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On the number of congruence classes of paths

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ABSTRACT

Let P_n denote the undirected path of length $n-1$. The cardinality of the set of congruence classes induced by the graph homomorphisms from P_n onto P_k is determined. This settles an open problem of Michels and Knauer [M. A. Michels, U. Knauer, The congruence classes of paths and cycles, Discrete Mathematics, 309 (2009) 5352–5359]. Our result is based on a new proven formula of the number of homomorphisms between paths.

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1. Introduction

We use standard notations and terminology of graph theory in [3] or [6, Appendix]. The graphs considered here are finite and undirected without multiple edges and loops. Given a graph G, we write $V(G)$ for the vertex set and $E(G)$ for the edge set. A *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that the images of adjacent vertices are adjacent. An *endomorphism* of a graph is a homomorphism from the graph to itself. Denote by $\mathrm{Hom}(G, H)$ the set of homomorphisms from G to H and by $\mathrm{End}(G)$ the set of endomorphisms of a graph G. For any finite set X we denote by $|X|$ the cardinality of X. A *path* with n vertices is a graph whose vertices can be labeled $v_1, \ldots, v_n$ so that v_i and v_j are adjacent if and only if $|i-j| = 1$; let P_n denote such a graph with $v_i = i$ for $1 \le i \le n$. Every endomorphism f on G induces a partition ρ of $V(G)$, also called *the congruence classes induced by* f, with vertices in the same block if they have the same image.

Let $\mathscr{C}(P_n)$ denote the set of endomorphism-induced partitions of $V(P_n)$, and let $|\rho|$ denote the number of blocks in a partition ρ. For example, if $f \in \mathrm{End}(P_4)$ is defined by $f(1) = 3, f(2) = 2, f(3) = 1, f(4) = 2$, then the induced partition ρ is $\{\{1\}, \{2, 4\}, \{3\}\}$ and $|\rho| = 3$.

The problem of counting homomorphisms from G to H is difficult in general. However, some algorithms and formulas for computing the number of homomorphisms of paths have been published recently (see [1,2,4]). In particular, Michels and Knauer [4] give an algorithm based on the *epispectrum* $\mathrm{Epi}(P_n)$ of a path P_n. They define $\mathrm{Epi}(P_n) = (l_1(n), \ldots, l_{n-1}(n))$, where

$$l_k(n) = |\{\rho \in \mathscr{C}(P_n) : |\rho| = n-k+1\}|. \tag{1.1}$$

Here a misprint in the definition of $l_k(n)$ in [4] is corrected.

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In [4], based on the first values of $l_k(n)$, Michels and Knauer speculated the following conjecture.

Conjecture 1. *There exists a polynomial $f_k \in \mathbb{Q}[x]$ with $\deg(f_k) = \lceil (k-2)/2 \rceil$ such that for a fixed n_k (most probably $n_k = 2k$) the equality $l_k(n) = f_k(n)$ holds for $n \geq n_k$.*

The aim of this paper is to confirm this conjecture by giving an explicit formula for the polynomial f_k. For this purpose, we shall prove a new formula for the number of homomorphisms from P_n to P_k, which is the content of the following theorem.

Theorem 2. *For any positive integers n and k,*

$$|\mathrm{Hom}(P_n, P_k)| = k \times 2^{n-1} - \sum_{i=0}^{n-2} 2^{n-1-i} \sum_{j \in \mathbb{Z}} \left(\binom{i}{\lceil \frac{i}{2} \rceil - j(k+1)} - \binom{i}{\lfloor \frac{i+k+1}{2} \rfloor - j(k+1)} \right). \tag{1.2}$$

From the above theorem we are able to derive the following main result.

