Extensions of the alternating group of degree 6 in the geometry of K3 surfaces

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Abstract

We shall determine the uniquely existing extension of the alternating group of degree 6 (being normal in the group) by a cyclic group of order 4, which can act on a complex K3 surface. © 2005 Elsevier Ltd. All rights reserved.

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1. Introduction

A K3 surface $X$ is a simply connected compact complex 2-dimensional manifold which admits a nowhere vanishing holomorphic 2-form $\omega_X$. As is well known, K3 surfaces form a 20-dimensional analytic family.

In our previous note [4], we have shown, among all K3 surfaces, the unique existence of the triplet $(F, \tilde{A}_6, \rho_F)$ of a complex K3 surface $F$ and its (faithful) finite group action $\rho_F : \tilde{A}_6 \times F \rightarrow F$ of $\tilde{A}_6$ on $F$, up to isomorphisms. The group $\tilde{A}_6$ is an extension of $A_6$ (being normal in $\tilde{A}_6$) by $\mu_4$, which has also been shown to be the unique maximum finite extension of $A_6$ in the automorphism groups of K3 surfaces. (Here and hereafter, we shall employ the notation of groups as in the list of notation at the end of the Introduction.) We have also described the target K3 surface $F$; it is isomorphic to the minimal resolution of the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ given...
by the following equation, where \([S : T], [X : Y : Z]\) are the homogeneous coordinates of \(\mathbb{P}^1 \times \mathbb{P}^2\):

\[
S^2(X^3 + Y^3 + Z^3) - 3(S^2 + T^2)XYZ = 0.
\]

However, as was remarked in [4], the action of \(\tilde{A}_6\) is so invisible in the equation that it seems hard to find the abstract group structure of the (uniquely existing) group \(A_6\) from the equation above and this remains unsolved to date.

The aim of this short note is to describe the group structure of \(\tilde{A}_6\) explicitly as an abstract group (Theorem 4.1).

In contrast to the fact that \(\text{Out}(A_n) \cong C_2\) when \(n \geq 3\) and \(n \neq 6\), the very special nature of \(A_6\), that the outer automorphism group \(\text{Out}(A_6)\) is isomorphic to a bigger group \(C_2^2\), makes our humble work non-trivial and interesting. Indeed, corresponding to the three involutions, the automorphism group \(\text{Aut}(A_6)\) has three index 2 subgroups \(A_6 < G < \text{Aut}(A_6)\), which are \(S_6, \text{PGL}(2, 9)\) and \(M_{10}\). (See for instance [9, Chapter 3], [3, Chapters 10.11] or [2, Pages 4, 5]. See also Section 1.) According to Suzuki [9, Page 300], this extraordinary property of \(A_6\) also makes the classification of simple groups deep and difficult.

We first notice that there are exactly four isomorphism classes of \(A_6, \mu_4\) corresponding to the four normal proper subgroups \(A_6, S_6, \text{PGL}(2, 9)\) and \(M_{10}\) of \(\text{Aut} A_6\) (Theorem 2.3). This is purely group-theoretic and should be known to experts. We then determine which one our \(A_6\) is. The part here involves geometric arguments on \(K3\) surfaces such as the representation of the group action on the Picard lattice of a \(K3\) surface and its constraint from projective geometry of a target \(K3\) surface (Propositions 3.2 and 4.2). It turns out that our \(A_6\) is the one which arises from the last normal subgroup \(M_{10}\), the Mathieu group of degree 10 (Theorem 4.1).

It might be of interest that \(K3\) surfaces could distinguish \(M_{10}\) from \(S_6\) and \(\text{PGL}(2, 9)\) in this way.

**Notation.** We denote the symmetric group of degree \(n\), the alternating group of degree \(n\), the cyclic group of order \(n\), and the multiplicative group of order \(n\) (in \(\mathbb{C}^\times\)) as \(S_n, A_n, C_n\) and \(\mu_n \cong C_n\), respectively. Then \(\mu_n = \langle \zeta_n \rangle\) where \(\zeta_n = \exp(2\pi i/n)\). We denote the direct product of \(m\) copies of \(C_n\) as \(C_n^m\).

We denote the projective linear group (resp. projective special linear group) of \(\mathbb{P}(\mathbb{F}_q^{\otimes n})\) over the finite field \(\mathbb{F}_q\) of \(q\) elements as \(\text{PGL}(n, q)\) (resp. \(\text{PSL}(n, q)\)).

