J. Math. Anal. Appl. 370 (2010) 431–452

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Locally minimal topological groups 1 ☆

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ARTICLE INFO

Article history:
Received 15 October 2009
Available online 13 April 2010
Submitted by R. Gornet

To the memory of Ivan Prodanov (1935–1985)

Keywords:
Locally minimal group
Minimal group
Group without small subgroups
Group uniformly free from small subgroups
Pseudo-convex set
GTG set
Locally GTG group
Locally bounded group
Bounded group

ABSTRACT

The aim of this paper is to go deeper into the study of local minimality and its connection to some naturally related properties. A Hausdorff topological group (G,τ) is called locally minimal if there exists a neighborhood U of 0 in τ such that U fails to be a neighborhood of zero in any Hausdorff group topology on G which is strictly coarser than τ. Examples of locally minimal groups are all subgroups of Banach–Lie groups, all locally compact groups and all minimal groups. Motivated by the fact that locally compact NSS groups are Lie groups, we study the connection between local minimality and the NSS property, establishing that under certain conditions, locally minimal NSS groups are metrizable. A symmetric subset of an abelian group containing zero is said to be a GTG set if it generates a group topology in an analogous way as convex and symmetric subsets are unit balls for pseudonorms on a vector space. We consider topological groups which have a neighborhood basis at zero consisting of GTG sets. Examples of these locally GTG groups are: locally pseudoconvex spaces, groups uniformly free from small subgroups (UFSS groups) and locally compact abelian groups. The precise relation between these classes of groups is obtained: a topological abelian group is UFSS if and only if it is locally minimal, locally GTG and NSS. We develop a universal construction of GTG sets in arbitrary non-discrete metric abelian groups, that generates a strictly finer non-discrete UFSS topology and we characterize the metrizable abelian groups admitting a strictly finer non-discrete UFSS group topology. Unlike the minimal topologies, the locally minimal ones are always available on "large" groups. To support this line, we prove that a bounded abelian group G admits a non-discrete locally minimal and locally GTG group topology iff $|G| \geqslant \mathfrak{c}$.

1. Introduction

Minimal topological spaces have been largely studied in the literature [7]. Minimal topological groups were introduced independently by Choquet, Doïtchinov [14] and Stephenson [36]: a Hausdorff topological group (G,τ) is called minimal if there exists no Hausdorff group topology on G which is strictly coarser than τ. The major problem that determined the theory of minimal abelian groups was establishing *precompactness* of the abelian minimal groups (Prodanov–Stoyanov's theorem [13, Theorem 2.7.7]; for recent advances in this field see [9,10,13]).

☆ The first named author was partially supported by MTM 2008-04599. The third author was partially supported by SRA, grants P1-0292-0101 and J1-9643-0101. The other authors were partially supported by supported by Spanish MICINN MTM2009-14409-C02-01.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.04.026

Generalizations of minimality were recently proposed by various authors. Relative minimality and co-minimality were introduced by Megrelishvili in [24] (see also [11,34]). The notion of local minimality (see Definition 2.1) was introduced by Morris and Pestov in [26] (see also Banakh [5]). A stronger version of this notion was used in [12] to characterize the locally compact subgroups of infinite products of locally compact groups.

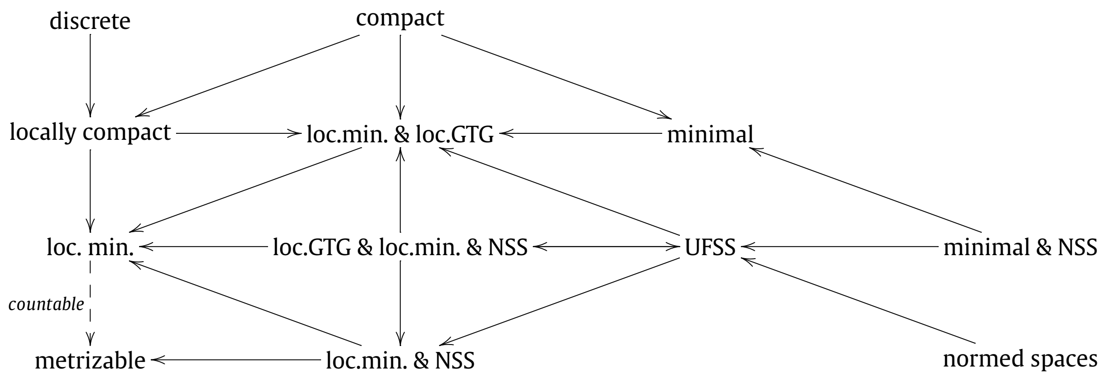
We start Section 2 with some permanence properties of local minimality (with respect to taking closed or open subgroups). We prove in Theorem 2.8 that $nw(G) = w(G)$ for every locally minimal group (in particular, all countable locally minimal groups are metrizable). Section 2.2 is dedicated to the NSS groups. Let us recall, that a topological group (G, τ) is called *NSS group* (resp., *NSnS group*) if a suitable zero neighborhood contains only the trivial (resp., normal) subgroup. The relevance of the NSS property comes from the fact that it characterizes the Lie groups within the class of locally compact groups. Since local minimality generalizes local compactness, it is quite natural to investigate local minimality combined with the NSS property. It turns out that locally minimal abelian NSS groups are metrizable (Proposition 2.13), which should be compared with the classical fact that locally compact NSS groups are Lie groups (hence, metrizable). We do not know whether "abelian" can be removed here (cf. Question 6.7).

Section 3 is dedicated to a property, introduced by Enflo [15] that simultaneously strengthens local minimality and the NSS property. A Hausdorff topological group is UFSS (Uniformly Free from Small Subgroups) if its topology is generated by a single neighborhood of zero in a natural analogous way as the unit ball of a normed space determines its topology (a precise definition is given in Definition 3.1 below). In Proposition 3.8 we show that locally minimal NSnS precompact groups are UFSS (hence minimal NSS abelian groups are UFSS). Local minimality presents a common generalization of local compactness, minimality and UFSS. Since the latter property is not sufficiently studied, in contrast with the former two, we dedicate Section 3.2 to a detailed study of the permanence properties of this remarkable class. We show in Proposition 3.12 that UFSS is stable under taking subgroups, extensions (in particular, finite products), completions and local isomorphisms.

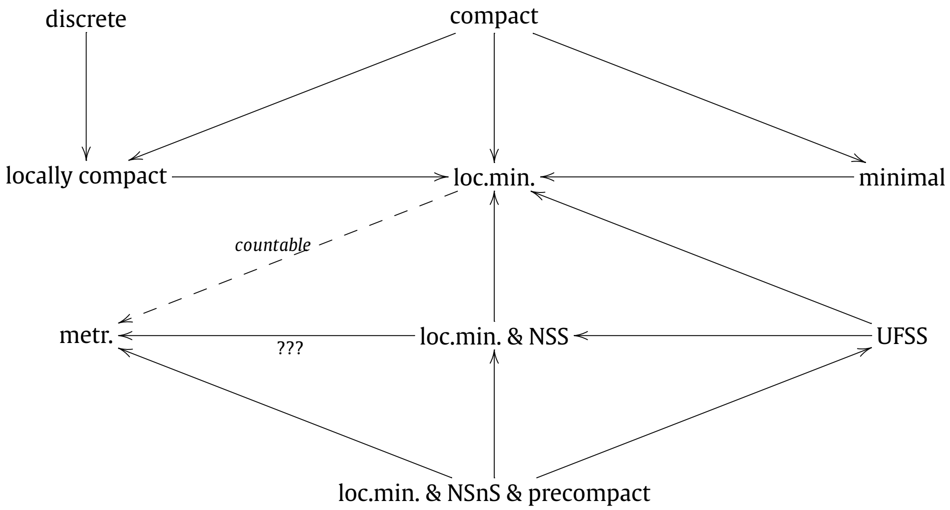
In Section 4 we introduce the concept of a GTG set that, roughly speaking, is a symmetric subset U of a group G containing 0, with an appropriate convexity-like property (i.e., these sets are generalizations of the symmetric convex sets in real vector spaces, see Definition 4.2). A topological group is called locally GTG, if it has a base of neighborhoods of 0 that are GTG sets. Since locally precompact abelian groups, as well as UFSS groups, are locally GTG, this explains the importance of this new class. On the other hand, minimal abelian groups are precompact, so minimal abelian groups are both locally minimal and locally GTG. We prove in Theorem 5.10 that a Hausdorff abelian topological group is UFSS iff it is locally minimal, NSS and locally GTG. According to a theorem of Hewitt [20], the usual topologies on the group $\mathbb{T}$ and the group $\mathbb{R}$ have the property that the only strictly finer locally compact group topologies are the discrete topologies. Since locally minimal locally GTG topologies generalize the locally compact group topologies, it would be natural to ask whether the groups $\mathbb{T}$ and $\mathbb{R}$ admit stronger non-discrete locally minimal locally GTG topologies. In Corollary 4.24 we give a strongly positive answer to this question for the large class of all non-totally disconnected locally compact metrizable abelian groups and for the stronger property of UFSS topologies. To this end we develop, in Theorem 4.21, a universal construction of GTG sets in arbitrary non-discrete metric abelian groups, that generates a strictly finer non-discrete UFSS topology.

The description of the *algebraic structure* of locally minimal abelian groups seems to be an important problem. Its solution for the class of compact groups by the end of the fifties of the last century brought a significant development of the theory of infinite abelian groups. This line was followed later also in the theory of minimal groups, but here the problem is still open even if solutions in the case of many smaller classes of abelian groups are available [9, §§4.3, 7.5], [13, chapter 5]. Unlike the minimal topologies, the locally minimal ones are always available on "large" groups. To support this line, we prove in Theorem 5.18 that a bounded abelian group G admits a non-discrete locally minimal and locally GTG group topology iff $|G| \geqslant \mathfrak{c}$ (and this occurs precisely when G admits a non-discrete locally compact group topology). Analogously, in another small group (namely, $\mathbb{Z}$), the non-discrete locally minimal and locally GTG group topologies are not much more than the minimal ones (i.e., they are either UFSS or have an open minimal subgroup, see Example 5.16). This line will be pursued further and in more detail in the forthcoming paper [4] where we study also the locally minimal groups that can be obtained as extensions of a minimal group via a UFSS quotient group.

In the next diagram we collect all implications between all properties introduced so far:



All the implications denoted by a solid arrow are true for arbitrary abelian groups, those that require some additional condition on the group are given by dotted arrows accompanied by the additional condition in question. We give separately in the next diagram only those arrows that are valid for all, not necessarily abelian, topological groups.



We dedicate this paper to the memory of Ivan Prodanov, whose ideas and results form the core of the theory of minimal groups.

1.1. Notation and terminology

The subgroup generated by a subset X of a group G is denoted by $\langle X\rangle$, and $\langle x\rangle$ is the cyclic subgroup of G generated by an element $x \in G$. The order of $x \in G$ is denoted by $o(x)$. The abbreviation $K \leqslant G$ is used to denote a subgroup K of G.

We use additive notation for a not necessarily abelian group, and denote by 0 its neutral element. We denote by $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{P}$ the sets of positive natural numbers, non-negative integers and primes, respectively; by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, and by $\mathbb{T}$ the unit circle group which is identified with $\mathbb{R}/\mathbb{Z}$. The cyclic group of order $n > 1$ is denoted by $\mathbb{Z}(n)$. For a prime p the symbol $\mathbb{Z}(p^\infty)$ stands for the quasicyclic p-group and $\mathbb{Z}_p$ stands for the p-adic integers.

The *torsion part* $t(G)$ of an abelian group G is the set $\{x \in G\colon\ nx = 0 \text{ for some } n \in \mathbb{N}\}$. Clearly, $t(G)$ is a subgroup of G. For any $p \in \mathbb{P}$, the *p-primary component* G_p of G is the subgroup of G that consists of all $x \in G$ satisfying $p^n x = 0$ for some positive integer n. For every $n \in \mathbb{N}$, we put $G[n] = \{x \in G\colon\ nx = 0\}$. We say that G is *bounded* if $G[n] = G$ for some $n \in \mathbb{N}$. If $p \in \mathbb{P}$, the *p-rank* of G, $r_p(G)$, is defined as the cardinality of a maximal independent subset of $G[p]$ (see [32, Section 4.2]). The group G is *divisible* if $nG = G$ for every $n \in \mathbb{N}$, and *reduced*, if it has no divisible subgroups beyond $\{0\}$. The *free rank* $r(G)$ of the group G is the cardinality of a maximal independent subset of G. The *socle* of G, $Soc(G)$, is the subgroup of G generated by all elements of prime order, i.e. $Soc(G) = \bigoplus_{p\in\mathbb{P}} G[p]$.

We denote by $\mathcal{V}_\tau(0)$ (or simply by $\mathcal{V}(0)$) the filter of neighborhoods of the neutral element 0 in a topological group (G, τ). Neighborhoods are not necessarily open.

For a topological group G we denote by $\widetilde{G}$ the Raĭkov completion of G. We recall here that a group G is *precompact* if $\widetilde{G}$ is compact (some authors prefer the term “totally bounded”).

We say a topological group G *is linear* or *is linearly topologized* if it has a neighborhood basis at 0 formed by open subgroups.

The cardinality of the continuum 2^ω will be also denoted by $\mathfrak{c}$. The *weight* of a topological space X is the minimal cardinality of a basis for its topology; it will be denoted by $w(X)$. The *netweight* of X is the minimal cardinality of a network in X (that is, a family $\mathcal{N}$ of subsets of X such that for any $x \in X$ and any open set U containing x there exists $N \in \mathcal{N}$ with $x \in N \subseteq U$). The netweight of a space X will be denoted by $nw(X)$. The *character* $\chi(X, x)$ (resp. *pseudocharacter* $\psi(X, x)$) of a space X at a point x is the minimal cardinality of a basis of neighborhoods of x (resp. a family of open neighborhoods of x whose intersection is $\{x\}$); if X is a homogeneous space, it is the same at every point and we denote it by $\chi(X)$ (resp. $\psi(X)$). The *Lindelöf number* $l(X)$ of a space X is the minimal cardinal κ such that any open cover of X admits a subcover of cardinality not greater than κ.

