Equi-isoclinic planes of Euclidean spaces

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Dedicated to the memory of J.J. Seidel

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ABSTRACT

In a Euclidean space, a \( p \)-set of equi-isoclinic planes is a set of \( p \) isoclinic planes of which each pair has the same non-zero angle.
The Euclidean 4-space \( E^4 \) contains a unique congruence class of quadruples of equi-isoclinic planes, whereas quintuples of equi-isoclinic planes do not exist in \( E^4 \).

In the following a method is given to derive sets of equi-isoclinic planes in Euclidean spaces. We find again the well-known sets of equi-isoclinic planes of \( E^4 \). The quadruples of equi-isoclinic planes in \( E^5 \) are derived. It turns out that \( E^5 \) contains one congruence class of such quadruples which are not flat quadruples and one congruence class of quintuples of equi-isoclinic planes, whereas sextuples of equi-isoclinic planes do not exist in \( E^5 \).

It appears that the symmetry group of that quintuple is isomorphic to the symmetric group \( S_5 \).

1. INTRODUCTION

Two planes in Euclidean 4-space \( E^4 \) have two angles. They are the stationary values of the angle between the lines \( l \) and \( m \), if \( m \) runs through one plane and \( l \) runs through the other plane. If the two angles are equal, then the planes are said to be isoclinic. In [7] the authors pose the problem of finding the maximum number \( v_2(n, r) \) of equi-isoclinic \( n \)-subspaces in \( E^r \) with the parameter \( \lambda \), that is, of pairwise isoclinic \( n \)-subspaces in \( E^r \) with the same angle \( \varphi \), \( \cos^2 \varphi = \lambda \), and the maximum number \( v(n, r) \) of equi-isoclinic \( n \)-subspaces in \( E^r \). For instance, they prove

\[(1) \quad (n - r \lambda) v_2(n, r) \leq r(1 - \lambda).\]

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They also determine the values of \( v_\lambda(n, 2n) \) for all \( n \) and \( \lambda \), by use of the Hurwitz matrix equations. In particular, they prove \( v(2, 4) = 4 \). In [1] and [7] all the values of \( v_\lambda(n, r) \) are determined for some even integers \( r \).

In this paper results on the plane problem of Lemmens and Seidel are improved.

The central problem of the present paper is to determine \( v(2, 5) \). It turns out that \( v(2, 5) = 5 \). That is, in \( E^5 \) the maximum number of pairwise isoclinic planes with the same angle \( \varphi \), is 5 with \( \cos^2 \varphi = \frac{1}{4} \).

The planes denoted by \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5 \) are generated respectively by

\[
\begin{align*}
e_{11} &= (1, 0, 0, 0, 0), \\
e_{12} &= \left( \frac{1}{2}, 0, \frac{\sqrt{3}}{2}, 0, 0 \right), \\
e_{21} &= \left( \frac{1}{2}, 0, \frac{1}{2\sqrt{3}}, 0, \frac{\sqrt{2}}{3} \right), \\
e_{22} &= \left( 0, \frac{1}{2}, 0, \frac{\sqrt{3}}{2}, 0 \right), \\
e_{31} &= \left( \frac{1}{2}, 0, \frac{1}{2\sqrt{3}}, 0, \frac{\sqrt{2}}{3} \right), \\
e_{32} &= \left( 0, \frac{1}{2}, 0, \frac{\sqrt{3}}{2}, 0 \right), \\
e_{41} &= \left( \frac{1}{2}, 0, -\frac{1}{\sqrt{3}}, \frac{1}{2}, -\frac{1}{\sqrt{6}} \right), \\
e_{42} &= \left( 0, \frac{1}{2}, 0, -\frac{1}{\sqrt{2}} \right), \\
e_{51} &= \left( \frac{1}{2}, 0, -\frac{1}{\sqrt{3}}, \frac{1}{2}, -\frac{1}{\sqrt{6}} \right), \\
e_{52} &= \left( 0, \frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{\sqrt{2}} \right).
\end{align*}
\]

Moreover it appears that \( E^5 \) contains a unique congruence class of quintuples of equi-isoclinic planes.

In Section 2, a reformulation of the plane problem, in terms of matrices, is given.

In Section 3, we proceed the investigation of congruence classes of triangles of equi-isoclinic planes in \( E^6 \). It is seen that there are two families of congruence classes of triangles of equi-isoclinic planes in \( E^6 \) denoted by \( T_{\lambda, \omega}^{(1)} \) and \( T_{\lambda, \omega}^{(2)} \), depending on two and one real parameters, respectively. For a triangle \( T_{\lambda, \omega}^{(1)} \), the parameters satisfy

\[(2) \quad \cos \omega \geq \frac{1}{2\lambda\sqrt{\lambda}} (3\lambda - 1),\]

equality holding if and only if the triangle spans a subspace \( E^4 \). For a triangle \( T_{\lambda}^{(2)} \), we have

\[(3) \quad 0 < \lambda \leq \frac{1}{4}.\]

If \( \lambda = \frac{1}{4} \) then \( T_{\lambda}^{(2)} \) spans a subspace \( E^5 \) (Theorem 1).

It is shown that there is a one-to-one correspondence between triangles \( T_{\lambda, \omega}^{(1)} \) in \( E^6 \) and equilateral triangles of a complex projective space \( \mathbb{C}P^2 \) with edge \( \arccos \sqrt{\lambda} \) and shape invariant \( \cos \omega \). For the shape invariant we refer to [1].