Theorem 3. *If $n \geq 2k$, then*

$$l_k(n) = \binom{n-1}{\lceil \frac{k}{2} \rceil - 1} + \binom{n-1}{\lfloor \frac{k}{2} \rfloor - 1}. \tag{1.3}$$

Equivalently, the above formula can be rephrased as follows

$$l_{2k}(n) = 2\binom{n-1}{k-1}, \qquad l_{2k+1}(n) = \binom{n}{k}. \tag{1.4}$$

When $n \geq 2k$, Theorem 3 shows immediately that $l_k(n)$ is a polynomial in n of degree $\lceil (k-2)/2 \rceil$. This proves Conjecture 1. In particular, we have $l_1(n) = 1, l_2(n) = 2, l_3(n) = n, l_4(n) = 2(n-1), l_5(n) = \frac{1}{2}n(n-1)$ and $l_6(n) = (n-1)(n-2)$, which coincide with the conjectured values in [4] after shifting the index by 1.

In the next section, we first recall some basic counting results about the lattice paths and then prove Theorem 2. In Section 3, we give the proof of Theorem 3.

2. The number of homomorphisms between paths

One can enumerate homomorphisms from P_n to P_k by picking a fixed point as image of 1 and moving to vertices which are adjacent to this vertex, as

$$f \in \mathrm{Hom}(P_n, P_k) \Leftrightarrow \forall x \in \{1, \ldots, n-1\} : \{f(x), f(x+1)\} \in E(P_k).$$

Hence, one can describe all possible moves through the edge structure of the two paths.

For $1 \leq j \leq k$, let

$$\mathrm{Hom}^j(P_n, P_k) = \{f \in \mathrm{Hom}(P_n, P_k) : f(1) = j\}. \tag{2.1}$$

Obviously, we have

$$|\mathrm{Hom}^j(P_n, P_k)| = |\{f \in \mathrm{Hom}(P_n, P_k) : f(n) = j\}|. \tag{2.2}$$

Definition 1. A *lattice path* of length n is a sequence $(\gamma_0, \ldots, \gamma_n)$ of points γ_i in the plan $\mathbb{Z} \times \mathbb{Z}$ for all $0 \leq i \leq n$ and such that $\gamma_{i+1} - \gamma_i = (1, 0)$ (east-step) or $(0, 1)$ (north-step) for $1 \leq i \leq n-1$.

As shown by Arworn [1], we can encode each homomorphism $f \in \mathrm{Hom}^1(P_n, P_k)$ by a lattice path $\gamma = (\gamma_0, \ldots, \gamma_{n-1})$ in $\mathbb{N} \times \mathbb{N}$ between the lines $y = x$ and $y = x - k + 1$ as follows:

- $\gamma_0 = (0, 0)$, and for $j = 1, \ldots, n-1$,
- $\gamma_{j+1} = \gamma_j + (1, 0)$ if $f(j) > f(j-1)$,
- $\gamma_{j+1} = \gamma_j + (0, 1)$ if $f(j) < f(j-1)$.

For example, if the images of successive vertices of $f \in \mathrm{Hom}(P_{15}, P_{11})$ are

1, 2, 3, 2, 3, 4, 5, 4, 3, 4, 5, 6, 5, 6, 5,

then the corresponding lattice path is given by Fig. 1.

Definition 2. For nonnegative integers n, m, t, s, let $\mathcal{L}(n, m)$ be the set of all the lattice paths from the origin to (n, m) and $\mathcal{L}(n, m; t, s)$ the set of lattice paths in $\mathcal{L}(n, m)$ that stay between the lines $y = x + t$ and $y = x - s$ (being allowed to touch them), where $n + t \geq m \geq n - s$.

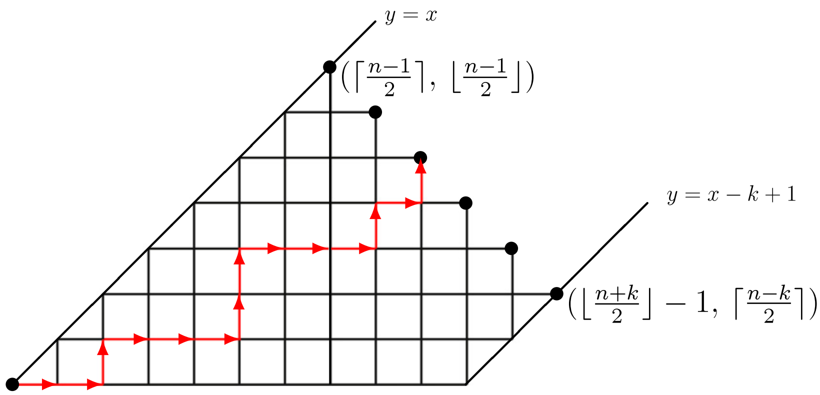

Fig. 1. A lattice path from (0, 0) to (9, 5) that stays between lines $y = x$ and $y = x - k + 1$, where $n = 15$ and $k = 11$.