\(M_{10}\) is the Mathieu group of degree 10, which is defined to be the pointwise stabilizer subgroup of \(\{11, 12\}\) of the Mathieu group \(M_{12}\) of degree 12 under the natural action of \(M_{12}\) on the twelve element set \(\{1, 2, \ldots, 12\}\). (See for instance [3, Chapters 10,11].)

We write \(G = A.B\) when \(G\) fits in the exact sequence

\[
1 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 1.
\]

(So, for given \(A\) and \(B\), there are in general several isomorphism classes of groups of the form \(A.B\).) For instance \(\text{Aut}(A_6) = A_6.C_2^5\). We always regard \(A\) as a normal subgroup of \(A.B\).

When the exact sequence above splits, \(G\) is a semi-direct product of \(A\) and \(B\) which we shall write as \(A : B\). (Again, for given \(A\) and \(B\), there are in general several isomorphism classes of groups of the form \(A : B\).) We always regard \(A\) as a normal subgroup of \(A : B\) and \(B\) as a subgroup of \(A : B\).

We denote the commutator subgroup of \(G\) and the center of \(G\) as \([G, G]\) and \(\text{Cent}(G)\), respectively.
2. The isomorphism classes of $A_6.\mu_4$

Let $G$ be a group of the form $A_6.\mu_4$, i.e. a group which fits in the exact sequence

$$1 \longrightarrow A_6 \longrightarrow G \overset{\alpha}{\longrightarrow} \mu_4 \longrightarrow 1.$$ 

The goal of this section is to determine the isomorphism classes of such $G$ (Theorem 2.3).

We notice that both $G$ and $\text{Aut}(A_6)$ have exactly one subgroup which is isomorphic to $A_6$. In order to make our argument clear, we also recall the following very well-known fact (see for instance pages 4–5 of [2]):

**Lemma 2.1.** The group $\text{Aut}(A_6) \simeq A_6.C_2^2$ has exactly one subgroup which is isomorphic to $S_6$, $\text{PGL}(2,9)$, and $M_{10}$. These three groups are mutually non-isomorphic even as abstract groups and satisfy

$$A_6 < S_6, \text{PGL}(2,9), M_{10} < \text{Aut}(A_6),$$

where the inclusions are of index 2. In terms of the natural (conjugacy) action on $A_6$, these three subgroups are also distinguished as follows: $S_6$ switches the conjugacy classes $5A$ and $5B$ of $A_6$, but not $3A$ and $3B$; $\text{PGL}(2,9)$ switches the conjugacy classes $3A$ and $3B$ of $A_6$ but not $5A$ and $5B$; and $M_{10}$ switches both. (The notation here is also taken from [2]; see also the table in Proposition 3.2.)

So, for instance one can speak of the subgroup $M_{10}$ of $\text{Aut}(A_6)$ without any ambiguity.

Since $A_6$ is a normal subgroup of $G$, it follows that $c(g)(a) = gag^{-1} \in A_6$ for $g \in G$ and $a \in A_6$. We have then a natural group homomorphism

$$c : G \longrightarrow \text{Aut}(A_6); g \mapsto c(g); c(g)(a) = gag^{-1}.$$ 

Set

$$N := c(G)$$

and consider the homomorphism

$$\tilde{c} := (c, \alpha) : G \longrightarrow N \times \mu_4 < \text{Aut}(A_6) \times \mu_4; g \mapsto (c(g), \alpha(g)).$$

We define the natural projections

$$p_1 : N \times \mu_4 \longrightarrow N; (x, y) \mapsto x, p_2 : N \times \mu_4 \longrightarrow \mu_4; (x, y) \mapsto y.$$ 

**Lemma 2.2.** (1) $N$ is either $A_6$, $S_6$, $\text{PGL}(2,9)$ or $M_{10}$.

(2) $\tilde{c}$ is injective.

(3) $\alpha^{-1}(\mu_2) \simeq A_6 \times \mu_2$ and $\tilde{c}(\alpha^{-1}(\mu_2)) = A_6 \times \mu_2$ in $N \times \mu_4$. In particular, if $\tilde{h} \in G$ and $c(\tilde{h}) \in N \setminus A_6$, then $\alpha(\tilde{h}) = \pm \xi_4$.

