JOURNAL OF COMPUTER AND SYSTEM SCIENCES 19, 163–178 (1979)

Least Fixed Point of a Functor

JIŘÍ ADÁMEK

Faculty of Electrical Engineering, Technical University, Prague, Czechoslovakia

AND

VÁCLAV KOUBEK

Faculty of Mathematics and Physics, Charles University, Prague, Czechoslovakia

Received April 28, 1978; revised May 23, 1979

INTRODUCTION

Fixed points of a functor $F: \mathscr{K} \to \mathscr{K}$ are objects X of $\mathscr{K}$ such that $FX \simeq X$. They arise, e.g., in Scott's approach to data types as lattices D satisfying $D \simeq R \times [S \to D]$: These are fixed points of $F(?) = R \times [S \to ?]$. Scott remarks in [13] that a general theory of fixed points of functors would be of a value; the aim of the present paper is to lay foundations for such a theory.

We introduce the least fixed point (LFP) of a functor $F: \mathscr{K} \to \mathscr{K}$. We exhibit a construction of the LFP, generalizing the Knaster–Tarski formula $\text{lub}\{F^n(0)\}_{n\in\omega}$: lubs are substituted by well-ordered colimits and n is allowed to be an arbitrary ordinal. Related LFP constructions, always restricted to $n \in \omega$, have been considered by various authors [6, 11, 13, 14]. The advantage of the present approach is its effectiveness: Whenever a functor F has a fixed point then our construction stops, yielding the LFP of F.

Least fixed points are closely related to free algebras. In fact, the LFP construction, studied in the present paper, is a result of application of the free-algebra construction, exhibited by the first author [1], to initial objects. (This application is an idea of Arbib [5].) We proceed in a converse order here, introducing free algebras as special fixed points.

We further investigate categories with the fixed-point property (with respect to endofunctor). Example: Sets of power $\leqslant \alpha$ and vector spaces of dimension $\leqslant \alpha$ form categories with the fixed-point property (for any cardinal α). On the other hand, we show that a complete category with the fixed-point property must be a preordered class. The problem of characterizing categories with the fixed-point property is difficult: already for posets it is an open problem of long standing.

Finally, we show that a result, concerning the construction of free algebras for set functors, is undecidable, since it is equivalent to the nonexistence of measurable cardinals.

Results of the present paper have been announced in [4].

0022-0000/79/050163–16$02.00/0

A. Fixed Points and LFP

1. Throughout this paper we assume that a category $\mathscr{K}$ and a functor $F\colon \mathscr{K} \to \mathscr{K}$ are given. An object X, isomorphic to its image FX, is called a *fixed point of* F (the first one to use this notion was probably Lambek [10]). We shall, sometimes, adopt a more rigorous attitude and we shall call a pair (X, v) a fixed point, if X is an object and v: $FX \to X$ is an isomorphism.

The *least fixed point* (LFP) of F is then defined as a fixed point (X_0, v_0) with the following universal property: For every fixed point (X, v) there exists a unique morphism $f\colon X_0 \to X$ such that $Ff = v^{-1} \cdot f \cdot v_0$.

$$\begin{array}{ccc} FX_0 & \xrightarrow{v_0} & X_0 \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FX & \underset{v^{-1}}{\overset{v}{\rightleftarrows}} & X \end{array}$$

This notion has been used by Arbib [6] and Wand [19].

Notice that, while a functor may have a number of distinct fixed points, the LFP is unique up to a (natural) isomorphism. For example, every object is a fixed point of the identity functor I of $\mathscr{K}$. But the LFP is precisely the *initial object* 0 characterized (up to isomorphism) by the property that each object X has precisely one morphism $f\colon 0 \to X$.

2. A category is said to be *chain-cocomplete* if each well-ordered diagram has a colimit. This is a property shared by a majority of "everyday" categories. For example the categories of ω-complete posets or of complete posets (with inf-preserving functions) are chain-cocomplete, in fact cocomplete. For chain-cocomplete categories we present a fixed-point construction. This construction is not applicable to categories of complete lattices (with sup- and inf-preserving functions) since they fail to be chain-cocomplete. For such categories the technique of D. Scott, based on the coincidence of certain limits and colimits, is quite different.

Let us remark that each chain-cocomplete category has an initial object 0—the colimit of the void diagram.

3. LFP *construction.* For every endofunctor $F\colon \mathscr{K} \to \mathscr{K}$ of a chain-cocomplete category $\mathscr{K}$ we define a transfinite chain of objects $W_i \in \mathscr{K}$ (for each ordinal i) and morphisms $W_{i,j}: W_i \to W_j$ $(i \leqslant j)$ subject to

$$W_{i,i} = 1_{W_i} \text{ and } W_{i,k} = W_{i,k} \cdot W_{i,j} \qquad \text{whenever} \quad i \leqslant j \leqslant k.$$

$$0 \xrightarrow{W_{0,1}} F0 \xrightarrow{FW_{0,1}} F^2 0 \xrightarrow{F^2 W_{0,1}} \cdots \to W_\omega = \operatorname*{colim}_{n<\omega} F^n 0 \xrightarrow{W_{\omega,\omega+1}} FW_\omega \xrightarrow{FW_{\omega,\omega+1}} \cdots.$$

First,

$$W_0 = 0, \qquad W_1 = F0, \qquad W_{0,1} : 0 \to F0 \text{ canonical.}$$

Then for each isolated step we have $W_{i,i+1} : W_i \to W_{i+1} = FW_i$ and we put

$$W_{i+2} = FW_{i+1} \qquad \text{and} \qquad W_{i+1,i+2} = FW_{i,i+1} : FW_i \to FW_{i+1} ;$$

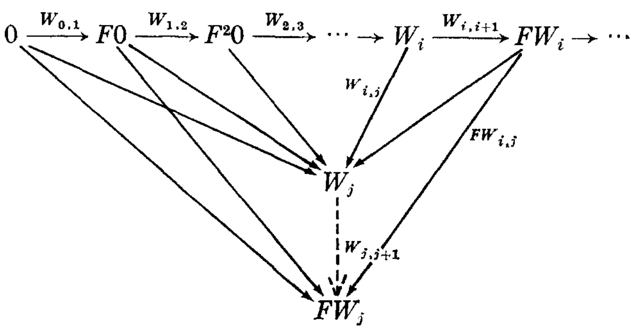
finally, given a limit ordinal j put

$$W_j = \operatorname*{colim}_{i<j} W_i \text{ with } W_{i,j}\colon W_i \to W_j \qquad (i < j) \text{ canonical}$$

and

$$W_{j+1} = FW_j ;$$

we have a compatible family of morphisms $FW_{i,j} \cdot W_{i,i+1} : W_i \to FW_j \ (i < j)$



which yields a unique morphism $W_{j,j+1} : W_j \to FW_j = W_{j+1}$ with

$$W_{j,j+1} \cdot W_{i,j} = FW_{i,j} \cdot W_{i,i+1} \qquad ((i < j).$$

The LFP construction is said to *stop after* α *steps* if $W_{\alpha,\alpha+1}$ is an isomorphism.