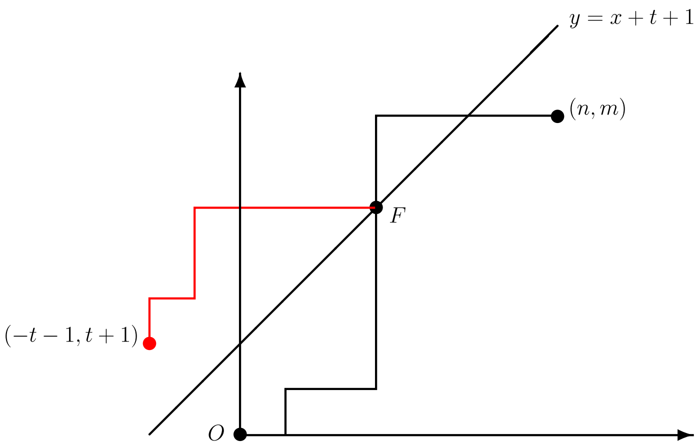

Fig. 2. Reflection of the segment of the path from O to the first reaching point F with respect to the line $y = x + t + 1$.

Lemma 4. *Let* $K = \min(\lfloor \frac{n+k}{2} \rfloor, n)$*, then*

$$|\mathrm{Hom}^1(P_n, P_k)| = \sum_{l=\lceil \frac{n-1}{2} \rceil}^{K-1} |\mathcal{L}(l, n-1-l; 0, k-1)|. \tag{2.3}$$

Proof. It follows from the above correspondence that each homomorphism from P_n to P_k is encoded by a lattice path in some $\mathcal{L}(\#E, \#N; 0, k-1)$, where $\#E$ is the number of east-steps and $\#N$ the number of north-steps. The path structures require that

$$\#E + \#N = n - 1, \qquad \#E - \#N \le k - 1, \qquad \#E - \#N \ge 0.$$

Therefore, we must have $\#E \ge (n-1)/2$, $\#E \le n - 1$ and $\#E \le (k + n - 2)/2$. □

To evaluate the sum in (2.3), we need a formula for the cardinality of $\mathcal{L}(n, m; t, s)$. First of all, each lattice path in $\mathcal{L}(n, m)$ can be encoded by a word of length $n + m$ on the alphabet $\{A, B\}$ with n letters A and m letters B. So, the cardinality of $\mathcal{L}(n, m)$ is given by the binomial coefficient $\binom{n+m}{n}$. Next, each lattice path in $\mathcal{L}(n, m)$ which passes above the line $y = x + t$ (or reaching the line $y = x + t + 1$) can be mapped to a lattice path from $(-t-1, t+1)$ to (n, m) by the *reflection* with respect to the line $y = x + t + 1$ (see Fig. 2). Hence, there are $\binom{n+m}{n+t+1}$ such lattice paths. Therefore, the number of lattice paths in $\mathcal{L}(n, m)$ which do not pass above the line $y = x + t$ (or not reaching the line $y = x + t + 1$), where $m \le n + t$, is given by

$$\binom{n+m}{n} - \binom{n+m}{n+t+1}.$$

By a similar reasoning, we can prove the following known result (see [5, Lemma 4A], for example). For the reader's convenience, we provide a sketch of the proof.

Lemma 5. *The cardinality of* $\mathcal{L}(n, m; t, s)$ *is given by*

$$|\mathcal{L}(n, m; t, s)| = \sum_{k\in\mathbb{Z}} \left(\binom{n+m}{n-k(t+s+2)} - \binom{n+m}{n-k(t+s+2)+t+1} \right), \tag{2.4}$$

where $\binom{n}{k} = 0$ *if* $k > n$ *or* $k < 0$.