On the other hand we see that for each triangle \( T_{\lambda}^{(2)} = \{ \Delta_1', \Delta_2', \Delta_3' \} \) there exist three lines \( p_1 \subset \Delta_1', p_2 \subset \Delta_2', p_3 \subset \Delta_3' \), which are equiangular with the angle \( \arccos \sqrt{\lambda} \).
It turns out that for each triangle $T = \{\Gamma_1, \Gamma_2, \Gamma_3\}$ of equi-isoclinic planes with the parameter $\lambda$, there exist three lines $p_1 \subset \Gamma_1$, $p_2 \subset \Gamma_2$, $p_3 \subset \Gamma_3$, which are equiangular with the angle $\arccos \sqrt{\lambda}$ if and only if $T$ is a triangle $T^{(1)}_\lambda$, $T^{(1)}_\lambda$ or $T^{(2)}_\lambda$ (Theorem 2).

In Section 4, let $\Gamma_1, \Gamma_2, \ldots, \Gamma_v$ be $v$ ($v \geq 4$) equi-isoclinic planes with the parameter $\lambda$. We give a necessary and sufficient condition, in terms of matrices, for the existence of $v$ lines $p_1 \subset \Gamma_1$, $p_2 \subset \Gamma_2$, $\ldots$, $p_v \subset \Gamma_v$, which are equiangular with the angle $\arccos \sqrt{\lambda}$ (Theorem 3).

In Section 5, quadruples of equi-isoclinic planes in Euclidean spaces $E^4$ and $E^5$ are derived. It appears that $E^5$ contains a unique congruence class of quadruples of equi-isoclinic planes which are not contained in a subspace $E^4$, and a unique congruence class of quintuples of equi-isoclinic planes (Theorems 4 and 5).

We close this paper with two interesting properties of the quintuple of $E^5$, which turns out to be regular, that is, its symmetry group is isomorphic to $S_5$ (Theorem 6), and on the other hand if one considers in $E^5$ the orthogonal subspaces of the direct sums of each pair of planes of the regular quintuple, one gets 10 equiangular lines with angle $\arccos \frac{1}{3}$, which form the maximum equilateral point-set in $E^5$ [8, Theorem 7]. Finally we discuss a conjecture on $v(2, 2r)$ for any $r$. It seems that for any $r \geq 2$, $v(2, 2r) = r^2$.

2. REFORMULATION OF THE PLANE PROBLEM

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_v$ each provided with an orthonormal basis, be $v$ equi-isoclinic planes with the parameter $\lambda$, which span a space $E^r$. Let the inner products of the basis of $\Gamma_i$ with the basis of $\Gamma_j$ be collected in the matrix $A_{ij}$.

Then the block matrix

$$A = \begin{pmatrix} I & A_{12} & \cdots & A_{1v} \\ A_{21} & I & \cdots & A_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ A_{v1} & A_{v2} & \cdots & I \end{pmatrix}$$

of order $2v$, is symmetric positive semi-definite and of rank $r$. Furthermore Lemmens and Seidel showed in [7] that

$$A_{ij} = \lambda I, \quad i \neq j = 1, \ldots, v.$$  

Conversely, to any matrix with these properties there exist $v$ equi-isoclinic planes in $E^r$ with the parameter $\lambda$. Thus, in order to investigate $v$-tuples of equi-isoclinic planes which span $E^r$ we ask for symmetric matrices $M = \lambda^{-1/2}(A - I_{2v})$ whose smallest eigenvalue has multiplicity $2v - r$.

Any such $M$-matrix $M$ with smallest eigenvalue $\mu_0$ leads to $v$ equi-isoclinic planes in $E^r$ with the parameter $\lambda = 1/\mu_0^2$.

A $M$-matrix $M$ is then partitioned into square blocks ($M_{ij}$) of order 2 with $M_{ii} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ for all $i = 1, \ldots, v$, and $M_{ij} \in O(2)$ for all $i \neq j$.  

207
However, \( v \)-tuples of equi-isoclinic planes are not characterized by single matrices but by the classes of such matrices under the equivalence relation generated by the following operations:

**Operation 1.** Multiplication of any block row by \( U \in O(2) \) and the corresponding block column by \( U^T \) (replacement of an orthonormal basis by another one has no effect on the plane).

**Operation 2.** Interchange of two block rows and simultaneously, of the corresponding block columns (the order of the planes is irrelevant).

Thus we arrive at the following formulation, in terms of matrices, of the plane problem.

**Problem 1.** For all positive integers \( r \), find the integers \( v \) and the classes under Operations 1 and 2 of \( M \)-matrices \( M \) with order \( 2v \) and zero blocks on the diagonal and orthonormal blocks elsewhere, whose smallest eigenvalue has multiplicity \( 2v - r \). What is this smallest eigenvalue?

For any given \( r \) what is the maximum \( v(2, r) \)?

3. ASSOCIATED MATRICES OF TRIANGLES

Triangles of equi-isoclinic planes are obtained from 3 block row symmetric \( M \)-matrices \( M = (M_{ij}) \).

