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Fuzzy Topological Spaces

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We start by discussing a class of special fuzzy topological spaces, that is, the product-induced spaces [5]. First, we show that every fuzzy topological space is topologically isomorphic with a certain topological space, and then proceed to prove every open fuzzy set is defined by some lower semicontinuous function. Taking this as the background, we introduce the concept of dual points [6], and thus establish a kind of neighborhood structure of fuzzy points such that the Q-neighborhood [4], which is one of the important notions in fuzzy topology, and the neighborhood are integrated in this structure. This neighborhood structure will be the core of our developing the theory of fuzzy topological spaces. We introduce the concept of strong quasi-discoincident [11], and so give a group of fuzzy separation properties which is a most natural generalization of the usual separation properties. Next, we introduce a kind of fuzzy metrics and use this metrics directly to discuss the fuzzy metric space. By means of fuzzy points, we define kinds of uniformities, discuss their fundamental properties and extend Weil's theorem on usual topology to fuzzy topological spaces, and hence obtain their separation character. Naturally, these fuzzy uniformities can still be characterized by a family of fuzzy metrics. Finally, we discuss the problem of fuzzy metrization on fuzzy topological spaces and obtain a fuzzy metrization theorem which contains the Nagata-Smirnov theorem as a special example.

I. PRELIMINARIES

Let $X = \{x\}$ be a set of points. A fuzzy set A is characterized by a membership function $A(x)$ from X to the unit interval $I = [0, 1]$. In particular, $X(x) \equiv 1$ and $\emptyset(x) \equiv 0$ are fuzzy sets in X. The family of all fuzzy sets in X is denoted by I^X.

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DEFINITION 1.1. Let A and B be fuzzy sets in X, then

$$A \subset B \Leftrightarrow A(x) \leqslant B(x), \qquad x \in X,$$

$$A = B \Leftrightarrow A(x) = B(x), \qquad x \in X,$$

$$A = B' \Leftrightarrow A(x) = 1 - B(x), \qquad x \in X,$$

where B' is the complement of B. For a family of fuzzy sets $\{A_\lambda : \lambda \in \Lambda\}$

$$C = \bigcup_{\lambda \in \Lambda} A_\lambda \Leftrightarrow C(x) = \sup_{\lambda \in \Lambda} A_\lambda(x), \qquad x \in X,$$

$$D = \bigcap_{\lambda \in \Lambda} A_\lambda \Leftrightarrow D(x) = \inf_{\lambda \in \Lambda} A_\lambda(x), \qquad x \in X.$$

In particular,

$$C = A \cup B \Leftrightarrow C(x) = \max[A(x), B(x)], \qquad x \in X,$$

$$D = A \cap B \Leftrightarrow D(x) = \min[A(x), B(x)], \qquad x \in X.$$

If $\min[A(x), B(x)] \neq 0$, for some $x \in X$, then A is said to be intersecting with B and if $A(x) + B(x) > 1$, then A is quasi-coincident with B.

DEFINITION 1.2. Suppose $T: X \to Y$ is a mapping and A is a fuzzy set in X, then the image TA is a fuzzy set in Y whose membership function is defined by

$$[TA](y) = \sup_{x \in T^{-1}(y)} A(x), \qquad T^{-1}(y) \neq \varnothing,$$

$$= 0, \qquad T^{-1}(y) = \varnothing.$$

If B is a fuzzy set in Y, then the inverse $T^{-1}B$ is a fuzzy set in X defined by

$$[T^{-1}B](x) = B(T(x)).$$

DEFINITION 1.3. A fuzzy point P in X is a special fuzzy set with membership function

$$P(x) = \alpha, \qquad x = x_0,$$

$$= 0, \qquad x \neq x_0,$$

where $0 < \alpha < 1$. P is said to have support x_0 value α and is denoted by $P_{x_0}^\alpha$ or $P(x_0, \alpha)$.

Let A be a fuzzy set in X, then $P_{x_0}^\alpha \subset A \Leftrightarrow \alpha \leqslant A(x_0)$ in particular, $P_{x_0}^\alpha \subset P_y^\beta \Leftrightarrow x_0 = y$, $\alpha \leqslant \beta$. A fuzzy point $P_{x_0}^\alpha$ is said to be in A, denoted $P_{x_0}^\alpha \in A$ iff $\alpha < A(x_0)$.

A fuzzy set with membership function

$$P(x) = 1, \qquad x = x_0,$$
$$= 0, \qquad x \neq x_0,$$

is called a crisp point, denoted $P^1_{x_0}$. For any fuzzy set A in X, we define the crisp point $P^1_{x_0} \subset A \Leftrightarrow A(x_0) = 1$ and $P^1_{x_0} \in A \Leftrightarrow A(x_0) = 1$.

The fuzzy point and crisp point are often referred to as a point.

DEFINITION 1.4. Let A be a fuzzy set in X, then the subset of X

$$\omega_\alpha(A) = \{x : A(x) \geqslant \alpha\}, \qquad \alpha \in (0, 1],$$

and

$$\sigma_\alpha(A) = \{x : A(x) > \alpha\}, \qquad \alpha \in [0, 1),$$

is called the weak α-cut and strong α-cut of A, respectively.

DEFINITION 1.5. Suppose $\mathbb{F}$ is a family of fuzzy sets in X, which satisfies the following axioms:

(T1) $0, 1 \in \mathbb{F}$,

(T2) if $A, B \in \mathbb{F}$, then $A \cap B \in \mathbb{F}$,

(T3) if $A_\lambda \in \mathbb{F}$, $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathbb{F}$,

then $\mathbb{F}$ is called a fuzzy topology for X and the pair $(X, \mathbb{F})$ is a fuzzy topological space.

Every member of $\mathbb{F}$ is called an $\mathbb{F}$-open fuzzy set (or simply open fuzzy set) and its complement is an $\mathbb{F}$-closed fuzzy set (or closed fuzzy set).

Let A be a fuzzy set in fts $(X, \mathbb{F})$. The closure $\bar{A}$ and interior A^0 of A are defined, respectively, by

$$\bar{A} = \bigcap \{B : B \supset A,\ B' \in \mathbb{F}\}$$

and

$$A^0 = \bigcup \{B : B \subset A,\ B \in \mathbb{F}\}.$$

DEFINITION 1.6. Let $\mathbb{F}$ be a fuzzy topology. A subfamily $\mathbb{B}$ of $\mathbb{F}$ is a base for $\mathbb{F}$ iff each member of $\mathbb{F}$ can be expressed as the union of some members of $\mathbb{B}$. A subfamily $\mathbb{S}$ of $\mathbb{F}$ is a subbase for $\mathbb{F}$ iff the family of finite intersections of members of $\mathbb{S}$ form a base for $\mathbb{F}$.

DEFINITION 1.7. A mapping $T: (X, \mathbb{F}) \to (Y, \mathbb{E})$ is called fuzzy continuous iff for each $A \in \mathbb{E}$ implies $T^{-1}A \in \mathbb{F}$.

DEFINITION 1.8. A mapping $T: (X, \mathbb{F}) \to (Y, \mathbb{E})$ is called fuzzy open (closed) iff for each $A \in \mathbb{F}$ ($A' \in \mathbb{F}$) implies $TA \in \mathbb{E}$ ($[TA]' \in \mathbb{E}$).

The foundational notions and the definitions mentioned above and other definitions used but not shown in this paper can be found in [1–6].

II. PRODUCT-INDUCED SPACES

In paper [5], we introduced a class of fts which is defined by a class of special product topologies $JX\theta$. This class of fts is very foundational and important.

DEFINITION 2.1. Let A be a fuzzy set in X. Then the subset $A^S = \{(x, \alpha): P_x^\alpha \in A, \alpha \in I_0\}$ in product set $X \times I_0$ is called the shape of the fuzzy set A. The family $\{A^S: A \in I^X\}$ of all shapes of fuzzy sets in X is denoted by $\mathbb{G}$, where $I_0 = (0, 1)$.

THEOREM 2.1. *The operator* $^S: I^X \to \mathbb{G}$ *is an isomorphism for* "$\bigcup$" *and finite* "$\bigcap$."

Proof. It is evident that the operator ^{S} is 1–1. Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of fuzzy sets in X, then

$$\begin{aligned}\bigcup_{\lambda \in \Lambda} A_\lambda^S &= \bigcup_{\lambda \in \Lambda} \{(x, \alpha): x \in X, 0 < \alpha < A_\lambda(x)\} \\ &= \{(x, \alpha): x \in X, 0 < \alpha < \sup_{\lambda \in \Lambda} A_\lambda(x)\} \\ &= \left[\bigcup_{\lambda \in \Lambda} A_\lambda\right]^S.\end{aligned}$$

Similarly, we can prove

$$\bigcap_{\lambda = 1, \ldots, n} A_\lambda^S = \left[\bigcap_{\lambda = 1, \ldots, n} A_\lambda\right]^S.$$

DEFINITION 2.2. A topology for the open interval $I_0 = (0, 1)$ is called a θ-topology iff its open sets family consists of some open intervals $(0, \alpha)$, where $\alpha \in [0, 1]$.

It is easily seen that

$$\theta_0 = \{\phi, I_0\},$$

$$\theta_I = \{(0, \alpha): \alpha \in [0, 1]\}$$

are θ-topologies.

DEFINITION 2.3. Let $\mathcal{T}$ be a topology for X and θ be a θ-topology for I_0. The family of fuzzy sets in X

$$\mathbb{F}_{\mathcal{T} \times \theta} = \{A : A \in I^X \cdot A^S \in \mathcal{T} \times \theta\}$$

is a fuzzy topology for X, which is called a product-induced topology, and $(X, \mathbb{F}_{\mathcal{T} \times \theta})$ is the product-induced space.

THEOREM 2.2. *The product-induced space* $(X, \mathbb{F}_{\mathcal{T} \times \theta})$ *and topological space* $(X \times I_0, \mathcal{T} \times \theta)$ *are topologically isomorphic, that is, there exists a* 1 1 *correspondence* ${}^S : \mathbb{F}_{\mathcal{T} \times \theta} \to \mathcal{T} \times \theta$ *such that*

(i) $0^S = \varnothing$, $1^S = X \times I_0$,

(ii) $(A \cap B)^S = A^S \cap B^S$,

(iii) $(\bigcup_{\lambda \in \Lambda} A_\lambda)^S = \bigcup_{\lambda \in \Lambda} A_\lambda^S$,

for any $A, B, A_\lambda \in \mathbb{F}_{\mathcal{T} \times \theta}$, $\lambda \in \Lambda$.

Proof. By Theorem 2.1 is trivial.

By the above theorem, it is easy for us to see that a statement involving only the topology and the operator "$\bigcup$" and finite "$\bigcap$" holds in $(X \times I_0, \mathcal{T} \times \theta)$, then the corresponding statement holds in $(X, \mathbb{F}_{\mathcal{T} \times \theta})$. For example, we have the followng.

THEOREM 2.3. *A product-induced space* $(X, \mathbb{F}_{\mathcal{T} \times \theta})$ *is fuzzy* C_I, C_{II} *and separable, respectively, iff the topological space* (X, J) *is* C_I, C_{II} *and separable.*

Let $(X, \mathcal{T})$ be a topological space. The family of all lower semicontinuous functions from $(X, \mathcal{T})$ to $[0, 1]$ forms a fuzzy topology for X and the corresponding fts is called semicontinuous fts or induced fts [3, 7, 8]. By the properties of lower semicontinuous functions, it is easy to prove the following proposition.

THEOREM 2.4. *The product-induced space* $(X, \mathbb{F}_{\mathcal{T} \times \theta_l})$ *is just the lower semicontinuous fts which is induced by topological space* $(X, \mathcal{T})$.

DEFINITION 2.4. Let $(X, \mathbb{F})$ be a fts, then the family of subsets in X

$$\{\sigma_\alpha(A) : A \in \mathbb{F}, \alpha \in [0, 1]\}$$

forms a subbase of some topology for X. This topology is called initial topology of $\mathbb{F}$, denoted by $\iota(\mathbb{F})$, and the corresponding topological space $(X, \iota(\mathbb{F}))$ is the initial topological space.

Obviously, the definition of $\iota(\mathbb{F})$ mentioned above and the corresponding definition in (8) are identical. Similarly, the topologies $\iota_\alpha(\mathbb{F})$, $\alpha \in [0, 1]$, for X induced in (10) can be defined by

$$\iota_\alpha(\mathbb{F}) = \{\sigma_\alpha(A) : A \in \mathbb{F}\}.$$

It is evident that we have

$$\iota(\mathbb{F}) = \sup_{\alpha \in [0,1]} \iota_\alpha(\mathbb{F}).$$

THEOREM 2.5. *Every open (closed) fuzzy set in a fts $(X, \mathbb{F})$ is a lower (upper) semicontinuous function from $(X, \iota(\mathbb{F}))$ to $[0, 1]$.*

Proof. It follows from the character of semicontinuous functions.

Due to Theorems 2.4. and 2.5. we obtain immediately the following important results.

THEOREM 2.6. *For any fuzzy topology $\mathbb{F}$ on X, we have*

$$\mathbb{F} \subset \mathbb{F}_{\mathscr{T} \times \theta_l},$$

where $\mathscr{T} = \iota(\mathbb{F})$.

Thus it can be seen that all fuzzy topologies for X can be classified by their initial topology $\mathscr{T} = \iota(\mathbb{F})$ and the finest member of each class is $\mathbb{F}_{\mathscr{T} \times \theta_l}$.

THEOREM 2.7. *For every fts $(X, \mathbb{F})$ there exists a topological space $(X \times I_0, \mathscr{U})$, $\mathscr{U} \subset \iota(\mathbb{F}) \times \theta_l$, such that $(X, \mathbb{F})$ and $(X \times I_0, \mathscr{U})$ are topologically isomorphic.*

Proof. Trivial from Theorems 2.2 and 2.6.

Let $(X, \mathbb{F})$ be a fts and $U \in \mathscr{T} = \iota(\mathbb{F})$, $\alpha \in (0, 1]$. The fuzzy set with shape $U \times (0, \alpha)$ is called a fundamental fuzzy set for $(X, \mathbb{F})$, denoted by N_U^α. If $N_U^\alpha \in \mathbb{F}$, then it is called a fundamental open fuzzy set. In particular, for product-induced space $(X, \mathbb{F}_{\mathscr{T} \times \theta})$, if $(0, \alpha) \in \theta$, then N_U^α is a fundamental fuzzy open set, and the family $\{N_U^\alpha : U \in \mathscr{T}, (0, \alpha) \in \theta\}$ forms a base of $\mathbb{F}_{\mathscr{T} \times \theta}$.

