Review of *Multidimensional Systems Theory,* edited by N. K. Bose*

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THE AREA

Few parts of application-oriented mathematics have benefited from the interaction with modern algebraic and analytic geometry as much as the area usually referred to as *multidimensional systems theory*. This field consists of the study of various topics in the theory of functions of several complex variables, motivated mostly by problems in network design and synthesis and by signal-processing applications. Because of finite realizability constraints, the focus is often on *rational* functions; this accounts for the strong algebraic flavor of papers in the area, and in particular the use of techniques and results from commutative algebra. A linear-algebraic component is introduced by the need to consider matrices whose entries are analytic or rational functions.

In "classical" systems and control theory, one studies ordinary differential equations

$$y^{(n)}(t)+a_1y^{(n-1)}(t)+\cdots+a_ny(t)=b_1u^{(n-1)}(t)+\cdots+b_nu(t), \quad (1)$$

where $a_1,\dots,a_n,b_1,\dots,b_n$ are fixed real numbers, and $u(\cdot)$, $y(\cdot)$ are respectively the *input* and *output* signals. The (scalar) independent variable t is interpreted as time. *Multidimensional* systems appear when dealing instead with partial differential (or difference) equations. The independent variables may now represent different space coordinates (as in image-processing applications), or perhaps mixed time and space variables (as in seismic data processing). Multidimensional models are also useful when studying certain types of functional differential equations in one independent variable, as delay-differential systems.

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Taking Laplace transforms in (1), we see that the input-output behavior of a classical system is characterized by the rational function

$$w(s)=\frac{b_1 s^{n-1}+\cdots+b_n}{s^n+a_1 s^{n-1}+\cdots+a_n}.$$

Questions of stability, in the sense that small input signals should produce small output signals, depend basically on the location of the singularities of $w(s)$. If we assume that the numerator and denominator of $w(s)$ are relatively prime, this means that stability is related to the location of the zeros of

$$Q(s)=s^n+a_1 s^{n-1}+\cdots+a_n.$$

For difference equations—which appear in digital signal processing applications—the situation is entirely analogous, except that z-transforms are used instead of Laplace transforms. When dealing with *multi*dimensional systems, Laplace or z-transforms result in rational functions in several variables. Even if we write such a function as a quotient of two relatively prime polynomials, it is clear that the locus of zeros of the denominator is no longer sufficient to characterize singularities, since this locus may intersect the set of zeros of the numerator. Moreover, the study of zero sets themselves is now highly nontrivial—indeed, such a study is precisely the subject matter of algebraic geometry.

Another set of problems, motivated by realizability issues for systems and networks, involves problems of factorization as well as various concepts of "positivity" of a rational function. Here again, the passage from classical systems to the multidimensional case changes the difficulty of the study in an essential way.

Much of the work in multidimensional systems has dealt with the search of "easily testable" conditions guaranteeing various types of stability and positiveness. In principle, most questions of this sort are decidable in the sense of computer science, since they can be stated in terms of the first-order theory of real closed fields. Unfortunately, this fact turns out to be misleading, since the computational complexities involved are worse than exponential. For instance, no "simple" tests are known for deciding if a polynomial $Q(z_1,\ldots,z_r)$ has no zeros in the closed polydisk

$$|z_1|\leqslant 1,\ldots,\qquad |z_r|\leqslant 1,$$

in contrast to the classical case (Routh-Hurwitz type tests). A number of tests have been proposed; see for instance [1], [2], and references there.

From the linear algebraist's point of view, matters become interesting when "multivariable" systems are considered. These are systems for which both inputs and outputs (u and y above) are vector-valued instead of scalar. Here one has to study properties (factorizations, positivity, etc.) of polynomial and rational *matrices*. A large amount of accumulated knowledge notwithstanding (see e.g. [6]), linear algebra over rings is still very much in its infancy, especially with regards to effective algorithms. For example, one basic difficulty is that the rank is no longer a complete invariant for the size of minimal factorizations. If $A(z)$ is a matrix of polynomials (each entry is a polynomial in $z_1,\dots,z_r$) of rank k, it is false in general that there exist polynomial matrices $B(z)$, $C(z)$, with k columns and rows respectively, such that $A(z)=B(z)C(z)$. Such *rank factorizations*, which are central in system design, exist for all A and k if and only if $r\leqslant 2$ (see [4]); in general one needs a larger intermediate dimension. (For networks—as opposed to control systems—factorization problems typically involve further positivity constraints.)