Each of these matrices can, by multiplication of the two last block columns by \( M_{12}^T \) and \( M_{13}^T \) respectively and corresponding block rows by \( M_{12} \) and \( M_{13} \) respectively, be brought into one of the following forms.

\[
\begin{pmatrix}
0 & I & I \\
I & 0 & R_\omega \\
I & R_{-\omega} & 0
\end{pmatrix}
\]

where \( R_\omega = \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix} \), with \( \omega \in [0, 2\pi] \) and \( I = I_2 \).

\[
\begin{pmatrix}
0 & I & I \\
I & 0 & S_\omega \\
I & S_{-\omega} & 0
\end{pmatrix}
\]

where \( S_\omega = \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix} \), \( \omega \in [0, 2\pi] \).

The \( M \)-matrix (I) satisfies

\[
M^3 = 3M + 2 \cos \omega \times I_6.
\]

Its characteristic polynomial \( C_1(X) \) is equal to

208
$$C_1(X) = (X^3 - 3X - 2\cos\omega)^2$$

$$= \left(X + 2\cos\left(\frac{\omega - \pi}{3}\right)\right)^2 \left(X + 2\cos\left(\frac{\omega + \pi}{3}\right)\right)^2 \left(X - 2\cos\left(\frac{\omega}{3}\right)\right)^2.$$ 

The smallest eigenvalue of $(I)$ is equal to $-2\cos\left(\frac{\omega - \pi}{3}\right)$ with multiplicity 2. Therefore $(I)$ leads to 3 equi-isoclinic planes which span a 4-space with the parameter \( \lambda = \frac{1}{4\cos^2\left(\frac{\omega - \pi}{3}\right)} \).

Now let \( \omega \) be a real such that \( \cos \omega > \frac{(\lambda - 1)}{2\lambda} \). Then the 3 equi-isoclinic planes form a triangle denoted by \( T^{(1)}_{\lambda,\omega} \) which lies in \( E^6 \). The planes denoted by \( \Delta_1, \Delta_2, \Delta_3 \) are generated respectively by

\[
\begin{align*}
    u_{11} &= (1, 0, 0, 0, 0, 0), & u_{12} &= (0, 1, 0, 0, 0, 0); \\
    u_{21} &= (\sqrt{\lambda}, 0, (1 - \lambda)^{1/2}, 0, 0, 0), & u_{22} &= (0, \sqrt{\lambda}, 0, (1 - \lambda)^{1/2}, 0, 0); \\
    u_{31} &= (\sqrt{\lambda}, 0, (1 - \lambda)^{-1/2}(\sqrt{\lambda}\cos\omega - \lambda), \sqrt{\lambda}(1 - \lambda)^{-1/2}\sin\omega, \\
          &\quad (1 - \lambda)^{-1/2}(1 - 3\lambda + 2\sqrt{\lambda}\cos\omega)^{1/2}, 0), \\
    u_{32} &= (0, \sqrt{\lambda}, -\sqrt{\lambda}(1 - \lambda)^{-1/2}\sin\omega, (1 - \lambda)^{-1/2}(\sqrt{\lambda}\cos\omega - \lambda), 0, \\
          &\quad (1 - \lambda)^{-1/2}(1 - 3\lambda + 2\sqrt{\lambda}\cos\omega)^{1/2}).
\end{align*}
\]

The condition (2) is represented in the rectangle \([0, 1] \times [-1, 1]\) where we represent the function \( f(\sqrt{\lambda}) = \frac{(\lambda - 1)}{2\lambda} \).

We circle the dimension of the Euclidean space spanned by the triangle of equi-isoclinic planes. Hatched is the region where there is no triangle of equi-isoclinic planes.

By a conjugation with \( R_{\omega/2} \) the matrix \( (II) \) can be brought into the form

$$
\begin{pmatrix}
0 & R_{\omega/2} & R_{\omega/2} \\
R_{-\omega/2} & 0 & S_0 \\
R_{-\omega/2} & S_0 & 0
\end{pmatrix}.
$$

By multiplication of the first block column by \( R_{\omega/2} \) and the first block row by \( R_{-\omega/2} \) this matrix is equivalent to

$$
(II)' \begin{pmatrix}
0 & I & I \\
I & 0 & S_0 \\
I & S_0 & 0
\end{pmatrix},
$$

whose characteristic polynomial \( C_2(X) \) is equal to

$$C_2(X) = (X^2 - 1)^2(X^2 - 4).$$

The smallest eigenvalue \(-2\) has multiplicity 1, thus \((II)' \) leads to a triangle of equi-isoclinic planes with the parameter \( \frac{1}{4} \) which spans a 5-space. If \((II)' \) is positive
definite, it leads to a triangle, denoted by $T^{(2)}$, of equi-isoclinic planes with the parameter $\lambda$, $0 < \lambda < \frac{1}{4}$, which spans a 6-space.

These planes are generated by

$$
\begin{align*}
    v_{11} &= (1, 0, 0, 0, 0, 0), & v_{12} &= (0, 1, 0, 0, 0, 0); \\
    v_{21} &= (\sqrt{\lambda}, 0, (1 - \lambda)^{1/2}, 0, 0, 0), & v_{22} &= (0, \sqrt{\lambda}, 0, (1 - \lambda)^{1/2}, 0, 0); \\
    v_{31} &= (\sqrt{\lambda}, 0, \sqrt{\lambda}(1 - \sqrt{\lambda})^{1/2}(1 + \sqrt{\lambda})^{-1/2}, 0, \\
            & \quad (1 - \sqrt{\lambda})^{1/2}(1 + \sqrt{\lambda})^{-1/2}(1 + 2\sqrt{\lambda})^{1/2}, 0), \\
    v_{32} &= (0, \sqrt{\lambda}, 0, -\sqrt{\lambda}(1 - \sqrt{\lambda})^{-1/2}(1 + \sqrt{\lambda})^{1/2}, 0, \\
            & \quad (1 - \sqrt{\lambda})^{-1/2}(1 + \sqrt{\lambda})^{1/2}(1 - 2\sqrt{\lambda})^{1/2}).
\end{align*}
$$

In Fig. 1 is represented the triangle $T^{(2)}_\lambda$ on the straight line $\omega = 0$. We have then proved the following theorem.