THEOREM 2.8. *The shape A^S of an open fuzzy set A in fts is an open set in $(X, \mathscr{T}) \times [0, 1]$, where $\mathscr{T} = \iota(\mathbb{F})$ and $[0, 1]$ is the subspace of number line.*

Proof. First, the shape $U \times (0, \alpha)$ of N_U^α is an open set in $(X \cdot \mathscr{T}) \times [0, 1]$. For any $A \in \mathbb{F}_{\mathscr{T} \times \theta_l}$, because all N_U^α forms a base for $\mathbb{F}_{\mathscr{T} \times \theta_l}$, so $A = \bigcup N_U^\alpha$. Hence A is an open set in $(X, \mathscr{T}) \times [0, 1]$. By Theorem 2.6, $A \in \mathbb{F}$ implies $A \in \mathbb{F}_{\mathscr{T} \times \theta_l}$, therefore A^S is an open set in $(X, \mathscr{T}) \times [0, 1]$.

DEFINITION 2.5. Let A be a fuzzy set in fts $(X, \mathbb{F})$. We define its weak 0-cut

$$\omega_0(A) = \bigcap \{W: \sigma_0(A) \subset W,\ W' \in \iota(\mathbb{F})\}.$$

THEOREM 2.9. *Let C be a closed fuzzy set in fts $(X, \mathbb{F})$, then set $C^S \cup \{(x, c(x)): x \in \omega_0(\iota)\}$ is a closed set in $(X, \mathscr{T}) \times [0, 1]$, $\mathscr{T} = \iota(\mathbb{F})$.*

Proof. Obvious form Theorem 2.8.

THEOREM 2.10. *Let $\theta_1 \supset \theta_2$. The mapping $T: (X, \mathbb{F}_{\mathscr{T} \times \theta_1}) \to (Y, \mathbb{F}_{\mathscr{U} \times \theta_2})$ is fuzzy continuous iff the mapping $T: (X, \mathscr{T}) \to (Y, \mathscr{U})$ is continuous.*

Proof. *Sufficiency.* Let N_U^α be a fundamental fuzzy set in $(Y, \mathbb{F}_{\mathscr{U} \times \theta_2})$. Obviously, we have

$$T^{-1}N_U^\alpha = N_{T^{-1}(U)}^\alpha.$$

Since T is continuous and $\theta_1 \supset \theta_2$ thus $T^{-1}(U) \in \mathscr{T}$, $(0, \alpha) \in \theta_1$ and so $T^{-1}N_U^\alpha \in \mathbb{F}_{\mathscr{T} \times \theta_1}$.

Let $B \in \mathbb{F}_{\mathscr{U} \times \theta_2}$, then there exists a family $\{N_{U_\lambda}^{\alpha_\lambda}: \lambda \in \Lambda\}$ of fundamental fuzzy sets in $\mathbb{F}_{\mathscr{U} \times \theta_2}$ such that

$$B = \bigcup_{\lambda \in \Lambda} N_{U_\lambda}^{\alpha_\lambda}.$$

Hence

$$T^{-1}B = \bigcup_{\lambda \in \Lambda} T^{-1}N_{U_\lambda}^{\alpha_\lambda} \in \mathbb{F}_{\mathscr{T} \times \theta_1}.$$

Necessity. For every $U \in \mathscr{U}$, it is obvious that $N_U^1 \in \mathbb{F}_{\mathscr{U} \times \theta_2}$. By fuzzy continuity of T, we have $T^{-1}N_U^1 = N_{T^{-1}(U)}^1 \in \mathbb{F}_{\mathscr{T} \times \theta_1}$, hence $T^{-1}(U) \in \mathscr{T}$.

THEOREM 2.11. *The mapping $T: (X, \mathbb{F}_{\mathscr{T} \times \theta}) \to (Y, \mathbb{F}_{\mathscr{U} \times \theta})$ is fuzzy open (closed) iff the mapping $T: (X, \mathscr{T}) \to (Y, \mathscr{U})$ is open (closed).*

Proof. We prove it in case of fuzzy open.

Sufficiency. If mapping T is fuzzy open, then for each fundamental open fuzzy set $N_U^1 \in \mathbb{F}_{\mathscr{T} \times \theta}$ the image $TN_U^1 \in \mathbb{F}_{\mathscr{U} \times \theta}$. It is easily seen that $TN_U^1 = N_{T(U)}^1$, thus $N_{T(U)}^1 \in \mathbb{F}_{\mathscr{U} \times \theta}$ and so $T(U) \in \mathscr{U}$. Hence the mapping T is open.

Necessity. Let $A \in \mathbb{F}_{\mathscr{T} \times \theta}$ and $P_y^\alpha \in TA$, take note of $0 < \alpha < TA(y)$, then there is an $x \in X$ such that $T(P_x^\alpha) = P_y^\alpha$ and $P_x^\alpha \in A$. Hence there is a fundamental fuzzy open set N_U^β such that $P_x^\alpha \in N_U^\beta \subset A$. Since mapping T is open, we have $T(U) \in \mathscr{U}$ and $TN_U^\beta = N_{T(U)}^\beta \in \mathbb{F}_{\mathscr{U} \times \theta}$. By $TN_U^\beta \subset TA$ we have

$$TA = \bigcup_{P_x^\alpha \in A} TN_U^\beta \in \mathbb{F}_{\mathscr{U} \times \theta}.$$

The mapping T is fuzzy open.

Two fuzzy topological spaces are said to be homeomorphic iff there exist a 1 1 and fuzzy continuous open mapping such that the image of one is exactly the other. Obviously, product-induced spaces $(X, \mathbb{F}_{\mathscr{T} \times \theta})$ and $(Y, \mathbb{F}_{\mathscr{U} \times \theta})$ are homeomorphic which is equivalent to $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ being homeomorphic.

III. Fuzzy Points and Level Sets

The fuzzy points and level sets of fuzzy sets are important tools for reserach on fuzzy topologies because any fuzzy set can be resolved vertically and levelly by means of them, respectively.

The fuzzy point is a kind of most simple and basic fuzzy set. For it the relation "$\subset$" is natural, and the relation "$\in$" has only subordinate status.

Definition 3.1. Let P_x^α, $0 < \alpha \leqslant 1$, be a point and A be a fuzzy set in fts $(X, \mathbb{F})$. A is called a neighborhood of P_x^α iff there is a open fuzzy set $B \in \mathbb{F}$, such that $P_x^\alpha \in B \subset A$.

Definition 3.2. Let P_x^α be a point and N_U^β a fundamental fuzzy set in fts $(X, \mathbb{F})$; If $P_x^\alpha \in N_U^\beta$ then N_U^β is called a neighborhood germ of P_x^α.

Theorem 3.1. *A fuzzy set A is a neighborhood of fuzzy point P_x^α, $\alpha \in (0, 1)$, in fts $(X, \mathbb{F})$ iff there exists an open fuzzy set $B \in \mathbb{F}$ and a neighborhood germ N_U^β of P_x^α such that*

$$P_x^\alpha \in N_U^\beta \subset B \subset A.$$

Proof. Obviously, from Theorem 2.5 and the property of lower semicontinuous function.

Definition 3.3. A fuzzy set A is called an S-neighborhood of point P_x^α in fts $(X, \mathbb{F})$ iff there is a neighborhood germ N_U^β of P_x^α and an open fuzzy set B such that

$$P_x^\alpha \in N_U^\beta \subset B \subset A.$$

Note that for the crisp point the neighborhood and the S-neighborhood may be different.

DEFINITION 3.4. Fuzzy point $P_x^{1-\alpha}$ is called a dual point of P_x^α. For crisp point P_x^1, we define its dual point the support x of P_x^1, denoted by P_x^0.

In our system, since the P_x^0 is not a point, the neighborhood has no meaning for P_x^0. However, it is convenient that we call the neighborhood of P_x^α, $0<\alpha\leqslant 1$, a neighborhood of P_x^0, and define $P_x^0\in A \Leftrightarrow A(x)>0$, $P_x^0\subset A \Leftrightarrow x\in\omega_0(A)$.

The notions of quasi-coincident and Q-neighborhood introduced by [4] are very important, and can perfectly deal with the question: the shape of the complement of a fuzzy set has to turn in topological space $X\times(0,1)$, such that we can transplant various definitions of closure on usual topology to fuzzy topology and make these concepts compatible with each other. We introduce the concept of dual point for the same object. The Q-neighborhood of a point is exactly the neighborhood of its dual point. Hence in our system the Q-neighborhood system and the neighborhood system are dual each other.

THEOREM 3.2. *Let A be a fuzzy set in fts $(X, \mathbb{F})$. The point $P_x^\alpha\in\bar{A}$ iff each neighborhood of its dual point $P_x^{1-\alpha}$ is quasi-coincident with A. The fuzzy point $P_x^\alpha\in A^0$ iff its dual point $P_x^{1-\alpha}\notin\overline{A'}$.*

Proof. The first half of the theorem is exactly Theorem 4.1. in [4]. Now we prove the second half.

If $P_x^{1-\alpha}\notin\overline{A'}$ then there is a neighborhood B of P_x^α quasi-coincident with A', i.e., $B\subset A$, and so $P_x^\alpha\in B\subset A$, hence $P_x^\alpha\in A^0$. Inversely, if $P_x^\alpha\in A^0$ then there is $B\in\mathbb{F}$ such that $P_x^\alpha\in B\subset A$, i.e., B is not quasi-coincident with A' (or B and A' are quasi-discoincident), hence $P_x^\alpha\notin\overline{A'}$.

In [3], for induced fuzzy topological spaces, i.e., $(X, \mathbb{F}_{\mathcal{T}\times\theta_l})$, Weiss proved the following proposition:

In fts $(X, \mathbb{F}_{\mathcal{T}\times\theta_l})$, the fuzzy set A is an open (or closed) fuzzy set iff the set $\sigma_\alpha(A)$ (or $\omega_\alpha(A)$) is $\mathcal{T}$-open (or $\mathcal{T}$-closed).

By this result and Theorem 2.6, we have

THEOREM 3.3. *If A is an open (or closed) fuzzy set in fts $(x, \mathbb{F})$, then $\sigma_\alpha(A)$ is an $\iota(\mathbb{F})$-open set (or $\omega_\alpha(A)$ is an $\iota(\mathbb{F})$-*closed set).

DEFINITION 3.5. Let a be a fuzzy set in *fts* $(X, \mathbb{F})$. The strong 1-cut is defined by

$$\sigma_1(A)=\bigcup\{U: N_U^1\subset A\},$$

where N_U^1 are the fundamental fuzzy sets in $(x, \mathbb{F})$.

THEOREM 3.4. *Let A be an open fuzzy set in fts $(X, \mathbb{F})$, then for each $\alpha \in [0, 1]$ we have*

$$\sigma_\alpha(A) = \bigcup \{U : P_x^\alpha \in N_U^\beta \subset A\},$$

where N_U^β are the fundamental fuzzy sets in $(X, \mathbb{F})$.

Proof. Obvious from the property of lower semicontinuous function.

IV. FUZZY SEPARATION

DEFINITION 4.1. A fts $(X, \mathbb{F})$ is a fuzzy T_0 space iff, for any two distinct points in $(X, \mathbb{F})$, at least one of them has a neighborhood which is not a neighborhood of the other.

THEOREM 4.1. *If fts $(X, \mathbb{F})$ is a fuzzy T_0 space, then the initial space $(X, \iota(\mathbb{F}))$* is a T_0 space.

Proof. Let x and y be two distinct points in X. Consider the fuzzy points P_x^α and P_y^α, $0 < \alpha < 1$. Due to $(X, \mathbb{F})$ being a fuzzy T_0 space, we might as well suppose that P_x^α has an open neighborhood B such that $P_y^\alpha \notin B$. Thus $x \in \sigma_\alpha(B) \in \iota(\mathbb{F})$ but $y \notin \sigma_\alpha(B)$ hence $(X, \iota(\mathbb{F}))$ is a T_0 space.

DEFINITION 4.2. A fts$(X, \mathbb{F})$ is a fuzzy T_1 space iff for any point P_x^α and $P_y^\beta \not\subset P_x^\alpha$ the dual point $P_y^{1-\beta}$ has a neighborhood which is not quasi-coincident with P_x^α.

THEOREM 4.2. *If fts $(X, \mathbb{F})$ is a fuzzy T_1 space, then the initial space $(X, \iota(\mathbb{F}))$ is a T_1 space.*

Proof. Let $x \neq y$; Consider points $P_x^{1/2}$ and $P_y^{1/2}$ then $P_y^{1/2}$ has a neighborhood A such that A and $P_x^{1/2}$ are not quasi-coincident. Thus $P_x^{1/2} \notin A$ and so $x \notin \sigma_{1/2}(A)$. Hence $(X, \iota(\mathbb{F}))$ is T_1.

THEOREM 4.3. *A fts $(X, \mathbb{F})$ is a T_1 space iff every point P_x^α is fuzzy closed.*

Proof. It is obvious from Theorem 3.2.

THEOREM 4.4. *A fts $(X, \mathbb{F})$ is a fuzzy T_1 space iff for any fuzzy set A, we have*

$$A = \bigcap \{B : A \subset B \in \mathbb{F}\}.$$

Proof. *Sufficiency.* For any P_x^α and $P_y^\beta \not\subset P_x^\alpha$, we have

$$P_y^{1-\beta} = \bigcap \{B : P_y^{1-\beta} \subset B \in \mathbb{F}\}.$$

Thus $P_y^{1-\beta}$ has an open neighborhood B such that B and P_x^α are not quasi-coincident and so $P_y^\beta \not\subset \overline{p_x^\alpha}$. Hence $\overline{p_x^\alpha} = p_x^\alpha$. By Theorem 4.3, the fts $(X, \mathbb{F})$ is a fuzzy T_1.

Necessity. For any fuzzy set A and point $P_x^\alpha \subset A$, there is $\beta < \alpha$ such that $P_x^\beta \not\subset A$. Since $(X, \mathbb{F})$ is a fuzzy T_1 space and the point $P_x^{1-\beta}$ is a closed fuzzy set, then $(P_x^{1-\beta})'$ is an open fuzzy set and $A \subset (P_x^{1-\beta})'$, $P_x^\alpha \not\subset (P_x^{1-\beta})'$. Let $B = (P_x^{1-\beta})'$ then

$$A = \bigcap \{B : A \subset B \in \mathbb{F}\}.$$

DEFINITION 4.3. A fts $(X, \mathbb{F})$ is a fuzzy T_2 space iff for any point P_x^α and $P_y^\beta \not\subset P_x^\alpha$ there are neighborhoods of P_x^α and $P_y^{1-\beta}$, respectively, which are not quasi-coincident each other.

THEOREM 4.5. *If fts $(X, \mathbb{F})$ is a fuzzy T_2 space, then $(X, \iota(\mathbb{F}))$ is a T_2 space.*

The proof is straightforward (cf. Theorem 4.2).

THEOREM 4.6. *A fts $(X, \mathbb{F})$ is a fuzzy T_2 space iff for any P_x^α we have*

$$P_x^\alpha = \bigcap \{C : P_x^\alpha \in C^0,\ C' \in \mathbb{F}\}.$$

Proof. Sufficiency. For any points P_x^α and $P_y^\beta \not\subset P_x^\alpha$, there is a closed neighborhood C of P_x^α such that $P_y^\beta \not\subset C$ hence $P_y^{1-\beta} \in C'$. Thus C^0 and C' are the neighborhoods of P_x^α and $P_y^{1-\beta}$, respectively, which are not quasi-coincident each other. Therefore the fts $(X, \mathbb{F})$ is a fuzzy T_2 space.