Sketch of proof. Let T and S be the lines $y = x + t + 1$ and $y = x - s - 1$, respectively. Let A_1 denote the set of lattice paths in $\mathcal{L}(n, m)$ reaching T at least once, regardless of what happens at any other step, and let A_2 denote the set of lattice paths in $\mathcal{L}(n, m)$ reaching T, S at least once in the order specified. Generally, let A_i denote the set of lattice paths in $\mathcal{L}(n, m)$ reaching $T, S, \ldots,$ alternatively (i times) at least once in the specified order. Let B_i be the set defined in the same way as A_i with S, T interchanged. A standard Inclusive–Exclusive principle argument yields:

$$|\mathcal{L}(n, m; t, s)| = \binom{n+m}{n} + \sum_{i\geq 1}(-1)^i(|A_i| + |B_i|). \tag{2.5}$$

As the symmetric point of (a, b) with respect to the line $y = x + c$ is $(b - c, a + c)$, by repeatedly applying the reflection principle argument, we obtain

$$|A_{2j}| = \binom{n+m}{n+j(t+s+2)}, \qquad |A_{2j+1}| = \binom{n+m}{n-j(t+s+2)-(t+1)},$$

and

$$|B_{2j}| = \binom{n+m}{n-j(t+s+2)}, \qquad |B_{2j+1}| = \binom{n+m}{n+j(t+s+2)-(s+1)}.$$

Substituting this into (2.5) leads to (2.4). □

Lemma 6. *For each positive integers n and k,*

$$|\mathrm{Hom}^1(P_n, P_k)| = \sum_{j\in\mathbb{Z}} \left(\binom{n-1}{\lceil \frac{n-1}{2} \rceil - j(k+1)} - \binom{n-1}{\lfloor \frac{n+k}{2} \rfloor - j(k+1)} \right). \tag{2.6}$$

Proof. Substituting (2.4) into (2.3) and exchanging the order of the summations,

$$\begin{aligned} |\mathrm{Hom}^1(P_n, P_k)| &= \sum_{j\in\mathbb{Z}} \sum_{l=\lceil \frac{n-1}{2} \rceil}^{K-1} \left(\binom{n-1}{l-j(k+1)} - \binom{n-1}{l+1-j(k+1)} \right) \\ &= \sum_{j\in\mathbb{Z}} \left(\binom{n-1}{\lceil \frac{n-1}{2} \rceil - j(k+1)} - \binom{n-1}{K-j(k+1)} \right). \end{aligned} \tag{2.7}$$

Now, if $n \geq k$, then $K = \lfloor \frac{n+k}{2} \rfloor$,

$$\binom{n-1}{K-j(k+1)} = \binom{n-1}{\lfloor \frac{n+k}{2} \rfloor - j(k+1)}; \tag{2.8}$$

if $k > n$, then $K = n$, since

$$\binom{n-1}{n-j(k+1)} = \binom{n-1}{\lfloor \frac{n+k}{2} \rfloor - j(k+1)} = 0,$$

Eq. (2.8) is also valid. Hence (2.7) and (2.6) are equal. □

Proof of Theorem 2. For $f \in \mathrm{Hom}(P_{i+1}, P_k)$ with $i = 1, \ldots, n-1$, consider the following three cases.

(i) If $f(i) = 1$, then $f(i+1) = 2$ and there are $|\mathrm{Hom}^1(P_i, P_k)|$ such homomorphisms.
(ii) If $f(i) = k$, then $f(i+1) = k-1$ and there are $|\mathrm{Hom}^k(P_i, P_k)|$ such homomorphisms.
(iii) If $f(i) = j$ with $j \in \{2, 3, \ldots, k-1\}$, then $f(i+1) = j-1$ or $j+1$ and there are $2|\mathrm{Hom}^j(P_i, P_k)|$ such homomorphisms.