**Theorem 1.** (1) There exist in $E^6$ two families of congruence classes of equi-isoclinic planes denoted by $T^{(1)}_{\lambda,\omega}$ and $T^{(2)}_{\lambda}$.  

(2) The parameters $\lambda$ and $\omega$ of $T^{(1)}_{\lambda,\omega}$ satisfy:

$$
\cos \omega \geq \frac{3\lambda - 1}{2\lambda \sqrt{\lambda}}.
$$
equality holding if and only if $T^{(1)}_{\lambda,0} \subset E^4$.

(3) The parameter $\lambda$ of $T^{(2)}_{\lambda}$ satisfies:

$$0 < \lambda \leq \frac{1}{4}, \quad \lambda = \frac{1}{4} \text{ if and only if } T^{(2)}_{\lambda} \subset E^5.$$
4.2 Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_v$ be $v$ equi-isoclinic planes with the parameter $\lambda$. Lemmens and Seidel gave in [7] a necessary condition for the existence of $v$ lines $p_1 \subset \Gamma_1, p_2 \subset \Gamma_2, \ldots, p_v \subset \Gamma_v$, which are equiangular with the angle $\arccos \sqrt{\lambda}$.

Let $M = (M_{ij})$ with $M_{ii} = 0$ for all $i = 1, \ldots, v$, and $M_{1j} = I$ for all $j = 2, \ldots, v$, be an associated $M$-matrix of the set of the $v$ planes.

**Theorem 3.** Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_v$ be $v$ $(v \geq 4)$ equi-isoclinic planes with the parameter $\lambda$. There exist $v$ lines $p_1 \subset \Gamma_1, p_2 \subset \Gamma_2, \ldots, p_v \subset \Gamma_v$, which are equiangular with the angle $\arccos \sqrt{\lambda}$ if and only if an associated $M$-matrix is such that $M_{ij} \in \{\pm R_0, \pm S_0\}$ for all $i < j, i = 2, \ldots, v - 1, j = 3, \ldots, v$.

**Proof.** Let $M$ be an $M$-matrix which satisfies the above condition, omitting the second column of each block column and the corresponding rows we obtain a $B$-matrix $B$ of order $v$ with zero on the diagonal and $\pm 1$ elsewhere which is associated to $v$ equiangular lines $p_1 \subset \Gamma_1, p_2 \subset \Gamma_2, \ldots, p_v \subset \Gamma_v$, with the angle $\arccos \sqrt{\lambda}$ [8].

Conversely, let $e_{11}, \ldots, e_{v1}$ be $v$ unit vectors which generate $p_1, \ldots, p_v$, respectively. Now let $e_{12}, \ldots, e_{v2}$ be $v$ unit vectors such that $(e_{11}, e_{12}), \ldots, (e_{v1}, e_{v2})$ are orthonormal basis of $\Gamma_1, \ldots, \Gamma_v$, respectively. Consider the associated $M$-matrix $M = (M_{ij})$ with elements $M_{ii} = 0$ for all $i = 1, \ldots, v$, and $M_{1j} = I$ for all $j = 2, \ldots, v$. Following Theorem 2 each subtriangle contained in the set $\Gamma_1, \ldots, \Gamma_v$ is congruent to $T^{(1)}_{\lambda, 0}, T^{(1)}_{\lambda, \pi}$ or $T^{(2)}_{\lambda}$. Therefore the $M$-matrix $M$ satisfies $M_{ij} \in \{\pm R_0, S_0\}$ for all $i < j, i = 2, \ldots, v - 1, j = 3, \ldots, v$. Now omitting the second column of each block column and the corresponding rows, we obtain a matrix of order $v$ which is associated to the set $p_1, \ldots, p_v$. Hence this matrix has zero on the diagonal and $\pm 1$ elsewhere, and thus $\cos \omega = \pm 1$. Therefore $\omega = 0$ or $\pi$ and the theorem follows. $\square$

5. QUADRUPLES

Quadruples of equi-isoclinic planes are obtained from 4 block row symmetric $M$-matrices.

Each of these matrices can, by suitable changing of block rows (and corresponding block columns) and multiplication of certain block columns by elements of $O(2)$ (and corresponding block rows by the respective transpose of those elements of $O(2)$), be brought into one of the following forms.

\[
(A) \begin{pmatrix}
0 & I & I & I \\
I & 0 & R_{\omega_1} & R_{\omega_2} \\
I & R_{-\omega_1} & 0 & R_{\omega_3} \\
I & R_{-\omega_2} & R_{-\omega_3} & 0
\end{pmatrix}, \quad (B) \begin{pmatrix}
0 & I & I & I \\
I & 0 & R_{\omega_1} & R_{\omega_2} \\
I & R_{-\omega_1} & 0 & S_0 \\
I & R_{-\omega_2} & S_0 & 0
\end{pmatrix},
\]
In the following we investigate quadruples of equi-isoclinic planes in four- and five-dimensional Euclidean spaces. We first prove the following lemma.