**Proof.** Since $A_6$ is simple and non-commutative, the restriction $c|A_6$ is injective. Thus


Moreover, we have that $N \neq A_6.C_2^2$; otherwise $c : A_6.\mu_4 \simeq A_6.C_2^2$ is an isomorphism and $\mu_4 \simeq C_2^2$ (by the uniqueness of $A_6$ in $A_6.\mu_4$), a contradiction. Thus, we obtain (1).

Let $g \in \text{Ker}(\tilde{c})$. Then, $\alpha(g) = 1$, i.e. $g \in A_6$. Since $A_6$ is simple and non-commutative, it follows that $\text{Cent}(A_6) = \{1\}$. Thus $g = 1$ by $c(g) = 1$, and we obtain (2).
Since \( \mu_2 \) is a normal subgroup of \( \mu_4 \) of index 2 and \( \alpha \) is surjective, it follows that \( \alpha^{-1}(\mu_2) \) is a normal subgroup of \( G \) of index 2. Thus \( [\alpha^{-1}(\mu_2) : A_6] = 2 \). Take \( g \in G \) such that \( \alpha(g) = \zeta_4 \). Then \( \alpha^{-1}(\mu_2) = \langle A_6, g^2 \rangle \). Since \( [c(G) : A_6] \leq 2 \) by (1), we have \( c(g^2) \in A_6 \). Thus, \( c(g^2h^{-1}) = 1 \) for some \( h \in A_6 \). Set \( f := g^2h^{-1} \). Then \( \tilde{\alpha}(f) = (1, -1) \), whence \( \text{ord } f = 2 \) by (2), and \( fa = af \) for each \( a \in A_6 \). Thus

\[
\alpha^{-1}(\mu_2) = \langle A_6, f \rangle = A_6 \times \langle f \rangle \simeq A_6 \times \mu_2.
\]

This implies

\[
A_6 \times \mu_2 < \tilde{\alpha}(\alpha^{-1}(\mu_2)) < N \times \mu_2
\]
in \( N \times \mu_4 \), and hence \( A_6 \times \mu_2 = \tilde{\alpha}(\alpha^{-1}(\mu_2)) \), because the orders are the same. \( \square \)

The four candidates for \( N \) in Lemma 2.2(1) give four different group structures on \( G = A_6, \mu_4 \), which are all semi-direct products, indeed:

\textbf{Theorem 2.3.} There are exactly four possible group structures of \( G \), up to isomorphism. More explicitly, \( G \) is isomorphic to the following subgroup of \( N \times \mu_4 \) corresponding to each of the four candidates for \( N < \text{Aut}(A_6) \) as in (2.2)(1):

1. If \( N = A_6 \), then \( G = A_6 \times \mu_4 \). In this case, we set \( \tilde{g} = (1, \zeta_4) \). Then \( \tilde{g} \) is an order 4 element and \( G = A_6 \times \langle \tilde{g} \rangle \).
2. If \( N = S_6 \), then \( G = A_6 \cdot \langle \tilde{g} \rangle = A_6 : \langle \tilde{g} \rangle \), where \( \tilde{g} = (g, \zeta_4) \) and \( g = (1, 2) \in S_6 \).
3. If \( N = \text{PGL}(2, 9) \), then \( G = A_6 \cdot \langle \tilde{g} \rangle = A_6 : \langle \tilde{g} \rangle \), where \( \tilde{g} = (h^5, \zeta_4) \) for some element \( h \in \text{PGL}(2, 9) \) of order 10.
4. If \( N = M_{10} \), then \( G = A_6 \cdot \langle \tilde{g} \rangle = A_6 : \langle \tilde{g} \rangle \), where \( \tilde{g} = (g, \zeta_4) \) and \( g \) is an order 4 element of \( M_{10} \setminus A_6 \).

In each case, the semi-direct product structure is the natural one. We denote the groups in (1), (2), (3), (4) as \( A_6(4), S_6(2), \text{PGL}(2, 9)(2), M_{10}(2) \) respectively.

\textbf{Remark 2.4.} In the first three cases, \( \tilde{g}^2a = a\tilde{g}^2 \) for each \( a \in A_6 \) while this is not true in the last case. This difference between the first three cases and the last case is crucial in the proof of our main Theorem 4.1.

\textbf{Proof.} It is clear that the four groups \( A_6(4), S_6(2), \text{PGL}(2, 9)(2), M_{10}(2) \) satisfy the required conditions, i.e, \( G \) is of the form \( A_6, \mu_4 \) and \( c(G) = N \).