By a *character* on an abelian topological group G it is commonly understood a continuous homomorphism from G into the unit circle group $\mathbb{T}$.

Let U be a symmetric subset of a group $(G, +)$ such that $0 \in U$, and $n \in \mathbb{N}$. We define $(1/n)U := \{x \in G\colon\ kx \in U\ \forall k \in \{1, 2, \dots, n\}\}$ and $U_\infty := \{x \in G\colon\ nx \in U\ \forall n \in \mathbb{N}\}$.

Recall that a non-empty subset U of a real vector space is *starlike* whenever $[0, 1]U \subseteq U$. Note that if U is starlike and symmetric then $(1/n)U = \frac{1}{n}U$; in general, for symmetric U: $(1/n)U = \bigcap_{k=1}^{n} \frac{1}{k}U$.

All unexplained topological terms can be found in [16]. For background on abelian groups, see [17] and [32].

2. Local minimality

2.1. The notion of a locally minimal topological group

In this section we recall the definition and basic examples of locally minimal groups, and prove that for locally minimal groups the weight and the netweight coincide.

Definition 2.1. A Hausdorff topological group (G,τ) is *locally minimal* if there exists a neighborhood V of 0 such that whenever $\sigma \leqslant \tau$ is a Hausdorff group topology on G such that V is a σ-neighborhood of 0, then $\sigma = \tau$. If we want to point out that the neighborhood V witnesses local minimality for (G,τ) in this sense, we say that (G,τ) is *V-locally minimal.*

Remark 2.2. As mentioned in [12], one obtains an equivalent definition replacing "V is a σ-neighborhood of 0" with "V has a non-empty σ-interior" above.

It is easy to see that if local minimality of a group G is witnessed by some $V \in \mathcal{V}_\tau(0)$, then every smaller $U \in \mathcal{V}_\tau(0)$ witnesses local minimality of G as well.

Example 2.3. Examples for locally minimal groups:

(a) If G is a minimal topological group, G is locally minimal [G witnesses local minimality of G].
(b) If G is a locally compact group, G is locally minimal [every compact neighborhood of zero witnesses local minimality of G [12]].
(c) It is easy to check that a normed space (E,τ) with unit ball B is B-locally minimal.

We start with some permanence properties of locally minimal groups.

Proposition 2.4. *A group having an open locally minimal subgroup is locally minimal.*

Proof. Let H be a locally minimal group witnessed by $U \in \mathcal{V}_H(0)$ and suppose that H is an open subgroup of the Hausdorff group (G,τ). Then U is a neighborhood of 0 in G. Assume that σ is a Hausdorff group topology on G coarser than τ such that U is a neighborhood of 0 in (G,σ). Then $\tau|_H \geqslant \sigma|_H$ and since U is a neighborhood of 0 in $(H,\sigma|_H)$, we obtain $\tau|_H = \sigma|_H$. Since U is a neighborhood of 0 in (G,σ), the subgroup H is open in σ and hence $\sigma = \tau$. □

In the other direction we can weaken the hypothesis "open subgroup" to the much weaker "closed subgroup", but we need to further impose the restraint on H to be central.

Proposition 2.5. *Let G be a locally minimal group and let H be a closed central subgroup of G. Then H is locally minimal.*

Proof. Let τ denote the topology of G and let $V_0 \in \mathcal{V}_{(G,\tau)}(0)$ witness local minimality of (G,τ). Choose $V_1 \in \mathcal{V}_{(G,\tau)}(0)$ such that $V_1 + V_1 \subseteq V_0$. We show that $V_1 \cap H$ witnesses local minimality of H. Suppose σ is a Hausdorff group topology on H coarser than $\tau|_H$ such that $V_1 \cap H$ is σ-neighborhood of 0. It is easy to verify that the family of sets $(U+V)$ where U is a σ-neighborhood of 0 in H and V is a τ-neighborhood of 0, forms a neighborhood basis of a group topology τ' on G which is coarser than τ. Let us prove that τ' is Hausdorff: Therefore, observe that for a subset $A \subseteq H$ we have $\overline{A}^\tau \subseteq \overline{A}^\sigma$, since H is closed in τ. Hence we obtain $\overline{\{0\}}^{\tau'} = \bigcap\{U+V\colon\ U \in \mathcal{V}_{(H,\sigma)}(0),\ V \in \mathcal{V}_{(G,\tau)}\} = \bigcap_{U\in\mathcal{V}_{(H,\sigma)}(0)}\bigcap_{V\in\mathcal{V}_{(G,\tau)}(0)} U+V = \bigcap_{U\in\mathcal{V}_{(H,\sigma)}(0)} \overline{U}^\tau \subseteq \bigcap_{U\in\mathcal{V}_{(H,\sigma)}(0)} \overline{U}^\sigma = \{0\}$ since σ was assumed to be Hausdorff.

Moreover, if $W \in \mathcal{V}_\sigma(0)$ such that $W \subseteq V_1 \cap H$, then $W + V_1 \subseteq V_0$ implies that $V_0 \in \mathcal{V}_{(G,\tau')}(0)$. By the choice of V_0 this yields $\tau' = \tau$. Hence $\sigma = \tau$. □

Corollary 2.6. *An open central subgroup U of a topological group G is locally minimal iff G itself is locally minimal.*

These results leave open the question on whether "central" can be omitted in the above corollary and Proposition 2.5 (see Question 6.8).

The question whether the product of two minimal (abelian) groups is again minimal was answered negatively by Doïtchinov in [14] where he proved that $(\mathbb{Z},\tau_2)\times(\mathbb{Z},\tau_2)$ is not minimal although the 2-adic topology τ_2 on the integers is minimal. We will show in Proposition 5.17 that $(\mathbb{Z},\tau_2)\times(\mathbb{Z},\tau_2)$ is not even locally minimal.

Next we are going to see some cases where metrizability can be deduced from local minimality. We start with a generalization to locally minimal groups of the following theorem of Arhangel'skij: $w(G) = nw(G)$ for every minimal group; in particular, every minimal group with countable netweight is metrizable. For that we need the following result from [1]:

Lemma 2.7. *Let κ be an infinite cardinal and let G be a topological group with*

(a) $\psi(G) \leqslant \kappa$;
(b) *G has a subset X with* $\langle X\rangle = G$ *and* $l(X) \leqslant \kappa$.

Then for every family of neighborhoods $\mathcal{B}$ *of the neutral element* 0 *of G with* $|\mathcal{B}| \leqslant \kappa$ *there exists a coarser group topology* τ' *on G such that* $w(G, \tau') \leqslant \kappa$ *and every* $U \in \mathcal{B}$ *is a* τ'*-neighborhood of* 0.

Theorem 2.8. *For a locally minimal group* (G, τ) *one has* $w(G) = nw(G)$. *In particular, every countable locally minimal group is metrizable.*

Proof. Let $\kappa = nw(G)$ and let $\mathcal{N}$ be a network of G of size κ. Then also $\psi(G) \leqslant \kappa$ as

$$\bigcap\{G \setminus \overline{B}\colon\ 0 \notin \overline{B},\ B \in \mathcal{N}\} = \{0\}.$$

Moreover, the Lindelöf number $l(G)$ of G is $\leqslant \kappa$. Indeed, if $G = \bigcup_{i\in I} U_i$ and each U_i is a non-empty open set, then by the definition of a network for every $x \in G$ there exist $i_x \in I$ and $B_x \in \mathcal{N}$ such that $x \in B_x \subseteq U_{i_x}$. [For $z \in G$ we choose $y \in Y$ such that $B_z = B_y$ and obtain $z \in B_z = B_y \subseteq U_{i_y}$.] Let $\mathcal{N}_1 = \{B_x\colon\ x \in G\}$ and $Y \subseteq G$ such that the assignment $Y \to \mathcal{N}_1$, defined by $Y \ni x \mapsto B_x$, is bijective. Then $|Y| \leqslant \kappa$ and $G = \bigcup_{y\in Y} U_{i_y}$. This proves $l(G) \leqslant \kappa$.

To end the proof of the theorem apply Lemma 2.7 taking $X = G$ and any family $\mathcal{B}$ of size κ of τ-neighborhoods of 0 containing U as a member and witnessing $\psi(G) \leqslant \kappa$ (i.e., $\bigcap \mathcal{B} = \{0\}$). This gives a Hausdorff topology $\tau' \leqslant \tau$ on G satisfying the conclusion of the lemma. By the local minimality of (G, τ) we conclude $\tau' = \tau$. In particular, $w(G, \tau) \leqslant \kappa$. Since always $nw(G) \leqslant w(G)$, this proves the required equality $w(G) = nw(G)$.

Now suppose that G is countable. Then $nw(G) = \omega$, so the equality $w(G) = nw(G)$ implies that G is second countable, in particular metrizable. □

Remark 2.9.

(a) The fact that every countable locally minimal group (G, τ) is metrizable admits also a straightforward proof. Indeed, let $\{x_n\colon\ n \in \mathbb{N}\} = G \setminus \{0\}$ and let U_0 be a neighborhood of 0 such that G is U_0-locally minimal. Then there exists a sequence of symmetric neighborhoods of zero (U_n) satisfying $U_n + U_n \subseteq U_{n-1}$, $x_n \notin U_n$, $U_n \subseteq \bigcap_{k=1}^{n-1}(x_k + U_{n-1} - x_k)$ for all $n \in \mathbb{N}$. Since $\bigcap_{n=1}^{\infty} U_n = \{0\}$, the family (U_n) forms a base of neighborhoods of 0 of a metrizable group topology $\sigma \leqslant \tau$ on G with $U_0 \in \sigma$. Hence $\tau = \sigma$ is metrizable.
(b) A similar direct proof shows that every locally minimal abelian group (G, τ) of countable pseudocharacter is metrizable. Here “abelian” cannot be removed, since examples of minimal (necessarily non-abelian) groups of countable pseudocharacter and arbitrarily high character (in particular, non-metrizable) were built by Shakhmatov [33].

2.2. Groups with no small (normal) subgroups

In this subsection we show that groups with no small (normal) subgroups are closely related to locally minimal groups and study some of their properties.

Definition 2.10. A topological group (G, τ) is called *NSS group* (No Small Subgroups) if a suitable neighborhood $V \in \mathcal{V}(0)$ contains only the trivial subgroup.

A topological group (G, τ) is called *NSnS group* (No Small normal Subgroups) if a suitable neighborhood $V \in \mathcal{V}(0)$ contains only the trivial normal subgroup.

The distinction between NSS and NSnS will be necessary only when we consider non-abelian groups (or non-compact groups, see Remark 3.6 below).

Example 2.11. Examples for NSS and non-NSS groups:

(a) The unit circle $\mathbb{T}$ is an NSS group.
(b) Montgomery and Zippin’s solution to Hilbert’s fifth problem asserts that every locally compact NSS group is a Lie group.
(c) Any free abelian topological group on a metric space is a NSS group [27].
(d) A dichotomy of Hausdorff group topologies on the integers: Any Hausdorff group topology τ on the integers is NSS if and only if it is not linear. Indeed, suppose that τ is not NSS; let U be a closed neighborhood of 0. By assumption, U contains a non-trivial closed subgroup H which is of the form $n\mathbb{Z}$ ($n \geqslant 1$). Since $\mathbb{Z}/n\mathbb{Z}$ is a finite Hausdorff group, it is discrete and hence $n\mathbb{Z}$ is open in $\mathbb{Z}$. This shows that τ is linear.
(e) A group G is *topologically simple* if G has no proper closed normal subgroups. Every Hausdorff topologically simple group is NSnS. [Suppose that G is topologically simple and Hausdorff and let $U \neq G$ be a closed neighborhood of 0. Let N be a normal subgroup of G contained in U. Then $\overline{N}$ is also a closed subgroup of G contained in U and hence

$\{0\} = \overline{N} = N$. So G is an NSnS group. Actually a stronger property is true: if G is Hausdorff and every closed normal subgroup of G is finite, then G is NSnS (this provides a proof of item (a)).] The infinite permutation group $G = S(\mathbb{N})$ is an example of a topologically simple group [13, 7.1.2].

We omit the easy proof of the next lemma:

Lemma 2.12.

(a) *The classes of NSnS groups and NSS groups are stable under taking finite direct products and finer group topologies.*
(b) *The class of NSS groups is stable under taking subgroups.*
(c) *The class of NSnS groups is stable under taking dense subgroups.*
(d) *No infinite product of non-trivial groups is NSnS.*

Recall that a SIN group (SIN stands for Small Invariant Neighborhoods) is a topological group G such that for every $U \in \mathcal{V}(0)$ there exists $V \in \mathcal{V}(0)$ with $-x + V + x \subseteq U$ for all $x \in G$.

Proposition 2.13. *Every locally minimal SIN group G is metrizable provided it is NSnS.*

Proof. Let us assume that (G, τ) is V-locally minimal and NSnS, where V is a neighborhood of 0 in (G, τ) containing no non-trivial normal subgroups. Since τ is a group topology, it is possible to construct inductively a sequence (V_n) of symmetric neighborhoods of 0 in τ which satisfy $V_n + V_n \subseteq V_{n-1}$ (where $V_0 := V$) and $-x + V_n + x \subseteq V_{n-1}$ for all $x \in G$.

Let σ be the group topology generated by the neighborhood basis $(V_n)_{n\in\mathbb{N}}$. Obviously, σ is coarser than τ and $V \in \mathcal{V}_\sigma(0)$. In order to conclude that $\sigma = \tau$, it only remains to show that σ is a Hausdorff topology, which is equivalent to $\bigcap_{n\in\mathbb{N}} V_n = \{0\}$. This is trivial, since the intersection is a normal subgroup contained in V. $\square$

Example 2.14. One cannot relax the "SIN" condition even when G is minimal. Indeed, for every infinite set X the symmetric group $G = S(X)$ is minimal and NSnS. On the other hand, $S(X)$ is metrizable only when X is countable [13, §7.1]. Note that this group strongly fails to be NSS, as $\mathcal{V}(0)$ has a base consisting of open subgroups (namely, the pointwise stabilizers of finite subsets of X).