Summarizing, we get

$$|\mathrm{Hom}(P_{i+1}, P_k)| = |\mathrm{Hom}^1(P_i, P_k)| + 2\sum_{j=2}^{k-1} |\mathrm{Hom}^j(P_i, P_k)| + |\mathrm{Hom}^k(P_i, P_k)|.$$

Since $|\mathrm{Hom}(P_i, P_k)| = \sum_{j=1}^{k} |\mathrm{Hom}^j(P_i, P_k)|$ and $|\mathrm{Hom}^1(P_i, P_k)| = |\mathrm{Hom}^k(P_i, P_k)|$, it follows that

$$|\mathrm{Hom}(P_{i+1}, P_k)| = 2|\mathrm{Hom}(P_i, P_k)| - 2|\mathrm{Hom}^1(P_i, P_k)|.$$

By iteration, we derive

$$\begin{aligned} |\mathrm{Hom}(P_n, P_k)| &= 2^{n-1}|\mathrm{Hom}(P_1, P_k)| - \sum_{i=1}^{n-1} 2^{n-i}|\mathrm{Hom}^1(P_i, P_k)| \\ &= k \times 2^{n-1} - \sum_{i=1}^{n-1} 2^{n-i}|\mathrm{Hom}^1(P_i, P_k)|. \end{aligned} \tag{2.9}$$

Plugging (2.6) into (2.9), we obtain (1.2). □

Remark. The key point in the above proof is to reduce the counting problem of $|\mathrm{Hom}(P_n, P_k)|$ to $|\mathrm{Hom}^1(P_i, P_k)|$ for $i = 1, \ldots, n-1$. Arworn and Wojtylak [2] give a formula for $|\mathrm{Hom}(P_n, P_k)| = \sum_{j=1}^{k} |\mathrm{Hom}^j(P_n, P_k)|$ without using this reduction. Moreover, their expression for $|\mathrm{Hom}^j(P_n, P_k)|$ depends on the parity of $n - j$:

$$|\mathrm{Hom}^j(P_n, P_k)| = \begin{cases} \displaystyle\sum_{t=-n+1}^{n-1} (-1)^t \sum_{u=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{n-1}{\frac{n-j-1}{2} + u + \lceil \frac{(k+1)t}{2} \rceil} & \text{if } n-j \text{ is odd,} \\ \displaystyle\sum_{t=-n+1}^{n-1} (-1)^t \sum_{u=0}^{\lceil \frac{k-1}{2} \rceil} \binom{n-1}{\lfloor \frac{n-j-1}{2} \rfloor + u + \lfloor \frac{(k+1)t}{2} \rfloor} & \text{if } n-j \text{ is even.} \end{cases} \tag{2.10}$$

Note that Eq. (2.6) unifies the two cases in (2.10) when $j = 1$.

When $k = n$, we can deduce a simple formula for the number of endomorphisms of P_n (see http://oeis.org/A102699) by applying two binomial coefficient identities.

Lemma 7. *For $m \geq 1$, the following identities hold*

$$\sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-1-2k} = m\binom{2m}{m}, \tag{2.11}$$

$$\sum_{k=0}^{m-1} \binom{2k+1}{k} 2^{2m-1-2k} = (m+1)\binom{2m+1}{m} - 2^{2m}. \tag{2.12}$$

Proof. We prove (2.11) by induction on m. Clearly (2.11) is true for $m = 1$. If it is true for $m \geq 1$, then for $m+1$, the left-hand side after cutting out the last term, can be written as

$$\begin{aligned} 2^2 \sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-1-2k} + 2\binom{2m}{m} &= 4m\binom{2m}{m} + 2\binom{2m}{m} \\ &= (m+1)\binom{2m+2}{m+1}. \end{aligned}$$

Thus (2.11) is proved. Similarly we can prove (2.12). □

Proposition 8. *For $n \geq 1$,*

$$|\mathrm{End}(P_n)| = \begin{cases} (n+1)2^{n-1} - (2n-1)\binom{n-1}{(n-1)/2} & \text{if } n \text{ is odd,} \\ (n+1)2^{n-1} - n\binom{n}{n/2} & \text{if } n \text{ is even.} \end{cases} \tag{2.13}$$