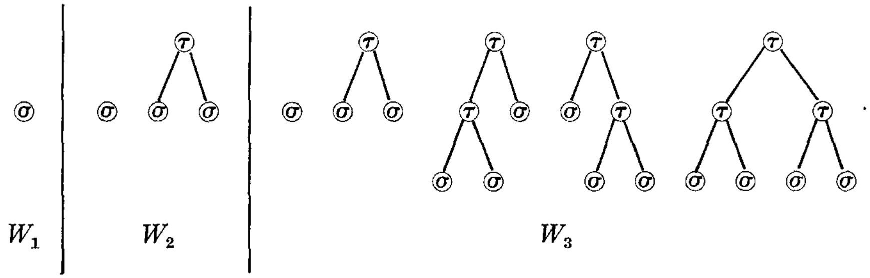
4. EXAMPLE. Let Σ be a ranked alphabet, i.e., a set, equipped by an arity function ar: $\Sigma \to \{0, 1, 2,\ldots\}$. Define a set functor F_Σ, assigning to each set X the disjoint union of its Cartesian powers, each power related to a letter $\sigma \in \Sigma$ (the 0th power X^0 means a singleton set):

$$F_\Sigma X = \coprod_{\sigma\in\Sigma} X^{\operatorname{ar}(\sigma)}.$$

It is clear how to define (naturally) F_Σ on morphisms to obtain a functor. Denoting $\Sigma_0 = \{\sigma \in \Sigma;\ \operatorname{ar}(\sigma) = 0\}$, we have

$$W_0 = \varnothing ; \qquad W_1 = \coprod_{\sigma\in\Sigma} \varnothing^{\operatorname{ar}(\sigma)} = \Sigma_0 ; \qquad W_2 = \coprod_{\sigma\in\Sigma} \Sigma_0^{\operatorname{ar}(\sigma)}; \ \cdots.$$

The nth step W_n can be clearly interpreted as the set of all Σ trees (i.e., trees labeled by Σ so that arities agree with the number of successors) of length smaller than n. For example, for $\Sigma = \{\sigma, \tau\}$ with $\operatorname{ar}(\sigma) = 0$, $\operatorname{ar}(\tau) = 2$:



Then $W_\omega \cong \bigcup_{n=0}^{\infty} W_n$ is the set of all Σ trees and $W_{\omega,\omega+1}$ is a bijection, hence the LFP construction stops after ω steps.

Allowing infinite arities, the LFP construction for F_Σ always stops but later than after ω steps; W_i is the set of Σ trees of length smaller than i.

5. PROPOSITION *If the* LFP *construction stops after* α *steps then* W_α *is the* LFP *with respect to* $W_{\alpha,\alpha+1}^{-1} : FW_\alpha \to W_\alpha$.

Proof. Given a fixed point (X, v), define $f_i : W_i \to X$ by induction to yield a compatible family ($f_i = f_j \cdot W_{i,j}$ for each $i \leqslant j$).

$f_0 : 0 \to X$ is canonical.

$f_{i+1} = v \cdot Ff_i : FW_i = W_{i+1} \to X$.

f_γ (γ a limit ordinal) is determined by the fact that f_i ($i \leqslant \gamma$) form a compatible family.

Particularly, for $f = f_\alpha$ we have $v \cdot Ff = f_{\alpha+1}$ and so

$$v \cdot Ff = fW_{\alpha,\alpha+1}^{-1},$$

hence $Ff = v^{-1} \cdot f \cdot W_{\alpha,\alpha+1}^{-1}$.

The uniqueness of f is easy: Given f: $W_\alpha \to X$ with $Ff = v^{-1} \cdot f \cdot W_{\alpha,\alpha+1}^{-1}$, we have $f \cdot W_{\alpha,0} = f_0$ (because 0 is an initial object) and it suffices to show that $f \cdot W_{i,\alpha} = f_i$ implies $f \cdot W_{i+1,\alpha} = f_{i+1}$ (then, also $f \cdot W_{\gamma,\alpha} = f_\gamma$ for a limit ordinal γ and, by induction, $f = f \cdot W_{\alpha,\alpha} = f_\alpha$):

$$\begin{aligned} f \cdot W_{i+1,\alpha} &= f \cdot W_{\alpha,\alpha+1}^{-1} \circ W_{i+1,\alpha+1} = f \cdot W_{\alpha,\alpha+1}^{-1} \cdot FW_{i,\alpha} \\ &= v \cdot Ff \cdot FW_{i,\alpha} = v \cdot Ff_i \\ &= f_{i+1}. \end{aligned}$$

Thus, if the LFP construction stops then F has LFP. But not conversely.

6. EXAMPLE. Let $\mathscr{K}$ be the category of sets and let $F = P^*$ be the "infinite-powerset" functor defined as follows:

$P^* \varnothing = N$, the set of natural numbers;

$P^*X = \{T \subset X;\ T \text{ infinite}\} \cup \{*\}$ for $X \neq \varnothing$.

Given a mapping $f\colon X \to Y$, we have three possibilities:

$X \neq \varnothing$, then $P^*f(T) = f(T)$ if $f(T)$ is infinite, $P^*f(T) = * = P^*f(*)$ if $f(T)$ is finite;

$X = Y = \varnothing$, then $P^*f = \mathrm{id}_N$;

$X = \varnothing \neq Y$, then $P^*f(n) = *$ for $n \in N$.