**Lemma 1.** Let \( \{ \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \} \) be a quadruple of equi-isoclinic planes such that each of its subtriangles spans a 4-space. Then the quadruple spans a 4-space.

**Proof.** Note that two isoclinic planes span a 4-space. Thus \( \Gamma_3 \) and \( \Gamma_4 \) lie in the 4-space spanned by \( \Gamma_1 \) and \( \Gamma_2 \). \( \square \)

5.1 Now we ask for quadruples of equi-isoclinic planes in a 4-space. To that end let’s assume that the subtriangles are in a 4-space. Then the principal submatrices of order 6 of the associated M-matrix \( (A) \) are of type (I) and hence the real numbers \( \omega_1, \omega_2, \omega_3 \) satisfy

\[
\cos \omega_1 = \cos \omega_2 = \cos \omega_3 = f(\sqrt{\lambda}).
\]

Therefore, we arrive to two distinct cases using our equivalence relation.

*Case 1*. \( \omega_1 = \omega_2 = \omega_3 \).

*Case 2*. \( \omega_1 = -\omega_2 = \omega_3 \).

**Case 1.** The M-matrix \( (A) \) satisfies

\[
M^4 - 6M^2 - 8 \cos \omega \times M - (1 + 2 \cos 2\omega)I_8 = 0.
\]

Its characteristic polynomial \( C_A(X) \) is equal to

\[
C_A(X) = \left( X^4 - 6X^2 - 8 \cos \omega \times X - (1 + 2 \cos 2\omega) \right)^2
\]

\[
= \left( X + 1 + 2 \sin \frac{\omega}{2} \right)^2 \left( X + 1 - 2 \sin \frac{\omega}{2} \right)^2 \left( X - 1 + 2 \cos \frac{\omega}{2} \right)^2
\]

\[
\times \left( X - 1 - 2 \cos \frac{\omega}{2} \right)^2.
\]

The smallest eigenvalue \( \mu_0 \) is equal to \( \mu_0 = -1 - 2 \sin \frac{\omega}{2} \), with multiplicity 2. Hence the matrix \( (A) \) leads to a quadruple of equi-isoclinic planes with the parameter 

\[
\frac{1}{(1 + 2 \sin \frac{\omega}{2})^2}, \omega \in [0, 2\pi],
\]

which spans a 6-space.

**Case 2.** The M-matrix \( (B) \) satisfies

\[
M^4 - 6M^2 - 8 \cos^3 \omega \times M + (9 - 12 \cos^2 \omega)I_8 = 0.
\]

Its characteristic polynomial \( C'_A(X) \) is equal to
\[ C'_A(X) = (X^4 - 6X^2 - 8\cos^3 \omega \times X + (9 - 12\cos^2 \omega))^2 \]
\[ = \left(X + 2\cos\left(\frac{\omega - \pi}{3}\right)\right)^2 \left(X + 2\cos\left(\frac{\omega + \pi}{3}\right)\right)^2 \]
\[ \times \left(X - \cos \omega + \sqrt{3 + \cos^2 \omega}\right)^2 \left(X - \cos \omega - \sqrt{3 + \cos^2 \omega}\right)^2. \]

If \( \omega \in [0, \frac{\pi}{2}] \) then the smallest eigenvalue \( \mu_0 \) is equal to \( \mu_0 = -2\cos(\omega - \frac{\pi}{3}) \), if \( \omega \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \) then \( \mu_0 = \cos \omega - \sqrt{3 + \cos^2 \omega} \) and if \( \omega \in \left[\frac{3\pi}{2}, 2\pi\right[ \) then \( \mu_0 = -2\cos(\omega + \frac{\pi}{3}) \). The associated matrix of one of the subtriangles is equivalent to
\[
\begin{pmatrix}
0 & I & I \\
I & 0 & R_{3\omega} \\
I & R_{-3\omega} & 0
\end{pmatrix}
\]

Since this triangle spans a 4-space then \( \cos 3\omega = \cos \omega \) and hence \( \omega = 0, \frac{\pi}{2}, \pi \) or \( \frac{3\pi}{2} \). For \( \omega = \frac{\pi}{2} \) or \( \frac{3\pi}{2} \) it is easily seen that the multiplicity of \( \mu_0 = -\sqrt{3} \) is 4. The corresponding matrices lead to the same congruence class of quadruples of equi-isoclinic planes with the parameter \( \frac{1}{3} \) which spans a 4-space. The planes are given in [7] and independently in [4] as a regular quadruple of a complex projective line \( \mathbb{C}P^1 \), that is its symmetry group is isomorphic to the symmetric group \( S_4 \).

Following Fig. 1, formula (1) with \( n = 2, r = 4, \lambda = \frac{1}{3} \) and the above investigation, we arrive to the following results obtained in [7] by use of the Hurwitz matrix equations.

**Theorem 4.** \( v_1(2, 4) = 2 \) if \( 0 \leq \lambda < \frac{1}{4} \), \( v_2(2, 4) = 3 \) if \( \frac{1}{4} \leq \lambda < \frac{1}{3} \), \( v_{1/3}(2, 4) = 4 \), \( v_{1}(2, 4) = 3 \) if \( \frac{1}{3} < \lambda < 1 \), \( v_{1}(2, 4) = 1 \).