Let us next show that \( G \) is isomorphic to the group \( A_6(4), S_6(2), \text{PGL}(2, 9)(2), M_{10}(2) \), if \( N = A_6, S_6, \text{PGL}(2, 9), M_{10} \) respectively.

If \( c(G) = A_6 \), then \( \tilde{\alpha} : G \rightarrow A_6 \times \mu_4 \) is an isomorphism. Indeed, \( \tilde{\alpha} \) is injective by Lemma 2.2(2) and \( |G| = |A_6 \times \mu_4| \).

Consider the case \( N = S_6 \) or \( \text{PGL}(2, 9) \). Since \( c(G) = N \), there is an element \( \tilde{g} \) of \( G \) such that \( c(\tilde{g}) \in N \setminus A_6 \), \( \text{ord}(c(\tilde{g})) = 2 \) and \( \alpha(\tilde{g}) = \pm \zeta_4 \). Indeed, by Lemma 2.2(3), one can take a preimage of \( g \) in (2) and (3) as \( \tilde{g} \). Moreover, \( \zeta_4 \mapsto -\zeta_4 \) gives a group automorphism of \( \mu_4 \); the isomorphism class of \( G \) does not depend on the choice of the sign of \( \alpha(\tilde{g}) \). Therefore we may also adjust to \( \alpha(\tilde{g}) = \zeta_4 \) for a chosen \( g \). Since \( \tilde{\alpha} \) is injective, this also implies \( \text{ord}(\tilde{g}) = 4 \) and consequently \( G = A_6 : \langle \tilde{g} \rangle \) as claimed.

Finally consider the case \( N = M_{10} \). Note that \( M_{10} \setminus A_6 \) has no order 2 element and the order 4 elements of \( M_{10} \setminus A_6 \) form one conjugacy class of \( M_{10} \). (See for instance, [2].) Let \( g \in M_{10} \setminus A_6 \) be an order 4 element and \( \tilde{g} \in G \) be an element such that \( c(\tilde{g}) = g \). Then \( \alpha(\tilde{g}) = \pm \zeta_4 \) by
Lemma 2.2(3). Now, as in the cases (2) and (3), we may adjust to $\alpha(g) = \zeta_4$ for a chosen $g$, and $G = A_6 : \langle \tilde{g} \rangle$ as claimed in (4).

Since $A_6$ is the unique subgroup of $G$, the image $N$ of the homomorphism $c$ is uniquely determined by $G$. Thus, the four groups $A_6(4), S_6(2), \text{PGL}(2, 9)(2), M_{10}(2)$ are not isomorphic to one another. □

3. Some lattice representations

In this section, we recall some known facts about $K3$ surface from [1] and about $K3$ surfaces admitting an $A_6$-action from [4]. For details, please refer to the cited references and the references therein.

By a $K3$ surface, we mean a simply connected compact complex surface $X$ admitting a nowhere vanishing global holomorphic 2-form $\omega_X$. $K3$ surfaces form a 20-dimensional analytic family. The second cohomology group $H^2(X, \mathbb{Z})$ together with its cup product becomes an even unimodular lattice of index $(3, 19)$ and is isomorphic to the so-called $K3$ lattice

$$ L := U^{\oplus 3} \oplus E_8^{\oplus 2}, $$

where $U$ is the even unimodular hyperbolic lattice of rank 2 and $E_8$ is the negative definite even unimodular lattice of rank 8. We denote the Néron–Severi lattice of $X$ as $S(X)$. This is a primitive sublattice of $H^2(X, \mathbb{Z})$ generated by the (first Chern) classes of line bundles. The rank of $S(X)$ is called the Picard number of $X$, and is denoted as $\rho(X)$. We have $0 \leq \rho(X) \leq 20$. Denote by $T(X)$ the transcendental lattice of $X$, i.e. the minimum primitive sublattice whose $C$-linear extension contains the class $\omega_X$, or equivalently $T(X) = S(X)^\perp$ in $H^2(X, \mathbb{Z})$. If $X$ is projective, then $S(X)$ is of index $(1, r(X) - 1)$ (and vice versa), $S(X) \cap T(X) = \{0\}$ and $S(X) \oplus T(X)$ is a finite-index sublattice of $H^2(X, \mathbb{Z})$.