Proof. When $k = n$, Theorem 2 becomes

$$|\mathrm{End}(P_n)| = n \times 2^{n-1} - \sum_{i=0}^{n-2} 2^{n-1-i} \times \binom{i}{\lceil \frac{i}{2} \rceil}. \tag{2.14}$$

By Lemma 7, if n is even, say $n = 2m$, then

$$\sum_{i=0}^{n-2} 2^{n-1-i} \times \binom{i}{\lceil \frac{i}{2} \rceil} = \sum_{k=0}^{m-2} \binom{2k+1}{k} 2^{2m-2-2k} + \sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-1-2k}$$
$$= 2m \binom{2\,m-1}{m-1} - 2^{2\,m-1} + m \binom{2\,m}{m};$$

if n is odd, say $n = 2m + 1$, then

$$\sum_{i=0}^{n-2} 2^{n-1-i} \times \binom{i}{\lceil \frac{i}{2} \rceil} = \sum_{k=0}^{m-1} \binom{2k+1}{k} 2^{2m-1-2k} + \sum_{k=0}^{m-1} \binom{2k}{k} 2^{2m-2k}$$
$$= (m+1) \binom{2\,m+1}{m} - 2^{2\,m} + 2m \binom{2\,m}{m}.$$

Substituting these into (2.14) we obtain the desired result. □

3. Proof of Theorem 3

We first establish three lemmas. For any $n \geq 1$, let $[n] = \{1, \ldots, n\}$, which is $V(P_n)$. Denote by $\mathfrak{S}_n$ the set of permutations of $[n]$. For $1 \leq k \leq n$, denote by $\mathrm{Epi}(P_n, P_k)$ the set of epimorphisms from P_n to P_k, namely,

$$\mathrm{Epi}(P_n, P_k) = \{f \in \mathrm{Hom}(P_n, P_k) : f([n]) = [k]\}. \tag{3.1}$$

Lemma 9. *For* $1 \leq k \leq n - 1$,

$$l_k(n) = |\mathrm{Epi}(P_n, P_{n-k+1})|/2. \tag{3.2}$$

Proof. Let $r = n - k + 1$. Denote by $\mathrm{End}_r(P_n)$ the subset of endomorphisms in $\mathrm{End}(P_n)$ such that $|f([n])| = r$ and $\mathcal{L}_k(n)$ the set of partitions induced by endomorphisms in $\mathrm{End}_r(P_n)$. By definition (see (1.1)), the integer $l_k(n)$ is the cardinality of $\mathcal{L}_k(n)$.

For each $f \in \mathrm{End}_r(P_n)$, if $f([n]) = \{a, a+1, \ldots, a+r-1\}$ for some integer $a \in [n-r+1]$, we define $\bar{f} \in \mathrm{Epi}(P_n, P_r)$ by $\bar{f}(x) = f(x) - a + 1$. Then f and $\bar{f}$ induce the same partition in $\mathcal{L}_k(n)$. Hence, we can consider $\mathcal{L}_k(n)$ as the set of partitions induced by epimorphisms in $\mathrm{Epi}(P_n, P_r)$.

If $\{A_1, \ldots, A_r\}$ is a partition of $[n]$ induced by an $f \in \mathrm{Epi}(P_n, P_r)$, then, we can assume that $\min(A_1) \leq \min(A_2) \leq \cdots \leq \min(A_r)$. Hence, we can identify f with a permutation $\sigma \in \mathfrak{S}_r$ by $f(A_{\sigma(i)}) = i$ for $i \in [r]$. Moreover, two blocks A_i and A_j are adjacent in the arrangement $A_{\sigma(1)} \cdots A_{\sigma(r)}$ if and only if there are two consecutive integers α and β such that $\alpha \in A_i$ and $\beta \in A_j$. We show that there are exactly two such permutations for a given induced partition.