This functor has a unique fixed point: singleton set. Hence, this is the LFP of P^*. Yet, the LFP construction does not stop: We have $W_1 = P^*\varnothing = N$ and, whenever W_i is an infinite set then P^*W_i has a greater power than W_i—thus, W_i cannot be a fixed point of P.

The reason why LFP construction fails for the functor P^* is that P^* does not preserve monos (the void map $\varnothing \to \{*\}$ is lifted to the constant map $N \to \{*\}$). Before discussing how preservation of monos influences the situation we shall introduce a broader view of F-algebras.

B. Fixed Points and Free Algebras

7. Given a functor $F\colon \mathscr{K} \to \mathscr{K}$, an ***F*-algebra** is a pair (Q, d) consisting of an object Q of $\mathscr{K}$ and a morphism $d\colon FQ \to Q$. A homomorphism between F-algebras $f\colon (Q, d) \to (Q', d')$ is a morphism $f\colon Q \to Q'$ in $\mathscr{K}$ such that the square

$$\begin{array}{ccc} FQ & \xrightarrow{\;d\;} & Q \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FQ' & \xrightarrow[\;d'\;]{} & Q' \end{array}$$

commutes. For example, let $F = F_\Sigma$ be as in Example 4, then an F_Σ-algebra is a set Q together with a map $d\colon \coprod_{\sigma\in\Sigma} Q^{\mathrm{ar}(\sigma)} \to Q$, which can be viewed as a collection of operations

$$Q^{\mathrm{ar}(\sigma)} \to Q \qquad (\sigma \in \Sigma).$$

In that sense, F_Σ-algebras are just universal algebras of type Σ. The above notion of homomorphism coincides with that from universal algebra.

8. A ***free F*-algebra**, generatee by an object I, is an F-algebra $(I^\#, \varphi)$ together with a morphism $s\colon I \to I^\#$ (insertion of generators) which is universal in the following sense: Every diagram

$$\begin{array}{ccccc} FI^\# & \xrightarrow{\;\sigma\;} & I^\# & \xleftarrow{\;s\;} & I \\ & & & \swarrow{\scriptstyle f} & \\ FQ & \xrightarrow[\;d\;]{} & Q & & \end{array}$$

has a unique commutative fill-in $f^{\#}$:

$$\begin{array}{ccccc} FI^{\#} & \xrightarrow{\varphi} & I^{\#} & \xleftarrow{s} & I \\ {\scriptstyle Ff^{\#}}\downarrow & & \downarrow{\scriptstyle f^{\#}} & \swarrow{\scriptstyle f} & \\ FQ & \xrightarrow[d]{} & Q & & \end{array}$$

Free F-algebras play an important role in the theory of machines in a category due to Arbib and Manes [7]. We call F a *varietor* (*input process* in [7]) provided that for every object I there exists the free algebra $(I^{\#}, \varphi)$.

9. The free algebra over the initial object 0 (provided that both exist) is the algebra having exactly one homomorphism into any other F-algebra. Thus, this is the *initial algebra*, i.e., the initial object in the category of F-algebras. If $(0^{\#}, \varphi)$ is the initial algebra then $0^{\#}$ is a fixed point, indeed the LFP, see [6]; thus, it is natural to ask:

(a) Does the existence of the LFP guarantee the existence of an initial algebra?

(b) Does the existence of a fixed point guarantee the existence of the LFP (and the stop of the LFP construction)?

In general, the answer is negative:

EXAMPLE. LFP exists but initial algebra does not. Let $\mathscr{K}$ be the category of partial semigroups. Denote by $(I, *)$ the (total) one-element semigroup and define $F: \mathscr{K} \to \mathscr{K}$ on objects by

$$F(X, \cdot) = (I, *) \qquad \text{if the operation } \cdot \text{ is anywhere defined (i.e. } \cdot \neq \varnothing);$$
$$F(X, \varnothing) = (\exp X, \varnothing).$$

There is a unique way of defining F on morphisms provided that

$$Ff = \exp f \qquad \text{for each} \quad f: (X, \varnothing) \to (Y, \varnothing).$$

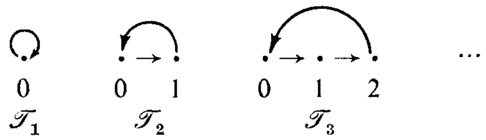
Then F has a unique (up to isomorphism) fixed point, which is its LFP, viz., $(I, *)$. Yet F has no initial algebra. Indeed, if an initial algebra existed it would have to coincide with the LFP. But, given any set X and any map $d: \exp X \to X$ we have an F algebra

$$((X, \varnothing), d) \qquad [\text{since } d: F(X, \varnothing) \to (X, \varnothing)]$$

and there exists no homomorphism from $(I, *)$ to $(X, \varnothing)$, of course.

EXAMPLE. Fixed points exist but LFP does not. Let $\mathscr{K}$ be the category of graphs (i.e., pairs $\mathscr{G} = (V, E)$, where V is a set and $E \subset V \times V$) and graph morphisms (i.e., maps $f: (V, E) \to (V', E')$ such that $(x, y) \in E$ implies $(f(x), f(y)) \in E'$). Denote by $\mathscr{K}_n$

the class of all graphs which contain a cycle of length n, i.e., which have a subgraph, isomorphic to $\mathscr{T}_n$:

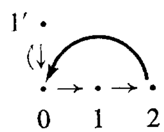


It is easy to verify that given a graph morphism $f: \mathscr{G} \to \mathscr{G}'$ then $\mathscr{G} \in \mathscr{K}_1$ implies $\mathscr{G}' \in \mathscr{K}_2$ $\mathscr{G} \in \mathscr{K}_2$ implies $\mathscr{G}' \in \mathscr{K}_1 \cup \mathscr{K}_2$, and $\mathscr{G} \in \mathscr{K}_3$ implies $\mathscr{G}' \in \mathscr{K}_1 \cup \mathscr{K}_3$.