Necessity. Let $P_y^\beta \not\subset P_x^\alpha$, then P_x^α and $P_y^{1-\beta}$ have neighborhoods A and B, respectively, which are not quasi-coincident each other. That is, $P_x^\alpha \in A \subset B'$ and $P_y^\beta \not\subset B'$. Let $C = B'$, then C is a closed neighborhood of P_x^α, we have

$$P_x^\alpha = \bigcap \{C : P_x^\alpha \in C^0,\ C' \in \mathbb{F}\}.$$

It is evident that fuzzy T_2 implies fuzzy T_1 and fuzzy T_1 implies fuzzy T_0.

DEFINITION 4.4. A fts $(X, \mathbb{F})$ is fuzzy regular iff, for any point P_x^α and its any open neighborhood A, there is a fuzzy set B such that

$$P_x^\alpha \in B^0 \subset \bar{B} \subset A,$$

and A is a neighborhood of B.

A fuzzy set A is said to be neighborhood of B iff A is a neighborhood of every point $P(x, B(x))$, $B(x) > 0$, $x \in X$.

THEOREM 4.7. *If fts $(X, \mathfrak{F})$ is fuzzy regular, then the initial space $(X, \iota(\mathfrak{F}))$ is regular.*

Proof. Since the family $\{\sigma_\alpha(A): A \in \mathfrak{F}, \alpha \in (0, 1)\}$ forms a base of $\iota(\mathbb{F})$ we must only show that, for any $x \in X$ and its any open neighborhood $U = \bigcap_{i=1,\ldots,n} \{\sigma_{\alpha_i}(A_i): A_i \in \mathbb{F}, \alpha_i \in (0, 1)\}$, there exists an open set V and closed set W in $(X, \iota(\mathfrak{F}))$ such that

$$x \in V \subset W \subset U.$$

In fact, by the fuzzy regular of $(X, \mathfrak{F})$, for any fuzzy point $P_x^{\alpha_i}$ and its any open neighborhood A_i there is an open fuzzy set B_i such that

$$P_x^{\alpha_i} \in B_i \subset \overline{B_i} \subset A_i,$$

and A_i is a neighborhood of $\overline{B_i}$, so we have

$$x \in \sigma_{\alpha_i}(B_i) \subset \omega_{\alpha_i}(\overline{B_i}) \subset \sigma_{\alpha_i}(A_i).$$

Let $V = \bigcap_{i=1,\ldots,n} \sigma_{\alpha_i}(B_i)$ and $W = \bigcap_{i=1,\ldots,n} \omega_{\alpha_i}(\overline{B_i})$. Obviously, they are open and closed sets in $(X, \iota(\mathbb{F}))$, respectively, and we have

$$x \in V \subset W \subset U.$$

Therefore, $(X, \iota(\mathbb{F}))$ is regular.

THEOREM 4.8. *A fts $(X, \mathfrak{F})$ is fuzzy regular iff for any closed fuzzy set A we have*

$$A = \bigcap \{C: P(x, A(x)) \in C^0, x \in X, C' \in \mathbb{F}\}.$$

Proof. Sufficiency. Let B be an open fuzzy set in $(X, \mathfrak{F})$ and $P_x^\alpha \in B$. It is clear that $P_x^{1-\alpha} \not\in B'$. Let $A = B'$, then there is a closed neighborhood C of A such that $P_x^{1-\alpha} \not\in C$, and we have

$$P_x^\alpha \in C' \subset C^{0\prime} \subset B,$$

and B is a neighborhood of $C^{0\prime}$. Hence $(X, \mathbb{F})$ is fuzzy regular.

Necessity. Let A be a closed fuzzy set in $(X, \mathbb{F})$ and $P_x^\alpha \not\in A$, then $P_x^{1-\alpha} \in A'$ and there is an open fuzzy set B such that

$$P_x^{1-\alpha} \in B \subset \overline{B} \subset A',$$

and A' is a neighborhood of $\bar{B}$. Let $C = B'$, obviously C is a closed neighborhood of A and $P_x^\alpha \not\subset C$, hence we have

$$A = \bigcap \{C : P(x, A(x)) \in C^0, x \in X, C' \in \mathbb{F}\}.$$

DEFINITION 4.5. The fuzzy sets A and B in $(X, \mathbb{F})$ are said to be strong non-quasi-coincident (or strong quasi-discoincident) iff, for any $x \in X$, $A(x) + B(x) \leqslant 1$, and if $A(x) + B(x) = 1$, then either $A(x) = 1$ or $B(x) = 1$.

THEOREM 4.9. *A fts $(X, \mathbb{F})$ is fuzzy regular iff, for any point P_x^α and any closed fuzzy set C which is strong non-quasi-coincident with P_x^α, there are neighborhoods A and B of P_x^α and C, respectively, such that A and B are not quasi-coincident.*

Proof. Sufficiency. Let C be a closed fuzzy set and point $P_x^\alpha \in C' \in \mathbb{F}$, obviously P_x^α and C are strong quasi-discoincident, then there exist neighborhoods A and B of P_x^α and C, respectively, such that A and B are not quasi-coincident and so

$$P_x^\alpha \in A \subset B' \subset C',$$

and C' is a neighborhood of B'. Hence $(X, \mathbb{F})$ is fuzzy regular.

Necessity. Suppose point P_x^α and closed fuzzy set C are strong quasi-discoincident, i.e., $P_x^\alpha \in C'$, then, by the fuzzy regular property, there is an open fuzzy set A such that

$$P_x^\alpha \in A \subset \bar{A} \subset C',$$

and C' is a neighborhood of B'. Let $B = \bar{A}'$, then A and B are neighborhoods of P_x^α and C, respectively, and they are quasi-discoincident.

A fuzzy regular T_0 space is called a fuzzy T_3 space. It is easy to show that a fuzzy T_3 space is a fuzzy T_2 space. Indeed, let $(X, \mathbb{F})$ be a fuzzy T_3 space, give any $P_y^\beta \not\subset P_x^\alpha$, of course they are different by fuzzy T_3 property, we might as well let $A \in \mathbb{F}$ and $P_x^\alpha \in A$, $P_y^\beta \notin A$. then there exists an open fuzzy set B such that

$$P_x^\alpha \in B \subset \bar{B} \subset A.$$

Let $C = B'$, then C is a neighborhood of $P_y^{1-\beta}$ and is not quasi-coincident with B. That is, fts $(X, \mathbb{F})$ is a fuzzy T_2 space.

DEFINITION 4.6. A fts $(X, \mathbb{F})$ is fuzzy normal iff for any closed fuzzy set C and its any open neighborhood B in $(X, \mathbb{F})$ there is a fuzzy set A such that

$$C \subset A^0 \subset \bar{A} \subset B,$$

and A^0 is a neighborhood of C, B is a neighborhood of $\bar{A}$.

We can easily prove the following statement (cf. Theorem 4.9).

THEOREM 4.10. *A fts $(X, \mathbb{F})$ is fuzzy normal iff for any two strong quasi-discoincident closed sets C and D there are their open neighborhoods A and B, respectively, such that A and B are quasi-discoincident.*

DEFINITION 4.7. Let $(X, \mathbb{F})$ be a fts and D a dense subset of the interval $[0, 1]$. The family $\{O_d : d \in D\}$ of open fuzzy sets in $(X, \mathbb{F})$ is called a scale of fuzzy open sets iff $\bar{O}_{d_1} \subset O_{d_2}$ and O_{d_2} is a neighborhood of $\bar{O}_{d_1}$ for any pair $d_1 < d_2$ in D.

Now we generalize Uryshon's lemma on usual topology to the fuzzy topology.

THEOREM 4.11. *A fts $(X, \mathbb{F})$ is fuzzy normal iff for any closed fuzzy set B and its any open neighborhood A, there exists a scale of fuzzy open sets $\{O_d : d \in D\}$ such that $B \subset O_d \subset A$, for every $d \in D$.*

Proof. Sufficiency. Obvious.

Necessity. Given any closed fuzzy set B and its open neighborhood A in $(X, \mathbb{F})$. Since $(X, \mathbb{F})$ is fuzzy normal there is an open fuzzy set O_0 such that

$$B \subset O_0 \subset \bar{O}_0 \subset A,$$

moreover O_0 and A are the neighborhoods of B and $\bar{O}_0$, respectively.

Let D be the set of diadic rationals and $D_n = \{m/2^n : m < 2^n, m = 2k - 1, k = 1, 2, \ldots\}$ then $D = \bigcup_{n=1,2,\ldots} D_n$. We construct a scale of fuzzy open sets as follows: Let $O_1 = A$, we can choose an open set $O_{1/2}$ such that

$$\bar{O}_0 \subset O_{1/2} \subset \bar{O}_{1/2} \subset O_1$$

and $O_{1/2}$, O_1 are the neighborhoods of $\bar{O}_0$, $\bar{O}_{1/2}$, respectively. Suppose for every member d in $D_1 \cup D_2 \cup \cdots \cup D_n$ the O_d is already defined such that $\bar{O}_{d'} \subset O_{d''}$ and $O_{d''}$ is a neighborhood of $\bar{O}_{d'}$ for any pair $d' < d''$. For any $d \in D_{n+1}$ there are adjacent elements d' and d'' of $D_1 \cup \cdots \cup D_n$ such that $d = (d' + d'')/2$. Using the fuzzy normal we can choose an open fuzzy set O_d satisfying

$$\bar{O}_{d'} \subset O_d \subset \bar{O}_d \subset O_{d''},$$

moreover O_d, $O_{d''}$ are the neighborhoods of $\bar{O}_{d'}$, $\bar{O}_d$, respectively. By the induction, we obtain a scale of fuzzy open sets $\{O_d : d \in D\}$ such that $B \subset O_d \subset A$, moreover O_d and A are the neighborhoods of B and $\bar{O}_d$.

A fuzzy normal and fuzzy T_1 space is called a fuzzy T_4 space. Obviously, fuzzy T_4 implies fuzzy T_3.

DEFINITION 4.8. A fts $(X, \mathfrak{F})$ is fuzzy completely normal iff for any set A and its any neighborhood B satisfying the condition $P(x, A(x)) \in B$, there is an open fuzzy set O such that

$$A \subset O \subset \bar{O} \subset B,$$

moreover O is a neighborhood of A and $P(x, \bar{O}(x)) \in B$, $x \in X$.

Fuzzy sets A and B are said to be fuzzy separated iff A and $\bar{B}$, and at the same time $\bar{A}$ and B, are strong quasi-discoincident.

THEOREM 4.12. *A fts $(X, \mathfrak{F})$ is completely normal iff for any pair of fuzzy separated fuzzy sets A and B, there exist quasi-discoincident open fuzzy sets O_A and O_B being the neighborhoods of A and B, respectively.*

Proof. Sufficiency. Let fuzzy set B be a neighborhood of A satisfying the condition $P(x, \bar{A}(x)) \in B$, then $\bar{A}$ and B' and at the same time A and $\overline{B'}$ are strong quasi-discoincident. Thus the fuzzy sets A and B' have, respectively, quasi-discoincident open neighborhoods O_A and $O_{B'}$, and so $O_A \subset O'_{B'}$. Observe that $O'_{B'} \subset B$ is a closed fuzzy set, we have $\bar{O}_A \subset O'_{B'}$ and $A \subset O_A \subset \bar{O}_A \subset B$, moreover O_A is a neighborhood of A and $P(x, \bar{O}_A(x)) \in B$.

Necessity. Let fuzzy sets A and B be fuzzy separated, then $A \subset \bar{B}'$, $P(x, A(x)) \in \bar{B}'$ and $\bar{A} \subset B'$, $P(x, \bar{A}(x)) \in B'$, $x \in X$. Since $(X, \mathfrak{F})$ is completely normal there exists an open fuzzy set O such that

$$A \subset O \subset \bar{O} \subset B',$$

O is a neighborhood of A and $P(x, \bar{O}(x)) \in B'$. Let $O_A = O$ and $O_B = \bar{O}'$ then O_A and O_B are quasi-discoincident neighborhoods of A and B, respectively. Theorem 4.12 is proved.

Fuzzy completely normal T_1 space is called fuzzy T_5 space. Obviously, fuzzy T_5 space is fuzzy T_4 space.

So far, we have given a group of increasing fuzzy separation axioms: fuzzy T_i, $i = 0, 1, 2, 3, 4, 5$.

THEOREM 4.13. *Product-induced space $(X, \mathfrak{F}_{\mathscr{T} \times \theta_I})$ possesses any one of the separate properties mentioned above iff the topological space $(X, \mathscr{T})$ has corresponding properties.*

Proof. As an example, we only give the proof of fuzzy completely normal space.

Sufficiency. Suppose fuzzy sets A and B satisfying the condition of Definition 4.8, then for each $0 < \alpha < 1$ we have $\omega_{0\alpha}(A) \subset \sigma_\alpha(B^0)$, $\omega_\alpha(\bar{A}) \subset \omega_\alpha(B)$. Let $A_\alpha = \omega_\alpha(A)$, $B_\alpha = \omega_\alpha(B)$, then it is easy to show that

$A_\alpha \subset B^0_\alpha$, $\bar A_\alpha \subset B_\alpha$. Since $(X, \mathcal{T})$ is completely normal there exists an open set U_α such that

$$A_\alpha \subset U_\alpha \subset \bar U_\alpha \subset B_\alpha.$$

Let $O = \bigcup_{0<\alpha<1} N^{\alpha}_{U_\alpha}$ then $\bar O = \bigcup_{0<\alpha<1} N^{\alpha}_{U_\alpha}$ (cf. [6]). It is evident that

$$A \subset O \subset \bar O \subset B,$$

moreover O is a neighborhood of A and $P(x, \bar O(x)) \in B$. Hence $(X, \mathbb{F}_{\mathcal{T}\times\theta_I})$ is fuzzy completely normal.

Necessity. Obvious.

The fts $(Y, \mathbb{E})$ is a subspace of fts $(X, \mathbb{F})$ if Y is a subset of X and $\mathbb{E}$ is the family of all restriction of members of $\mathbb{F}$ to Y [13].

It is easy to prove that the subspace of a fuzzy completely normal space is also fuzzy completely normal. By Theorem 4.13, it is easy to show the following:

THEOREM 4.14. *Product-induced space $(X, \mathbb{F}_{\mathcal{T}\times\theta_I})$ is fuzzy completely normal iff its every subspace is fuzzy normal.*

In every fuzzy separate axiom discussed above instead of the neighborhood by S-neighborhood we shall obtain a group of S-separate axioms [14] denoted by ST_0, ST_1, S-regular...etc. Generally, the S-separation is stronger than the corresponding fuzzy separation.