Let $(X, G, \rho)$ be a triplet consisting of a $K3$ surface $X$, a finite group $G$ and a faithful action $\rho : G \times X \longrightarrow X$. Then $G$ has a 1-dimensional representation on $H^0(X, \Omega^2_X) = C\omega_X$ given by $g^*\omega_X = \alpha(g)\omega_X$, and we have an exact sequence, called the basic sequence:

$$ 1 \longrightarrow G_N := \text{Ker} \alpha \longrightarrow G \xrightarrow{\alpha} \mu_1 \longrightarrow 1. $$

The importance of the basic sequence was first noticed by Nikulin [6]. We call $G_N$ the symplectic part and $\mu_1 := \langle \zeta_1 \rangle$ (resp. $I$) the transcendental part (resp. the transcendental value) of the action $\rho$. We note that if $A_6, \mu_4$ acts faithfully on a $K3$ surface then $G_N \simeq A_6$ and the transcendental part is isomorphic to $\mu_4$. This follows from the fact that $A_6$ is simple and also maximum among all finite groups acting on a $K3$ surface symplectically. This is a result of Mukai [5]. We also note that $X$ is projective if $I \geq 2$ [6].

We say that two triplets $(X, G, \rho)$ and $(X', G', \rho')$ are isomorphic if there are a group isomorphism $f : G' \simeq G$ and an isomorphism $\varphi : X' \simeq X$ such that the following diagram commutes:

$$ \begin{array}{ccc}
G \times X & \xrightarrow{\rho} & X \\
\downarrow{\text{f}} & & \downarrow{\varphi} \\
G' \times X' & \xrightarrow{\rho'} & X'
\end{array} $$

The main result of [4] is the following:

Theorem 3.1. Let $G$ be a finite group acting faithfully on a $K3$ surface $X$. Assume that $A_6 < G$ and $I \geq 2$, where $I$ is the transcendental value. Then we have $G_N = A_6$, $I = 2$ or 4, and rank $L^{G_N} = 3$. In particular, $S(X)^{G_N} = \mathbb{Z}I$, where $I$ is an ample class, and rank $T(X) = 2$. 
Consider the case having maximum $I = 4$, i.e. the case where $G$, which we shall denote as $\tilde{A}_6$, is a group of the form $A_6.\mu_4$. Then there is a unique triplet $(F, \tilde{A}_6, \rho_F)$ consisting of a K3 surface $F$ and a faithful group action $\rho_F : \tilde{A}_6 \times F \rightarrow F$ of $\tilde{A}_6$ on $F$ up to isomorphism. In particular, the isomorphism class of $\tilde{A}_6$ is unique. Moreover the K3 surface $F$ has Picard number 20 and is uniquely characterized by the following equivalent conditions:

1. $F$ is the K3 surface whose transcendental lattice $T(F) = \mathbb{Z}(t_1, t_2)$ has the intersection matrix
$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}.$$  

2. $F$ is the minimal resolution of the double cover $\overline{F}$ of the (rational) elliptic modular surface $E$ with level 3 structure. The double cover is branched along two of a total of 4 singular fibres of the same type $I_3$ and $\overline{F}$ has 6 ordinary double points.

3. $F$ is the minimal resolution of the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ given by the following equation, where $([S : T], [X : Y : Z])$ are coordinates of $\mathbb{P}^1 \times \mathbb{P}^2$:
$$S^2(X^3 + Y^3 + Z^3) - 3(S^2 + T^2)XYZ = 0.$$  

In the course of proof, we have also shown the following fact:

**Proposition 3.2.** Let $X$ be a K3 surface of Picard number 20. Assume that $X$ admits a faithful action of $G_N = A_6$. Then, as $A_6$-modules, one has the irreducible decomposition
$$S(X) \otimes \mathbb{C} = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_6.$$  

In this description, we used notation in [2] for irreducible characters/representations of $A_6$ as in the table below.

<table>
<thead>
<tr>
<th></th>
<th>1A</th>
<th>2A</th>
<th>3A</th>
<th>3B</th>
<th>4A</th>
<th>5A</th>
<th>5B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>8</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>$(1 - \sqrt{5})/2$</td>
<td>$(1 + \sqrt{5})/2$</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>8</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>$(1 + \sqrt{5})/2$</td>
<td>$(1 - \sqrt{5})/2$</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>9</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>10</td>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Proof.** Since many ingredients and some ideas in the next section have already appeared in the proof of this proposition, we give the proof for the reader’s convenience.

