 2}.\) If \(\alpha < 1,\) then \(h''(\xi) > 0,\) \(\xi > 0\) and \(h''(\xi) \xi(-\rho - 1) - \rho h'(\xi) = \alpha \xi^{\alpha - 1} (ap + \alpha - 1) \geq 0,\) whenever \(\rho \geq (1 - \alpha)/\alpha.\) (Note \((1 - \alpha)/\alpha < 1,\) whenever \(\alpha > 1/2.\) Similarly if \(\alpha > 1,\) then \(\rho \geq (\alpha - 1)/\alpha.\) (Note \((\alpha - 1)/\alpha < 1\) for \(\alpha \geq 1.\) Of course, when \(\alpha = 1,\) we have normality. Hence, for \(\frac{1}{2} < \alpha < 1, f\) is TP\(_2\) for \(1 > \rho \geq (1 - \alpha)/\alpha;\) for \(\alpha = 1, f\) is TP\(_2\) for \(1 > \rho \geq 0;\) for \(\alpha > 1, f\) is TP\(_2\) for \(1 > \rho \geq (\alpha - 1)/\alpha.\) Clearly \((\alpha - 1)/\alpha\) can be made arbitrarily close to 0 for values of \(\alpha \geq 1.\)

Example 3.2. (A bivariate t-distribution). Let \(f(x, y)\) be given by (2.4). Then \(h''(\xi) = c(\frac{1}{2}(v + 2) v^{-2} (1 + \xi/v)^{-2} \geq 0.\) It follows that \(h''(\xi) \xi(-\rho - 1) - \rho h'(\xi) \geq 0\) whenever \(-\xi + \rho v \geq 0\) which is not true for sufficiently large \(\xi.\) Therefore, \(f\) is not TP\(_2\) for any values of \(\rho\) and \(v.\)

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