Starting from a partition $\{A_1, \ldots, A_r\}$ of $[n]$ induced by an $f \in \mathrm{Epi}(P_n, P_r)$, we arrange step by step the blocks $A_1, \ldots, A_i$ for $2 \leq i \leq r$ such that A_i is adjacent to the block A_j containing $\min(A_i) - 1$ and $j < i$. Since $\min(A_1) = 1$ and $\min(A_2) = 2$, there are two ways to arrange A_1 and A_2: A_1A_2 or A_2A_1. Suppose that the first i (≥ 2) blocks have been arranged as $W_i := A_{\sigma_i(1)} \cdots A_{\sigma_i(i)}$ with $\sigma_i \in \mathfrak{S}_i$, then $\min(A_{i+1}) - 1$ must belong to $A_{\sigma_i(1)}$ or $A_{\sigma_i(i)}$ because any two adjacent blocks in W_i should stay adjacent in all the W_j for $i \leq j \leq r$. Hence there is only one way to insert A_{i+1} in W_i: at the left of W_i (resp. right of W_i) if $\min(A_{i+1}) - 1 \in A_{\sigma_i(1)}$ (resp. $A_{\sigma_i(i)}$) for $i \geq 2$. As there are two possibilities for $i = 2$ we have thus proved that there are exactly two corresponding epimorphisms in $\mathrm{Epi}(P_n, P_r)$ for a given induced partition with r blocks. For example, starting from the induced partition $\{\{1, 3, 5, 9\}, \{2, 4, 10\}, \{6, 8\}, \{7\}, \{11\}\}$ of $V(P_{11})$, we obtain the two corresponding arrangements:

$$\{7\}\{6, 8\}\{1, 3, 5, 9\}\{2, 4, 10\}\{11\} \quad \text{and} \quad \{11\}\{2, 4, 10\}\{1, 3, 5, 9\}\{6, 8\}\{7\}.$$

This is the desired result. □

Lemma 10. *For* $1 \leq k \leq n$,

$$l_k(n) = \frac{1}{2}|\mathrm{Hom}(P_n, P_{n-k+1})| - |\mathrm{Hom}(P_n, P_{n-k})| + \frac{1}{2}|\mathrm{Hom}(P_n, P_{n-k-1})|.$$

Proof. By definition we have $\mathrm{Hom}(P_n, P_k) \setminus \mathrm{Epi}(P_n, P_k) = A \cup B$, where

$$A = \{f \in \mathrm{Hom}(P_n, P_k) : f([n]) \subseteq [k-1]\},$$
$$B = \{f \in \mathrm{Hom}(P_n, P_k) : f([n]) \subseteq [k] \setminus [1]\}.$$

Hence

$$|\mathrm{Hom}(P_n, P_k)| - |\mathrm{Epi}(P_n, P_k)| = |A| + |B| - |A \cap B|. \tag{3.3}$$

Since $|A| = |B| = |\mathrm{Hom}(P_n, P_{k-1})|$, and

$$|A \cap B| = |\{f \in \mathrm{Hom}(P_n, P_k) : f([n]) \subseteq [k-1] \setminus [1]\}| = |\mathrm{Hom}(P_n, P_{k-2})|,$$

we derive from (3.3) that

$$|\mathrm{Epi}(P_n, P_k)| = |\mathrm{Hom}(P_n, P_k)| - 2|\mathrm{Hom}(P_n, P_{k-1})| + |\mathrm{Hom}(P_n, P_{k-2})|.$$

The result follows then by applying Lemma 9. □

It follows from Lemma 10 and Theorem 2 that

$$l_k(n) = \sum_{i=0}^{n-2} 2^{n-i-2} \sum_{j\in\mathbb{Z}} (-A_{i,j} + 2B_{i,j} - C_{i,j}), \tag{3.4}$$

where

$$A_{i,j} = A^+_{i,j} - A^-_{i,j}, \qquad B_{i,j} = B^+_{i,j} - B^-_{i,j}, \qquad C_{i,j} = C^+_{i,j} - C^-_{i,j},$$

with

$$A^+_{i,j} = \binom{i}{\lceil \frac{i}{2} \rceil - j(n-k+2)}, \qquad A^-_{i,j} = \binom{i}{\lfloor \frac{i+n-k}{2} \rfloor + 1 - j(n-k+2)},$$
$$B^+_{i,j} = \binom{i}{\lceil \frac{i}{2} \rceil - j(n-k+1)}, \qquad B^-_{i,j} = \binom{i}{\lfloor \frac{i+n-k-1}{2} \rfloor + 1 - j(n-k+1)},$$
$$C^+_{i,j} = \binom{i}{\lceil \frac{i}{2} \rceil - j(n-k)}, \qquad C^-_{i,j} = \binom{i}{\lfloor \frac{i+n-k-2}{2} \rfloor + 1 - j(n-k)}.$$