V. FUZZY METRIC SPACES

For convenience, we denote $\mathbb{P}_0 = \{P^\alpha_x : x \in X,\ \alpha \in (0, 1)\}$, $\mathbb{P} = \mathbb{P}_0 \cup \{P'_x : x \in X\}$ and $\mathbb{P}_* = \mathbb{P} \cup \{P^0_x : x \in X\}$.

DEFINITION 5.1. A fuzzy metric for a set X is a mapping $e: \mathbb{P}_* \times \mathbb{P}_* \to [0, \infty)$ which is continuous for membership grade and satisfies, for all P^α_x, P^β_y, $P^\gamma_z \in \mathbb{P}_*$, the following axioms:

(M1) If $P^\beta_y \subset P^\alpha_x$, then $e(P^\alpha_x, P^\beta_y) = 0$.

(M2) $e(P^\alpha_x, P^\gamma_z) \leqslant e(P^\alpha_x, P^\beta_y) + e(P^\beta_y, P^\gamma_z)$.

(M3) $e(P^\alpha_x, P^\beta_y) = e(P^{1-\beta}_y, P^{1-\alpha}_x)$.

(M4) If $P^\beta_y \not\subset P^\alpha_x$, then $e(P^\alpha_x, P^\beta_y) > 0$.

A mapping is called continuous for membership grade iff for every fuzzy point $P^\alpha_x \in \mathbb{P}_0$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\beta - \alpha| < \delta$ implies $e(P^\alpha_x, P^\beta_y) < \varepsilon$.

This definition is a little different from that given in [15].

Essentially, the fuzzy metrics is a kind of special quasi-metrics. It is easy to verify that the mapping $e(P_x^\alpha, P_y^\beta) = \max[d(x, y), \beta - \alpha]$, which is given in [14], satisfies these axioms.

In the definition mentioned above, if (M4) (or (M3), (M4)) is omitted, then e is called a fuzzy pseudo-metrics (or fuzzy quasi-metric).

Let e be a fuzzy quasi-metrics for X, then, for any $P_x^\alpha \in \mathbb{P}_*$ and $\varepsilon > 0$, the

$$B_\varepsilon(P_x^\alpha) = \bigcup \{P_y^\beta : e(P_x^\alpha, P_y^\beta) < \varepsilon\}$$

is a fuzzy set, which is called an ε-open ball of P_x^α. Correspondingly, we call

$$B_\varepsilon(P_x^\alpha) = \bigcup \{P_y^\beta : e(P_x^\alpha, P_y^\beta) \leqslant \varepsilon\}$$

a fuzzy closed ball of P_x^α.

It is easy to verify that the family of all fuzzy open balls, corresponding to fuzzy (quasi-, pseudo-) metric e,

$$\mathbb{B} = \{B_\varepsilon(P_x^\alpha) : P_x^\alpha \in \mathbb{P}, \varepsilon > 0\}$$

forms a base of some fuzzy topology $\mathbb{F}_e$ for X. We call it fuzzy (quasi-, pseudo-) metric topology and $(X, \mathbb{F}_e)$ fuzzy (quasi-, pseudo-) metric space.

THEOREM 5.1. *Let $(X, \mathbb{F}_e)$ be a fuzzy quasi-metric space, then, for any point $P_x^\alpha \in \mathbb{P}$ and $\varepsilon > 0$, the fuzzy ε-open ball $B_\varepsilon(P_x^\alpha)$ is an open neighborhood of point P_x^α.*

Proof. It is sufficient to show $P_x^\alpha \in B_\varepsilon(P_x^\alpha)$. Indeed, if $\alpha = 1$, then $e(P_x^1, P_x^1) < \varepsilon$, i.e., $P_x^1 \subset B_\varepsilon(P_x^1)$ and hence $P_x^1 \in B_\varepsilon(P_x^\alpha)$. And if $0 < \alpha < 1$, since e is continuous for membership grade, then there is $\beta > \alpha$ such that $e(P_x^\alpha, P_x^\beta) < \varepsilon$, i.e., $P_x^\beta \subset \beta_\varepsilon(P_x^\alpha)$. Hence $P_x^\alpha \in B_\varepsilon(P_x^\alpha)$.

If we strengthen a little the condition of continuity for membership grade, just as we gave in [15], then one can further prove $B_\varepsilon(P_x^\alpha)$ to be an S-neighborhood of P_x^α.

THEOREM 5.2. *Let $(X, \mathbb{F}_e)$ be a fuzzy pseudo-metric space. If $0 < \beta = B_\varepsilon(P_x^\alpha)(y) < 1$ then $e(P_x^\alpha, P_y^\beta) = \varepsilon$.*

Proof. Let β_n, $n = 1, 2, \ldots$, be a strictly increasing sequence and convergent to β. For any $n' < n''$, by axiom (M2), we have $e(P_x^\alpha, P_y^{\beta_{n'}}) \leqslant e(P_x^\alpha, P_y^{\beta_{n''}}) < \varepsilon$. Therefore

$$\lim_{n \to \infty} e(P_x^\alpha, P_y^{\beta_n}) = \varepsilon^* \leqslant \varepsilon.$$

First we show that $e(P_x^\alpha, P_y^\beta) = \varepsilon^*$. Since e is continuous for membership grade and $\beta < 1$ and remarked axiom (M3), we have

$$\lim_{n \to \infty} e(P_y^{\beta_n}, P_y^\beta) = 0.$$

By axiom (M2), we have

$$e(P_x^\alpha, P_y^{\beta_n}) \leqslant e(P_x^\alpha, P_y^\beta) \leqslant e(P_x^\alpha, P_y^{\beta_n}) + e(P_y^{\beta_n}, P_y^\beta),$$

for any n. Hence $e(P_x^\alpha, P_y^\beta) = \varepsilon^*$.

And then, we prove $\varepsilon = \varepsilon^*$. If not, then $\varepsilon > \varepsilon^*$. Remember that e continuous for membership grade, there exists $\delta > 0$ such that $\beta^* - \beta < \delta$ implies $e(P_y^\beta, P_y^{\beta^*}) < \varepsilon - \varepsilon^*$ for $\varepsilon - \varepsilon^* > 0$ and point P_y^β. Choose β^* satisfying $\beta < \beta^* < \beta + \delta$ then

$$e(P_x^\alpha, P_y^{\beta^*}) \leqslant e(P_x^\alpha, P_y^\beta) + e(P_y^\beta, P_y^{\beta^*}) < \varepsilon$$

and so $P_y^{\beta^*} \in B_\varepsilon(P_x^\alpha)$. This fact contradicts with the definition of β. Therefore $\varepsilon = \varepsilon^*$, i.e., $e(P_x^\alpha, P_y^\beta) = \varepsilon$.

Generally, $e(P_x^\alpha, P_y^\beta) = \varepsilon$ does not imply $P_y^\beta \subset B_\varepsilon(P_x^\alpha)$ and $P_y^\beta \subset \overline{B_\varepsilon(P_x^\alpha)}$.

THEOREM 5.3. *Let* $(X, \mathbb{F}e)$ *be a fuzzy pseudo-metric space, if* $P_x^\alpha = \bigcup_{\lambda \in \Lambda} P_x^{\alpha_\lambda}$ *then* $B_\varepsilon(P_x^\alpha) = \bigcup_{\lambda \in \Lambda} B_\varepsilon(P_x^{\alpha_\lambda})$ *for any* $\varepsilon > 0$.

Proof. Given any $y \in X$ and let $\beta = B_\varepsilon(P_x^\alpha)(y)$, $\beta_\lambda = B_\varepsilon(P_x^{\alpha_\lambda})(y)$, $\lambda \in \Lambda$, we prove $\beta = \sup_{\lambda \in \Lambda} \beta_\lambda$. Suppose $\sup_{\lambda \in \Lambda} \beta_\lambda = \beta^*$, we have $\beta_\lambda < \beta$ and so $\sup_{\lambda \in \Lambda} \beta_\lambda \leqslant \beta$, i.e., $\beta^* \leqslant \beta$ because $P_y^{\beta_\lambda} \in B_\varepsilon(P_x^\alpha)$. But the inequality cannot hold. Otherwise $\beta^* < \beta$ then $e(P_x^\alpha, P_y^{\beta^*}) < \varepsilon$. On the other side, $e(P_x^\alpha, P_y^{\beta^*}) \geqslant e(P_x^\alpha, P_y^{\beta_\lambda})$ for any $\lambda \in \Lambda$. And

$$\inf_{\lambda \in \Lambda} e(P_x^{\alpha_\lambda}, P_x^\alpha) = 0,$$

because e is continuous for membership grade. By axiom (M2) and Theorem 5.2, we have

$$e(P_x^{\alpha_\lambda}, P_y^{\beta_\lambda}) - e(P_x^{\alpha_\lambda}, P_x^\alpha) \leqslant e(P_x^\alpha, P_y^{\beta_\lambda}) \leqslant e(P_x^{\alpha_\lambda}, P_y^{\beta_\lambda})$$

and

$$e(P_x^{\alpha_\lambda}, P_y^{\beta_\lambda}) = \varepsilon, \qquad \lambda \in \Lambda.$$

Hence $\sup_{\lambda \in \Lambda} e(P_x^\alpha, P_y^{\beta_\lambda}) = \varepsilon$ and so $e(P_x^\alpha, P_y^{\beta^*}) \geqslant \varepsilon$. This is in contradiction to $e(p_x^\alpha, p_y^{\beta^*}) < \varepsilon$. Therefore $\beta^* = \beta$.

THEOREM 5.4. *Let* $(X, \mathbb{F}_e)$ *be a fuzzy pseudo-metric space, then for any*

P_x^α and $\varepsilon^* > \varepsilon > 0$ we have $\overline{B_\varepsilon(P_x^\alpha)} \subset B_{\varepsilon^*}(P_x^\alpha)$ and $B_{\varepsilon^*}(P_x^\alpha)$ is a neighborhood of $\overline{B_\varepsilon(P_x^\alpha)}$.

Proof. Given any $P_z^\gamma \subset \overline{B_\varepsilon(P_x^\alpha)}$, then each neighborhood of $P_z^{1-\gamma}$ and $B_\varepsilon(P_x^\alpha)$ are quasi-coincident. If $0 < \gamma < 1$, then $B_\delta(P_z^{1-\gamma})$ is quasi-coincident with $B_\varepsilon(P_x^\alpha)$ for any $\delta > 0$. That is, there is a point $P_y^\beta \subset B_\delta(P_z^{1-\gamma})$ and $P_y^\beta \not\subset B_\varepsilon(P_x^\alpha)'$, so that $e(P_z^{1-\gamma}, P_y^\beta) \leqslant \delta$ and $e(P_y^\beta, P_x^{1-\alpha}) = e(P_x^\alpha, P_y^{1-\beta}) < \varepsilon$. Hence

$$e(P_x^\alpha, P_z^\gamma) \leqslant e(P_z^{1-\gamma}, P_y^\beta) + e(P_y^\beta, P_x^{1-\alpha}) < \varepsilon + \delta.$$

Since $\delta > 0$ is arbitrary, we have $e(P_x^\alpha, P_z^\gamma) \leqslant \varepsilon$. If $\gamma = 1$, instead of $P_z^{1-\gamma}$ by any P_z^ζ, $\zeta > 0$ in the above discussion, we have $e(P_x^\alpha, P_z^{1-\zeta}) \leqslant \varepsilon$ and so $e(P_x^\alpha, P_z^\gamma) \leqslant \varepsilon$. This shows $P_z^\gamma \in B_{\varepsilon^*}(P_x^\alpha)$ hence $\overline{B_\varepsilon(P_x^\alpha)} \subset B_{\varepsilon^*}(P_x^\alpha)$, moreover $B_{\varepsilon^*}(P_x^\alpha)$ is a neighborhood of $\overline{B_\varepsilon(P_x^\alpha)}$.

THEOREM 5.5. *In fuzzy pseudo-metric space, any fuzzy ε-closed ball $B_{\bar\varepsilon}(P_x^\alpha) = \bigcup \{P_y^\beta : e(P_x^\alpha, P_y^\beta) \leqslant \varepsilon\}$ is a closed fuzzy set and*

$$B_{\bar\varepsilon}(P_x^\alpha) = \bigcap_{\varepsilon^* > \varepsilon} B_{\varepsilon^*}(P_x^\alpha).$$

Proof. Given any $P_z^\gamma \subset \overline{B_{\bar\varepsilon}(P_x^\alpha)}$, by the proof of Theorem 5.4, we have $e(P_x^\alpha, P_z^\gamma) \leqslant \varepsilon$, i.e., $p_z^\gamma \subset B_{\bar\varepsilon}(P_x^\alpha)$. Hence $\overline{B_{\bar\varepsilon}(P_x^\alpha)} \subset B_{\bar\varepsilon}(P_x^\alpha)$ and so $B_{\bar\varepsilon}(P_x^\alpha)$ is a closed fuzzy set.

If $P_y^\beta \subset \bigcap_{\varepsilon^* > \varepsilon} B_{\varepsilon^*}(P_x^\alpha)$ then $e(P_x^\alpha, P_y^\beta) \leqslant \varepsilon^*$ for any $\varepsilon^* > \varepsilon$ so that $e(P_x^\alpha, P_y^\beta) \leqslant \varepsilon$, i.e., $P_y^\beta \subset B_{\bar\varepsilon}(P_x^\alpha)$, hence $\bigcap_{\varepsilon^* > \varepsilon} B_{\varepsilon^*}(P_x^\alpha) \subset B_{\bar\varepsilon}(P_x^\alpha)$. Thus we have

$$B_{\bar\varepsilon}(P_x^\alpha) = \bigcap_{\varepsilon^* > \varepsilon} B_{\varepsilon^*}(P_x^\alpha).$$

Since $B_\varepsilon(P_x^\alpha) \subset B_{\bar\varepsilon}(P_x^\alpha)$ and $B_{\bar\varepsilon}(P_x^\alpha)$ is a closed fuzzy set, we have

$$\overline{B_\varepsilon(P_x^\alpha)} \subset B_{\bar\varepsilon}(P_x^\alpha),$$

but they may be different.

THEOREM 5.6. *Every fuzzy pseudo-metric space is a fuzzy C_I space.*

Proof. Obvious.

THEOREM 5.7. *Every fuzzy pseudo-metric space $(X, \mathbb{F}_e)$ is fuzzy regular.*

Proof. Given any open fuzzy set A and point $P_x^\alpha \in A$, by Theorems 5.1 and 5.4, we have

$$P_x^\alpha \in B_{\varepsilon/2}(P_x^\alpha) \subset \overline{B_{\varepsilon/2}(P_x^\alpha)} \subset B_\varepsilon(P_x^\alpha) \subset A,$$

moreover $B_{\varepsilon/2}(P_x^\alpha)$ and A are the neighborhoods of P_x^α and $\overline{B_{\varepsilon/2}(P_x^\alpha)}$, respectively. Hence $(X, \mathfrak{T}_e)$ is fuzzy regular.