THE BOOK

The present book consists of a collection of articles by various authors. The editor's goal, stated in the introduction, was to provide an overview of some current research topics in multidimensional systems. On that basis the book is quite successful. By and large the articles are very clearly written, and extensive references are provided. The main criticism one could make is the lack of cross-referencing between articles. Ideally, the editor should have coordinated the contributions a bit. On the other hand, the first paper, "Trends in multidimensional systems theory," by Bose himself, includes an overview of the rest, drawing some connections. The article reviews the current (ca. 1984) status of various questions involving problems of stability and/or factorization. The writing could benefit from some polishing, but it is informative. The next article, also by the editor, consists basically of an annotated (and very useful) bibliography of papers on Padé-type approximation in the multivariate (and even matrix) case.

The paper "Causal and weakly causal 2-D filters with applications in stabilization," by J. P. Guiver and Bose, begins by reviewing definitions of causality for "2-D" (picture-processing) systems. The "right" definition of causality is not obvious in this context. For classical ("1-D") systems, there is a natural such notion: the future must not affect the past; for multidimensional systems, various definitions are possible, depending on the goals of the study. Typically, one is interested in recursive implementations of filters, and hence in causality with respect to a cone in $\Re^2$ which plays the role of the "past." Once causality has been properly defined, and a notion of stability has been introduced, it is possible to study the question of feedback stabilization of unstable systems. The authors present an excellent exposition of various results in this area, using a coprime-factorization approach. In the time-honored tradition of reviewers' advertising, I'll add the reference [5] to those given in the paper. This reference clarifies the meaning of the various stabilizability hypotheses in state-space terms.

Chapter 4, "Stabilization of linear spatially-distributed continuous-time and discrete-time systems," by E. W. Kamen, deals with systems for which input and output signals depend on both continuous and discrete variables. This situation arises for instance when discretizing only one variable (the "time") in a system described by a p.d.e. with two independent variables. In such examples, it is often the case that there is enough decay along the spatial (nontime) coordinate; one may then study this type of distributed system as a *system over a ring*, meaning a control system analogous to those studied in the classical finite-dimensional linear theory, but where the coefficient matrices have entries now over a ring of operators. Specifically, the author studies state space systems corresponding to controlled differential equations in the Banach space $l^\infty(\mathbf{Z}, \Re^n)$, where the right-hand side contains operators in $l^1(\mathbf{Z}, \Re^n)$. The exposition is clear, and includes a careful review of reachability and stabilizability and local/global questions. It should be remarked that Kamen was responsible, in the early 1970s for the development of the theory of systems over rings as applied to distributed systems. His contributions dealing with delay systems are well known, and relate naturally to the material in this chapter. See the recent textbook [3] for an exposition of many aspects of the theory of systems over rings as well as further references.

The next article is "Linear shift-variant multidimensional systems," by H. M. Valenzuela and Bose. A shift-variant filter is the multidimensional analogue of a time-varying linear system [the coefficients in Equation (1) are functions of time instead of constants]. Such models appear when considering, for instance, blurring due to lens aberration. To obtain a reconstruction of the original image, one needs to "deconvolve" or more precisely, to pass the observed picture through an inverse system. The authors develop the basic theory of shift-variant systems in the state and input-output senses, and present results on system inversion. I found the two examples, dealing with applications to image deconvolution, to be the most interesting part of the paper.

The book turns more purely algebraic with the next article. Chapter 6 is "Gröbner bases: an algorithmic method in polynomial ideal theory," by B. Buchberger. Many, if not most, problems in multidimensional system theory require at some point or another the calculation of ideal (or module) bases. For instance, "solve" a homogeneous system of linear equations over a polynomial ring means "give a general solution," i.e., a basis of a suitable nullmodule. The method of Gröbner bases, introduced by Buchberger about 20 years ago, has proved to be well suited to computer implementation. As opposed to the older technique of G. Hermann (ca. 1926), the Gröbner-basis method has a computational complexity that depends on the particular ideal being studied (instead of a fixed running time that depends on the number of variables and the degrees of the polynomials appearing in the problem statement). Its worst-case complexity is still double exponential in the number of variables, however, a fact that would not surprise anyone that has used the corresponding MACSYMA package. (We have been told by Professor F. Mora that the next release of REDUCE will incorporate a much-improved Gröbner package, which should result in at least an order of magnitude speedup over the current MACSYMA version.) The article presents an excellent exposition of the theory of Gröbner bases, as well as the various heuristics that can be (and are) used to make it more efficient. This is the best exposition on the topic that I have seen.

The book closes with a short article by J. P. Guiver on solvability conditions for linear equations over a polynomial ring in two variables, and an article listing various open problems. This last article may be useful to those contemplating starting research in the area.

On the whole, the book is worth reading and provides a good introduction to current research in an active area of applied mathematics.

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