Define $F: \mathscr{K} \to \mathscr{K}$ on objects as follows:

$$
\begin{aligned}
F\mathscr{G} &= \mathscr{T}_1 && \text{if } \mathscr{G} \in \mathscr{K}_1, \\
F\mathscr{G} &= \mathscr{T}_2 && \text{if } \mathscr{G} \in \mathscr{K}_2 - (\mathscr{K}_1 \cup \mathscr{K}_3), \\
F\mathscr{G} &= \mathscr{T}_3 && \text{if } \mathscr{G} \in \mathscr{K}_3 - (\mathscr{K}_1 \cup \mathscr{K}_2); \\
F\mathscr{G} &= \mathscr{T}_{2,3} && \text{if } \mathscr{G} \in \mathscr{K}_2 \cap \mathscr{K}_3,
\end{aligned}
$$

where $\mathscr{T}_{2,3}$ is the graph



$\mathscr{T}_{2,3}$

$$F\mathscr{G} = (\exp V, \varnothing) \qquad \text{if} \quad \mathscr{G} = (V, E) \in \mathscr{K} - (\mathscr{K}_1 \cup \mathscr{K}_2 \cup \mathscr{K}_3).$$

Given a graph morphism $f: \mathscr{G} \to \mathscr{G}'$, define Ff as follows:

(a) $\mathscr{G} \in \mathscr{K} - (\mathscr{K}_1 \cup \mathscr{K}_2 \cup \mathscr{K}_3)$,
$Ff = \exp f$ if $\mathscr{G}' \in \mathscr{K} - (\mathscr{K}_1 \cup \mathscr{K}_2 \cup \mathscr{K}_3)$,
$Ff =$ const 0 if $\mathscr{G}' \in \mathscr{K}_1 \cup \mathscr{K}_2 \cup \mathscr{K}_3$,

(b) $\mathscr{G} \in \mathscr{K}_1$ [implies $\mathscr{G}' \in \mathscr{K}_1$],
$Ff = \mathrm{id}_{\{0\}}$,

(c) $\mathscr{G} \in \mathscr{K}_2$ [implies $\mathscr{G}' \in \mathscr{K}_1 \cup \mathscr{K}_2$],
$Ff =$ const 0 if $\mathscr{G}' \in \mathscr{K}_1$,
$Ff = \mathrm{id}_{\{0,1\}}$ if $\mathscr{G}' \in \mathscr{K}_2 - (\mathscr{K}_1 \cup \mathscr{K}_3)$,
$Ff =$ embedding $\mathscr{T}_2 \to \mathscr{T}_{2,3}$ if $\mathscr{G}' \in \mathscr{K}_3$,

(d) analogously for $\mathscr{G} \in \mathscr{K}_3$.

A mechanical checking shows that F is a correctly defined functor. It has, up to isomorphism, exactly four fixed points, viz.,

$$\mathscr{T}_1, \qquad \mathscr{T}_2, \qquad \mathscr{T}_3, \qquad \text{and} \qquad \mathscr{T}_{2,3}.$$

Since none of these admits a homomorphism into each of the other three, F has no LFP.

10. Definition. A class C of morphisms is said to be *chain-cocomplete* if for every ordinal γ and every chain of C-morphisms

$$W_{i,j} : W_i \to W_j \text{ in } C \qquad (i \leqslant j < \gamma)$$

there is a colimit, say W_γ and $W_{i,\gamma} : W_i \to W_\gamma$, such that

(i) $W_{i,\gamma} \in C$ for each $i < \gamma$;

(ii) for every compatible family $U_i : W_i \to U$ in C the factorizing morphism $h: W_\gamma \to U$ $[h \cdot W_{i,\gamma} = U_i]$ is in C, too.

Examples. All monos form a chain-cocomplete class in a number of categories, including sets, complete posets, topological spaces, varieties of finitary algebras, etc. They fail to form a chain-cocomplete class, e.g., in the category of compact Hausdorff spaces. See [18] for details.

Note. If C is a chain-cocomplete class then

(0) $\mathscr{K}$ has an initial object 0 with each $0 \to X$ in C.

Proof. Apply (ii) in the definition to the void chain.

(1) C is closed to isomorphisms.

Proof. Apply (i) to the chain $1_A : A \to A$ of length 1: Given an isomorphism $v: A \to B$ then B together with v is the colimit of this chain.

(2) C is closed to composition.

Proof. Apply (i) to the chain $f: A \to B$, $g: B \to C$ of length 2. Its colimit is C with colimit injections $g \cdot f: A \to C$ (!), $g: B \to C$, and $1: C \to C$.

11. Theorem. *Let $\mathscr{M}$ be a chain-cocomplete class of monos in a category $\mathscr{K}$ and let $\mathscr{K}$ be $\mathscr{M}$-well powered. The following conditions are equivalent for any functor $F: \mathscr{K} \to \mathscr{K}$, preserving $\mathscr{M}$ (i.e., such that $m \in \mathscr{M}$ implies $Fm \in \mathscr{M}$):*

(i) *F has a fixed point;*

(ii) *F has the* LFP;

(iii) *the initial F-algebra exists;*

(iv) *the* LFP *construction stops.*

Proof. Only (i) $\Rightarrow$ (iii) & (iv) has to be proved, of course. Thus, let (X, v) be a fixed point of F. To see that the LFP construction stops, we shall construct a compatible family $h_i : W_i \to X$ of $\mathscr{M}$-monos. Since $\mathscr{K}$ is $\mathscr{M}$-well powered, this will guarantee that two of these subobjects of X, say h_α, $h_{\alpha'}$ ($\alpha < \alpha'$) are isomorphic, from which it follows that $W_{\alpha,\alpha'}$ (and $W_{\alpha,\alpha+1}$) is an isomorphism—hence, the construction stops and yields the LFP. It is proved in [1] that if this is the case, then the LFP is the initial F-algebra.

We shall construct h_i by induction in i, proving also, as we proceed, that $W_{t,i+1}$ is also in $\mathscr{M}$ for each $t \leqslant i + 1$. First, both $h_0 : 0 \to X$ and $W_{0,1} : 0 \to F0$ are in $\mathscr{M}$. Second, given $h_i \in \mathscr{M}$ we define $h_{i+1} = v \cdot Fh_i$.

$$\begin{array}{ccccc} W_i & \xrightarrow{W_{i,i+1}} & W_{i+1} = FW_i & \xrightarrow{FW_{i,i+1}} & FW_{i+1} \\ \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle Fh_i} & & \\ X & \xleftarrow[v]{} & FX & & \end{array}$$

Since $h_i \in \mathscr{M}$ implies $Fh_i \in \mathscr{M}$ we see that $h_{i+1} \in \mathscr{M}$ (for, by the above note, $\mathscr{M}$ is closed to composition with isomorphisms). Further,

$$W_{t,i+1} = (FW_{t,i}) \cdot W_{t,t+1} \in \mathscr{M}.$$

Finally, given a limit ordinal γ and given $h_i : W_i \to X$ in $\mathscr{M}$ for each $i < \gamma$, we define $h_\gamma : W_\gamma \to X$ by $h_\gamma \cdot W_{i,\gamma} = h_i$ for each $i < \gamma$. Since $\mathscr{M}$ is a chain-cocomplete class, we have both $h_\gamma \in \mathscr{M}$ and $W_{\gamma,\gamma+1} \in \mathscr{M}$.