THEOREM 5.8. *Every fuzzy pseudo-metric space $(X, \mathfrak{F}_e)$ is fuzzy completely normal and so also is fuzzy normal.*

Proof. Let A be a fuzzy set in $(X, \mathfrak{F}_e)$ and B its neighborhood satisfying condition $P(x, A(x)) \in B$, $x \in X$. We must prove that there is an open fuzzy set O such that

$$A \subset O \subset \bar{O} \subset B,$$

moreover O is a neighborhood of A and $P(x, \bar{O}(x)) \in B$, $x \in X$.

Since B is a neighborhood of A, there is an $\varepsilon = \varepsilon(P_x^\alpha)$ such that $B_\varepsilon(P_x^\alpha) \subset B^0$ for any point $P_x^\alpha \in A$. Let $O_A = \bigcup \{B_{\varepsilon/2}(P_x^\alpha): P_x^\alpha \in A, \varepsilon = \varepsilon(P_x^\alpha)\}$, then O_A is an open neighborhood of A. Since $P(x, \bar{A}(x)) \in B$, $x \in X$, the $\bar{A}'$ is an open neighborhood of B', and so we can define an open neighborhood $O_{B'}$ of B' similarly.

Now we prove the fuzzy sets O_A and $O_{B'}$ are quasi-discoincident. In fact, if O_A and $O_{B'}$ are quasi-coincident, then there exists a fuzzy point $P_z^\gamma \in A$ such that P_z^γ and $O_{B'}$ are quasi-coincident, i.e., $P_z^{1-\gamma} \in O_{B'}$. That is, there are points $P_x^\alpha \in A$ and $P_y^\beta \not\subset B$ satisfying $e(P_x^\alpha, P_z^\gamma) < \varepsilon(P_x^\alpha)/2$ and $e(P_z^{1-\beta}, P_z^{1-\gamma}) < e(P_x^\alpha)/2$. By axioms (M2) and (M3), we have

$$e(P_x^\alpha, P_y^\beta) \leqslant e(P_x^\alpha, P_z^\gamma) + e(P_z^\gamma, P_y^\beta) < \max[\varepsilon(P_x^\alpha), \varepsilon(P_y^{1-\beta})].$$

On the other side, since $P_x^\alpha \in A$, $P_y^\beta \not\subset B$ then $e(P_x^\alpha, P_y^\beta) \geqslant \varepsilon(P_x^\alpha)$, $e(P_y^{1-\beta}, P_x^{1-\alpha}) \geqslant \varepsilon(P_y^{1-\beta})$ and so we have

$$e(P_x^\alpha, P_y^\beta) \geqslant \max[\varepsilon(P_x^\alpha), \varepsilon(P_y^{1-\beta})].$$

This is a contradiction. So that O_A and $O_{B'}$ are quasi-discoincident, i.e., $O_A \subset O'_{B'}$, hence $A \subset O_A \subset O'_{B'} \subset B$, $P(x, \overline{O'_{B'}}(x)) \in B$, $x \in X$. Remarked that $O'_{B'}$ is a closed fuzzy set, we have

$$A \subset O_A \subset \bar{O}_A \subset B,$$

moreover O_A is a neighborhood of A and $P(x, \bar{O}_A(x)) \in B$, $x \in X$.

THEOREM 5.9. *If $(X, \mathfrak{F}_e)$ is a fuzzy pseudo-metric space, then the initial topological space $(X, \iota(\mathfrak{F}_e))$ is a pseudo-metric space.*

Proof. For any $\alpha \in [0, 1]$, we define a mapping $d_\alpha: X \times X \to [0, \infty)$

$$d_\alpha(x, y) = e(P_x^\alpha, P_y^\alpha) + e(P_x^{1-\alpha}, P_y^{1-\alpha}),$$

obviously d_α is a pseudo-metrics for X.

Let $\{\alpha_i : i = 1, 2, \ldots,\}$ be a dense subset of $[0, 1]$ and $\mathscr{T}_i$ the metric topologies for X induced by d_{α_i}. We can prove that

$$\iota(\mathbb{F}_c) = \sup_{i=1,2,\ldots,} \mathscr{T}_i,$$

hence $(X, \iota(\mathbb{F}_c))$ is a pseudo-metric space.

In fact, let $U_i(x, \varepsilon) = \{y : d_{\alpha_i}(x, y) < \varepsilon\}$ then one can easily show that

$$U_i(x, \varepsilon) \subset \sigma_{\alpha_i}[B_\varepsilon(P_x^{\alpha_i})] \cap \sigma_{1-\alpha_i}[B_\varepsilon(P_x^{1-\alpha_i})]$$

and

$$U_i(x, \varepsilon) \supset \sigma_{\alpha_i}[B_{\varepsilon/2}(P_x^{\alpha_i})] \cap \sigma_{1-\alpha_i}[B_{\varepsilon/2}(P_x^{1-\alpha_i})].$$

Therefore $\{U_i(x, \varepsilon) : x \in X, \varepsilon > 0, i = 1, 2, \ldots,\}$ and $\{\sigma_{\beta_i}[B_\varepsilon(P_x^{\beta_i})] : \beta_i = \alpha_i$ or $1 - \alpha_i$, $i = 1, 2, \ldots$, $x \in X$, $\varepsilon > 0\}$ are equivalent. Moreover they are all the subbases of $\iota(\mathbb{F}_c)$, because $\{\alpha_i : i = 1, 2, \ldots,\}$ is dense in $[0, 1]$.

From the proof mentioned above, we can easily see that, if $(X, \mathbb{F}_c)$ is a fuzzy pseudo-metric space, then $(X, \iota_{1/2}(\mathbb{F}_c))$ is a pseudo-metric space.

THEOREM 5.10. *Every fuzzy metric space $(X, \mathbb{F}_c)$ is a fuzzy T_1 space.*

Proof. Let $A = (P_y^\beta)'$ be the complement of P_y^β, then

$$\begin{aligned} A(x) &= 1 - \beta, & \chi &= y, \\ &= 1, & \chi &\neq y. \end{aligned}$$

Given any point $P_x^\alpha \in A$, then $P_y^{1-\beta} \not\subset P_x^\alpha$. By axiom (M4), we have $e(P_x^\alpha, P_y^{1-\beta}) > 0$. Hence there exists $\varepsilon > 0$ such that $P_y^{1-\beta} \not\subset B_\varepsilon(P_x^\alpha)$ and so

$$A = \bigcup_{P_x^\alpha \in A} B_\varepsilon(P_x^\alpha).$$

Therefore A is an open fuzzy set. It follows that any point P_y^β is a closed fuzzy set. By Theorem 4.3, the fuzzy metric space $(X, \mathbb{F}_c)$ is a fuzzy T_1 space.

THEOREM 5.11. *Every fuzzy metric space $(X, \mathbb{F}_c)$ is fuzzy T_5,* and so is fuzzy T_i space, $i = 0, 1, 2, 3, 4$.

Proof. Obvious from Theorems 5.8 and 5.10.

Let $(X, \mathbb{F}_c)$ be a fuzzy metric space. If each ε-open ball $B_\varepsilon(P_x^\alpha)$ in $(X, \mathbb{F}_c)$ is an S-neighborhood of point P_x^α, then we call $(X, \mathbb{F}_c)$ S-metric space.

VI. Fuzzy Uniform Spaces

In [17] we introduced a kind of fuzzy uniformity. Consider a class of nonvoid family $\mathscr{C}$ of subsets U of product set $\mathbb{P}_* \times \mathbb{P}_*$ satisfying the following conditions:

(c1) If $(P_x^\alpha, P_y^\beta) \in U$, then $(P_x^\alpha, P_y^\gamma) \in U$ for every $0 \leqslant \gamma < \beta$.

(c2) If $P_x^\alpha = \bigcup_\lambda P_x^{\alpha_\lambda}$ then $P_y^{\beta^*} = \bigcup_\lambda P_y^{\beta_\lambda^*}$ for every $y \in X$, where $\beta^* = \sup\{\beta : (P_x^\alpha, P_y^\beta) \in U\}$, $\beta_\lambda^* = \sup\{\beta_\lambda : (P_x^{\alpha_\lambda}, P_y^{\beta_\lambda}) \in U\}$.

For any $U \in \mathscr{C}$, $A \in I^X$ and $P_x^\alpha \in \mathbb{P}$ we define

$$U(P_x^\alpha) = \bigcup \{P_y^\beta : (P_x^\alpha, P_y^\beta) \in U\}$$

and

$$U(A) = \bigcup \{P_y^\beta : (P_x^\alpha, P_y^\beta) \in U, P_x^\alpha \subset A\}.$$

Obviously, $U(P_x^\alpha) \in I^X$, $U(A) \in I^X$ and

$$U\left(\bigcup_\lambda P_x^{\alpha_\lambda}\right) = \bigcup_\lambda U(P_x^{\alpha_\lambda}),$$

$$U(A) = \bigcup_{x \in X} U(P(x, A(x))).$$

Generally, for a family $\{A_\lambda : A_\lambda \in I^X, \lambda \in \Lambda\}$ we can prove that

$$U\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right) = \bigcup_{\lambda \in \Lambda} U(A_\lambda).$$

For any U, $V \in \mathscr{C}$ the composition operator $\circ$ is defined by

$$V \circ U = \{(P_x^\alpha, P_z^\gamma) : \exists P_y^\beta \in \mathbb{P}_*, \text{ s.t. } (P_x^\alpha, P_y^\beta) \in U, (P_y^\beta, P_z^\gamma) \in V\}.$$

Obviously, $V \circ U(A) = V[U(A)]$ for any $A \in I^X$.

The inverse of $U \in \mathscr{C}$ is defined by

$$U^{-1} = \{(P_x^\alpha, P_y^\beta) : (P_y^{1-\beta}, P_x^{1-\alpha}) \in U\}.$$

If $U^{-1} = U$, then U is said to be symmetric. It is easy to verify that

$$(V \circ U)^{-1} = U^{-1} \circ V^{-1}.$$

Definition 6.1. The nonvoid family $\mathscr{U} \subset \mathscr{C}$ of subsets of $\mathbb{P}_* \times \mathbb{P}_*$ is called a fuzzy uniformity for X iff the following axioms are satisfied:

(U1) If $U \in \mathscr{U}$ then $(P_x^\alpha, P_x^\alpha) \in U$ for any $P_x^\alpha \in \mathbb{P}_*$, moreover, if $P_x^\alpha \in \mathbb{P}_0$ then there exists $\delta = \delta(P_x^\alpha) > 0$ such that $(P_x^\alpha, P_x^{\alpha+\delta}) \in U$.

(U2) If $U \in \mathscr{U}$, then there is a $V \in \mathscr{U}$ such that $V \circ V \subset U$.

(U3) If $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$.

(U4) If U, $V \in \mathscr{U}$, then $U \cap V \in \mathscr{U}$.

(U5) If $U \in \mathscr{U}$ and $U \subset V$ then $V \in \mathscr{U}$.

This definition is different from all definitions given in [17] and [18].

Next we shall see that any fuzzy metric space defines a compatible fuzzy uniformity.

The nonvoid subfamily $\mathscr{B}$ of a fuzzy uniformity $\mathscr{U}$ is called a base of $\mathscr{U}$ iff for any $U \in \mathscr{U}$ there is a $V \in \mathscr{B}$ such that $V \subset U$.

The nonvoid subfamily $\mathscr{B}$ of $\mathscr{C}$ is a base of some fuzzy uniformity iff it satisfies axioms (U1)–(U3) and

(U4)′ If U, $V \in \mathscr{B}$ then there is a $W \in \mathscr{B}$ such that $W \subset U \cap V$.

All symmetric members of a fuzzy uniformity $\mathscr{U}$ form a base of $\mathscr{U}$.

A nonvoid subfamily S of a fuzzy uniformity $\mathscr{U}$ is called a subbase of $\mathscr{U}$ iff all finite intersections of members of S form a base of $\mathscr{U}$.

If the nonvoid subfamily $S \in \mathscr{C}$ satisfies axioms (U1)–(U3), then it forms a subbase for some fuzzy uniformity for X. In particular, the union of any collection of fuzzy uniformities for X is the subbase for a fuzzy uniformity for X.

Definition 6.2. Let $\mathscr{U}$ be a fuzzy uniformity for X, then a family of fuzzy sets

$$\mathbb{F}_{\mathscr{U}} = \{A : \forall P_x^\alpha \in A, \exists U \in \mathscr{U}, \text{ s.t. } U(P_x^\alpha) \subset A\}$$

is a fuzzy topology of X, which is called fuzzy uniform topology.

It is easy to verify that $\mathbb{F}_{\mathscr{U}}$ is really a fuzzy topology for X.

Theorem 6.1. *Let A be a fuzzy set in fuzzy uniform topological space* $(X, \mathbb{F}_{\mathscr{U}})$, then its interior

$$A^0 = \bigcup \{P_x^\alpha : \exists U \in \mathscr{U}, \text{ s.t. } U(P_x^\alpha) \subset A\}.$$

Proof. Let $A^* = \bigcup\{P_x^\alpha : \exists U \in \mathscr{U}, \text{ s.t. } U(P_x^\alpha) \subset A\}$, then $A^* \subset A$. Since for any open fuzzy set $B \subset A$ implies $B \subset A^*$, it is sufficient to prove $A^* \in \mathbb{F}_{\mathscr{U}}$. Indeed, given any point $P_x^\alpha \in A^*$, there is a $U \in \mathscr{U}$ such that $U(P_x^\alpha) \subset A$. By axiom (U2), we can choose $V \in \mathscr{U}$ such that $V \circ V \subset U$. If $P_y^\beta \subset V(P_x^\alpha)$, then $V(P_y^\beta) \subset V \circ V(P_x^\alpha) \subset U(P_x^\alpha) \subset A$, that is, $P_y^\beta \subset A^*$. Hence $V(P_x^\alpha) \subset A$ and so $A^* \in \mathbb{F}_{\mathscr{U}}$.

THEOREM 6.2. *Let $(X, \mathbb{F}_{\mathscr{U}})$ be a fuzzy uniform topological space and give any $P_x^\alpha \in \mathbb{P}$, $U \in \mathscr{U}$, then $U(P_x^\alpha)$ is a neighborhood of P_x^α.*

Proof. Choose $V \in \mathscr{U}$ such that $V \circ V \subset U$. By Theorem 6.1 and axiom (U1), we have $V(P_x^\alpha) \subset U(P_x^\alpha)^0$ and $P_x^\alpha \in V(P_x^\alpha) \subset U(P_x^\alpha)^0$. Hence $U(P_x^\alpha)^0$ is a neighborhood of point P_x^α.

Generally, $U(P_x^\alpha)$ is not an S-neighborhood of P_x^α. If $U(P_x^\alpha)$ is an S-neighborhood of P_x^α for every $U \in \mathscr{U}$ and $P_x^\alpha \in \mathbb{P}$, then $\mathscr{U}$ is said to have property S and $(X, \mathbb{F}_{\mathscr{U}})$ is an S-uniform topological space.