Remark 2.15. We gratefully acknowledge the referee's remark that Theorem 2.8 and Proposition 2.13 can be generalized as follows:

For a non-discrete locally minimal group G we have (the statements with prime hold under the additional assumption that G is *NSnS*)

$$\text{(a)}\quad \chi(G) = \psi(G) \cdot \mathrm{inv}(G), \qquad \text{(a}'\text{)}\quad \chi(G) \overset{\text{NSnS}}{=} \mathrm{inv}(G),$$

$$\text{(b)}\quad w(G) = \psi(G) \cdot \mathrm{ib}(G), \qquad \text{(b}'\text{)}\quad w(G) \overset{\text{NSnS}}{=} \mathrm{ib}(G),$$

where $\mathrm{inv}(G)$ denotes the *invariance number* of G, which is the smallest infinite cardinal number κ such that for each neighborhood $U \in \mathcal{V}_{(G,\tau)}$ there exists a family $\mathcal{W}$ of neighborhoods of 0 of cardinality $\leqslant \kappa$ such that for every $x \in G$ there exists $W \in \mathcal{W}$ such that $x + W - x \subseteq U$, and $\mathrm{ib}(G)$, the *index of boundedness*, is the smallest infinite cardinal κ such that for every neighborhood $U \in \mathcal{V}_{(G,\tau)}$ there exists a subset $F \subseteq G$ of cardinality $\leqslant \kappa$ such that $G = F + U$.

In all equalities "$\geqslant$" is easy to see. (a) and (a′) can be deduced from the following modification of (5.1.9) in [3]: Every locally minimal group can be embedded into a product $\prod_{i\in I} G_i$ of topological groups where the character of each G_i is bounded by $\mathrm{inv}(G)$ and $|I| \leqslant \psi(G)$. In case G is additionally NSnS, (5.1.23) in [3] allows to choose I finite.

(b) and (b′) are consequences of (a) and (a′) and the following facts: $w(G) = \chi(G) \cdot \mathrm{ib}(G)$ and $\mathrm{inv}(G) \leqslant \mathrm{ib}(G)$.

As shown in the proof of Theorem 2.8, $\psi(G) \leqslant nw(G)$ and $l(G) \leqslant nw(G)$. In (5.1.3) in [3] it is shown that $\mathrm{ib}(G) \leqslant l(G)$. Combining these results we obtain $\psi(G) \cdot \mathrm{ib}(G) \leqslant nw(G) \leqslant w(G)$ so that (b) improves Theorem 2.8.

Remark 2.16.

(a) The completion of a NSS group is not NSS in general: For example the group $\mathbb{T}^{\mathbb{N}}$ is monothetic, i.e. it has a dense subgroup H algebraically isomorphic to $\mathbb{Z}$. Since the completion of a linear group topology is again linear, and the product topology on $\mathbb{T}^{\mathbb{N}}$ is not linear, H is not linear either. So Example 2.11(d) implies that H is NSS. But H is dense in $\mathbb{T}^{\mathbb{N}}$ which is not NSS.
(b) It was a problem of I. Kaplansky whether the NSS property is preserved under taking arbitrary quotients. A counter-example was given by S. Morris [25] and Protasov [30]; the latter proved that NSS is preserved under taking quotients with respect to discrete normal subgroups.

(c) In contrast with the NSS property, a subgroup of an NSnS group need not be NSnS. Indeed, take the permutation group $G = S(\mathbb{N})$. Let $\mathbb{N} = \bigcup_n F_n$ be a partition of the naturals into finite sets F_n such that each F_n has size 2^n. Let σ_n be a cyclic permutation of length 2^n of the finite set F_n and let σ be the permutation of $\mathbb{N}$ that acts on each F_n as σ_n. Obviously, σ is a non-torsion element of G, so it generates an infinite cyclic subgroup $C \cong \mathbb{Z}$. For convenience identify C with $\mathbb{Z}$. Then, while G is NSnS by Example 2.11(e), the induced topology of C coincides with the 2-adic topology of $C = \mathbb{Z}$, so it is linear and certainly non-NSnS. Indeed, a subbasic neighborhood of the identity element $\mathrm{id}_{\mathbb{N}}$ in C has the form $U_x = C \cap \mathrm{Stab}_x$, where Stab_x is the stabilizer of the point $x \in \mathbb{N}$. If $x \in F_n$, then obviously all powers of σ^{2^n} stabilize x, so U_x contains the subgroup $V_n = \langle \sigma^{2^n} \rangle$. This proves that the induced topology of $C \cong \mathbb{Z}$ is coarser than the 2-adic topology. Since the latter is minimal [13, 2.5.6], we conclude that C has the 2-adic topology.

3. Groups uniformly free from small subgroups

3.1. Local minimality and the UFSS property

We have seen (Example 2.3(c)) that all normed spaces are locally minimal when regarded as topological abelian groups. The following group analog of a normed space was introduced by Enflo [15]; we will show in Facts 3.3(a) that every such group is locally minimal:

Definition 3.1. A Hausdorff topological group (G, τ) is *uniformly free from small subgroups* (UFSS for short) if for some neighborhood U of 0, the sets $(1/n)U$ form a neighborhood basis at 0 for τ.

Neighborhoods U satisfying the condition described in Definition 3.1 will be said to be *distinguished*. It is easy to see that any neighborhood of zero contained in a distinguished one is distinguished, as well.

Obviously, discrete groups are UFSS. Now we give some non-trivial examples.

Example 3.2.

(a) $\mathbb{R}$ is a UFSS group with respect to $[-1, 1]$.
(b) $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a UFSS group with respect to $\mathbb{T}_+$, the image of $[-1/2, 1/2]$ under the quotient map $\mathbb{R} \to \mathbb{T}$.
(c) A topological vector space is UFSS as a topological abelian group if and only if it is locally bounded. In particular every normed space is a UFSS group.
Recall that a subset B of a (real or complex) topological vector space E is usually referred to as *bounded* if for every neighborhood of zero U in E there exists $\alpha > 0$ with $B \subseteq \lambda U$ for every λ with $|\lambda| > \alpha$, and *balanced* whenever $\lambda B \subseteq B$ for every λ with $|\lambda| \leqslant 1$. The space E is *locally bounded* if it has a bounded neighborhood of zero. It is straightforward that any locally bounded space is UFSS when regarded as a topological abelian group, and any of its bounded neighborhoods of zero is a distinguished neighborhood. Conversely, if a topological vector space is UFSS as a topological abelian group, then any distinguished balanced neighborhood of zero is bounded in this sense.
This, of course, includes unit balls of normed spaces, but there are some important non-locally-convex examples as well (see Example 3.5(b)).
(d) Every Banach–Lie group is UFSS [26, Theorem 2.7].

Facts 3.3.

(a) Every UFSS group with distinguished neighborhood U is U-locally minimal [26, Proposition 2.5]. Indeed, one can see that a UFSS group (G, τ) with distinguished neighborhood U has the following property, which trivially implies that (G, τ) is U-locally minimal: if $\mathcal{T}$ is a group topology on G such that U is a $\mathcal{T}$-neighborhood of 0, then $\tau \leqslant \mathcal{T}$.
(b) All UFSS groups are NSS groups.

Next we give some examples of NSS groups that are not UFSS.

Example 3.4.

(a) Consider the group $\mathbb{R}^{(\mathbb{N})} = \{(x_n) \in \mathbb{R}^{\mathbb{N}}\colon\ x_n = 0 \text{ for almost all } n \in \mathbb{N}\}$, endowed with the box topology, which admits as a basis of neighborhoods of zero the following family of sets:

$$U_{(\varepsilon_n)} := \left\{(x_n) \in \mathbb{R}^{(\mathbb{N})}\colon\ |x_n| < \varepsilon_n\ \forall n \in \mathbb{N}\right\}, \quad (\varepsilon_n)_{n\in\mathbb{N}} \in (0, \infty)^{\mathbb{N}}.$$

This group is not metrizable, hence it cannot be a UFSS group. On the other hand, any of the neighborhoods $U_{(\varepsilon_n)}$ contains only the trivial subgroup, so it is a NSS group.
(b) All free abelian topological groups on a metric space are NSS groups (see Example 2.11(c)).
Take a non-locally compact metric space X, then $A(X)$ is NSS, but not UFSS (indeed, if $A(X)$ is a k-space for some metrizable X, then X is locally compact by [2, Proposition 2.8]).

Example 2.3, Facts 3.3 and Example 3.2 give us a strong motivation to study locally minimal groups, which put under the same umbrella three extremely relevant properties as minimality, UFSS and local compactness.

Example 3.2 shows that a locally minimal abelian group need not be precompact, in contrast with Prodanov–Stoyanov's theorem. We see in the following example that actually there exist abelian locally minimal groups without non-trivial continuous characters.

Example 3.5.

(a) According to a result of W. Banaszczyk [6], every infinite-dimensional Banach space E has a discrete and free subgroup H such that the quotient group E/H admits only the trivial character. E/H is locally isomorphic with E, hence it is a Banach–Lie group and then UFSS.

(b) Fix any $s \in (0,1)$ and consider the topological vector space L^s of all classes of Lebesgue measurable functions f on $[0,1]$ (modulo almost everywhere equality) such that $\int_0^1 |f|^s\,d\lambda$ is finite, with the topology given by the following basis of neighborhoods of zero:

$$U_r = \left\{ f\colon \int_0^1 |f|^s\,d\lambda \leqslant r \right\}, \quad r > 0.$$

(Following a customary abuse of notation, we use here (and in Example 5.4 and Remark 6.6) the same symbol to denote both a function and its class under the equivalence relation of almost everywhere equality.) In [8] it was proved that L^s has no non-trivial continuous linear functionals. It is known that every character defined on the topological abelian group underlying a topological vector space can be lifted to a continuous linear functional on the space [35]. Thus as a topological group, L^s has trivial dual. On the other hand L^s is a locally bounded space (note that for every $r > 0$ one has $U_1 \subseteq r^{-1/s} U_r$), hence it is a UFSS group (Example 3.2(c)).

Remark 3.6. It is a well-known fact (see for instance [37, 32.1]) that for every compact group K and $U \in \mathcal{V}(0)$ there exists a closed normal subgroup N of K contained in U such that K/N is a Lie group, hence UFSS. This implies that the following assertions are equivalent:

(a) K is UFSS,
(b) K is NSS,
(c) K is NSnS,
(d) K is a Lie group.

In case K is abelian, they are equivalent to: K is a closed subgroup of a finite-dimensional torus.

(The same equivalences are known to be true for locally compact groups which are either connected or abelian.)

In order to extend the above equivalences to locally minimal precompact groups, we need the following lemma:

Lemma 3.7. *Let (G,τ) be a precompact group. Then the following are equivalent*:

(a) *(G,τ) is NSnS.*
(b) *For every $U \in \mathcal{V}(0)$ there exists a continuous injective homomorphism $f: G \to L$ such that L is a compact Lie group and $f(U)$ is a neighborhood of 0 in $f(G)$.*
(c) *There exist a compact Lie group L and a continuous injective homomorphism $f: G \to L$.*
(d) *G admits a coarser UFSS group topology.*
(e) *(G,τ) is NSS.*

In case G is abelian these conditions are equivalent to the existence of a continuous injective homomorphism $G \to \mathbb{T}^k$ for some $k \in \mathbb{N}$.

Proof. To prove that (a) implies (b) assume that (G,τ) is NSnS and fix a $U \in \mathcal{V}(0)$. Let W be a neighborhood of 0 in the completion K of G such that $W \cap G$ contains no non-trivial normal subgroups and $(W+W) \cap G \subseteq U$. As in Remark 3.6 there exists a closed normal subgroup N of K contained in W such that $L = K/N$ is a Lie group. As $N \cap G = \{0\}$ by our choice of W, the canonical homomorphism $q: K \to L$ restricted to G gives a continuous injective homomorphism $f = q\restriction_G\colon G \to L$. Observe that

$$f(U) \supseteq q\big((W+W) \cap G\big) \supseteq q(N+W) \cap q(G)$$

as $N \subseteq W$. Finally, the latter set is a neighborhood of 0 in $f(G)$ as $N + W \in \mathcal{V}(0_K)$.

(b) $\Rightarrow$ (c) is trivial. (c) $\Rightarrow$ (d) is a consequence of the fact that every Lie group is UFSS. (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (a) are trivial. $\square$

Proposition 3.8. *For a locally minimal precompact group G the following are equivalent*:

(a) *G is NSnS*;
(b) *G is NSS*;
(c) *G is UFSS*;
(d) *G is isomorphic to a dense subgroup of a compact Lie group.*

Proof. The implication (a) $\Rightarrow$ (d) follows from (a) $\Rightarrow$ (b) in Lemma 3.7, since the local minimality of G and (b) from Lemma 3.7 imply that $G \to L$ is an embedding. Note that a compact subgroup of a compact Lie group is closed, so a Lie group itself. (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are trivial. □

Remark 3.9. For locally minimal precompact abelian groups, condition (d) of Proposition 3.8 can be replaced by: G is isomorphic to a subgroup of a torus $\mathbb{T}^n$, $n \in \mathbb{N}$. Note that the class of locally minimal precompact abelian groups contains all minimal abelian groups, due to the deep theorem of Prodanov and Stoyanov which states that such groups are precompact.

Remark 3.10. Proposition 3.8 shows very neatly the differences between minimality and UFSS. While all (dense) subgroups of a torus $\mathbb{T}^n$ are UFSS, the minimal among the dense subgroups of $\mathbb{T}^n$ are those that contain the socle $Soc(\mathbb{T}^n)$.