Recall that the order structure of $A_6$ is as follows:

<table>
<thead>
<tr>
<th>conjugacy class</th>
<th>1A</th>
<th>2A</th>
<th>3A</th>
<th>3B</th>
<th>4A</th>
<th>5A</th>
<th>5B</th>
</tr>
</thead>
<tbody>
<tr>
<td>cardinality</td>
<td>1</td>
<td>45</td>
<td>40</td>
<td>40</td>
<td>90</td>
<td>72</td>
<td>72</td>
</tr>
</tbody>
</table>

Moreover, by [6], the number of the fixed points of the symplectic action is as follows:

<table>
<thead>
<tr>
<th>$\text{ord}(g)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>X^g</td>
<td>$</td>
<td>X</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>
Set $\tilde{H}(X, Z) = H^0(X, Z) \oplus H^2(X, Z) \oplus H^4(X, Z)$. Now, by applying the topological Lefschetz fixed point formula for $G_N = A_6$, we calculate that
\[
\text{rank } \tilde{H}(X, Z)^{A_6} = \frac{1}{|A_6|} \sum_{g \in A_6} \text{tr}(g^* | \tilde{H}(X, Z)) = \frac{1}{360} (24 + 8 \cdot 45 + 6 \cdot 80 + 4 \cdot 90 + 4 \cdot 144) = 5.
\]
Since $G_N$ is trivial on $H^0(X, Z)$, $H^4(X, Z)$ and $T(X)$, one has $S(X)^{A_6} = Zl$ and $l$ is an ample primitive class of $X$. (Here we used two standard facts: one is that any K3 surface of Picard number 20 is projective (cf. [8]), because $S(X) \otimes R = H^{1,1}(X, R)$ and therefore $S(X)$ is of signature $(1, 19)$, and the other is that if $h$ is an ample class of $X$ and $G$ is a finite group acting on $X$, then $\sum g \in G g^* h$ is also an ample class which is invariant under $G$.)

Thus the irreducible decomposition of $S(X)$ by $A_6$ must be of the following form:
\[
S(X) \otimes C = \chi_1 \oplus a_2 \chi_2 \oplus a_3 \chi_3 \oplus a_4 \chi_4 \oplus a_5 \chi_5 \oplus a_6 \chi_6 \oplus a_7 \chi_7.
\]
where $a_i$ are non-negative integers. Let us determine $a_i$‘s. As in (2), using the topological Lefschetz fixed point formula and the fact that rank $T(X) = 2$, we have
\[
\chi_{\text{top}}(X^g) = 4 + \text{tr}(g^* | S(X))
\]
for $g \in A_6$. Running $g$ through the 7-conjugacy classes of $A_6$ and calculating both sides based on Nikulin’s table and the character table above, we obtain the following system of equations:
\[
\begin{align*}
20 &= 1 + 5(a_2 + a_3) + 8(a_4 + a_5) + 9a_6 + 10a_7, \\
4 &= 1 + (a_2 + a_3) + a_6 - 2a_7, \\
2 &= 1 + (2a_2 - a_3) - (a_4 + a_5) + a_7, 2 = 1 + (-a_2 + 2a_3) - (a_4 + a_5) + a_7, \\
0 &= 1 - (a_2 + a_3) + a_6, \\
0 &= 1 + \left( \frac{1 - \sqrt{5}}{2} a_4 + \frac{1 + \sqrt{5}}{2} a_5 \right) - a_6, 0 = 1 + \left( \frac{1 + \sqrt{5}}{2} a_4 + \frac{1 - \sqrt{5}}{2} a_5 \right) - a_6.
\end{align*}
\]
Now, we get the result by solving this system of Diophantine equations. □

4. Determination of the isomorphism class of $\tilde{A}_6$

Let $\tilde{A}_6$ be a group of the form $A_6, \mu_4$ which can act on a K3 surface $X$. Among the four candidates $A_6(4), S_6(2), \text{PGL}(2, 9)(2)$, and $M_{10}(2)$ (Theorem 2.3), only one is isomorphic to $\tilde{A}_6$ (Theorem 3.1). 

The aim of this section is to determine the isomorphism class of this $\tilde{A}_6$:

Theorem 4.1. $\tilde{A}_6$ is isomorphic to the group $M_{10}(2)$.

Proof. First we observe the following general result, which will also give us a strong geometric constraint on the pair $(X, \tilde{A}_6)$.