THEOREM 6.3. *Let $(X, \mathbb{F}_{\mathscr{U}})$ be a fuzzy uniform topological space and A, $B \in I^X$. If $U(A) \subset B$ for some $U \in \mathscr{U}$, then $A \subset B^0$ and B^0 is a neighborhood of A.*

Proof. For any point $P_x^\alpha \subset A$, $U(P_x^\alpha)$ is a neighborhood of P_x^α, i.e., $P_x^\alpha \in U(P_x^\alpha)^0 \subset U(A)^0 \subset B^0$, hence $A \subset B^0$ and B^0 is a neighborhood of A.

THEOREM 6.4. *Let $(X, \mathbb{F}_{\mathscr{U}})$ be a fuzzy uniform topological space and A, $B \in I^X$. If $U(A) \subset B$ for some $U \in \mathscr{U}$ then $\bar{A} \subset B^0$ and B^0 is a neighborhood of $\bar{A}$.*

Proof. Given any point $P_x^\alpha \subset \bar{A}$, then every neighborhood of $P_x^{1-\alpha}$ is quasi-coincident with A. Let $V \in \mathscr{U}$ be symmetric and $V \circ V \subset U$. If $0 < \alpha < 1$, then $V(P_x^{1-\alpha})$ and A are quasi-coincident. Hence there is $P_y^\beta \in A$ such that $(P_x^{1-\alpha}, P_y^{1-\beta}) \in V$ and so $(P_y^\beta, P_x^\alpha) \in V$, i.e., $P_x^\alpha \subset V(P_y^\beta) \subset V(A)$. If $\alpha = 1$, then for any $0 < \varepsilon < 1$, by the above discussion, we have $P_x^{1-\varepsilon} \subset U(A)$. Note that $\varepsilon > 0$ is arbitrary, so that $P_x^1 \subset V(A)$. Therefore $\bar{A} \subset V(A)$. Recall $V[V(A)] \subset U(A) \subset B$, by Theorem 6.3, we have $V(A) \subset B^0$ and B^0 is a neighborhood of $V(A)$. Hence $\bar{A} \subset B^0$ and B^0 is a neighborhood of $\bar{A}$.

THEOREM 6.5. *Let $(X, \mathbb{F}_{\mathscr{U}})$ be a fuzzy uniform topological space and U, V, $W \in \mathscr{U}$. If $W \circ V \subset U$, then $\overline{V(A)} \subset U(A)^0$ and $U(A)^0$ is a neighborhood of $\overline{V(A)}$ for any $A \in I^X$.*

Proof. Obvious from Theorem 6.4.

THEOREM 6.6. *If $(X, \mathbb{F}_{\mathscr{U}})$ is a fuzzy uniform topological space, then $(X, \iota(\mathbb{F}_{\mathscr{U}}))$ is a uniform topological space.*

Proof. Let D be the set of all rationals in $(0, 1)$. For any $d \in D$, $U \in \mathscr{U}$ and $x \in X$, by Theorem 6.2 and axiom (U1), we have $U(P_x^d)^0(x) > d$, hence $U^d(x) = \sigma_\alpha(U(P_x^\alpha))$ is a neighborhood of x in $(X, \iota(\mathbb{F}_{\mathscr{U}}))$.

Consider the subset of $X \times X$

$$U^d = \{(x, y) : y \in U^d(x), x \in X\}$$

then

$$\mathscr{U}_D = \{U^d : U \in \mathscr{U}, d \in D\}$$

forms a subbase for some uniformity. Indeed, we easily verify that for $\mathscr{U}_D$ the axioms (a)-(c) [20] are satisfied.

(a) If $U^d \in \mathscr{U}_D$, then $\Delta \subset U^d$.

By the definition of U, obvious.

(b) For each $U^d \in \mathscr{U}_D$, then set $(U^d)^{-1}$ contains a member of $\mathscr{U}_D$.

Let $V \in \mathscr{U}$ and $V \circ V \subset U$, then

$$\begin{aligned}(V^{-1})^{1-d}(x) &= \sigma_{1-d}[V^{-1}(P_x^{1-d})] \\ &= \sigma_{1-d}\{P_y^{1-\beta} : (P_y^\beta, P_x^d) \in V\} \\ &= \{y : 1-\beta > 1-d, (P_y^\beta, P_x^d) \in V\} \\ &= \{y : \beta < d, P_x^d \subset V(P_y^\beta)\} \\ &\subset \{y : P_x^d \subset V(P_y^d)\} \\ &\subset \{y : P_x^d \in U(p_y^d)\} \\ &= \{y : x \in U^d(y)\} = (U^d)^{-1}(x).\end{aligned}$$

Hence

$$(U^d)^{-1} \supset (V^{-1})^{1-d} \in \mathscr{U}_D.$$

(c) For each $U^d \in \mathscr{U}_D$ there is $V^d \in \mathscr{U}_D$ such that $V^d \circ V^d \subset U^d$.

Let $U \in \mathscr{U}$, then there is $V \in \mathscr{U}$ such that $V \circ V \subset U$ and so $(V \circ V)^d \subset U^d$. It is easy to prove that $V^d \circ V^d \subset (V \circ V)^d$ hence $V^d \circ V^d \subset U^d$.

The uniformity $\iota(\mathscr{U})$ generated by $\mathscr{U}_D$ induces a uniform topology $\mathscr{T}_{\mathscr{U}}$. We shall prove $\mathscr{T}_{\mathscr{U}} = \iota(\mathbb{F}_{\mathscr{U}})$.

By general topology [20], we know $\{U^d(x) : U^d \in \mathscr{U}_D\}$ forming a subbase for the neighborhood system of x relative to $\mathscr{T}_{\mathscr{U}}$. Remember $U^d(x) = \sigma_\alpha[U(P_x^\alpha)]$, the strong d^*-cut $\sigma_{d^*}[U(P_x^d)]$ is a neighborhood of x and $\sigma_{d^*}[U(P_x^\alpha)] \subset U^d(x)$ for any $d^* \in D$, $d < d^* < U(P_x^\alpha)(x)$. Hence all $\sigma_{d^*}[U(P_x^\alpha)]$, $U \in \mathscr{U}$, also constitutes a subbase for the neighborhood system of x relative to $\mathscr{T}_{\mathscr{U}}$. On the other hand $\{\sigma_\alpha[U(P_x^\alpha)] : \alpha \in (0, 1), U \in \mathscr{U}\}$ is a subbase for the neighborhood system of x relative to $\iota(\mathbb{F}_{\mathscr{U}})$. Therefore we need only to show that each $\sigma_\alpha[U(P_x^\alpha)]$ contains some $\sigma_{d^*}[U(P_x^d)]$ in order to prove $\mathscr{T}_{\mathscr{U}} = \iota(\mathbb{F}_{\mathscr{U}})$. Indeed, by (C2) and the density of D, we can choose $d \in D$ such that $d < \alpha$, $U(P_x^d)(x) > \alpha$. Hence there is $d^* \in D$, $\alpha < d^* < U(P_x^\alpha)(x)$ such that $\sigma_{d^*}[U(P_x^d)] \subset \sigma_\alpha[U(P_x^\alpha)]$.

From the proof mentioned above we can easily see that:

If $\mathscr{U}$ has countable, base, then $\mathscr{U}_D$ has countable base and so $\iota(\mathscr{U})$ also has countable base.

The topological space $(X, \iota_{1,2}(\mathbb{F}_{\mathscr{U}}))$ is a uniform topological space.

THEOREM 6.7. *Let $\mathscr{U}$ be a fuzzy uniformity for X, then the closure*

$$\bar{A} = \bigcap_{U \in \mathscr{U}} U(A)$$

for any $A \in I^X$.

Proof. First, we prove $\bar{A} \subset \bigcap_{U \in \mathscr{U}} U(A)$. Given any point $P_x^\alpha \subset \bar{A}$ then each neighborhood of $P_x^{1-\alpha}$ is quasi-coincident with A. Let $U \in \mathscr{U}$ be symmetrical. If $0 < \alpha < 1$, then $U(P_x^{1-\alpha})$ and A are quasi-coincident, hence there is $P_y^\beta \in A$ such that $P_y^{1-\beta} \subset U(P_x^{1-\alpha})$ and so $P_x^\alpha \subset U(P_y^\beta) \subset U(A)$. If $\alpha = 1$, by the above discussion, we have $P_x^{1-\varepsilon} \subset U(A)$ for any $0 < \varepsilon < 1$ and so $P_x^1 \subset U(A)$. Since all symmetric members of $\mathscr{U}$ form its base, therefore $P_x^\alpha \subset \bigcap_{U \in \mathscr{U}} U(A)$ and so $\bar{A} \subset \bigcap_{U \in \mathscr{U}} U(A)$.

Next, we prove $\bigcap_{U \in \mathscr{U}} U(A) \subset \bar{A}$. If $P_x^\alpha \subset \bigcap_{U \in \mathscr{U}} U(A)$, then there is a point $P_y^\beta \subset A$ such that $(P_y^\beta, P_x^\alpha) \in U$ for any symmetrical $U \in \mathscr{U}$, and so $(P_x^{1-\alpha}, P_y^{1-\beta}) \in U$. If $0 < \alpha < 1$, then $U(P_x^{1-\alpha})$ and P_y^β are quasi-coincident. If $\alpha = 1$, then $U(P_x^\varepsilon)$ and P_y^β are quasi-coincident for all $0 < \varepsilon < 1$. Therefore every neighborhood of $P_x^{1-\alpha}$ is quasi-coincident with A, that is, $\bigcap_{U \in \mathscr{U}} U(A) \subset \bar{A}$.

THEOREM 6.8. *A fuzzy uniform topological space $(X, \mathbb{F}_{\mathscr{U}})$ is a fuzzy T_1 space iff for any point $P_x^\alpha \in \mathbb{P}$ we have*

$$\bigcap_{U \in \mathscr{U}} U(P_x^\alpha) = P_x^\alpha.$$

Proof. If $(X, \mathbb{F}_{\mathscr{U}})$ is a fuzzy T_1 space, then $\overline{P_x^\alpha} = p_x^\alpha$. By Theorem 6.7, we have $\bigcap_{U \in \mathscr{U}} U(P_x^\alpha) = \overline{P_x^\alpha} = P_x^\alpha$. On the contrary, if $\bigcap_{U \in \mathscr{U}} U(P_x^\alpha) = P_x^\alpha$, then $\overline{P_x^\alpha} = P_x^\alpha$, that is, $(X, \mathbb{F}_{\mathscr{U}})$ is a fuzzy T_1 space.

A fuzzy uniformity $\mathscr{U}$ is called separate iff $\bigcap_{U \in \mathscr{U}} U = \{(P_x^\alpha, P_x^\alpha): P_x^\alpha \in \mathbb{P}_*]$. From Theorem 6.8, $\mathscr{U}$ is separate iff $\mathbb{F}_{\mathscr{U}}$ is fuzzy T_1 topology.

VII. CHARACTERIZATION OF FUZZY UNIFORM SPACES

In this section, first, we introduce the fuzzy separation axiom (T) [21] and show that it is the character of fuzzy uniform space. Then we give a fuzzy metrization theorem and prove that every uniformity is characterized by a family of fuzzy pseudo-metrics.

DEFINITION 7.1. A fts $(X, \mathbb{F})$ is called a fuzzy (T) space [21] iff for any $A \in \mathbb{F}$ and any point $P_\xi^\gamma \in A$, there exists a scale of fuzzy open sets $\{O_d: d \in D\}$ such that $P_\xi^\gamma \in O_d \subset A$ for every $d \in D$.

It is easy to show that the particular choice of the index set D is of no importance for fuzzy (T) space.

Let $\{O_d: d \in D\}$ be a scale of fuzzy open sets and $O_d^* = \bar{O}'_{1-d}$. It is easy to prove that $\{O_d^*: d \in D\}$ is also a scale of fuzzy open sets which is called a dual scale of $\{O_d: d \in D\}$.

THEOREM 7.1. *A fts $(X, \mathbb{F})$ is a fuzzy uniform topological space iff it is a fuzzy* (T) *space.*

Proof. Sufficiency. Suppose $(X, \mathbb{F})$ is a fuzzy (T) space then for every fuzzy set $A \in \mathbb{F}$ and point $P_\xi^\gamma \in A$ there exists a scale of fuzzy open sets $\{O_d: d \in D\}$ such that $P_\xi^\gamma \in O_d \subset A$ for every $d \in D$.

Consider the function $f: \mathbb{P}_* \to [0, 1]$

$$\begin{aligned} f(P_x^\alpha) &= \inf\{d: P_x^\alpha \in O_d\}, && P_x^\alpha \in O_1, \\ &= 1, && P_x^\alpha \notin O_1, \end{aligned}$$

and the function $e: \mathbb{P}_* \times \mathbb{P}_* \to [0, 1]$

$$e(P_x^\alpha, P_y^\beta) = \max[f(P_y^\beta) - f(P_x^\alpha), 0].$$

It is easy to verify that e is a fuzzy quasi-metric for X. Similarly, for the dual scale $\{O_d^*: d \in D\}$ we can define the corresponding function f^* and e^*. And e^* is also a fuzzy quasi-metric for X.

We now prove that

$$e^*(P_x^\alpha, P_y^\beta) = e(P_y^{1-\beta}, P_x^{1-\alpha}),$$

for every $P_x^\alpha, P_y^\beta \in \mathbb{P}_*$. In fact, if $P_x^\alpha \in O_1^*$, then

$$\begin{aligned} f^*(P_x^\alpha) &= \inf\{d: P_x^\alpha \in O_d^*\} \\ &= \inf\{d: P_x^\alpha \in \bar{O}'_{1-d}\} \\ &= \inf\{d: P_x^{1-\alpha} \not\subset \bar{O}_{1-d}\} \\ &= 1 - \sup\{1-d: P_x^{1-\alpha} \not\subset \bar{O}_{1-d}\} \\ &= 1 - \inf\{1-d: P_x^{1-\alpha} \in O_{1-d}\} = 1 - f(P_x^{1-\alpha}), \end{aligned}$$

and if $P_x^\alpha \notin O_1^*$, then $f^*(P_x^\alpha) = 1$ and $P_x^{1-\alpha} \subset \bar{O}_0$, $f(P_x^{1-\alpha}) = 0$. Hence $f^*(P_x^\alpha) = 1 - f(P_x^{1-\alpha})$. Thus

$$\begin{aligned} e^*(P_x^\alpha, P_y^\beta) &= \max[f^*(P_y^\beta) - f^*(P_x^\alpha), 0] \\ &= \max[f(P_x^{1-\alpha}) - f(P_y^{1-\beta}), 0] = e(P_y^{1-\beta}, P_x^{1-\alpha}). \end{aligned}$$

For every $\varepsilon > 0$, we consider the subset of $\mathcal{P}_* \times \mathcal{P}_*$

$$B_\varepsilon = \{(P_x^\alpha, P_y^\beta) : e(P_x^\alpha, P_y^\beta) < \varepsilon\},$$

and

$$B_\varepsilon^* = \{(P_x^\alpha, P_y^\beta) : e^*(P_x^\alpha, P_y^\beta) < \varepsilon\}.$$

It is easy to see that $B_\varepsilon^{-1} = B_\varepsilon^*$. Next, we shall prove that all B_ε, B_ε^*, $\varepsilon > 0$, form a base of some fuzzy uniformity for X.