Indeed, $Soc(\mathbb{T}^n)$ is dense and every closed non-trivial subgroup N of $\mathbb{T}^n$ is still a Lie group, so has non-trivial torsion elements (i.e., meets $Soc(\mathbb{T}^n)$). Therefore, by [13, Theorem 2.5.1] a dense subgroup H of $\mathbb{T}^n$ is minimal iff H contains $Soc(\mathbb{T}^n)$. In particular, there is a smallest dense minimal subgroup of $\mathbb{T}^n$, namely $Soc(\mathbb{T}^n)$.

Example 3.11. Let τ be a UFSS precompact topology on $\mathbb{Z}$. Then $(\mathbb{Z}, \tau)$ is a dense subgroup of a group of the form $\mathbb{T}^k \times \mathbb{Z}(m)$, where $k, m \in \mathbb{N}$, $k > 0$. Indeed, by Proposition 3.8 and Remark 3.9 $(\mathbb{Z}, \tau)$ is isomorphic to a subgroup of some finite-dimensional torus $\mathbb{T}^n$. Then the closure C of $\mathbb{Z}$ in $\mathbb{T}^n$ will be a monothetic compact abelian Lie group. So the connected component $c(C) \cong \mathbb{T}^k$ for some $k \in \mathbb{N}$, $k > 0$ and $C/c(C)$ is a discrete monothetic compact group, so $C/c(C) \cong \mathbb{Z}(m)$ for some $m \in \mathbb{N}$, so $C \cong \mathbb{T}^k \times \mathbb{Z}(m)$ since $c(C)$ splits as a divisible subgroup of C.

3.2. Permanence properties of UFSS groups

In the next proposition we collect all permanence properties of UFSS groups we can verify.

Proposition 3.12. *The class of UFSS groups has the following permanence properties*:

(a) *If G is a dense subgroup of $\widetilde{G}$ and G is UFSS, then $\widetilde{G}$ is UFSS.*
(b) *Every subgroup of a UFSS group is UFSS.*
(c) *Every finite product of UFSS groups is UFSS.*
(d) *Every group locally isomorphic to a UFSS group is UFSS.*
(e) *If an abelian topological group G has a closed subgroup H such that both H and G/H are UFSS, then G is UFSS as well.*

Proof. (a) Let G be a UFSS group with distinguished neighborhood U. Note that closures in $\widetilde{G}$ of the neighborhoods of 0 in G form a basis of the neighborhoods of 0 in $\widetilde{G}$. Let W be a symmetric neighborhood of 0 in G which satisfies $G \cap (\overline{W} + \overline{W}) \subseteq U$.

Let us prove that

$$(1/n)\overline{W} \subseteq \overline{(1/n)U} \quad \forall n \in \mathbb{N}.$$

To this end fix $x \in (1/n)\overline{W}$. This means $x, 2x, \ldots, nx \in \overline{W}$. Hence there exists a sequence (x_k) in W which tends to x and the sequences (jx_k) converge to $jx \in \overline{W}$ for $j \in \{1, \ldots, n\}$. We may assume that $jx_k - jx \in \overline{W}$ for all $j \in \{1, \ldots, n\}$ and all $k \in \mathbb{N}$, which implies $jx_k \in G \cap (\overline{W} + \overline{W}) \subseteq U$ for all $k \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$. This implies $x_k \in (1/n)U$ for all $k \in \mathbb{N}$ and hence $x \in \overline{(1/n)U}$.

The inclusion $(1/n)\overline{W} \subseteq \overline{(1/n)U}$ assures that the sets $(1/n)\overline{W}$ form a neighborhood basis of 0 in $\widetilde{G}$; i.e. $\overline{W}$ is a distinguished neighborhood for $\widetilde{G}$.

(b) to (d) are easy to see.

(e) By assumption, there exists a neighborhood W of 0 in G such that $\pi(W + W)$, and $(W + W) \cap H$, are distinguished neighborhoods of zero in G/H and H, respectively, where $\pi : G \to G/H$ denotes the canonical projection.

According to a result of Graev ([18] or (5.38)(e) in [21]), G is first countable, since H and G/H have this property.

Let us show that

$$\forall (x_n) \quad \text{with } x_n \in (1/n)W \quad \Longrightarrow \quad x_n \xrightarrow{\tau} 0, \tag{1}$$

where τ is the original topology on G. Since $\pi((1/n)W) \subseteq (1/n)\pi(W)$, $\pi(W)$ is a distinguished neighborhood of zero in G/H and G is first countable, there exists a sequence (h_n) in H such that $x_n - h_n \to 0$.

For $n_0 \in \mathbb{N}$, there exists $n_1 \geqslant n_0$ such that for all $n \geqslant n_1$ we have

$$h_n = x_n + (h_n - x_n) \in \big((1/n)W + (1/n_0)W\big) \cap H \subseteq \big((1/n_0)W + (1/n_0)W\big) \cap H \subseteq (1/n_0)\big((W + W) \cap H\big).$$

Since the sets $(1/n)((W+W) \cap H)$ form a basis of zero neighborhoods in H, the sequence (h_n) tends to 0 and hence (x_n) tends to 0 as well.

Condition (3) implies that the family $((1/n)W)$ is a basis of zero neighborhoods for G. Indeed, fix $U \in \mathcal{V}_\tau(0)$ and suppose $(1/n)W \nsubseteq U$ for every $n \in \mathbb{N}$. Select $x_n \in (1/n)W$, $x_n \notin U$. According to (3) the sequence (x_n) converges to zero, which contradicts $x_n \notin U$ $\forall n \in \mathbb{N}$. □

Remark 3.13.

(a) Items (b) and (c) imply that finite suprema of UFSS group topologies are still UFSS. In the next section we will introduce the locally GTG topologies which, at least in the NSS case, can be characterized as arbitrary suprema of UFSS group topologies (see Definition 5.1 and Theorem 5.7).
(b) Item (c) follows also from (e). Let us note, that it cannot be strengthened to countably infinite products. Indeed, any infinite product of non-indiscrete groups (e.g., copies of $\mathbb{T}$) fails to be NSS, so cannot be UFSS either.

The rest of the subsection is dedicated to a very natural property that was missing in Proposition 3.12, namely stability under taking quotients and continuous homomorphic images. It follows from item (d) of this proposition that a quotient of a UFSS group with respect to a discrete subgroup is UFSS. Actually it has been shown in [28] (Proposition 4.5) that every Hausdorff abelian UFSS group is a quotient group of a subgroup of a Banach space. However, as we see in the next example, a Hausdorff quotient of a UFSS group need not be UFSS.

Example 3.14. Let $\{e_n\colon n \in \mathbb{N}\}$ denote the canonical basis of the Hilbert space ℓ^2. Consider the closed subgroup $H := \langle\{\frac{1}{n}e_n\colon n \in \mathbb{N}\}\rangle$ of ℓ^2. Let us denote by B the unit ball in ℓ^2 and by $\pi : \ell^2 \to \ell^2/H$ the canonical projection. For an arbitrary $\varepsilon > 0$, we will show that $\pi(\varepsilon B)$ contains a non-trivial subgroup. This will imply that the quotient ℓ^2/H is not NSS and, in particular, is not UFSS.

Let $k_0 \in \mathbb{N}$ such that $\sum_{k>k_0} \frac{1}{k^2} < 4\varepsilon^2$. Let S be the linear hull of the set $\{e_k\colon k > k_0\}$. We will obtain

$$\pi(S) \subseteq \pi(\varepsilon B).$$

Indeed, fix $x = (x_n) \in S$. For $n > k_0$, there exists $k_n \in \mathbb{Z}$ such that $|x_n - \frac{k_n}{n}| \leqslant \frac{1}{2n}$. Since $h := \sum_{n>k_0} \frac{k_n}{n} e_n \in H$ and $\|x - h\| \leqslant \sqrt{\sum_{n>k_0} (\frac{1}{2n})^2} < \varepsilon$, we obtain: $\pi(x) = \pi(h + (x-h)) = \pi(x-h) \in \pi(\varepsilon B)$ and hence $\pi(S) \subseteq \pi(\varepsilon B)$.

The next corollary shows that the class of *precompact* UFSS groups is closed under taking arbitrary quotients.

Corollary 3.15. *If G is a precompact UFSS group, then every continuous homomorphic image of G is UFSS.*

Proof. Let $f : G \to G_1$ be a continuous surjective homomorphism. It can be extended to the respective compact completions $f' : \widetilde{G} \to \widetilde{G}_1$ of G and G_1 respectively. Since f is surjective and each group is dense in its completion, the compactness of $\widetilde{G}$ yields that f' is surjective. Moreover, f' is open by the open mapping theorem. Hence $\widetilde{G}_1$ is isomorphic to a quotient of $\widetilde{G}$. By Proposition 3.12(a) $\widetilde{G}$ is UFSS, hence (Remark 3.6) $\widetilde{G}$ is a Lie group. Then $\widetilde{G}_1$ is a Lie group as well, so UFSS. This proves that G_1 is UFSS. □

4. GTG sets and UFSS topologies

4.1. General properties of GTG subsets

Vilenkin [40] introduced locally quasi-convex groups while generalizing the notion of a locally convex space. His definition is inspired on the description of closed symmetric subsets of vector spaces given by the Hahn–Banach theorem.

Next we present a new generalization of locally convex spaces in the setting of topological groups which we will call *locally GTG groups* where GTG abbreviates **g**roup **t**opology **g**enerating (set). Similarly to the notion of a convex set (that depends only on the linear structure of the topological vector space structure, but not on its topology), the notion of a GTG set depends only on the algebraic structure of the group. In particular, it does not use any dual object at all, whereas the notion of quasi-convex set of a topological group G depends on the topology of G via the continuity of the characters to be used for the definition of the polar.

The class of locally GTG groups will be shown to contain all locally quasi-convex groups, all locally pseudoconvex spaces and all UFSS groups. As we will see, it fits very well in the setting of locally minimal groups as it gives a connection between locally minimal groups and minimal groups (Theorem 5.12). Moreover, we are not aware of any locally minimal group not having this property (see Question 6.2).

Recall that a subset A of a vector space E is called *pseudoconvex* if $[0,1]A \subseteq A$ and $A + A \subseteq cA$ for suitable $c > 0$. One may assume that $c \in \mathbb{N}$. (Indeed, choose $\mathbb{N} \ni n > c$, then $cA \subseteq nA$ as $ca = (c/n)na \in [0,1]nA \subseteq nA$ for all $a \in A$.) Hence the set A is pseudoconvex iff $[0,1]A \subseteq A$ and for some $n \in \mathbb{N}$, $\frac{1}{n}A + \frac{1}{n}A \subseteq A$. If A is symmetric, this already implies that $(\frac{1}{n}A)$ forms a neighborhood basis of a not necessarily Hausdorff group topology: $\frac{1}{nm}A + \frac{1}{nm}A = \frac{1}{m}(\frac{1}{n}A + \frac{1}{n}A) \subseteq \frac{1}{m}A$. A standard argument shows that scalar multiplication is also continuous.

It is well known that the unit balls of the vector spaces ℓ^s where $0 < s < 1$ are pseudoconvex but not convex. The same can be said of their natural finite-dimensional counterparts ℓ^s_n, with $n \geqslant 2$. Nevertheless, by far not all symmetric subsets of a vector space are pseudoconvex, as we see in the next example.

Example 4.1. The subsets of $\mathbb{R}^2$: $U = ([-1,1] \times \{0\}) \cup (\{0\} \times [-1,1])$ and $V = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ are symmetric and not pseudoconvex. Observe that $[0,1]U \subseteq U$; $\frac{1}{n}U = ([-1/n, 1/n] \times \{0\}) \cup (\{0\} \times [-1/n, 1/n])$; $[0,1]V \subseteq V$ and $\frac{1}{n}V = V$.

Definition 4.2. Let G be an abelian group and let U be a symmetric subset of G such that $0 \in U$. We say that U is a *group topology generating* subset of G ("GTG subset of G" for short) if the sequence of subsets $\{(1/n)U\colon n \in \mathbb{N}\}$ is a basis of neighborhoods of zero for a (not necessarily Hausdorff) group topology $\mathcal{T}_U$ on G.

In case U is a GTG set in G, $\mathcal{T}_U$ is the coarsest group topology on G such that U is a neighborhood.

We do not know whether the following natural converse is true: Let G be an abelian group and U a symmetric subset of G which contains zero and such that there exists the coarsest group topology on G for which U is a neighborhood of zero. Then U is a GTG set.

Example 4.3.

(a) Every symmetric distinguished neighborhood of zero in a UFSS group is a GTG set.
(b) Every subgroup of a group G is a GTG subset of G.

Proposition 4.4. *A symmetric subset $U \subseteq G$ of an abelian group G is a GTG subset if and only if*

$$\exists m \in \mathbb{N} \quad \text{with } (1/m)U + (1/m)U \subseteq U. \qquad (*)$$

Moreover, if U is a GTG set, $U_\infty = \bigcap_{n=1}^{\infty} (1/n)U$ is the $\mathcal{T}_U$-closure of $\{0\}$ and in particular, it is a closed subgroup and a G_δ subset of $(G, \mathcal{T}_U)$.

Proof. The given condition is obviously necessary. Conversely, to prove that addition is continuous, we are going to see that $(1/mn)U + (1/mn)U \subseteq (1/n)U$ $\forall n \in \mathbb{N}$. Fix $x, y \in (1/mn)U$ and observe that $jx, jy \in (1/m)U$ for all $1 \leqslant j \leqslant n$. This implies $j(x+y) = jx + jy \in (1/m)U + (1/m)U \subseteq U$ for all $1 \leqslant j \leqslant n$ and hence $x + y \in (1/n)U$.

If U is a GTG set, then $\mathcal{T}_U$ is a group topology of G, hence $U_\infty = \overline{\{0\}}^{\mathcal{T}_U}$ is a subgroup of G. □

Proposition 4.4 gives the possibility to define a GTG set in a more precise way. Namely, one can introduce the following invariant for a symmetric subset $U \subseteq G$ of an abelian group G with $0 \in U$

$$\gamma(U) := \min\{m \in\colon (1/m)U + (1/m)U \subseteq U\}$$

with the usual convention $\gamma(U) = \infty$ when no such m exists. According to Proposition 4.4, U is a GTG set iff $\gamma(U) < \infty$. Let us call $\gamma(U)$ the *GTG-degree* of U, it obviously measures the GTG-ness of the symmetric set U containing 0. Clearly, U has GTG-degree 1 precisely when U is a subgroup. (Compare this with the *modulus of concavity* defined in [31, 3.1].)