**Corollary 1.** \( v(2, 4) = 4 \).

### 5.2
We now look for quadruples of equi-isoclinic planes which span a 5-space. Necessarily each of these quadruples contains a subtriangle whose associated \( M \)-matrix of order 6 is of type \( (I)' \) and \( \lambda = \frac{1}{4} \). Indeed, if all the submatrices of order 6 associated to the subtriangles are all of type \( (I) \), then as a consequence to Lemma 1 the quadruple lies in 4-space, which contradicts our assumption. The condition \( \lambda = \frac{1}{4} \) is obvious. On the other hand, if a subtriangle of such a quadruple is of type \( T_{\lambda, \omega}^{(1)} \) then necessarily \( \omega = \pi \) because \( \lambda = \frac{1}{4} \). The following types are obtained:

**Case 1.** \( (B) \) with \( \omega_1 = \omega_2 = \pi \).

**Case 2.** \( (C) \) with \( \omega_1 = \pi \).

**Case 3.** \( (D) \).

**Case 1.** The characteristic polynomial \( C_B(X) \) is equal to
\[ C_B(X) = (X + 3)(X - 1)^4(X + 1)(X - \sqrt{5})(X + \sqrt{5}). \]
The smallest eigenvalue $\lambda_0 = -3$ has multiplicity 1, thus the matrix leads to a quadruple of equi-isoclinic planes with the parameter $\frac{1}{9}$ which spans a 7-space.

Case 2. Put $S_{o2} = \left( \begin{smallmatrix} g & b \\ b & -a \end{smallmatrix} \right)$ with $a^2 + b^2 = 1$. The characteristic polynomial $C_C(X)$ is equal to

$$C_C(X) = X^8 - 12X^6 + (4 + 4a)X^5 + (42 - 4a)X^4 - 24(1 + a)X^3 + (-36 + 24a - 4b^2)X^2 + (20 + 20a + 8b^2)X + 4a^2 - 20a + 1.$$ 

The rest in the euclidean division of $C_C(X)$ by $(X + 2)^3$ is equal to

$$R(X) = (-292 - 248a - 4b^2)X^2 + (-1036 - 908a + 8b^2)X + 4a^2 - 788a - 863.$$ 

Observe that for any $a$ and $b$ such that $a^2 + b^2 = 1$, the rest does not vanish. That is, the matrix $(C)$ does not lead to a quadruple which spans a 5-space.

Case 3. Put $S_{o1} = \left( \begin{smallmatrix} g & s \\ s & -c \end{smallmatrix} \right)$ and $S_{o2} = \left( \begin{smallmatrix} g & b \\ b & -a \end{smallmatrix} \right)$ with $c^2 + s^2 = a^2 + b^2 = 1$. The characteristic polynomial $C_D(X)$ is equal to

$$C_D(X) = X^8 - 12X^6 + (42 - 4a - 4c - 4sb - 4ac)X^4 + (-52 + 8c + 8a + 8ac + 8sb)X^2 + 9 - 8sbc - 4c - 4a + 8asbc + 4sb + 4c^2 - 4ac - 8asb + 8s^2a + 4s^2b^2 + 4a^2 + 8b^2c + 4a^2c^2.$$ 

If the parameter $\lambda$ equals $\frac{1}{4}$ and the quadruple spans a 5-space then $C_D(X)$ must be equal to $C(X) = (X + 2)^3(X - 2)^3(X^2 - \alpha^2), \alpha \in \mathbb{R}$, because $C_D(X)$ is even. Therefore $C_D(X) = C(X)$ if and only if $\alpha = 0; a + c + ac + sb = \frac{3}{2};$

$$9 - 4(a + c + ac - sb) - 8sb \left( a + c - ac - \frac{sb}{2} \right) + 4c^2 + 8s^2a + 4a^2 + 8b^2c + 4a^2c^2 = 0.$$ 

Now replacing in the last equation $s^2$ by $1 - c^2$, $b^2$ by $1 - a^2$ and $sb$ by $-\frac{3}{2} - a - c - ac$ we obtain

$$3 + 4ac + 6a + 6c + 4a^2 + 4c^2 = 0.$$ 

Which is equivalent to

$$3(a + c + 1)^2 + (a - c)^2 = 0.$$ 

Hence $a = c = -\frac{1}{2}$ and necessarily $s = \frac{\sqrt{3}}{2}, b = -\frac{\sqrt{3}}{2}$ or $s = -\frac{\sqrt{3}}{2}$ and $b = \frac{\sqrt{3}}{2}$. That is, $\omega_1 = \frac{2\pi}{3}$ and $\omega_2 = \frac{4\pi}{3}$ or $\omega_1 = \frac{4\pi}{3}$ and $\omega_2 = \frac{2\pi}{3}$. By a conjugation with $S_0$, the first matrix can be brought into the second one. Thus we arrive to the following.

**Theorem 5.** There is a unique congruence class of quadruples of equi-isoclinic planes which span a 5-space. And the parameter is equal to $\frac{1}{4}$. 215
6. QUINTUPLES

6.1 As a consequence to (1) the upper bound of \( v_{1/4}(2, 5) \) is 5. In the following we show that this upper bound is reached. Following the above results two possibilities can occur, quintuples with the parameter \( \frac{1}{3} \) or \( \frac{1}{4} \). First, we prove the nonexistence of quintuples of equi-isoclinic planes with the parameter \( \frac{1}{3} \) which span \( E^5 \). If there exists a quintuple of equi-isoclinic planes with the parameter \( \frac{1}{3} \) which spans \( E^5 \), then all its subquadruples of equi-isoclinic planes with the parameter \( \frac{1}{3} \) span a 4-space. Indeed, there is no quadruple with the parameter \( \frac{1}{3} \) which spans a subspace \( E^5 \). More precisely, all the subquadruples contained in that quintuple are congruent. As a consequence to Section 4 and [3], the quintuple of planes is nothing but an \( F \)-regular quintuple of points, with zero-shape invariant and distance \( \arccos \frac{1}{\sqrt{3}} \), in the complex projective space \( \mathbb{C}P^4 \). Moreover this quintuple is 4-regular [3], which is in contradiction with the proposition of [3], where it is shown the nonexistence of such quintuples in \( \mathbb{C}P^4 \).