Proposition 4.2. Let $Y$ be a K3 surface and $H = H_N : \langle \tilde{h} \rangle$ be a finite subgroup of $\text{Aut}(Y)$, of transcendental part $\langle \tilde{h} \rangle \cong \mu_{2^n}$ with $n \geq 2$. Assume that rank $S(Y)^{H_N} = 1$ and $\mu = a \tilde{a}$ for all $a \in H_N$. Here $\tilde{a} := \tilde{h}^{2^{n-1}}$. Then:

1. $\chi_{\text{top}}(Y^\mu) < 0$. More precisely, $Y^\mu$ is an ample irreducible smooth curve on $Y$. 
(2) Let \( \sigma \) be an element of \( H_N \) such that \( \text{ord} \, \sigma \geq 3 \). Then \( \chi_{\text{top}}(Y^{i \sigma}) \geq 0 \).

Here \( \chi_{\text{top}}(Y^a) \) is the topological Euler number of the fixed locus \( Y^a \) of \( a \in H \).

**Proof.** Since \( \iota^*\omega_Y = -\omega_Y \), the action of \( \iota \) on \( Y \) is locally linearizable at \( P \in Y^i \) as

\[
\begin{pmatrix}
1 & 0 \\
0 & -1 \\
\end{pmatrix}.
\]

Set \( C := Y^i \). Note that \( C \neq \emptyset \). Indeed, otherwise \( \langle \tilde{h} \rangle \simeq \mu_{2^n} \) would act freely on \( Y \). However, then \( 2^i \mid \chi(O_Y) = 2 \), a contradiction. Thus \( C \neq \emptyset \) and therefore \( C \) is a smooth curve (from the local description above), possibly reducible. Since \( \iota a = a \iota \) for each \( a \in H_N \), the curve \( C \) is stable under the action of \( H_N \). Thus the class \( [C] \in S(Y) \) is \( H_N \)-invariant. Since the transcendental part of \( H \) is non-trivial, \( S \) is projective [6]. Since rank \( S(Y)^{H_N} = 1 \), there is then an ample class \( l \) such that \( S(Y)^{H_N} = Zl \). Thus \( C \) is an ample curve. Therefore, \( (C^2) > 0 \) and \( C \) is connected and hence irreducible (by the smoothness). Now we have by the adjunction formula

\[
\chi_{\text{top}}(C) = 2 - 2g(C) = -(K_Y + C.C) = -(C^2) < 0.
\]

Let us show (2). Set \( \tau = \iota \sigma \). Note that \( Y^\tau \subset Y^{r^2} \). We have \( \tau^2 = \sigma^2 \in H_N \setminus \{id\} \), because \( \iota \sigma = \sigma \iota \) and \( \text{ord} \, \sigma \geq 3 \). Thus, the set \( Y^{r^2} \) is a finite set as in Nikulin’s table in Proposition 3.2. Hence \( Y^\tau \) is also a finite set, and therefore \( \chi_{\text{top}}(Y^\tau) \geq 0 \). \( \square \)

Let us return to the proof of Theorem 4.1. Set \( G = \tilde{A}_6 \). It suffices to show that \( G \) is not isomorphic to \( A_6(4) \), \( S_6(2) \), \( \text{PGL}(2, 9)(2) \).

From now on, assuming to the contrary that \( G \) is isomorphic to one of these three groups, we shall derive a contradiction.

Let \( \tilde{g} \in G \) be an order 4 element chosen in Theorem 2.3. Set \( \iota := \tilde{g}^2 \). Recall that \( G = A_6 : \langle \tilde{g} \rangle \), \( G_N = A_6 \), and \( S(X)^{A_6} = Zl \) (Theorems 2.3 and 3.1). Moreover, by Remark 2.4, \( \iota a = a \iota \) for all \( a \in A_6 \), because we are now assuming (to the contrary) that \( G \) is isomorphic to \( A_6(4) \), \( S_6(2) \), or \( \text{PGL}(2, 9)(2) \). So, we can apply Proposition 4.2 to our \((X, G)\).