Proposition 4.5. *A symmetric subset A of a vector space E which satisfies $[0,1]A \subseteq A$ is GTG iff it is pseudoconvex.*

Proof. By assumption, $[0,1]A \subseteq A$. This implies that $(1/n)A = \frac{1}{n}A$. In the introduction to this section we have already shown that a symmetric set A is pseudoconvex if and only if it satisfies $\frac{1}{n}A + \frac{1}{n}A \subseteq A$ for some $n \in \mathbb{N}$. Since $(1/n)A = \frac{1}{n}A$, it is a consequence of Proposition 4.4 that A is pseudoconvex iff it is GTG. □

Example 4.6. The subsets of $U = ([-1,1] \times \{0\}) \cup (\{0\} \times [-1,1])$ and $V = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ of Example 4.1 are not GTG sets.

Remark 4.7. Let U be a symmetric subset of an abelian group G with $0 \in U$. We analyze the behavior of the sequence $(1/n)U$ in the following cases of interest:

(a) If $U_\infty = \{0\}$, then U is a GTG set iff $(G, \mathcal{T}_U)$ is UFSS.

(b) Now assume that $(1/m)U = U_\infty$ for some m. Then U is GTG iff U_∞ is a subgroup. It is clear that $(1/m)U = U_\infty$ is a union of cyclic subgroups.
We know (Proposition 4.4) that if U is a GTG set, then U_∞ must be a subgroup. But in this circumstance, we can invert the implication. Indeed, if $U_\infty = (1/m)U$ is a subgroup, then obviously $(1/m)U + (1/m)U \subseteq (1/m)U \subseteq U$ holds true, so that U is a GTG set. This fact explains once more why the subset $V = V_\infty$ from Example 4.1 is not a GTG set (simply it is not a subgroup).

(Note that we are not considering here the third possibility: $U_\infty \neq \{0\}$ yet the chain $(1/m)U$ does not stabilize.)

Remark 4.8. Let U be a symmetric subset of a group G. Then the following holds true:

(a) $(1/n)((1/m)U) = (1/m)((1/n)U)$ for all $n, m \in \mathbb{N}$.
(b) For symmetric subsets A and B of G and $k \in \mathbb{N}$ we have: $(1/k)A + (1/k)B \subseteq (1/k)(A + B)$.
(c) The following assertions are equivalent:
  (i) U is a GTG set in G.
  (ii) For every $k \in \mathbb{N}$ the set $(1/k)U$ is a GTG set in G.
  (iii) There exists $k \in \mathbb{N}$ such that $(1/k)U$ is a GTG set in G.
  In this case $\mathcal{T}_U = \mathcal{T}_{(1/k)U}$ for every $k \in \mathbb{N}$.

Proof. (a) and (b) are straightforward.
(c)
(i) $\Rightarrow$ (ii): Suppose that $(1/m)U + (1/m)U \subseteq U$. This yields $(1/m)(1/k)U + (1/m)(1/k)U \overset{\text{(a)}}{=} (1/k)(1/m)U + (1/k)(1/m)U \overset{\text{(b)}}{\subseteq} (1/k)[(1/m)U + (1/m)U] \subseteq (1/k)U$ and hence the assertion follows from Proposition 4.4.
(ii) $\Rightarrow$ (iii) is trivial.
(iii) $\Rightarrow$ (i): Let m be such that $(1/m)((1/k)U) + (1/m)((1/k)U) \subseteq (1/k)U$. Since $(1/mk)U \subseteq (1/m)((1/k)U)$ we deduce

$$(1/mk)U + (1/mk)U \subseteq (1/k)U \subseteq U$$

and the assertion is a consequence of Proposition 4.4.
Finally, assume that U is a GTG set. From $(1/mk)U \subseteq (1/m)((1/k)U) \subseteq (1/m)U$, we obtain the equality of the topologies $\mathcal{T}_U = \mathcal{T}_{(1/k)U}$. □

Next we give investigate under which conditions intersections and products of GTG sets are GTG.

Lemma 4.9.

(a) *Inverse images of GTG sets by group homomorphisms are GTG. More precisely, if $\phi : G \to H$ is a homomorphism and $A \ni 0$ is a symmetric subset of H, then $\gamma(\phi^{-1}(A)) \leqslant \gamma(A)$. If $A \subseteq \phi(G)$ then $\gamma(\phi^{-1}(A)) = \gamma(A)$.*
(b) *If $\{A_i : i \in I\}$ is a family of GTG sets of a group G and the subset $\{\gamma(A_i) : i \in I\}$ of $\mathbb{N}$ is bounded, then also $\bigcap_{i\in I} A_i$ is a GTG subset of G. In particular, the intersection of any finite family of GTG sets of G is a GTG set of G.*
(c) *Let $(G_i)_{i\in I}$ be a family of groups and let A_i be a subset of G_i for every $i \in I$. The set $A := \prod_{i\in I} A_i \subseteq \prod_{i\in I} G_i$ is a GTG set of $G := \prod_{i\in I} G_i$ iff all A_i are GTG sets and the subset $\{\gamma(A_i) : i \in I\}$ of $\mathbb{N}$ is bounded. In particular,*
  (c_1) *if I is finite then $\prod_{i\in I} A_i$ is GTG iff all the sets A_i are GTG,*
  (c_2) *for an arbitrary index set I, U is a GTG set of a group G iff U^I is a GTG set of G^I.*

Proof. (a) is a consequence of the identity $(1/m)\phi^{-1}(A) = \phi^{-1}((1/m)A)$.
(b) It is straightforward to prove that $(1/m)\bigcap_{i\in I} A_i = \bigcap_{i\in I}(1/m)A_i$. By our hypothesis we may choose m so large that $(1/m)A_i + (1/m)A_i \subseteq A_i$ for all $i \in I$ and obtain $(1/m)\bigcap_{i\in I} A_i + (1/m)\bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I} A_i$. The assertion follows from Proposition 4.4.
(c) follows easily from (a), Proposition 4.4 and the equality $(1/n)\prod_{i\in I} A_i = \prod_{i\in I}(1/n)A_i$. □

Example 4.10.

(a) Let $\mathbb{P}$ be the set of all positive primes. For each $p \in \mathbb{P}$ we define the symmetric subset of $\mathbb{Z}$

$$U_p = \{0\} \cup \left\{\pm 2^{n_2} 3^{n_3} \cdots p^{n_p} : n_2, n_3, \ldots, n_p \in \mathbb{N} \cup \{0\}\right\}.$$

Note that for $p, q \in \mathbb{P}$, we have $(1/q)U_p = U_p$ for $q \leqslant p$ and $(1/q)U_p = \{0\}$ otherwise. This implies that for every $p \in \mathbb{P}$, U_p is a GTG set, $(U_p)_\infty = \{0\}$ and $U_p + U_p \nsubseteq U_p$. Hence $p < \gamma(U_p)$. The subset $U = \prod_{p\in\mathbb{P}} U_p \subseteq \mathbb{Z}^{\mathbb{P}}$ is symmetric and satisfies $U_\infty = \prod_{p\in\mathbb{P}}(U_p)_\infty = \{0\}$, but it is not a GTG set by Lemma 4.9(c).
Define $V_p := U_p \times \prod_{q\in\mathbb{P}, q\neq p} \mathbb{Z}$. Then for every $p \in \mathbb{P}$ the sets V_p are GTG, however, their intersection $\bigcap_{p\in\mathbb{P}} V_p = U$ is not GTG as shown above.

(b) A simpler example of a non-GTG intersection of GTG sets can be obtained from the set U of Example 4.1: it is the intersection of all $\|\cdot\|_{1/n}$-unit balls U_n in $\mathbb{R}^2$, for $n \in \mathbb{N}$.
(c) If U_n is the subset of $G_n = \mathbb{R}^2$, as in (b), then $\gamma(U_n) \to +\infty$. Therefore, $U = \prod_{n\in\mathbb{N}} U_n$ is not a GTG set in $G = (\mathbb{R}^2)^{\mathbb{N}}$, according to item (c) of Lemma 4.9.

The next proposition gives an intuitive idea about GTG sets:

Proposition 4.11. *If G is a compact connected abelian group and U is a GTG set of G with Haar measure* 1*, then $U = G$.*

Proof. For every positive n the map $f_n : G \to G$ defined by $f_n(x) = nx$ is a surjective continuous endomorphism (such a group G is always divisible, see e.g. [21, 24.25]). Since every surjective continuous endomorphism is measure preserving [19], one has $\mu(f_n^{-1}(U)) = \mu(U) = 1$. Therefore, also

$$U_\infty = \bigcap_n f_n^{-1}(U)$$

has measure 1. Since U_∞ is a subgroup, this is possible only when $U_\infty = G$. This yields $U = G$. □

4.2. Construction of GTG sets and UFSS topologies

Now we shall propose a general construction for building infinite GTG sets in abelian groups. In case the group is complete metric, the GTG set can be chosen compact and totally disconnected.

Remark 4.12. In the construction we shall need the following sets of sequences of integers:

$$\mathfrak{Z} = \mathbb{Z}^{\mathbb{N}_0}, \qquad K_m = \left\{(k_j) \in \mathfrak{Z}\colon \sum_{j=0}^{\infty} \frac{|k_j|}{2^j} \leqslant \frac{1}{2^m}\right\} \quad \text{for } m \in \mathbb{Z}, \quad \text{and} \quad \mathfrak{P} = \prod_{j=0}^{\infty} \{0, \pm 1, \pm 2, \pm 3, \dots, \pm 2^{j+2}\}.$$

(a) Obviously, $K_m \subseteq \mathfrak{P}$ when $m \geqslant -2$, and $K_m + K_m \subseteq K_{m-1}$, for $m \in \mathbb{Z}$.
(b) We use also the direct sum $\mathfrak{Z}_0 = \bigoplus_{\mathbb{N}_0} \mathbb{Z}$. For $(a_n) \in \mathfrak{Z}_0$ and any sequence (x_n) of elements of G the sum $\sum_{j=0}^{\infty} a_j x_j$ makes sense and will be used in the sequel. In this way, every element $\mathbf{x} = (x_n) \in G^{\mathbb{N}_0}$ gives rise to a group homomorphism $\varphi_{\mathbf{x}} : \mathfrak{Z}_0 \to G$ defined by $\varphi_{\mathbf{x}}((a_n)) := \sum_{j=0}^{\infty} a_j x_j$ for $(a_n) \in \mathfrak{Z}_0$.
(c) $\mathfrak{Z}$ will be equipped with the product topology, where $\mathbb{Z}$ has the discrete topology with basic open neighborhoods of 0 the subgroups

$$W_n = \{(k_j) \in \mathfrak{Z}\colon k_0 = k_1 = \dots = k_n = 0\},$$

$n \in \mathbb{N}_0$. Thus, $\mathfrak{P}$ is a compact zero-dimensional subspace of $\mathfrak{Z}$. Let us see that K_m is closed in $\mathfrak{P}$ for $m \geqslant -2$, hence a compact zero-dimensional space on its own account. Indeed, pick $\xi = (k_j)_{j\geqslant 0} \in \mathfrak{P} \setminus K_m$. Then $\sum_{j\geqslant 0} \frac{|k_j|}{2^j} > \frac{1}{2^m}$, so $\sum_{j=0}^{n} \frac{|k_j|}{2^j} > \frac{1}{2^m}$ for some index n. Hence the neighborhood $(\xi + W_n) \cap \mathfrak{P}$ of ξ misses the set K_m.

A sequence $(x_n)_{n\geqslant 0}$ in G will be called *nearly independent*, if it satisfies

$$\sum_{j=0}^{\infty} a_j x_j = 0 \quad \Longrightarrow \quad (a_n) = 0 \quad \text{for all } (a_n) \in \mathfrak{P} \cap \mathfrak{Z}_0. \tag{4}$$

This term is motivated by the fact, that usually a sequence $(x_n)_{n\geqslant 0}$ in G is called independent, if $\ker \varphi_{\mathbf{x}} = 0$.

Claim 4.13. *If G is an abelian group and $\mathbf{x} = (x_n)$ is a nearly independent sequence of G, then $\varphi_{\mathbf{x}} \restriction_{K_{-1} \cap \mathfrak{Z}_0} : K_{-1} \cap \mathfrak{Z}_0 \to G$ is injective.*

Proof. Assume that $\varphi_{\mathbf{x}}((k_j)) = \varphi_{\mathbf{x}}((l_j))$ for $(k_j) \in K_{-1}$ and $(l_j) \in K_{-1}$. Then $0 = \sum_{j=0}^{n} (k_j - l_j) x_j$ and $|k_j - l_j| \leqslant 2^{j+2}$, combined with near independence, imply $k_j = l_j$ for all j. □

The following lemma reveals a sufficient condition under which an abelian group G admits a non-discrete UFSS group topology, namely the existence of a nearly independent sequence. The necessity of this condition will be established at a later stage (see Corollary 4.25).

Lemma 4.14. *Let G be an abelian group and let $\mathbf{x} = (x_n)$ be a nearly independent sequence of G. Then the set $X := \varphi_{\mathbf{x}}(K_0 \cap \mathfrak{Z}_0)$ is a GTG subset of G with $\gamma(X) = 2$. More precisely,*

$$\left(1/2^m\right)X = \varphi_{\mathbf{x}}(K_m \cap \mathfrak{Z}_0) \quad \left(= \left\{ \sum_{j=0}^{n} k_j x_j \colon n \in \mathbb{N},\ k_j \in \mathbb{Z},\ \sum_{j=0}^{n} \frac{|k_j|}{2^j} \leqslant \frac{1}{2^m} \right\} \right), \tag{5}$$

$X_\infty = \{0\}$ and (x_n) tends to 0 in $\mathcal{T}_X$, so $\mathcal{T}_X$ is a non-discrete UFSS topology.