Lemma 11. *For $n \geq 2k$,*

$$\sum_{j\in\mathbb{Z}} (-A_{i,j} + 2B_{i,j} - C_{i,j}) = \left\{ \binom{i+1}{\lfloor \frac{i+n-k}{2} \rfloor + 1} + \binom{i+1}{\lfloor \frac{i+n-k}{2} \rfloor - n + k} \right\} - 2\left\{ \binom{i}{\lfloor \frac{i+n-k-1}{2} \rfloor + 1} + \binom{i}{\lfloor \frac{i+n-k-1}{2} \rfloor - n + k} \right\}. \tag{3.5}$$

Proof. Since $0 \leq k \leq \frac{n}{2}$, we have $\frac{n}{2} \leq n-k \leq n-1$. Therefore,

(1) if $j < 0$, then $\lceil \frac{i}{2} \rceil - j(n-k) \geq \lceil \frac{i}{2} \rceil + n - k \geq \lceil \frac{i}{2} \rceil + \frac{n}{2} \geq \lceil \frac{i}{2} \rceil + \frac{i}{2} + 1 \geq i+1$ because $i \leq n-2$. Similarly we have $\lfloor \frac{i+n-k-2}{2} \rfloor + 1 - j(n-k) \geq i+1$. Hence, all the summands $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ vanish;
(2) if $j > 0$, then $\lceil \frac{i}{2} \rceil - j(n-k) \leq \lceil \frac{i}{2} \rceil - (n-k) \leq \lceil \frac{i}{2} \rceil - \lceil \frac{n}{2} \rceil \leq -1$ because $i \leq n-2$. Hence, all $A^+_{i,j}$, $B^+_{i,j}$ and $C^+_{i,j}$ vanish;
(3) if $j \geq 2$, then $\lfloor \frac{i+n-k}{2} \rfloor + 1 - j(n-k+2) \leq \lfloor \frac{n-2+n-k}{2} \rfloor + 1 - 2(n-k+2) \leq \frac{3}{2}k - n - 5 \leq -1$. Similarly we have $\lfloor \frac{i+n-k-1}{2} \rfloor + 1 - j(n-k+1) \leq -1$ and $\lfloor \frac{i+n-k-2}{2} \rfloor + 1 - j(n-k) \leq -1$, so all $A^-_{i,j}$, $B^-_{i,j}$ and $C^-_{i,j}$ vanish.

It follows that the summation over $j \in \mathbb{Z}$ in (3.5) reduces to

$$-A^-_{i,0} + 2B^-_{i,0} - C^-_{i,0} - A^-_{i,1} + 2B^-_{i,1} - C^-_{i,1}.$$

Using $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ to combine $A^-_{i,0}$ with $C^-_{i,0}$ and $A^-_{i,1}$ with $C^-_{i,1}$, respectively, we derive the desired formula. □

Now, we are in position to prove Theorem 2. When $n \geq 2k$, by Lemma 11, the summands in (3.4) can be written as

$$2^{n-i-2} \sum_{j\in\mathbb{Z}} (-A_{i,j} + 2B_{i,j} - C_{i,j}) = D_{i+1} - D_i,$$

where

$$D_i = 2^{n-i-1} \left\{ \binom{i}{\lfloor \frac{i+n-k-1}{2} \rfloor + 1} + \binom{i}{\lfloor \frac{i+n-k-1}{2} \rfloor - n + k} \right\}.$$

Substituting this into (3.4) we obtain

$$l_k(n) = \sum_{i=0}^{n-2} (D_{i+1} - D_i) = D_{n-1},$$

which is clearly equivalent to (1.3).

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