Let us recall the irreducible decomposition of \( S(X) \) in Proposition 3.2. Since \( X \) has an ample \( G \)-invariant class \( l \), we have \( \tilde{g}^*l = l \) and hence \( \tilde{g}^* \mid \chi_1 = id \). Since \( G = A_6 : \langle \tilde{g} \rangle \), we have also \( \tilde{g}^*(\chi_6) = \chi_6 \) and either \( \tilde{g}^*(\chi_2) = \chi_2 \) and \( \tilde{g}^*(\chi_3) = \chi_3 \) or \( \tilde{g}^*(\chi_2) = \chi_3 \) and \( \tilde{g}^*(\chi_3) = \chi_2 \). Since \( \iota = \tilde{g}^2 \), we have \( \iota^*(\chi_i) = \chi_i \) for each \( i = 1, 2, 3, 6 \). Since \( \iota a = a \iota \) for all \( a \in A_6 \), it follows that \( \iota^* \mid \chi_i \) are scalar multiplications by Schur’s lemma. Moreover, since \( \iota \) is of order 2, we have

\[
\iota^* \mid \chi_1 = 1, \iota^* \mid \chi_i = (-1)^{n_i} id_{\chi_i}
\]

for some \( n_i \in \mathbb{Z} \) for each \( i = 2, 3, 6 \). We have also that \( \iota^* \mid H^0(X) \oplus H^4(X) = id \), \( \iota^* \mid T(X) = -id_{T(X)} \) and rank \( T(X) = 2 \) by Theorem 3.1. Thus, by the topological Lefschetz formula, we obtain

\[
\chi_{\text{top}}(X^i) = 1 + 5 \cdot ((-1)^{n_2} + (-1)^{n_3}) + 9 \cdot (-1)^{n_6}.
\]

We have \( \chi_{\text{top}}(X^i) < 0 \) by Proposition 4.2(1). Thus

\[
((-1)^{n_2}, (-1)^{n_3}, (-1)^{n_6})
\]

must be one of

\[
(-1, 1, -1), (1, -1, -1), (-1, -1, -1).
\]
Consider first the case \((-1,1,-1)\) (resp. \((1,-1,-1)\)). Take an order 3 element \(\sigma\) of \(A_6\) from the conjugacy class 3A (resp. 3B). Note that \((\iota \sigma)^* | H^0(X) \oplus H^4(X) = id\), \((\iota \sigma)^* | T(X) = -1\) and rank \(T(X) = 2\). Then by the topological Lefschetz formula and by the character table, we calculate
\[
\chi_{\text{top}}(X^{\iota \sigma}) = \text{tr}((\iota \sigma)^* | S(X)) = 1 - 2 - 1 + 0 = -2 < 0,
\]
a contradiction to Proposition 4.2(2).

Consider next the case \((-1,-1,-1)\). Since \(\tilde{g}^* (\chi_6) = \chi_6\) and \(\iota^* | \chi_6 = -id_{\chi_6}\), it follows that the eigenvalues of \(g^* | \chi_6 = \pm \zeta_4\) and \(\text{tr}(g^* | \chi_6) = (9 - 2n) \zeta_4\), where \(n\) is the multiplicity of \(-\zeta_4\). Note that \(\text{tr}(g^* | T(X)) = \zeta_4 - \zeta_4 = 0\).

So, if \(\tilde{g}^*(\chi_2) = \chi_3\), then \(\text{tr}(g^* | \chi_2 \oplus \chi_3) = 0\), and
\[
\chi_{\text{top}}(X^g) = 2 + \text{tr}(g^* | S(X)) = 3 + (9 - 2n) \cdot \zeta_4 \not\in \mathbb{Z},
\]
a contradiction to the obvious fact that \(\chi_{\text{top}}(X^g) \in \mathbb{Z}\).

If \(\tilde{g}^*(\chi_2) = \chi_2\), then for the same reason as above, the eigenvalues of
\[
\tilde{g}^* | \chi_2 \oplus \chi_3 \oplus \chi_6
\]
are \(\pm \zeta_4\). Let \(n\) be the multiplicity of \(-\zeta_4\). Using \(\dim \chi_2 \oplus \chi_3 \oplus \chi_6 = 19\), we have
\[
\chi_{\text{top}}(X^g) = 2 + \text{tr}(g^* | S(X)) = 3 + (19 - 2n) \zeta_4 \not\in \mathbb{Z},
\]
again a contradiction.

This completes the proof of Theorem 4.1. \(\square\)

**Remark 4.3.** According to Mukai [5], there are three finite non-commutative simple groups which can act faithfully on a \(K\) surface. They are \(A_6\), \(\text{PSL}(2,7)\) and \(A_5\). The extension problems on \(\text{PSL}(2,7)\) and \(A_5\) have been treated in [7] and [11,12] respectively.

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**References**