Proof. The inclusion $\supseteq$ in (5) is obvious. We prove the following stronger version of the reverse inclusion by induction:

$$\text{if } x = \sum_{j=0}^{n} k_j x_j \in X \text{ with } (k_j) \in K_0, \quad \text{then } x \in \left(1/2^m\right)X \quad \Longrightarrow \quad (k_j) \in K_m. \tag{6}$$

For $m = 0$ the assertion is trivial. So suppose (6) holds true for m and let $x = \sum_{j=0}^{n} k_j x_j \in (1/2^{m+1})X$, with $(k_j) \in K_0$. Since $x, 2x \in (1/2^m)X$, the induction hypothesis gives $(k_j) \in K_m$. Moreover, there exists a representation $2x = \sum_{j=0}^{n} l_j x_j$ with $(l_j) \in K_m$, i.e., $\sum_{j=0}^{n} \frac{|l_j|}{2^j} \leqslant \frac{1}{2^m}$. (Observe that without loss of generality we may assume that the upper index for the summation is equal for x and $2x$.) Then $\varphi_{\mathbf{x}}((2k_j)) = \varphi_{\mathbf{x}}((l_j))$ with $(2k_j), (l_j) \in K_{-1}$, so Claim 4.13 gives $2k_j = l_j$ for all j and hence $\sum_{j=0}^{n} \frac{|k_j|}{2^j} = \sum_{j=0}^{n} \frac{|l_j|}{2^{j+1}} \leqslant \frac{1}{2^{m+1}}$. This proves (6), and consequently also (5). Obviously, (6) yields also $X_\infty = \{0\}$.

For $m = 1$ Eq. (5) and $K_1 + K_1 \subseteq K_0$ give $(1/2)X + (1/2)X \subseteq X$. Hence $\gamma(X) \leqslant 2$, and consequently X is a GTG set and $\mathcal{T}_X$ is a UFSS topology.

For a fixed $N \in \mathbb{N}$ the definition of X and (5) give $x_n \in (1/2^N)X$ for all $n \geqslant N$, so $\{x_n \colon n \geqslant N\} \subseteq (1/2^N)X$. This shows that $x_n \to 0$ in $\mathcal{T}_X$ and so $\mathcal{T}_X$ is not discrete.

Finally, to prove that $\gamma(X) \geqslant 2$ it suffices to observe that $\gamma(X) = 1$ would imply that X is a subgroup, so $X = X_\infty$. Now $X_\infty = \{0\}$ contradicts the non-discreteness of $\mathcal{T}_X$. □

Let (G, d) be a metric abelian group, let v be the group seminorm associated to the metric d (i.e., $v(x) = d(x, 0)$ for $x \in G$) and let $B_\varepsilon = \{x \in G \colon v(x) \leqslant \varepsilon\}$ be the closed disk with radius ε around 0. For a nearly independent sequence (x_n) of G and a non-negative $n \in \mathbb{Z}$ let

$$\varepsilon_n := \min\left\{ v\left(\sum_{j=0}^{n} a_j x_j \right) \colon |a_j| \leqslant 2^{j+2},\ (a_j) \neq (0) \right\} > 0. \tag{7}$$

We call the sequence (x_n) *almost independent*, if the inequality

$$2^{n+3} v(x_{n+1}) < \varepsilon_n \leqslant v(x_n) \tag{8}$$

holds. Note that $\varepsilon_n \leqslant v(x_n)$ obviously follows from the definition of ε_n. Moreover, every almost independent sequence (rapidly) converges to 0 in (G, d).

It is straightforward to prove that a subsequence of a nearly (respectively almost) independent sequence is again nearly (respectively almost) independent. The motivation to introduce the sharper notion of almost independent sequence is given in the lemma below. First we need to isolate a property that will be frequently used in the sequel:

Claim 4.15. *If (G, d) is a metric group and (x_n) is an almost independent sequence of G, then $\varphi_{\mathbf{x}}(K_m \cap W_n \cap \mathfrak{Z}_0) \subseteq B_{\frac{v(x_n)}{2^{m+2}}}$ for any $m \in \mathbb{Z}$ and $n \geqslant 0$.*

Proof. We have to prove that $v(\sum_{j=n+1}^{k} k_j x_j) < \frac{1}{2^{m+2}} v(x_n)$, whenever $(k_j) \in K_m$. This follows applying (8) to the term $2^j v(x_j)$ in

$$v\left(\sum_{j=n+1}^{k} k_j x_j \right) \leqslant \sum_{j=n+1}^{k} |k_j| v(x_j) = \sum_{j=n+1}^{k} \frac{|k_j|}{2^j} 2^j v(x_j) < \sum_{j=n+1}^{k} \frac{|k_j|}{2^j} \frac{1}{4} v(x_{j-1}) \leqslant \frac{1}{4} v(x_n) \sum_{j=n+1}^{k} \frac{|k_j|}{2^j} \leqslant \frac{1}{2^{m+2}} v(x_n).$$

□

Lemma 4.16. *Let (G, d) be a metric group and let (x_n) be an almost independent sequence of G. Then*

(a) *the non-discrete UFSS topology $\mathcal{T}_X$ generated by the GTG set X of G corresponding to (x_n) as in Lemma 4.14, is finer than the original topology of G;*
(b) *the subsequence (x_{2n}) is still almost independent, and for the GTG set Y of G corresponding to (x_{2n}) as in Lemma 4.14, $\mathcal{T}_X < \mathcal{T}_Y$.*

Proof. (a) We have to prove that for a given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $(1/2^m)X \subseteq B_\varepsilon$. Since $x_n \to 0$ in the metric topology, there exists $m \in \mathbb{N}$ such that $\frac{v(x_{m-1})}{2^{m+2}} < \varepsilon$. As $K_m \subseteq W_{m-1}$ and $(1/2^m)X = \varphi_{\mathbf{x}}(K_m \cap \mathfrak{Z}_0)$, from Claim 4.15 we obtain $(1/2^m)X = \varphi_{\mathbf{x}}(K_m \cap \mathfrak{Z}_0) = \varphi_{\mathbf{x}}(K_m \cap W_{m-1} \cap \mathfrak{Z}_0) \subseteq B_\varepsilon$.

(b) By Lemma 4.14, $\mathcal{T}_Y$ is a UFSS topology on G. Since $Y \subseteq X$, we trivially have $\mathcal{T}_Y \supseteq \mathcal{T}_X$. It remains to be shown that $\mathcal{T}_Y$ is strictly finer than $\mathcal{T}_X$. Let us prove that the $\mathcal{T}_X$ null-sequence (x_{2n+1}) does not converge to 0 in $\mathcal{T}_Y$. It is enough to show that $\{x_{2n+1}\colon n \in \mathbb{N}\} \cap Y = \emptyset$. So assume $x_{2m+1} = \sum_{j=0}^n k_j x_{2j}$ for some $m \in \mathbb{N}$ and $(k_j) \in K_0$. As $x_{2m+1} \in \varphi_{\mathbf{x}}(K_0)$ as well, this contradicts Claim 4.13. □

In the next theorem we show that the set X from the previous lemmas, corresponding to an almost independent sequence of G, has a compact totally disconnected closure when G is complete.

Theorem 4.17. *Let (G, d) be a complete metric group and let (x_n) be an almost independent sequence of G. Then the closure $\widetilde{X}$ of the GTG set X corresponding to (x_n) as in Lemma 4.14, is compact and totally disconnected. Moreover, $\widetilde{X}$ is a GTG set with $\gamma(\widetilde{X}) = 2$, so $\mathcal{T}_{\widetilde{X}}$ is a non-discrete UFSS topology finer than the original topology of G.*

Proof. We intend to extend the map $\varphi_{\mathbf{x}}$ defined in item (b) of Remark 4.12 to a map $\varphi : \bigcup_{m\in\mathbb{Z}} K_m \to G$ by setting $\varphi((k_j)) = \sum_{j\geqslant 0} k_j x_j$ (the correctness of this definition is checked below). Furthermore, we show that $\varphi \restriction_{K_m}$ is continuous for each m, while $\varphi \restriction_{K_0}$ is injective. Since each K_m is a compact zero-dimensional space (Remark 4.12(c)), this will prove that $\widetilde{X} = \varphi(K_0)$ itself is a compact zero-dimensional space, while the subspaces $\varphi(K_m)$ with $m < 0$ are just compact.

For a fixed $(k_j)_{j\geqslant 0} \in K_m$ let $y_n = \sum_{j=0}^n k_j x_j \in G$. To see that (y_n) is a Cauchy sequence in G apply Claim 4.15 to get $v(y_k - y_n) \leqslant \frac{1}{2^{m+2}} v(x_n)$ for every pair $n \leqslant k$. Since $x_n \to 0$, this proves that (y_n) is a Cauchy sequence in G. Since G is complete, the limit $\lim y_n$ exists and $\varphi((k_j)) = \sum_{j\geqslant 0} k_j x_j$ and $\widetilde{X} := \varphi(K_0)$ make sense.

Since the norm function $v : G \to \mathbb{R}$ is continuous, we obtain from Claim 4.15, after passing to the limit:

$$\varphi(K_m \cap W_n) \subseteq B_{\frac{v(x_n)}{2^{m+2}}}. \tag{9}$$

Even if φ is not a homomorphism, one has

$$\varphi(\xi + \eta) = \varphi(\xi) + \varphi(\eta), \quad \text{whenever } \xi = (k_j),\ \eta = (l_j) \in K_m, \tag{10}$$

where $\xi + \eta = (k_j + l_j) \in K_{m-1}$.

Fix $m \in \mathbb{Z}$. In order to show that $\varphi \restriction_{K_m}$ is continuous, fix $(k_j) \in K_m$ and $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that $\frac{1}{2^{m+1}} v(x_n) < \varepsilon$. For $(l_j) \in (k_j) + W_n$ we have $\varphi((l_j)) \in \varphi((k_j)) + B_\varepsilon(0)$ by (9), which shows that φ is continuous.

In order to show that φ is injective, we show the following stronger statement, that will be necessary below:

Claim 4.18. *If $(k_j) \in K_{-1}$ and $(l_j) \in K_0$ with $(k_j) \neq (l_j)$, then $\varphi((k_j)) \neq \varphi((l_j))$.*

Proof. Assume for a contradiction that $\varphi((k_j)) = \varphi((l_j))$ with $(k_j) \neq (l_j)$. Fix m minimal with $k_m \neq l_m$. Then

$$(l_m - k_m)x_m = \sum_{j>m} k_j x_j - \sum_{j>m} l_j x_j$$

with $|k_j - l_j| \leqslant 3 \cdot 2^j < 2^{j+2}$ for all $j \geqslant m$ (as $k_j \leqslant 2^{j+1}$ and $l_j \leqslant 2^j$). Hence the definition of ε_m gives

$$\varepsilon_m \leqslant v\bigl((l_m - k_m)x_m\bigr) \leqslant |k_{m+1} - l_{m+1}| v(x_{m+1}) + v\Bigl(\sum_{j>m+1} k_j x_j\Bigr) + v\Bigl(\sum_{j>m+1} l_j x_j\Bigr). \tag{11}$$

To the second and the third term in the right hand side of (11) we may apply (9) with $m = 0$ (respectively $m = -1$) and $n = m + 1$ and obtain

$$v\Bigl(\sum_{j>m+1} k_j x_j\Bigr) + v\Bigl(\sum_{j>m+1} l_j x_j\Bigr) \leqslant \frac{1}{2} v(x_{m+1}) + \frac{1}{4} v(x_{m+1}) = \frac{3}{4} v(x_{m+1}). \tag{12}$$

Since $|k_{m+1} - l_{m+1}| \leqslant 3 \cdot 2^{m+1}$ yields $|k_{m+1} - l_{m+1}| v(x_{m+1}) \leqslant 3 \cdot 2^{m+1} v(x_{m+1})$, by (11) and (12), we get $\varepsilon_m \leqslant 3 \cdot 2^{m+1} v(x_{m+1}) + \frac{3}{4} v(x_{m+1})$. Along with (8), applied with $n = m$, we get $\varepsilon_m \leqslant 3 \cdot 2^{m+1} v(x_{m+1}) + \frac{3}{4} v(x_{m+1}) < 2^{m+3} v(x_{m+1}) < \varepsilon_m$, a contradiction. This proves Claim 4.18. □

From Claim 4.18 we conclude that $\varphi \restriction_{K_0}$ is a continuous bijective mapping. Since K_0 is compact, $\varphi \restriction_{K_0} : K_0 \to \widetilde{X} = \varphi(K_0)$ is a homeomorphism which implies in particular that $\widetilde{X}$ is compact and totally disconnected. Since φ is an extension of $\varphi_{\mathbf{x}}$ and since $\mathfrak{Z}_0 \cap K_0$ is dense in K_0, we deduce that $X = \varphi_{\mathbf{x}}(\mathfrak{Z}_0 \cap K_0) = \varphi(\mathfrak{Z}_0 \cap K_0)$ is dense in $\widetilde{X} = \varphi(K_0)$. Since the latter set is compact, it must be closed in G. Therefore, $\widetilde{X}$ coincides with the closure of X.

Next we claim that

$$\big(1/2^m\big)\widetilde{X} = \varphi(K_m) \quad \left(= \left\{ \sum_{j=0}^{\infty} k_j x_j \colon k_j \in \mathbb{Z},\ \sum_{j=0}^{\infty} \frac{|k_j|}{2^j} \leqslant \frac{1}{2^m} \right\}\right), \tag{13}$$

as in the case of the set X in Lemma 4.14.