12. In a category $\mathscr{K}$ with finite coproducts the above theorem yields a criterion on a functor $F: \mathscr{K} \to \mathscr{K}$ to be a varietor. For each object I in $\mathscr{K}$ denote by $F_{(I)} : \mathscr{K} \to \mathscr{K}$ the functor, defined by

$$F_{(I)}X = FX + I \qquad \text{and} \qquad F_{(I)}f = Ff + \mathrm{id}_I$$

(i.e., $F_{(I)}$ is a coproduct of F and the constant functor to I).

Observation. The free F-algebra, generated by an object I, is precisely the initial $F_{(I)}$-algebra.

Now, if F preserves a class $\mathscr{M}$ which is *additive* (i.e., given $m: A \to B$ and $m': A' \to B'$ in $\mathscr{M}$ then also $m + m': (A + A') \to (B + B')$ is in $\mathscr{M}$) then each of the functors $F_{(I)}$ preserves $\mathscr{M}$. Hence, we obtain the following result, originally proved in [18]:

COROLLARY. *Let $\mathscr{K}$ be a category with finite coproducts and let $\mathscr{M}$ be its additive, chain-cocomplete class of monos such that $\mathscr{K}$ is $\mathscr{M}$-well powered. A functor $F: \mathscr{K} \to \mathscr{K}$, preserving $\mathscr{M}$, is a varietor iff for every object I there exists an object X, isomorphic to $FX + I$ (i.e., a fixed point of $F_{(I)}$).*

13. An important feature of Dana Scott's fixed points is that every object has a "nice" embedding into a fixed point. Let us make a remark concerning this situation.

DEFINITION. We say that an object X is a *canonical subobject* of an object T if T is isomorphic to $X + Y$ for some object Y.

THEOREM. *Let $\mathscr{K}$, $\mathscr{M}$ be as in the above corollary and let $F\colon \mathscr{K} \to \mathscr{K}$ be a functor, preserving $\mathscr{M}$ and such that each object of $\mathscr{K}$ is a canonical subobject of a fixed point of F. If $\mathscr{K}$ has countable coproducts then F is a varietor.*

Proof. We shall use the above corollary: Given an object I, we form the coproduct I_ω of countably many copies of I; by hypothesis, there exists a fixed point X, isomorphic to $I_\omega + Y$ (for some object Y). Then

$$X \cong FX$$

and also, via $I_\omega + I \cong I_\omega$:

$$X + I \cong Y + I_\omega + I \cong Y + I_\omega \cong Y.$$

Hence, $FX \cong FX + I$ and we have

$$X \cong FX + I.$$

The converse fails even for very reasonable varietors. For example, let $\mathscr{K}$ be the category of partial commutative groupoids and let FX be the subgroupoid of X over all idempotents, $FX = \{y \in X;\ y \cdot y = y\}$. Then F preserves coproducts (hence it is a varietor with $I^{\#} = \coprod_{n=0} F^n I$) and preserves limits. Yet, no object other than a fixed point of F has a canonical embedding into a fixed point!

C. Fixed-Point Property of Categories

14. Example. Let Set (α) denote the category of sets of power $\leqslant \alpha$ (and mappings between them). For every α, Set (α) has the *fixed-point property*: Every endofunctor has a fixed point. Indeed, for any endofunctor F put $C_F = \{\beta;$ if card $X = \beta$ then card $FX \leqslant \beta\}$. Then $\alpha \in C_F$ hence $C_F \neq \varnothing$, and we can choose X_0 with card $X_0 = \min C_F$. Then card $FX_0 \leqslant$ card X_0 ($\in C_F$) and so either $FX_0 = \varnothing$ (in which case necessarily $X_0 = \varnothing$ is the LFP) or there is a retraction $r\colon X_0 \to FX_0$. Since Fr is also a retraction, there follows card $FFX_0 \leqslant$ card FX_0, i.e., card $FX_0 \in C_F$. Then card $FX_0 =$ card X_0, so that X_0 is a fixed point of F.

Example. The category T-Vect(α) of all vector spaces over a field T of dimension $\leqslant \alpha$ (and all linear mappings between them) has also the fixed-point property. The proof is the same.

Note. In T-Vect(α) the monomorphisms all split, hence they are preserved by every endofunctor. The class of all monos has all properties, required in Section 12 above. Hence, *every functor is a varietor*. In fact, with the exception of functors, equivalent to constant functors, every functor F has the property, mentioned in Section 13: Each space is a subspace of a fixed point.

The same is true for Set (α) but the argument must be more careful, see [12], because the monos from the void set do not split and are, generally, mapped by functors to nonmonos.

Our study of the fixed-point property of categories has been inspired by the result of Trnková [17] that every endofunctor of Set ($\aleph_0$) is a varietor.

15. We have seen in Section 14 a type of categories with the fixed-point property. Another type is ω-complete posets. We shall show now that the latter are the only "reasonable" categories with the fixed-point property. We denote by Set$^+$ the category of nonvoid sets and mappings.

PROPOSITION. *For every nonconstant set functor F there exists a cardinal α such that* $\operatorname{card} FX \geqslant \operatorname{card} X$ *for every set of power greater than α.*

In particular, the image $\{FX;\ X \text{ a set}\}$ of a nonconstant set functor F contains a proper class of pairwise nonisomorphic sets.

Proof. See [8].