If there exists a quintuple of equi-isoclinic planes with the parameter \( \frac{1}{4} \) which spans \( E^5 \), then all its subquadruples of equi-isoclinic planes with the parameter \( \frac{1}{4} \) span a 5-space. Indeed, there is no quadruple with the parameter \( \frac{1}{4} \) which spans a subspace \( E^4 \). More precisely, all the subquadruples contained in that quintuple are congruent and their associated \( M \)-matrices are equivalent to \( (D) \) with \( \omega_1 = \frac{2\pi}{3} \) and \( \omega_2 = \frac{4\pi}{3} \), as a consequence to Theorem 5.

All the symmetries contained in such matrices are \( S_0, S_{2\pi/3} \) and \( S_{4\pi/3} \). Now let \( M = (M_{ij})_{i,j} \) be a symmetric block matrix of order 10 such that \( M_{25} = S_0 \), \( M_{24} = S_{2\pi/3} \) and \( M_{34} = S_{4\pi/3} \).

**Lemma 2.** If in such a matrix a block row contains twice the same symmetry then the matrix can not lead to a quintuple spanning a 5-space.

**Proof.** If the matrix is such that \( M_{25} = S_0 \) or \( S_{2\pi/3} \) thus in each case it contains one principal submatrix of order 8 which is equivalent to type \( (D) \) with \( \omega_1 = \omega_2 \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\} \). Using calculation done in case 3 of Section 5 with \( a = c \) and \( b = s \) it follows that \(-2\) can not be an eigenvalue with multiplicity 3 and thus the associated quadruple does not span a 5-space and the lemma is proved.

As a consequence to Lemma 2 it turns out that the \( M \)-matrix with the above properties necessarily satisfies \( M_{25} = S_{4\pi/3}, M_{35} = S_{2\pi/3} \) and \( M_{45} = S_0 \).

Its characteristic polynomial \( C_5(X) \) is equal to

\[
C_5(X) = (X + 2)^5(X - 2)^5.
\]

This implies that the matrix \( M \) is associated to a quintuple of equi-isoclinic planes with the parameter \( \frac{1}{4} \) which spans a 5-space. Then 5 is the maximum number of equi-isoclinic planes in \( E^5 \).

Hence from this investigation follow the results.
Theorem 6. \( v_\lambda(2, 5) = 2 \) if \( 0 \leq \lambda < \frac{1}{4} \), \( v_{1/4}(2, 5) = 5 \), \( v_\lambda(2, 5) = 3 \) if \( \frac{1}{4} < \lambda < \frac{1}{3} \), \( v_{1/3}(2, 5) = 4 \), \( v_\lambda(2, 5) = 3 \) if \( \frac{1}{3} < \lambda < 1 \), \( v_1(2, 5) = 1 \).

Corollary 2. \( v(2, 5) = 5 \).

Remark. In case \( 0 < \lambda < \frac{1}{4} \) and \( \frac{1}{4} < \lambda < 1 \) the equi-isoclinic planes span a 4-space. That is, for these values of \( \lambda \) there is no set of equi-isoclinic planes with the parameter \( \lambda \), which span \( E^5 \).

Theorem 7. The symmetry group of the quintuple of equi-isoclinic planes which span \( E^5 \) is isomorphic to the symmetric group \( S_5 \).

Proof. The ordered 10-tuples \( (e_{11}, e_{12}, e_{21}, e_{22}, e_{31}, e_{32}, e_{41}, e_{42}, e_{51}, e_{52}) \) and

\[
\begin{pmatrix}
    e'_{21} \\
    e'_{22}
\end{pmatrix} = S_{2\pi/3} \begin{pmatrix}
    e_{21} \\
    e_{22}
\end{pmatrix}, \quad
\begin{pmatrix}
    e'_{31} \\
    e'_{32}
\end{pmatrix} = R_{2\pi/3} \begin{pmatrix}
    e_{31} \\
    e_{32}
\end{pmatrix},
\begin{pmatrix}
    e'_{41} \\
    e'_{42}
\end{pmatrix} = \begin{pmatrix}
    e_{41} \\
    e_{42}
\end{pmatrix},
\begin{pmatrix}
    e'_{51} \\
    e'_{52}
\end{pmatrix} = S_{4\pi/3} \begin{pmatrix}
    e_{51} \\
    e_{52}
\end{pmatrix},
\end{pmatrix}
\]

have equal Gram matrices. Then there exists \( U \in O(5) \) such that \( Ue_{ij} = e'_{ij} \) for all \( i, j, i = 1, \ldots, 5, \ j = 1, 2 \). Hence \( (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5) \) and \( (\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_1) \) are congruent ordered quintuples. This isometry induces the cycle \( (1, 2, 3, 4, 5) \) of the symmetric group \( S_5 \).