The inclusion $\supseteq$ in (13) is obvious. We prove the following stronger version of the reverse inclusion by induction:

$$\text{if } x = \sum_{j=0}^{\infty} k_j x_j, \text{ with } (k_j) \in K_0, \quad \text{then } x \in \big(1/2^m\big)\widetilde{X} \quad \Longrightarrow \quad (k_j) \in K_m. \tag{14}$$

For $m = 0$ the assertion is trivial. So suppose (14) holds true for m and let $x = \varphi((k_j))$, with $(k_j) \in K_0$ belong to $(1/2^{m+1})\widetilde{X}$. Since $x, 2x \in (1/2^m)\widetilde{X}$, by the induction hypothesis, $(k_j) \in K_m$. Moreover, there exists a representation $2x = \varphi((l_j))$ with $(l_j) \in K_m$. Then $\varphi((2k_j)) = \varphi((l_j))$. Since $(2k_j) \in K_{-1}$ and $(l_j) \in K_0$, from Claim 4.18 we conclude that $2k_j = l_j$ for all j and hence $\sum_{j=0}^{\infty} \frac{|k_j|}{2^j} \leqslant \frac{1}{2^{m+1}}$. This proves (14), and consequently also (13).

In particular, from (13) we get $(1/2)\widetilde{X} = \varphi(K_1)$. Since $K_1 + K_1 \subseteq K_0$ from Remark 4.12(a), this combined with (13), gives $(1/2)\widetilde{X} + (1/2)\widetilde{X} \subseteq \widetilde{X}$. Hence $\gamma(\widetilde{X}) \leqslant 2$.

It remains to note that (14) implies also $\widetilde{X}_\infty = \{0\}$. Hence $\mathcal{T}_{\widetilde{X}}$ is a UFSS topology coarser than $\mathcal{T}_X$ (as $X \subseteq \widetilde{X}$), so it is non-discrete. In particular, $\widetilde{X} \neq \{0\} = \widetilde{X}_\infty$, so $\gamma(\widetilde{X}) = 2$. From (9) and (13) we conclude that $\mathcal{T}_{\widetilde{X}}$ is finer than the original topology of G. $\square$

In order to characterize those abelian metrizable groups which admit a (strictly) finer UFSS group topology, we need the following definition which will characterize these groups.

Definition 4.19. An abelian topological group G is called *locally bounded* if there exists some $n \in \mathbb{N}$ such that the subgroup $G[n] = \{x \in G\colon\ nx = 0\}$ is open.

Remark 4.20. G is locally bounded iff it has a neighborhood U in which all elements are of bounded order. Obviously, a metric abelian topological group G is not locally bounded iff there exists a null-sequence $x_n \to 0$ such that $o(x_n) \to \infty$.

A locally compact abelian group G is locally bounded iff it has an open compact subgroup of finite exponent. Indeed, assume that G is a locally compact, locally bounded abelian group. For suitable $n \in \mathbb{N}$ the subgroup $G[n]$ is open. By the structure theorem for locally compact abelian groups, $G[n]$ contains a compact open subgroup K. It is clear that K is open in G and of finite exponent. The converse implication is trivial.

Theorem 4.21. *Let (G, d) be an abelian, metrizable, non-discrete group. The following assertions are equivalent*:

(i) *G is not locally bounded*;
(ii) *there exists a finer non-discrete UFSS group topology $\mathcal{T}_X$ on G*;
(iii) *there exists a strictly finer non-discrete UFSS group topology $\mathcal{T}_Y$ on G*;
(iv) *there exists an almost independent sequence in G.*

Proof. (iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i): Let G be locally bounded, this means there exists $n \geqslant 1$ such that the subgroup $G[n]$ is open. Suppose there exists a UFSS topology $\mathcal{T}_X$ on G with distinguished neighborhood X which is finer than the topology τ induced by the metric. We may assume that $X \subseteq G[n]$, because otherwise we can replace X by $X \cap G[n]$. Then $(1/n)X = X_\infty$. Since we assumed $\mathcal{T}_X$ to be finer than the original topology and hence Hausdorff, $\{0\} = X_\infty = (1/n)X$. This implies that $\mathcal{T}_X$ is discrete. So the only finer UFSS group topology on G is the discrete one.

(iv) $\Rightarrow$ (iii): this is covered by Lemma 4.16.

(i) $\Rightarrow$ (iv): Assume now that G is not locally bounded. We have to show that there exists an almost independent sequence (x_n) of elements in G. This will be done by induction. For $n = 0$ condition (4) is equivalent to $3x_0 \neq 0 \neq 4x_0$. So fix an element $x_0 \in G$ of order greater than 4. Assume that $x_0, \ldots, x_n$ have already been chosen to satisfy (4). Define ε_n by (7). Then choose x_{n+1} with $\nu(x_{n+1}) < \varepsilon_n/2^{n+3}$ and $o(x_{n+1}) > 2^{n+3}$. To check that this works, let $x = \sum_{j=0}^{n+1} a_j x_j$ with $(a_j) \neq (0)$ and $|a_j| \leqslant 2^{j+2}$ for $j = 0, 1, \ldots, n+1$. If $a_0 = \cdots = a_n = 0$ then $x \neq 0$, since $|a_{n+1}| \leqslant 2^{n+3} < o(x_{n+1})$. Otherwise, we have $\nu(x) \geqslant \nu(\sum_{j=0}^{n} a_j x_j) - \nu(a_{n+1}x_{n+1}) \geqslant \varepsilon_n - 2^{n+3}\nu(x_{n+1}) > 0$, so $x \neq 0$. By the choice of each x_n, the sequence is almost independent. $\square$

Remark 4.22. Let us note that for the set Y constructed in the proof, the strictly finer non-discrete UFSS topology $\mathcal{T}_Y$ is still locally unbounded. So to the group $(G, \mathcal{T}_Y)$ the same construction can be applied to provide an infinite strictly increasing chain of non-discrete UFSS topologies $\mathcal{T}_Y = \mathcal{T}_{Y_0} < \mathcal{T}_{Y_1} < \cdots < \mathcal{T}_{Y_n} < \cdots$. Hence in the theorem one can also add a stronger property (v) claiming the existence of such a chain.

Corollary 4.23. *Let (G, d) be a complete abelian, metrizable non-locally-bounded group. Then there exists a compact totally disconnected GTG set X of G, such that $\mathcal{T}_X$ is a finer non-discrete UFSS group topology on G.*

Proof. According to the above theorem G admits an almost independent sequence (x_n). □

E. Hewitt [20] observed that the group $\mathbb{T}$ and the group $\mathbb{R}$ have the property that the only stronger locally compact group topologies are the discrete topologies. Since locally minimal topologies generalize the locally compact group topologies, this suggests the following question: *Do the groups $\mathbb{T}$ and $\mathbb{R}$ admit stronger non-discrete locally minimal topologies*? The next corollary answers this question in a strongly positive way: for the class of all non-totally disconnected locally compact metrizable abelian groups (in place of $\mathbb{T}$ and $\mathbb{R}$ only) and for the smaller class of UFSS topologies (in place of locally minimal topologies).

Corollary 4.24. *A locally compact abelian metrizable group G has a strict UFSS refinement iff G contains no open compact subgroup of finite exponent.*

This happens for example, if G is not totally disconnected.

Proof. The first assertion is obvious when G is discrete, so we assume that G is non-discrete in the sequel.

According to Theorem 4.21, G has a strict, non-discrete UFSS refinement iff G is not locally bounded, which, by Remark 4.20, is equivalent to the fact that G contains no open compact subgroup of finite exponent.

In order to prove the second statement it is sufficient to show that every group H which has an open compact subgroup K of finite exponent is totally disconnected. The connected component C of H is contained in K and hence bounded. On the other hand, as every compact abelian connected group, C is divisible. This implies that C is trivial and hence H is totally disconnected. □

Now the topology-free version of Theorem 4.21 follows:

Corollary 4.25. *For an abelian group G TFAE*:

(i) *G is not bounded*;
(ii) *G admits a non-discrete UFSS group topology*;
(iii) *there exists a nearly independent sequence in G*.

Proof. The implication (iii) ⇒ (ii) was proved in Lemma 4.14.

To prove the implication (ii) ⇒ (i) assume G admits a non-discrete UFSS group topology $\mathcal{T}$ with distinguished neighborhood U of 0. Then for every $n \in \mathbb{N}$ the set $(1/n)U$ is a $\mathcal{T}$-neighborhood of 0, hence $(1/n)U \neq \{0\}$. If nG were $\{0\}$ for some $n \in \mathbb{N}$, then $(1/n)U = U_\infty = \{0\}$ which is a contradiction. So G is unbounded.

To prove the implication (i) ⇒ (iii) pick a countable subgroup H of G that is still not bounded. Since H is countable, there exists an injective homomorphism $j : H \to \mathbb{T}^{\mathbb{N}}$. Denote by d the metric induced on H by this embedding. Then (H, d) is an infinite metric precompact group, hence it is not discrete. Moreover, for no $n \in \mathbb{N}$ the subgroup $H[n]$ is open. Indeed, if $H[n]$ were open, then by the precompactness of H it would have finite index in H. Hence $mH \subseteq H[n]$ for some $m \in \mathbb{N}$. Therefore, $mnH = 0$, a contradiction. This argument proves that no subgroup $H[n]$ $(n \in \mathbb{N})$ is open in H. Hence, (H, d) is not locally bounded. Then H contains an almost independent sequence (x_n) by the above theorem. Clearly, this is also a nearly independent sequence in H, and consequently, also in G. □

5. Locally GTG groups

5.1. Locally GTG groups and their properties

Definition 5.1 *(V. Tarieladze, oral communication).* We say that a Hausdorff topological abelian group G is *locally GTG* if it admits a basis of neighborhoods of the identity formed by GTG subsets of G.

Example 5.2.

(a) Every UFSS group is locally GTG. In particular $\mathbb{R}$ and $\mathbb{T}$ are locally GTG.
(b) Every locally convex space is locally GTG.
(c) Assume that G is a bounded abelian group with exponent m. If U is a GTG neighborhood of 0 in some group topology τ of G, then $U_\infty = (1/m)U$ is a τ-neighborhood of 0. Therefore,
    (c_1) (G, τ) is locally GTG precisely when (G, τ) is linearly topologized;
    (c_2) (G, τ) is UFSS precisely when (G, τ) is discrete.

Example 5.3. A topological vector space is said to be *locally pseudoconvex* if it has a basis of pseudoconvex neighborhoods of zero. A topological vector space is locally GTG as a topological abelian group if and only if it is locally pseudoconvex.

Proof. Applying Proposition 4.5, it suffices to show that a topological vector space which is locally GTG has a neighborhood basis consisting of balanced GTG sets. So fix a GTG neighborhood A and define $B := \{a \in A\colon\ [0,1]a \subseteq A\}$. It is straightforward to prove that $[0,1]B \subseteq B$ and it is a well-known fact that B is a neighborhood of zero. Let us prove that B is GTG. Since A was assumed to be GTG, there exists $n \in \mathbb{N}$ such that $(1/n)A + (1/n)A \subseteq A$. Observe that $(1/n)B = \frac{1}{n}B$. We shall show that $\frac{1}{n}B + \frac{1}{n}B \subseteq B$. So fix $a, b \in \frac{1}{n}B$ and $t \in [0,1]$. Let us see that $t(a+b)$ belongs to A: $t(a+b) = ta + tb \in \frac{1}{n}B + \frac{1}{n}B \subseteq (1/n)A + (1/n)A \subseteq A$. □

Example 5.4. Local GTG-ness may seem to be a too mild property, but there exist natural examples of abelian topological groups lacking it. Consider the topological vector space $G = L^0$ of all classes of Lebesgue measurable functions f on $[0,1]$ (modulo almost everywhere equality) with the topology of convergence in measure. This topology can be defined by the invariant metric

$$d(f,g) = \int_0^1 \min\{1, |f(t) - g(t)|\}\, dt$$

(for details see for instance [22, Ch. 2]). It is known that L^0 is not locally pseudoconvex and hence, by Example 5.3, it is not locally GTG as a topological group.

Here we collect several properties of locally GTG groups.

Proposition 5.5.

(a) *Every subgroup of a locally GTG group is locally GTG.*
(b) *A group with an open locally GTG subgroup is locally GTG.*
(c) *The product of locally GTG groups is locally GTG.*
(d) *Quotient groups of locally GTG groups need not be locally GTG.*
(e) *Every group locally isomorphic to a locally GTG group is locally GTG. In particular, if a topological group G admits a non-trivial locally GTG open subgroup, then G is locally GTG.*

Proof. (a) is a consequence of Lemma 4.9(a) and Example 4.3(b). (b) follows from the fact that any basis of neighborhoods of zero in the open subgroup is a basis of neighborhoods of zero in the whole group. (c) is a consequence of Lemma 4.9(c). (d) Let G be a Hausdorff group which is not locally GTG. G is a quotient of the free abelian topological group $A(G)$ [23]. The free locally convex space $L(G)$ is locally GTG according to Example 5.2(b). According to a result of Uspenskij and Tkachenko ([38] and [39]) the free abelian topological group $A(G)$ is a subgroup of $L(G)$ and hence, due to (a), also locally GTG. This proves (d). (e) is straightforward using Lemma 4.9(a). □

Now we obtain another large class of examples:

Example 5.6.

(a) Every precompact abelian group is locally GTG. Indeed, every precompact abelian group is (isomorphic to) a subgroup of a power of $\mathbb{T}$, so items (a) and (c) of Proposition 5.5 and item (a) of Example 5.2 apply.
(b) Every locally compact abelian group is locally GTG. Indeed, every locally compact abelian group has the form $G = \mathbb{R}^n \times G_0$, where $n \in \mathbb{N}$ and G_0 contains an open compact subgroup. Then G_0 is locally GTG by item (a) and item (e) of Proposition 5.5, while $\mathbb{R}^n$ is UFSS, so locally GTG. Now item (c) of Proposition 5.5 applies.

The connection between locally GTG and UFSS groups is the following:

Theorem 5.7.