LEMMA. *Let $\mathscr{K}$ be a category, for which a functor Φ: Set$^+ \to \mathscr{K}$ exists, which is not naturally equivalent to a constant functor. Then $\mathscr{K}$ has an endofunctor without fixed points.*

Proof. (1) There exists a map $f\colon X \to X$ with $\Phi f \neq 1_{\Phi X}$. Proof: Since Φ is not equivalent to a constant functor, there are two possibilities:

(a) Φh is not an isomorphism, for some $h\colon A \to B$ in Set$^+$. Since $A \neq \varnothing$, we can write $h = m \cdot e$, where $e\colon A \to C$ is a split epi and $m\colon C \to B$ is a split mono. If Φe is not an isomorphism, then we choose $r\colon C \to A$ with $e \cdot r = 1$ and we find out that $\Phi(r \cdot e) \neq 1$ (else $\Phi e = (\Phi r)^{-1}$); then we put $f = r \cdot e$. If Φe is an isomorphism, then Φm cannot be an isomorphism. We choose $p\colon B \to C$ with $p \cdot m = 1$ and we find out that $\Phi(m \cdot p) \neq 1$; then we put $f = m \cdot p$.

(b) $\Phi h \neq \Phi k$ for some $h, k\colon A \to B$ in Set$^+$. Then for the coequalizer e of h, k we have: Φe is not an isomorphism (else, $\Phi e \cdot \Phi h = \Phi e \cdot \Phi k$ implies $\Phi h = \Phi k$). Then we return to *a*).

(2) The image of Φ contains a proper class of pairwise nonisomorphic objects of $\mathscr{K}$. Proof: Consider the set functor $F = \hom(\Phi X, \Phi -)$, where $\Phi f \neq 1$ for some $f\colon X \to X$. Then F is nonconstant, because $Ff(1_{\Phi X}) = \Phi f$ while $F1_{\Phi X}(1_{\Phi X}) = 1_{\Phi X}$. By the preceding proposition we can choose a class $\mathscr{D}$ of sets such that the sets $\{FX;\ X \in \mathscr{D}\}$ are pairwise nonisomorphic. Hence, the objects $\{\Phi X;\ X \in \mathscr{D}\}$ are pairwise nonisomorphic in $\mathscr{K}$.

(3) For every cardinal t there exists a cardinal t^* such that

$$|\hom(\Phi X, \Phi M)| = t \text{ implies } |M| \leqslant t^*$$

for every set M. Proof: By the preceding proposition it is clear that for every nonconstant set functor F and every cardinal t, $\operatorname{card} FX > t$ whenever X is sufficiently great (card

$X > \alpha + t$) and we can choose t^* such that card $X > t^*$ implies card $FX \neq t$. Apply this rule to F above: For every t there is t^* such that

$$|FM| = |\hom(\Phi X, \Phi M)| = t \text{ implies } |M| \leqslant t^*.$$

(4) There exists a set functor T such that the functor $\Phi \cdot T \cdot \hom(\Phi X, -)$ has no fixed point. For each cardinal t we can choose an infinite cardinal $\lambda(t)$, greater than t^* above, in such a way that

$$t_1 < t_2 \quad \text{implies} \quad \lambda(t_2) > 2^{\lambda(t_1)} + 2^{t_2}. \tag{$*$}$$

By ($*$) it follows from [8] that a set functor T exists for which card $Z = t$ implies card $TZ = \lambda(t)$. Now, given an object Y in $\mathscr{K}$, put $M = \hom(\Phi X, Y)$, $t = |M|$. Then $|TM| = \lambda(t)$ implies

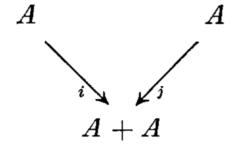
$$|\hom(\Phi X, \Phi TM)| \neq t$$

via (3) above. This proves that Y is not a fixed point of $\Phi \cdot T \cdot \hom(\Phi X, -)$: else, $t = |\hom(\Phi X, Y)| = |\hom(\Phi X, \Phi \cdot T \cdot \hom(\Phi X, Y))| = |\hom(\Phi X, \Phi M)|$. This concludes the proof.

THEOREM. *If a category $\mathscr{K}$ has the fixed-point property (i.e., if every functor $F\colon \mathscr{K} \to \mathscr{K}$ has a fixed point) and has powers or copowers of all objects, then $\mathscr{K}$ is a preordered class.*

Proof. Assuming that $\mathscr{K}$ is not a preordered class, it suffices to find a functor Φ: $\mathrm{Set}^+ \to \mathscr{K}$, nonequivalent to a constant functor. We shall assume that $\mathscr{K}$ has copowers; if it has powers we can work with its dual.

Thus, let $a, b\colon A \to B$ be distinct morphisms. We define Φ: $\mathrm{Set}^+ \to \mathscr{K}$ as the copowers of A: $\Phi M = \coprod_{m \in M} A_m$ with $A_m = A$; for $f\colon M \to N$ there is an induced morphism $\Phi f\colon \coprod_{m \in M} A_m \to \coprod_{n \in N} A_n$. To prove that Φ is indeed nonequivalent to a constant functor, it suffices to verify that in the copower $A + A$



i is not an isomorphism. Assume that, to the contrary, i is an isomorphism. Since we have a pair $1_A, 1_A\colon A \to A$, factorizing as $1_A = k \cdot i$, $1_A = k \cdot j$ for a unique $k\colon A + A \to A$, there follows that $k = i^{-1}$ and $j = k^{-1} = i$. This is a contradiction: For $a, b\colon A \to B$ there exists $c\colon A + A \to B$ with $a = c \cdot i = c \cdot j = b$, though $a \neq b$.

Note. Although only finite copowers were explicitly used in the argument above, the existence of finite copowers is not enough for the theorem to hold. Notice that Set (2^α) has all limits and colimits of diagrams with less then α objects; yet, Set (2^α) has the fixed-point property.

D. Free Algebras and Measurable Cardinals

16. The most important part of the LFP construction is the finitary part (W_i for $i \leqslant \omega$). In [1], a varietor F is called ***algorithmic*** if free algebras are obtained by the finitary part of the free-algebra construction, in other words, if for each $F_{(I)}$ the LFP construction stops after ω steps.

Every *finitary functor*, i.e., functor preserving filtered colimits, is an algorithmic varietor. In the case where $\mathscr{K}$ is the category of sets the converse seemed to be indicated by concrete examples: Every algorithmic varietor is finitary. Is it true?