On the other hand the ordered 10-tuples \( (e_{11}, e_{12}, e_{21}, e_{22}, e_{31}, e_{32}, e_{41}, e_{42}, e_{51}, e_{52}) \) and

\[
\begin{pmatrix}
    e''_{21} \\
    e''_{22}
\end{pmatrix} = \begin{pmatrix}
    e_{21} \\
    e_{22}
\end{pmatrix}, \quad
\begin{pmatrix}
    e''_{31} \\
    e''_{32}
\end{pmatrix} = S_0 \begin{pmatrix}
    e_{31} \\
    e_{32}
\end{pmatrix}, \quad
\begin{pmatrix}
    e''_{41} \\
    e''_{42}
\end{pmatrix} = S_{2\pi/3} \begin{pmatrix}
    e_{41} \\
    e_{42}
\end{pmatrix},
\begin{pmatrix}
    e''_{51} \\
    e''_{52}
\end{pmatrix} = S_{4\pi/3} \begin{pmatrix}
    e_{51} \\
    e_{52}
\end{pmatrix},
\end{pmatrix}
\]

have equal Gram matrices. Then there exists \( U' \in O(5) \) such that \( U'e_{ij} = e''_{ij} \) for all \( i, j, i = 1, \ldots, 5, \ j = 1, 2 \). Hence \( (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5) \) and \( (\Gamma_2, \Gamma_1, \Gamma_3, \Gamma_4, \Gamma_5) \) are congruent ordered quintuples. \( U' \) induces the transposition \( (1, 2) \) of the symmetric group \( S_5 \).

Since the cycle \( (1, 2, 3, 4, 5) \) and the transposition \( (1, 2) \) generate \( S_5 \) the symmetric group of our quintuple is isomorphic to \( S_5 \). \( \Box \)
Theorem 8. In $E^5$ the orthogonal subspaces of the direct sums of each pair of planes of the regular quintuple are 10 equiangular lines with angle $\arccos \frac{1}{3}$.

Proof. In $E^5$ the orthogonal subspaces of the direct sums $\Gamma_1 \oplus \Gamma_2$, $\Gamma_1 \oplus \Gamma_3$, $\Gamma_4 \oplus \Gamma_5$ are lines generated respectively by:

$$v_1^2 = (0, 0, 0, 0, 1), \quad v_1^3 = \left(0, 0, \frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right), \quad v_1^4 = \left(0, 0, \frac{\sqrt{2}}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

$$v_1^5 = \left(0, 0, \frac{\sqrt{2}}{3}, -\frac{6}{3}, \frac{1}{3}\right), \quad v_2^3 = \left(-\frac{\sqrt{2}}{3}, 0, 0, 0, \frac{1}{3}\right).$$

$$v_2^4 = \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3}, -\frac{1}{3}\right), \quad v_2^5 = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{1}{3}, -\frac{1}{3}\right).$$

$$v_3^4 = \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{1}{3}, -\frac{1}{3}\right), \quad v_3^5 = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3}, -\frac{1}{3}\right).$$

$$v_4^4 = \left(\frac{\sqrt{6}}{3}, 0, -\frac{\sqrt{2}}{3}, 0, \frac{1}{3}\right).$$

These lines are equiangular with angle $\arccos \frac{1}{3}$ and form the maximum equilateral point-set of $\mathbb{R}P^4$ discussed in [8]. □

Remark. We have just seen that in $E^5$ the maximum 5-tuple of equi-isoclinic planes with the parameter $\frac{1}{4}$ leads to the maximum equilateral 10-tuple of $\mathbb{R}P^4$, with edge $\arccos \frac{1}{3}$. This is false in general, for instance in $E^3$ if one considers the orthogonal subspaces of the direct sums of each pair of lines of the maximum equilateral 6-tuple, then we get a 15-tuple of lines which clearly can not be a set of equiangular lines in $E^3$.

6.2 Recently in [10], the authors constructed nine lines in $\mathbb{C}^3$ which are equiangular with the angle $\arccos \frac{1}{5}$.

The vertices of this 9-tuple produce 9 equi-isoclinic planes in $E^6$ with the parameter $\frac{1}{4}$. On the other hand, from (1) the upper bound of $v_{1/4}(2, 6)$ is equal to 9. Hence $v_{1/4}(2, 6) = 9$. Whereas, $v_3(2, 6) \leq 9$ for all $0 < \lambda \leq \frac{1}{4}$. Indeed, if $0 < \lambda < \frac{1}{3}$, $\frac{6(1-\lambda)}{2-6\lambda} \leq 9$ if and only if $0 < \lambda \leq \frac{1}{4}$. It seems that $v(2, 6) = 9$.

However the physicists conjecture that for any $r, r \geq 2$, we may construct $r^2$ lines in $\mathbb{C}^r$ which are equiangular with the angle $\arccos \frac{1}{\sqrt{r+1}}$. If their conjecture holds then $v_{1/(r+1)}(2, 2r) = r^2$, because from (1), $r^2$ is an upper bound of $v_{1/(r+1)}(2, 2r)$. We conjecture that $v(2, 2r) = r^2$.

It appears that the physicists’ conjecture holds for $r = 4$ [10] and for $r = 8$ [6].

Besides the work of [7] on this subject, there are two works in [5] and [9] where the authors investigate, independently with two distinct approaches, congruences classes of general triangles. Triangles of equi-isoclinic planes are also discussed in [5] and [9], where they are called semi-regular isoclinic. Their results are here improved.
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REFERENCES


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