First, verify B_ε and B_ε^* satisfying conditions (C1), (C2).

(C1) If $(P_x^\alpha, P_y^\beta) \in B_\varepsilon$ and $0 \leqslant \beta' < \beta$, then $e(P_x^\alpha, P_y^{\beta'}) \leqslant e(P_x^\alpha, P_y^\beta) < \varepsilon$, hence $(P_x^\alpha, P_y^{\beta'}) \in B_\varepsilon$.

(C2) Let $P_x^\alpha = \bigcup_\lambda P_x^{\alpha_\lambda}$, we have to prove $P_y^{\beta^*} = \bigcup_\lambda P_y^{\beta_\lambda^*}$, denoted by $\sup \beta_\lambda^* = \beta^0$; if $\beta^0 > \beta^*$, then there is a λ such that $\beta_\lambda^* > \beta^*$ and so there is $\beta_\lambda > \beta^*$. Hence we have $e(P_x^\alpha, P_y^{\beta^*}) \leqslant e(P_x^\alpha, P_y^{\beta_\lambda}) < \varepsilon$. This contradicts with the definition of β^*. If $\beta^0 < \beta^*$, then $e(P_x^\alpha, P_y^{\beta^0}) < \varepsilon$. On the other hand, for any λ, $e(P_x^\alpha, P_y^{\beta^0}) \geqslant e(P_x^\alpha, P_y^{\beta_\lambda^*})$. By axiom (M2), we have

$$e(P_x^{\alpha_\lambda}, P_y^{\beta_\lambda^*}) - e(P_x^{\alpha_\lambda}, P_x^\alpha) \leqslant e(P_x^\alpha, P_y^{\beta_\lambda^*}) \leqslant e(P_x^{\alpha_\lambda}, P_y^{\beta_\lambda^*}).$$

From the definition of e, it follows immediately that

$$\inf_\lambda e(P_x^{\alpha_\lambda}, P_x^\alpha) = 0$$

and for every λ, $\beta_\lambda^* > 0$, we have

$$e(P_x^{\alpha_\lambda}, P_y^{\beta_\lambda^*}) = \varepsilon.$$

Therefore

$$\sup_\lambda e(P_x^\alpha, P_y^{\beta_\lambda^*}) = \varepsilon$$

and so $e(P_x^\alpha, P_y^{\beta_0}) \geqslant \varepsilon$, which is in contradiction with the hypothesis $e(P_x^\alpha, P_y^{\beta^0}) < 0$, so that $\beta^* = \beta^0 = \sup_\lambda \beta_\lambda^*$.

Similarly, we can prove B^* satisfying (C1) and (C2).

Afterward, we verify B_ε, B_ε^*, $\varepsilon > 0$, satisfying axioms (U1)–(U3) and (U4)'.

(U1) Obvious.

(U2) For every B_ε, we have $B_{\varepsilon/2} \circ B_{\varepsilon/2} \subset B_\varepsilon$; indeed, if $(P_x^\alpha, P_z^\gamma) \in B_{\varepsilon/2} \circ B_{\varepsilon/2}$, then there is P_y^β such that $(P_x^\alpha, P_y^\beta) \in B_{\varepsilon/2}$, $(P_y^\beta, P_z^\gamma) \in B_{\varepsilon/2}$, hence $e(P_x^\alpha, P_z^\gamma) \leqslant e(P_x^\alpha, P_y^\beta) + e(P_y^\beta, P_z^\gamma) < \varepsilon$, i.e., $(P_x^\alpha, P_z^\gamma) \in B_\varepsilon$, so $B_{\varepsilon/2} \circ B_{\varepsilon/2} \subset B_\varepsilon$.

(U3) Obvious.

(U4)′ For any B_{ε_1} and B_{ε_2} we have $B_\varepsilon \subset B_{\varepsilon_1} \cap B_{\varepsilon_2}$, $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. For B^*_ε, a similar result can be obtained.

Thus all B_ε, B^*_ε, $\varepsilon > 0$. form a base of some fuzzy uniformity $\mathscr{U}(A, \xi, \gamma)$ for X.

For all pair (P^γ_ξ, A), $P^\gamma_\xi \in A$, the union of $\mathscr{U}(A, \xi, \gamma)$ forms a subbase of a fuzzy uniformity. This fuzzy uniformity

$$\mathscr{U} = \sup\{\mathscr{U}(A, \xi, \gamma): A \in \mathbb{F}, P^\gamma_\xi \in A, P^\gamma_\xi \in \mathbb{P}\}$$

is exactly what we want to find. Now we need only to show $\mathbb{F}_{\mathscr{U}} = \mathbb{F}$, where $\mathbb{F}_{\mathscr{U}}$ is a fuzzy topology for X induced by $\mathscr{U}$.

(a) $\mathbb{F} \subset \mathbb{F}_{\mathscr{U}}$.

Indeed, for every $A \in \mathbb{F}$ we have

$$A = \bigcup \{O_d: P^\gamma_\xi \in O_d \subset A, P^\gamma_\xi \in \mathbb{P}\}.$$

Given any $d \in D$ and $0 < \varepsilon < d$, we can find a point P^α_x such that $f(P^\alpha_x) = d - \varepsilon$, thus

$$B_\varepsilon(P^\alpha_x) = \{P^\beta_y: \max[f(P^\beta_y) - f(P^\alpha_x), 0] < \varepsilon\}$$

$$= \bigcup \{O_{d'}: d' < f(P^\alpha_x) + \varepsilon\}$$

$$= \bigcup \{O_{d'}: d' < d\} \subset O_d.$$

Since $B_\varepsilon(P^\alpha_x) \in \mathbb{F}_{\mathscr{U}}$ and $P^\gamma_\xi \in B_\varepsilon(P^\alpha_x)$, $A \in \mathbb{F}_{\mathscr{U}}$ hence $\mathbb{F} \subset \mathbb{F}_{\mathscr{U}}$.

(b) $\mathbb{F}_{\mathscr{U}} \subset \mathbb{F}$.

It is sufficient to show that for any $P^\gamma_\xi \in A \in \mathbb{F}$, $P^\alpha_x \in \mathbb{P}$ and $\varepsilon > 0$ the fuzzy sets $B_\varepsilon(P^\alpha_x)$ and $B^*_\varepsilon(P^\alpha_x)$ are the members of $\mathbb{F}$. Since $B_\varepsilon(P^\alpha_x) = \bigcup\{O_d: d < f(P^\alpha_x) + \varepsilon\}$ and $O_d \in \mathbb{F}$. thus $B_\varepsilon(P^\alpha_x) \in \mathbb{F}$. Similarly, we have $B^*_\varepsilon(P^\alpha_x) \in \mathbb{F}$, hence $\mathbb{F}_{\mathscr{U}} \subset \mathbb{F}$.

Necessity. Suppose $(X, \mathbb{F}_{\mathscr{U}})$ is a fuzzy uniform topological space. For any $A \in \mathbb{F}_{\mathscr{U}}$ and point $P^\gamma_\xi \in A$, we have to construct a scale of fuzzy open set $\{O_d: d \in D\}$ such that $P^\gamma_\xi \in O_d \subset A$ for every $d \in D$.

If $A \in \mathbb{F}_{\mathscr{U}}$, then there is $U \in \mathscr{U}$ such that

$$P^\gamma_\xi \in U(P^\gamma_\xi) \subset A.$$

Let $U_1 \in \mathscr{U}$ be symmetric and $U_1 \circ U_1 \subset U$, by Theorem 6.7, we have

$$\overline{U_1(P^\gamma_\xi)} \subset U(P^\gamma_\xi)^0 \subset A,$$

and $U(P^\alpha_x)^0$ is a neighborhood of $\overline{U_1(P^\alpha_x)}$. We can select a sequence $\{U_{2^{-n}}: n=1,2,\dots\}$ of symmetric members of $\mathcal{U}$ such that $U_{2^{-(n+1)}} \circ U_{2^{-(n+1)}} \subset U_{2^{-n}}$ and $U_{2^{-1}} \circ U_{2^{-1}} \subset U_1$. For every diadic rational $d = 2^{-n_1} + \cdots + 2^{-n_k}$, $0 < n_1 < \cdots < n_k$, we define

$$U_d = U_{2^{-n_k}} \circ \cdots \circ U_{2^{-n_1}}.$$

If $0 < d_1 < d_2 < 1$ then we can prove

$$U_{d_2 - d_1} \circ U_{d_1} \subset U_{d_2}.$$

By Theorem 6.7, for any point P^α_x, we have

$$\overline{U_{d_1}(P^\alpha_x)} \subset U_{d_2}(P^\alpha_x)^0,$$

and $U_{d_2}(P^\alpha_x)$ is a neighborhood of $\overline{U_{d_1}(P^\alpha_x)}$. Let $O_d = U_d(P^\alpha_x)^0$, then for any pair of positive diadic rational $d_1 < d_2$, we have $\bar{O}_{d_1} \subset O_{d_2}$ and O_{d_2} is a neighborhood of $\bar{O}_{d_1}$. Thus $\{O_d : d \in D\}$ is a scale of fuzzy open sets and for any $d \in D$ we have $P^\alpha_x \in O_d \subset A$ hence $(X, \mathfrak{F}_{\mathcal{U}})$ is a fuzzy (T) space. The theorem is proved.

A fuzzy (T) space is fuzzy regular and so a fuzzy uniform space is also fuzzy regular.

Let e be a fuzzy pseudo-metric, by the proof of the above theorem, we see that the family $\{B_{1/n} : n = 1, 2, \dots\}$ of subsets of $\mathbb{P}_* \times \mathbb{P}_*$ forms a countable base of some fuzzy uniformity for X, so that each fuzzy pseudo-metric induces a fuzzy uniformity $\mathcal{U}_e$. Conversely, we shall ask what is the condition of a fuzzy uniformity being a fuzzy pseudo-metric uniformity? The following theorem is exactly the answer to the question.

THEOREM 7.2. *A fuzzy uniform topological space $(X, \mathfrak{F}_{\mathcal{U}})$ is a fuzzy pseudo-metric space iff $\mathcal{U}$ has a countable base.*

Proof. Sufficiency. If $\mathcal{U}$ has a countable base, then we can select a monotonous decreasing base $\{U_{2^{-n}} : n = 0, 1, 2, \dots\}$ such that $U_{2^{-(n+1)}} \circ U_{2^{-(n+1)}} \subset U_{2^{-n}}$ (cf. Theorem 7.1). Denoted by D the all diadic rational in $[0, 1]$, we can define $U_d \in \mathcal{U}$ for every $d \in D$ such that for any $d_1, d_2 \in D$ and point P^α_x, we have $\overline{U_{d_1}(P^\alpha_x)} \subset U_{d_2}(P^\alpha_x)^0$, and $U_{d_2}(P^\alpha_x)^0$ is a neighborhood of $\overline{U_{d_1}(P^\alpha_x)}$.

Define a mapping $e: \mathbb{P}_* \times \mathbb{P}_* \to [0, 1]$

$$e(P^\alpha_x, P^\beta_y) = \inf\{d : P^\beta_y \subset U_d(P^\alpha_x)\} \qquad (P^\alpha_x, P^\beta_y) \in U_1,$$
$$= 1, \qquad (P^\alpha_x, P^\beta_y) \notin U_1.$$

We shall prove that it is a fuzzy pseudo-metric for X.

First, the mapping e is continuous for membership grade. Indeed, for any

$\varepsilon>0$ there is $d\in D$ such that $0<d<\varepsilon$. Consider $U_d\in\mathcal{U}$, by axiom (U1), there exists $\delta>0$ such that $(P_x^\alpha, P_x^{\alpha+\delta})\in U_d$ for any fuzzy point $P_x^\alpha\in\mathbb{P}_0$. Hence, if $\beta-\alpha<\delta$ then $(P_x^\alpha, P_x^\beta)\in U_d$ and so $e(P_x^\alpha, P_x^\beta)\leqslant d<\varepsilon$, that is, the mapping e is continuous for membership grade.

Now we are going to verify e satisfying the fuzzy pseudo-metric axioms:

(M1) Let $P_y^\beta\subset P_x^\alpha$, by axiom (U1) and property (C2), we have $(P_x^\alpha, P_x^\alpha)\in U_d$ for any $0<d<D$ and $(P_x^\alpha, P_y^\beta)\in U_d$. Since $d>0$ is arbitrary, $e(P_x^\alpha, P_y^\beta)=0$.

(M2) Given any $P_x^\alpha, P_y^\beta, P_z^\gamma\in\mathbb{P}_*$. If $e(P_x^\alpha, P_y^\beta)+e(P_y^\beta, P_z^\gamma)\geqslant 1$, then $e(P_x^\alpha, P_z^\gamma)\leqslant e(P_x^\alpha, P_y^\beta)+e(P_y^\beta, P_z^\gamma)$. If $e(P_x^\alpha, P_y^\beta)+e(P_y^\beta, P_z^\gamma)<1$, then we can select the suitable positive real numbers δ_1 and δ_2 such that $e(P_x^\alpha, P_y^\beta)+\delta_1=d_1$, $e(P_y^\beta, P_z^\gamma)+\delta_2=d_2$, $d_1+d_2<1$, $d_1, d_2\in D$. Thus $(P_x^\alpha, P_y^\beta)\in U_{d_1}$, $(P_y^\beta, P_z^\gamma)\in U_{d_2}$ and so $(P_x^\alpha, P_z^\gamma)\in U_{d_2}\circ U_{d_1}\subset U_{d_1+d_2}$, that is,

$$e(P_x^\alpha, P_z^\gamma)\leqslant e(P_x^\alpha, P_y^\beta)+e(P_y^\beta, P_z^\gamma)+\delta_1+\delta_2.$$

Since δ_1, δ_2 can be selected arbitrarily, hence

$$e(P_x^\alpha, P_z^\gamma)\leqslant e(P_x^\alpha, P_y^\beta)+e(P_y^\beta, P_z^\gamma).$$

(M3) Since $U_d=U_d^{-1}$, $d\in D$, then $(P_x^\alpha, P_y^\beta)\in U_d$ is equivalent with $(P_y^{1-\beta}, P_x^{1-\alpha})\in U_d$, hence

$$e(P_x^\alpha, P_y^\beta)=e(P_y^{1-\beta}, P_x^{1-\alpha}).$$

Finally, we show $\mathbb{F}_e=\mathbb{F}_{\mathcal{U}}$. In fact, they have the same base $\{U_{2^{-n}}(P_x^\alpha): P_x^\alpha\in\mathbb{P}, n=1, 2,\ldots\}$, so $\mathbb{F}_e=\mathbb{F}_{\mathcal{U}}$.

Necessity. Obvious.