For the case of sets the following conditions on F: Set $\to$ Set are equivalent (see [2]),

(i) F is finitary;

(ii) F preserves well-ordered unions (i.e., given a chain of sets W_i ($i < \alpha$), then $F(\bigcup_{i<\alpha} W_i) = \bigcup_{i<\alpha} Ft_i(FW_i)$, where $t_{i_0} : W_{i_0} \to \bigcup_{i<\alpha} W_i$ denotes the inclusion map);

(iii) for every set X and every point $x \in FX$ there exists a finite set $Y \subset X$ and a point $y \in FY$ with $x = Ft(y)$ (t: $Y \to X$ is the inclusion).

17. Recall that a cardinal α is measurable iff there exists a σ-additive $\{0, 1\}$-measure on a set X of power α; equivalently, if there exists an ultrafilter $\mathfrak{F}$ on X, which is free (i.e., not containing any singleton set) and is closed to countable intersections (i.e., $\bigcap_{n=1}^{\infty} A_n \in \mathfrak{F}$ whenever $A_n \in \mathfrak{F}$ for each n). The following lemma is folklore and we prove it here only because we do not know any reference.

Lemma. *Let there exist no measurable cardinal. Let $\mathfrak{F}$ be a filter on a set X, not containing any finite subset of X. Then there exist sets $X_0 \subset X_1 \subset X_2 \subset \cdots$ such that $X = \bigcup_{n=0}^{\infty} X_n$ and $X_n \notin \mathfrak{F}$ for each n.*

Proof. (A) Let $\mathfrak{F}$ be a free filter. Then $\mathfrak{F}$ is contained in a free ultrafilter $\mathfrak{G}$. By hypothesis, $\mathfrak{G}$ is not closed under countable intersections, hence there exist sets $A_n \in \mathfrak{G}$ with $\bigcap_{n=1}^{\infty} A_n \notin \mathfrak{G}$. Put $X_0 = \bigcap_{n=1}^{\infty} A_n$ and $X_k = X_0 \cup (X - \bigcap_{n=1}^{k} A_n)$, $k = 1, 2, 3,\ldots$. Then $X_0 \notin \mathfrak{G}$ and $X - \bigcap_{n=1}^{k} A_n \notin \mathfrak{G}$ (because $\bigcap_{n=1}^{k} A_n \in \mathfrak{G}$); since $\mathfrak{G}$ is an ultrafilter, necessarily $X_k \notin \mathfrak{G}$. Hence, for each k, $X_k \notin \mathfrak{F}$. And

$$\bigcup_{k=0}^{\infty} X_k = \left(\bigcap_{n=1}^{\infty} A_n\right) \cup \bigcup_{k=1}^{\infty} \left(X - \bigcup_{n=1}^{k} A_n\right) = X.$$

(B) $\mathfrak{F}$ is arbitrary. Let F be the intersection of all elements of $\mathfrak{F}$. If $F \in \mathfrak{F}$, then F cannot be finite and we can choose a sequence $f_0, f_1, f_2,\ldots$ of pairwise distinct elements of F. Put $X_n = X - \{f_n, f_{n+1}, f_{n+2},\ldots\}$: since $F = \cap\, \mathfrak{F}$ but X_n does not contain F, clearly $X_n \notin \mathfrak{F}$. Yet, $X = \bigcup_{n=0}^{\infty} X_n$ and $X_0 \subset X_1 \subset X_2 \subset \cdots$.

If $F \notin \mathfrak{F}$, put $X' = X - F$ ($\neq \varnothing$) and let $\mathfrak{F}' = \{T \subset X';\ T = S \cap X' \text{ for some } S \in \mathfrak{F}\}$. Then $\mathfrak{F}'$ is a free filter on X'. By part (A) there exist sets $X'_0 \subset X'_1 \subset X'_2 \subset \cdots$ with $X' = \bigcup_{n=0}^{\infty} X'_n$ and $X'_n \notin \mathfrak{F}'$. Then for $X_n = X'_n \cup F$ we have $X_0 \subset X_1 \subset X_2 \subset \cdots$, $X = \bigcup_{n=0}^{\infty} X_n$, and $X_n \notin \mathfrak{F}$ (the last follows from $X_n \cap X' = X'_n \notin \mathfrak{F}'$).

PROPOSITION. (*Set functors preserve finite nonvoid intersections.*) *Let F be an arbitrary set functor and let X be a set, x an element of FX. Then the collection*

$$\mathfrak{F}_x = \{Y;\ \varnothing \neq Y \subset X;\ x \in Ft(FY),\ t\colon Y \to X \text{ the inclusion map}\}$$

is either a filter on X or $\mathfrak{F}_x = \exp X - \{\varnothing\}$.

Proof. See [15].

18. THEOREM. *The nonexistence of measurable cardinals is equivalent to the validity of the following statement:*

(*) *A functor F*: Set $\to$ Set *is an algorithmic varietor iff it is finitary.*

Proof. Let there exist measurable cardinals. Then the following functor B: Set $\to$ Set is clearly nonfinitary:

BX is the set of all ultrafilters on X, closed to countable intersections (free or fixed);

given $f\colon X \to Y$ and an ultrafilter $\mathfrak{F} \in BX$ then

$$Bf(\mathfrak{F}) = \{R \subset Y; f^{-1}(R) \in \mathfrak{F}\}.$$

It is proved in [16] that B preserves countable colimits; then B is an algorithmic varietor Hence (*) fails.

Conversely, assuming that measurable cardinals do not exist, we shall prove (*), i.e., we shall verify that any functor F, which is not finitary, fails to be an algorithmic varietor. By (iii) in 16 there exists a point $x \in FX$ such that the collection

$$\mathfrak{F}_x = \{Y \subset Y;\ x \in Ft(FY),\ t\colon Y \to X \text{ inclusion}\}$$

contains no finite set. By the proposition in Section 17, $\mathfrak{F}_x$ is a filter and, since measurable cardinals do not exist, we have an increasing sequence of sets $X_n \subset X$, $X_n \notin \mathfrak{F}_x$ with $X = \bigcup_{n=0}^{\infty} X_n$.