(a) *If U is a GTG subset of an abelian group G, the quotient group $G_U := (G, \mathcal{T}_U)/U_\infty$ is UFSS when equipped with the quotient topology of $\mathcal{T}_U$.*
(b) *Every locally GTG group G can be embedded into a product of UFSS groups.*
(c) *A group topology τ on an abelian group G is a supremum of UFSS topologies on G iff τ is NSS and locally GTG.*
(d) *If a group topology τ on an abelian group G is a supremum of a family $\mathfrak{T} = \{\tau_i\colon\ i \in I\}$ of UFSS topologies on G, then τ is UFSS iff τ coincides with the supremum of a finite subfamily of $\mathfrak{T}$.*

Proof. (a) Let U be a GTG subset of G. Since U_∞ is the $\mathcal{T}_U$-closure of $\{0\}$, we can consider the Hausdorff quotient group $G_U = (G, \mathcal{T}_U)/U_\infty$, and the canonical epimorphism $\varphi_U : G \to G_U$.

Let $m \in \mathbb{N}$ be such that $(1/m)U + (1/m)U \subseteq U$. Let us show that for every $n \in \mathbb{N}$

$$(1/n)\varphi_U\big((1/m)U\big) \subseteq \varphi_U\big((1/n)U\big).$$

Indeed, fix an element $\varphi_U(x) \in (1/n)\varphi_U((1/m)U)$. Then for every $k \in \{1,\dots,n\}$, $kx \in (1/m)U + U_\infty \subseteq U$, hence $\varphi_U(x) \in \varphi_U((1/n)U)$.

This shows that G_U is a UFSS group with distinguished neighborhood $\varphi_U((1/m)U)$.

(b) Let $\mathcal{U}$ be a basis of neighborhoods of zero in G formed by GTG sets. The homomorphism

$$\Phi : G \to \prod_{U \in \mathcal{U}} G_U, \qquad \Phi(x) = \big(\varphi_U(x)\big)_{U \in \mathcal{U}}$$

is injective and continuous. Fix $U \in \mathcal{U}$, and let $m \in \mathbb{N}$ be such that $(1/m)U + (1/m)U \subseteq U$. Then $(1/m)U + U_\infty \subseteq U$, from which we deduce

$$\Phi(U) \supset \Phi(G) \cap \left(\left(\prod_{U' \in \mathcal{U} \setminus \{U\}} G_{U'}\right) \times \varphi_U\big((1/m)U\big)\right).$$

This implies that Φ is open onto its image.

(c) It is clear that every supremum of UFSS topologies is both NSS and locally GTG. Conversely, if G is locally GTG, its topology is the supremum of the family of topologies $\{\mathcal{T}_U\}_{U \in \mathcal{U}}$ where $\mathcal{U}$ is a basis of neighborhoods of zero. If moreover G is NSS, we may assume that no neighborhood in $\mathcal{U}$ contains non-trivial subgroups, and in particular the topologies $\mathcal{T}_U$ are UFSS.

(d) The sufficiency is obvious from Remark 3.13(a). To prove the necessity let us assume that $\tau = \sup \tau_i$ is UFSS. Then there exists a distinguished τ-neighborhood W of 0 such that $\tau = \mathcal{T}_W$. There exist a finite subset $J \subseteq I$ and τ_j-neighborhoods U_j of 0 for each $j \in J$ such that $\bigcap_{j \in J} U_j \subseteq W$. We can assume without loss of generality that U_j is a distinguished neighborhood of 0 in τ_j for each $j \in J$. Then $(1/n)W \in \sup_{j \in J} \tau_j$ for every n. Hence $\tau = \mathcal{T}_W \leqslant \sup_{j \in J} \tau_j$. The inequality $\tau = \sup_{j \in I} \geqslant \sup_{j \in J} \tau_j$ is trivial. This proves that $\tau = \sup_{j \in I} \tau_j$. □

Remark 5.8. Note that "NSS" is needed in (c) above; any non-metrizable compact abelian group is locally GTG (see Example 5.6(a)) but its topology is not a supremum of UFSS topologies.

Corollary 5.9. *The class of locally GTG abelian groups is stable under taking completions.*

Proof. By Theorem 5.7(b), every locally GTG group G can be embedded into a product $\prod_i G_i$ of UFSS groups G_i. By Proposition 3.12(a), the completion $\widetilde{G}_i$ of the UFSS group G_i is UFSS. So the completion $\widetilde{G}$ of G embeds into the product $P = \prod_i \widetilde{G}_i$ of UFSS groups. By Proposition 5.5(c) P is locally GTG, so $\widetilde{G}$ is locally GTG by Proposition 5.5(a). □

Theorem 5.10. *A Hausdorff abelian topological group (G, τ) is a UFSS group if and only if (G, τ) is locally minimal, locally GTG and NSS.*

Proof. Suppose that (G, τ) is a UFSS group with distinguished neighborhood U. Then (G, τ) is U-locally minimal according to Facts 3.3(a), locally GTG according to Example 5.2(a) and U does not contain any non-trivial subgroup.

Conversely, let (G, τ) be locally minimal, locally GTG and NSS. There exists a neighborhood of zero U which is a GTG set, witnesses local minimality and does not contain non-trivial subgroups. The group topology $\mathcal{T}_U$ generated by U is Hausdorff and coarser than τ; since U is one of its zero neighborhoods, it coincides with τ. □

5.2. Locally minimal, locally GTG groups

In this section we will give various properties of locally minimal locally GTG groups. Most of our results are based on the following proposition which allows us to find large, in appropriate sense, minimal subgroups in a locally minimal group.

Proposition 5.11. *(See [12].) Let G be a U-locally minimal group and let H be a closed central subgroup of G such that $H + V \subseteq U$ for some neighborhood V of 0 in G. Then H is minimal.*

Theorem 5.12. *If G is a U-locally minimal abelian group where U is a GTG set, then U_∞ is a minimal subgroup.*

Proof. Proposition 4.4 implies $U_\infty + (1/m)U \subseteq U$ for some $m \in \mathbb{N}$. Then, Proposition 5.11 immediately gives us that U_∞ is a minimal subgroup. □

One may ask whether GTG is needed in the above corollary (see Question 6.2). The problem is that without this assumption, the intersection U_∞ need not be a subgroup (although it is always a union of cyclic subgroups), as it happens in Example 5.4.

It easily follows from Theorem 5.12 that every locally minimal locally GTG abelian group contains a minimal, hence precompact, G_δ-subgroup (note that the subgroup U_∞ in Theorem 5.12 is a G_δ-set). Now we provide a different proof of this fact, that makes no recourse to local GTG-ness.

Proposition 5.13. *Every locally minimal abelian group contains a minimal, hence precompact, G_δ-subgroup.*

Proof. Let U witness local minimality of the group G. As in the proof of Proposition 2.13, it is possible to construct inductively a sequence (V_n) of symmetric neighborhoods of 0 in τ which satisfy $V_n + V_n \subseteq V_{n-1}$ (where $V_0 := U \cap -U$). It is easy to see that $H = \bigcap_{n\in\mathbb{N}} V_n$ is a subgroup of G, contained in each V_n. In particular, $H + V_1 \subseteq V_0 \subseteq U$. Now Proposition 5.11 immediately gives us that H is a minimal subgroup. □

Let us note that the minimal G_δ-subgroup obtained in this proof is certainly contained in the subgroup U_∞, provided U is a GTG set (as $H \subseteq U$ and U_∞ is the largest subgroup contained in U). However, this argument has the advantage to require weaker hypotheses.

The next corollary shows that non-metrizable complete locally minimal abelian groups contain large compact subgroups.

Corollary 5.14. *Every complete locally minimal abelian group contains a compact G_δ-subgroup.*

Proof. Follows directly from Proposition 5.13. □

Corollary 5.15. *Let (G, τ) be either*

(a) *a linearly topologized abelian group, or*
(b) *a bounded locally GTG abelian group.*

Then G is locally minimal iff G has an open minimal subgroup.

Proof. If G has an open minimal subgroup, then G is locally minimal (Proposition 2.4). Conversely, suppose that G is locally minimal.

(a) Let V be an open subgroup of G witnessing local minimality of G. Then $V + V \subseteq V$, so V is minimal by Proposition 5.11.

(b) Let G be U-locally minimal for a GTG neighborhood U. According to Theorem 5.12, U_∞ is a minimal subgroup of G. For the exponent m of G, we obtain $(1/m)U = U_\infty$ and hence U_∞ is open. □

If the algebraic structure of a group is sufficiently well understood, Theorem 5.12 helps to characterize locally minimal group topologies. As an example we describe the locally minimal locally GTG topologies on $\mathbb{Z}$. Let us recall that the minimal topologies on $\mathbb{Z}$ are precisely the p-adic ones (Prodanov [29]).

Example 5.16. Let $(\mathbb{Z}, \tau)$ be a locally minimal locally GTG group topology. Then either

(a) it is UFSS; or
(b) $(\mathbb{Z}, \tau)$ has an open minimal subgroup; more precisely, there exists a prime number p and $n \in \mathbb{Z}$ such that $(np^m\mathbb{Z})_{m\in\mathbb{N}}$ forms a neighborhood basis of the neutral element.

Indeed, if τ is not UFSS Theorem 5.10 gives that it is not NSS, and then, Example 2.11(d) says that τ is a non-discrete linear topology. We apply now Corollary 5.15 and we obtain that G contains an open minimal subgroup N. Let $N = n\mathbb{Z}$ for some $n \neq 0$. Then the minimality of N implies that for a suitable prime p, a neighborhood basis of 0 in $n\mathbb{Z}$ is given by the sequence of subgroups $(np^m\mathbb{Z})_{m\in\mathbb{N}}$ ((2.5.6) in [13]).

Proposition 5.17. *Products of locally minimal (abelian precompact) groups are in general not locally minimal, namely the group of integers with the 2-adic topology $(\mathbb{Z}, \tau_2)$ is minimal and hence locally minimal, but the product $(\mathbb{Z}, \tau_2) \times (\mathbb{Z}, \tau_2)$ is not locally minimal.*

Proof. Suppose that $(\mathbb{Z}, \tau_2) \times (\mathbb{Z}, \tau_2)$ is U-locally minimal. We may assume that $U = 2^n\mathbb{Z} \times 2^n\mathbb{Z}$. By Proposition 5.11, the closed subgroup U is minimal. But U is topologically isomorphic to $(\mathbb{Z}, \tau_2) \times (\mathbb{Z}, \tau_2)$, which yields a contradiction. □

According to Corollary 5.15(b) the bounded locally minimal locally GTG abelian groups have an open minimal subgroup. Now we use this fact to describe the bounded abelian groups that support a non-discrete locally minimal and locally GTG group topology:

Theorem 5.18. *Let G be a bounded abelian group. Then the following assertions are equivalent*:

(a) $|G| \geqslant \mathfrak{c}$;
(b) *G admits a non-discrete locally minimal and locally GTG group topology*;
(c) *G admits a non-discrete locally compact metrizable group topology.*

Proof. To prove the implication (a) ⇒ (c) use Prüfer's theorem to deduce that G is a direct sum of cyclic subgroups. Since G is bounded, there exists an $m > 1$ such that G has as a direct summand a subgroup $H \cong \bigoplus_{\mathfrak{c}} \mathbb{Z}(m) \cong \mathbb{Z}(m)^{\omega}$. Since $\mathbb{Z}(m)^{\omega}$ carries a metrizable compact group topology, one can build a non-discrete locally compact metrizable group topology on G by putting on H the topology transported by the isomorphism $H \cong \mathbb{Z}(m)^{\omega}$ and letting H to be an open subgroup of G.

(c) ⇒ (b) Let τ be a non-discrete locally compact group topology on the group G. According to Examples 2.3(b) and 5.6, τ is locally minimal and locally GTG.

(b) ⇒ (a) Assume that $|G| < \mathfrak{c}$. By Corollary 5.15 there exists an open minimal subgroup H of G. As $|H| < \mathfrak{c}$, we conclude that $r_p(H) < \infty$ for all primes p (see [13, Cor. 5.1.5]). Since H is a bounded abelian group we conclude that H is finite. Since H is open in G, G is discrete, a contradiction. □

6. Open questions

Question 6.1. Is the closure of every GTG set in a topological group again a GTG set?

Question 6.2. Is every locally minimal abelian group necessarily locally GTG?

According to Theorem 5.18, for a negative answer to Question 6.2 it suffices to build a non-discrete locally minimal group topology on an infinite bounded abelian group of size $< \mathfrak{c}$. To emphasize better the situation let us formulate this question in the following very specific case:

Question 6.3. Does the infinite Boolean group $\bigoplus_{\omega} \mathbb{Z}(2)$ admit a non-discrete locally minimal group topology? A positive answer to this question implies a negative answer to Question 6.2.

Actually, the following weaker version of Question 6.2 will still be useful for Theorem 5.12:

Question 6.4. If G is a U-locally minimal abelian group for some $U \in \mathcal{V}(0)$, does there exist a GTG neighborhood of 0 contained in U?

Theorem 5.10 suggests also another weaker version of Question 6.2:

Question 6.5. Is every locally minimal NSS abelian group necessarily locally GTG?

A positive answer to this question will modify the equivalence proved in Theorem 5.10 to equivalence between UFSS and the conjunction of local minimality and NSS.

Remark 6.6. Proposition 5.11 shows that the space in Example 5.4 cannot provide an answer to Question 6.2, since actually it is not locally minimal. [Suppose that for some $\varepsilon \in (0,1)$, W_ε witnesses local minimality of L^0. Let h be the characteristic function of $[0, \varepsilon/2]$, and H the subgroup $\langle h \rangle$ of L^0. H is discrete, hence it cannot be minimal; however, $H \subseteq W_{\varepsilon/2}$ and thus $H + W_{\varepsilon/2} \subset W_\varepsilon$, which contradicts Proposition 5.11.]

Question 6.7. Is every locally minimal NSS group metrizable? According to Proposition 2.13, this is true for abelian groups.

The next question is related to Proposition 2.5 and Corollary 2.6:

Question 6.8. Let H be a closed subgroup of a (locally) minimal group G. Is then H necessarily locally minimal?

Acknowledgments

It is a pleasure to thank Elena Martín Peinador for helpful discussions, and the referee for her/his careful reading and sharp and helpful comments.

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