Let α be a cardinal with card $FT \geqslant$ card T whenever T has a power greater than α (see the proposition in Section 15; F is clearly nonconstant). Choose a set I whose power is greater than both α and all card X_n (e.g., $I = X \times \alpha$). We shall verify that the FLP construction does not stop after ω_0 steps for $F_{(I)}$. In other words, we put

$$\begin{aligned}
&W_0 = F_{(I)}\varnothing = I,\\
&W_{n+1} = F_{(I)}W_n = I + FW_n,\\
&c_0 : I \to I + FW_n \text{ canonical},\\
&c_{n+1} = 1 + Fc_n : I + FW_n \to I + FW_{n+1},
\end{aligned}$$

and we shall verify that the construction does not stop after ω_0 steps. To do so, we shall exhibit a point in FW_{ω_0} which is not in the image of Fw_n for any n, where $w_n : W_n \to W_{\omega_0}$ $(n = 0, 1, 2,\ldots)$ is the colimit of $\{W_n\}$.

Since card $I >$ card X_0, there exists a one-to-one mapping $t_0 : X_0 \to I = W_0$. Since card $I > \alpha$, it is easy to see by induction

$$\begin{array}{ccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X \\ {\scriptstyle t_0}\downarrow & & {\scriptstyle t_1}\downarrow & & {\scriptstyle t_2}\downarrow & & & & {\scriptstyle t}\downarrow \\ W_0 & \xrightarrow[c_0]{} & W_1 & \xrightarrow[c_1]{} & W_2 & \xrightarrow[c_2]{} & \cdots & \longrightarrow & W_{\omega_0} \end{array}$$

that card$(W_n - \operatorname{Im} c_n) \geqslant$ card $I >$ card X_n for each n. Therefore there exist one-to-one mappings $t_n : X_n \to W_n$ such that

(a) for $x \in X_{n-1}$ always $t_n(x) = c_{n-1} \cdot t_{n-1}(x)$ and

(b) for $x \in X_n - X_{n-1}$ always $t_n(x)$ is out of the image of c_{n-1}.

Hence, we get a one-to-one mapping t: $X \to W_{\omega_0}$ with restrictions to X_n equal to $w_n \cdot t_n$. Supposing $Ft(x) \in \operatorname{Im} Fw_n$ for some n, we shall derive a contradiction, thus concluding the whole proof.

Let us apply the proposition in Section 17 to the point $y = Ft(x)$. First, we remark that $c_n : W_n \to W_{n+1}$ are all one to one (this is clear for c_0; since $I \neq \varnothing$, c_0 is then a split mono, hence $c_1 = 1 + Fc_0$ is a mono, etc.) and, therefore, so are $w_n : W_n \to W_{\omega_0}$. Since both $t(x)$ and $w_n(W_n)$ are elements of $\mathfrak{F}_y$, so is $t(x) \cap w_n(W_n) = t \cdot i(X_n)$, where i: $X_n \to X$ is the inclusion map. Hence, $y = F(t \cdot i)(x_0)$ for some $x_0 \in FX_n$. Since t is a split mono, Ft is one to one and so $Fi(x_0) = x$, because

$$Ft \cdot (Fi(x_0)) = y = Ft(x).$$

Thus, $x \in Fi(FX_n)$. This is a contradiction to $X_n \notin \mathfrak{F}_x$.

References

1. J. Adámek, Free algebras and automata realizations in the language of categories, *Comment. Math. Univ. Carolinae* **15** (1974), 589–602.
2. J. Adámek, Realization theory for automata in categories, *J. Pure Appl. Algebra* **9** (1977), 281–296.
3. J. Adámek and V. Koubek, Algebras and automata over a functor, *Kybernetia* **13** (1977), 245–260.
4. J. Adámek and V. Koubek, Remarks on fixed points of functors, *in* "Lecture Notes in Computer Science" No. 56, pp. 199–205, Springer–Verlag, Berlin/New York, 1977.
5. M. A. Arbib, Categorical notes on Scott's theory of computation, *in* "Proceedings, Conference on Categorical and Algebraic Methods in Computer Science and System Theory," Dortmund University, 1976.
6. M. A. Arbib, Free dynamics and algebraic semantics, *in* "Lecture Notes in Computer Science" No. 56, pp. 212–227, Springer–Verlag, Berlin/New York, 1977.
7. M. A. Arbib and E. G. Manes, Machines in a category, an expository introduction, *SIAM Rev.* **16** (1974), 163–192.
8. V. Koubek, Set functors, *Comment. Math. Univ. Carolinae* **12** (1971), 175–195.

9. V. KOUBEK AND J. REITERMAN, Categorial constructions of free algebras, colimits and completions of partial algebras, *J. Pure Appl. Algebra* **14** (1979), 195–231.
10. J. LAMBEK, A fixpoint theorem for complete category, *Math. Z.* **103** (1968), 151–161.
11. G. D. PLOTKIN AND M. B. SMYTH, The category-theoretic solution of recursive domain equations, *in* "Proceedings of the 18th IEEE Symposium on Foundations of Computer Science 1977."
12. V. POHLOVÁ, J. ADÁMEK, AND V. KOUBEK, The colimits in the generalized algebraic categories, *Acta Univ. Carolinae, Math. Phys.* **13** (1972), 29–40.
13. D. SCOTT, Continuous lattices, *in* "Toposes, Algebraic Geometry and Logic," pp. 97–136, Lecture Notes in Mathematics No. 274, Springer–Verlag, Berlin/New York, 1977.
14. M. B. SMYTH, "Category Theoretic Solution of Recursive Domain Equations," Theory of Computation Report No. 14, University of Warwick, 1976.
15. V. TRNKOVÁ, Some properties of set functors, *Comment. Math. Univ. Carolinae* **10** (1969), 323–352.
16. V. TRNKOVÁ, A descriptive classification of set functors, II, *Comment. Math. Univ. Carolinae* **12** (1971), 345–357.
17. V. TRNKOVÁ, On minimal realizations of behavior maps in categorical automata theory, *Comment. Math. Univ. Carolinae* **15** (1974), 555–566.
18. V. TRNKOVÁ, J. ADÁMEK, V. KOUBEK, AND J. REITERMAN, Free algebras, input processes and free monads, *Comment. Math. Univ. Carolinae* **16** (1973), 339–352.
19. M. WAND, "Fixed-Point Constructions in Order-Riched Categories," Technical Report No. 23, Indiana University, 1975.