From the above theorem, we obtain the following important result:

THEOREM 7.3. *Let* $\mathbb{F}_{e_n}$, $n=1, 2,\ldots$, *be fuzzy pseudo-metric topologies for* X, *then*

$$\mathbb{F}_e=\sup\{\mathbb{F}_{e_n}: n=1, 2,\ldots,\}$$

is also a fuzzy pseudo-metric topology.

THEOREM 7.4. *Let* $\mathcal{U}$ *be a fuzzy uniformity for* X, *then there exists a family of fuzzy pseudo-metric such that*

$$\mathcal{U}=\sup\{\mathcal{U}_e\}.$$

Proof. Given any $U\in\mathcal{U}$ we can select a sequence $\{U_{2^{-n}}: n=1, 2,\ldots,\}$ of symmetric members of $\mathcal{U}$ such that $U_{2^{-n}}\subset U$ and $U_{2^{-(n+1)}}\circ U_{2^{-(n+1)}}\subset U_{2^{-n}}$.

It is easy to see that $\{U_{2^{-n}}: n = 1, 2\cdots\}$ is a base of fuzzy uniformity $\mathscr{U}(U)$ for X. Since $U_{2^{-1}} \subset U$ we have $U \in \mathscr{U}(U)$. By Theorem 7.2, there is a fuzzy pseudo-metric $e = e(U)$ for X, such that the corresponding fuzzy pseudo-metric uniformity $\mathscr{U}_e = \mathscr{U}(U)$. Obviously $\mathscr{U}_e \subset \mathscr{U}$, hence $\sup\{\mathscr{U}_e\} \subset \mathscr{U}$. Conversely, for any $U \in \mathscr{U}$, there exists $e = e(U)$ such that $U \in \mathscr{U}_e$ and so $U \in \sup\{\mathscr{U}_e\}$, i.e., $\mathscr{U} \subset \sup\{\mathscr{U}_e\}$. Therefore $\mathscr{U} = \sup\{\mathscr{U}_e\}$ and the theorem is proved.

In Theorem 7.2, if, in addition, $\mathscr{U}$ is separated, then the following fuzzy metrization condition is easily obtained.

A fuzzy uniform space $(X, \mathfrak{F}_{\mathscr{U}})$ is fuzzy metrizable iff $\mathscr{U}$ is separated and has countable base.

VIII. Fuzzy Metrization Theorem

Definition 8.1. A family A of fuzzy sets in $(X, \mathfrak{F})$ is said to be fuzzy locally finite iff every point $P_x^\alpha \in \mathbb{P}$ has a neighborhood U which is quasi-coincident with at most a finite number of the members of A. The family A is σ-fuzzy locally finite iff it is the countable union of fuzzy locally finite families.

Theorem 8.1. *If $\{A_\lambda : \lambda \in \Lambda\}$ is a fuzzy locally finite family in $(X, \mathfrak{F})$, then*

$$\overline{\bigcup_{\lambda \in \Lambda} A_\lambda} = \bigcup_{\lambda \in \Lambda} \bar{A}_\lambda.$$

Proof. It is sufficient to show that $\overline{\bigcup_{\lambda \in \Lambda} A_\lambda} \subset \bigcup_{\lambda \in \Lambda} \bar{A}_\lambda$. Given any point $P_x^\alpha \subset \overline{\bigcup_{\lambda \in \Lambda} A_\lambda}$ then every neighborhood of $P_x^{1-\alpha}$ is quasi-coincident with $\bigcup_{\lambda \in \Lambda} A_\lambda$. When $0 < \alpha < 1$, $P_x^{1-\alpha}$ is a fuzzy point, and by the fuzzy locally finite property of $\{A_\lambda : \lambda \in \Lambda\}$, $P_x^{1-\alpha}$ has a neighborhood V which is quasi-coincident with at most a finite number of A_{λ_i}, $i = 1,\ldots, n$. But V is quasi-coincident with $\bigcup_{\lambda \in \Lambda} A_\lambda$, hence V is certainly quasi-coincident with $\bigcup_{i=1,\ldots,n} A_{\lambda_i}$. We now prove that every neighborhood of $P_x^{1-\alpha}$ is quasi-coincident with $\bigcup_{i=1,\ldots,n} A_{\lambda_i}$. It is sufficient to show that its every neighborhood which satisfies the condition $U \subset V$ is quasi-coincident with $\bigcup_{i=1,\ldots,n} A_{\lambda_i}$. Indeed, if $U \subset V$, then U is probably only quasi-coincident with A_{λ_i}, $i = 1,\ldots, n$. But U and $\bigcup_{\lambda \in \Lambda} A_\lambda$ are quasi-coincident, hence U and $\bigcup_{i=1,\ldots,n} A_{\lambda_i}$ are certainly quasi-coincident. Thus we have proved every neighborhood of $P_x^{1-\alpha}$ is quasi-coincident with $\bigcup_{i=1,\ldots,n} A_{\lambda_i}$ and so we have

$$P_x^\alpha \subset \overline{\bigcup_{i=1,\ldots,n} A_{\lambda_i}} = \bigcup_{i=1,\ldots,n} \bar{A}_{\lambda_i} \subset \bigcup_{\lambda \in \Lambda} \bar{A}_\lambda.$$

When $\alpha = 1$, instead of $P_x^{1-\alpha}$ by P_x^{ε}, $0 < \varepsilon < 1$ in the above discussion, we have $P_x^{1-\varepsilon} \subset \bigcup_{\lambda \in \Lambda} \bar{A}_\lambda$, therefore $P'_x \subset \bigcup_{\lambda \in \Lambda} \bar{A}_\lambda$, and so $\overline{\bigcup_{\lambda \in \Lambda}} \subset \bigcup_{\lambda \in \Lambda} \bar{A}_\lambda$.

THEOREM 8.2. *Suppose* $(X, \mathbb{F})$ *is fuzzy regular and has σ-fuzzy locally finite base, then it is fuzzy normal.*

Proof. Suppose closed fuzzy sets C and D are strong quasi-discoincident, $\mathbb{A}^n$, $n = 1, 2, \ldots$, are locally finite families of open fuzzy sets and $\mathbb{A} = \bigcup \{\mathbb{A}^n : n = 1, 2, \ldots, \}$ forms a base of $(X, \mathbb{F})$. By the fuzzy regular of $(X, \mathbb{F})$, for any $x, y \in X$, there are $A_x^m \in \mathbb{A}$ and $A_y^n \in \mathbb{A}^n$ such that

$$P_x^{C(x)} \in A_x^m \subset \overline{A_x^m} \subset D',$$

$$P_y^{D(x)} \in A_y^n \subset \overline{A_y^n} \subset C',$$

and D' and C' are the neighborhoods of $\overline{A_x^m}$ and $\overline{A_y^n}$, respectively. Denoted by

$$A_C^k = \bigcup \{A_x^m : m = k\},$$

$$A_D^k = \bigcup \{A_y^n : n = k\},$$

and let

$$O_C^k = A_C^k \cap \overline{A_D^1}' \cap \cdots \cap \overline{A_D^k}',$$

$$O_D^k = A_D^k \cap \overline{A_C^1}' \cap \cdots \cap \overline{A_C^k}',$$

then the open fuzzy sets $O_C = \bigcup_{k=1,2,\ldots} O_C^k$ and $O_D = \bigcup_{k=1,2,\ldots} O_D^k$ are quasi-discoincident, moreover O_C and O_D are the neighborhoods of C and D, respectively. As a result of Theorem 4.10 the fts $(X, \mathbb{F})$ is fuzzy normal.

THEOREM 8.3. *If fts* $(X, \mathbb{F})$ *is fuzzy regular and has σ-fuzzy locally finite base, then it is a fuzzy pseudo-metrizable space.*

Proof. We construct a countable family of fuzzy pseudo-metric and prove that the topology, which is generated by this family, is equal to $\mathbb{F}$.

Let the natural numbers m and n be fixed. Given any $A_i^n \in \mathbb{A}^n$, we consider the open fuzzy set

$$A_i = \bigcup \{A^m : A^m \in \mathbb{A}^m, P(x, \overline{A^m}(x)) \in A_i^n, x \in X\}.$$

since $\mathbb{A}^n$ is fuzzy locally finite, $\bar{A}_i \subset A_i^n$ and A_i^n is a neighborhood of $\bar{A}_i$.

Since $(X, \mathbb{F})$ is fuzzy regular and has σ-fuzzy locally finite base, by Theorem 8.2, it is also fuzzy normal. From Theorem 4.1, it follows that there exists a scale of fuzzy open sets $\{O_{i,d}: d \in D\}$ such that $\bar{A}_i \subset O_{i,d} \subset A_i^n$ for any $d \in D$. We define mapping $f_i: \mathbb{P}_* \to [0, 1]$

$$f_i(P_x^\alpha) = \inf\{d: P_x^\alpha \in O_{i,d}\}, \qquad P_x^\alpha \in O_{i,1},$$
$$= 1, \qquad P_x^\alpha \notin O_{i,1},$$

and mapping $g_i: \mathbb{P}_* \times \mathbb{P}_* \to [0, 1]$

$$g_i(P_x^\alpha, P_y^\beta) = \max[f_i(P_y^\beta) - f_i(P_x^\alpha), 0].$$

By the same reasoning as in the proof of Theorem 7.1, the mapping g_i is a fuzzy quasi-metric for X. Similarly, we can define f_i^* and g_i^* corresponding to the dual scale $\{O_{i,d}^*: d \in D\}$. Of course, g_i^* is also a fuzzy quasi-metric for X.

Consider the mapping $e_i: \mathbb{P}_* \times \mathbb{P}_* \to [0, 1]$

$$e_i(P_x^\alpha, P_y^\beta) = [g_i(P_x^\alpha, P_y^\beta) + g_i^*(P_x^\alpha, P_y^\beta)]/2,$$

note that

$$g_i^*(P_x^\alpha, P_y^\beta) = g_i(P_y^{1-\beta}, P_x^{1-\alpha}),$$

we easily verify that the mapping e_i is a fuzzy pseudo-metric for X. Let

$$e_{m,n}(P_x^\alpha, P_y^\beta) = \sup\{e_i(P_x^\alpha, P_y^\beta) \cdot A_i^n \in \mathbb{A}^n\},$$

we now prove that it is a fuzzy pseudo-metric for X.

When $\alpha < 1$ and $\beta < 1$ we consider points $P_x^{1-\alpha}$, $P_y^{1-\beta}$, and by the fuzzy locally finiteness of $\mathbb{A}^n$, there are, respectively, neighborhoods U_x and U_y of $P_x^{1-\alpha}$ and $P_y^{1-\beta}$ such that U_x and U_y are quasi-coincident with at most a finite number of A_i^n. Thus there are at most finite members of A_i^n such that $P_x^\alpha \subset A_i^n$ or $P_y^\beta \subset A_i^n$, that is, for only finitely many indices i we have $f_i(P_x^\alpha) < 1$ of $f_i(P_y^\beta) < 1$. Hence only for these i the inequalities $g_i(P_x^\alpha, P_y^\beta) \neq 0$ possibly hold.

When $\alpha = 1$ or $\beta = 1$, we choose $\varepsilon > 0$ suitably small and substitute P_x^ε or P_y^ε for $P_x^{1-\alpha}$ or $P_y^{1-\beta}$, and the above discussion is still effective. Therefore there are at most finitely many indices such that $g_i(P_x^\alpha, P_y^\beta) \neq 0$.

Note that $g_i^*(P_x^\alpha, P_y^\beta) = g_i(P_y^{1-\beta}, P_x^{1-\alpha})$ once again, for any pair P_x^α, $P_y^\beta \in \mathbb{P}_*$ there are only at most finitely many indices i such that $e_i(P_x^\alpha, P_y^\beta) \neq 0$. This shows that $e_{m,n}$ is a fuzzy pseudo-metric for X.

Now we have obtained a countable family of fuzzy pseudo-metric $\{e_{m,n}: m = 1, 2,..., \ n = 1, 2,...\}$. Denote by $\mathbb{F}_{e_{m,n}}$ the fuzzy pseudo-metric topology which is generated by $e_{m,n}$. By Theorem 7.3, the fuzzy topology

$$\mathbb{F}_e = \sup\{\mathbb{F}_{e_{m,n}}: m = 1, 2,..., n = 1, 2,...,\}$$

is a fuzzy pseudo-metric topology for X. So we need only show $\mathbb{F}_e = \mathbb{F}$.

(a) $\mathbb{F}_e \subset \mathbb{F}$.

For this reason we must show that every ε-open ball

$$B_\varepsilon(P_x^\alpha) = \bigcup \{P_y^\beta : e_{m,n}(P_x^\alpha, P_y^\beta) < \varepsilon\}$$

is an $\mathbb{F}$-neighborhood of point P_x^α for any m, n. That is, we have to choose an $\mathbb{F}$-neighborhood $N(P_x^\alpha)$ of P_x^α such that $N(P_x^\alpha) \subset B_\varepsilon(P_x^\alpha)$.

We know that there exist at most finitely many indices $i_1, \ldots, i_k$ such that $f_{i_1}(P_x^\alpha) < 1, \ldots, f_{i_k}(P_x^\alpha) < 1$. In scales of fuzzy open sets, which have indices i_l, $l = 1, \ldots, k$, respectively, we select $O_{i_l, d_l} \subset A_{i_l}^n$ such that $g_{i_l}(P_x^\alpha, P_y^\beta) < \varepsilon$ for every $P_y^\beta \in O_{i_l, d_l}$, $l = 1, \ldots, k$.

And there is an $\mathbb{F}$-open neighborhood U of P_x^α such that at most finite numbers of $A_{i_{k+1}}^n, \ldots, A_{i_{k+J}}^n$ are quasi-coincident U. Let $O_{i_{k+l}, d_{k+l}}^* \subset (\bar{A}_{i_{k+l}})'$ such that $g_{i_{k+l}}^*(P_x^\alpha, P_y^\beta) < \varepsilon$ for every $P_y^\beta \in O_{i_{k+l}, d_{k+l}}^*$, $l = 1, \ldots, J$.

Since $O_{i_1, d_1}, \ldots, O_{i_k, d_k}, O_{i_{k+1}, d_{k+1}}^*, \ldots, O_{i_{k+J}, d_{k+J}}^*$ are contained in $\mathbb{F}$, the fuzzy set

$$N(P_x^\alpha) = U \cap O_{i_1, d_1} \cap \cdots \cap O_{i_k, d_k} \cap O_{i_{k+1}, d_{k+1}}^* \cap \cdots \cap O_{i_{k+J}, d_{k+J}}^*$$

is an $\mathbb{F}$-open neighborhood of P_x^α and for every $P_y^\beta \in N(P_x^\alpha)$ we have $e_{m,n}(P_x^\alpha, P_y^\beta) < \varepsilon$. Hence $N(P_x^\alpha) \subset B_\varepsilon(P_x^\alpha)$, i.e., $B_\varepsilon(P_x^\alpha)$ is an $\mathbb{F}$-neighborhood of P_x^α. Therefore $\mathbb{F}_e \subset \mathbb{F}$.

(b) $\mathbb{F} \subset \mathbb{F}_e$.

Given any $B \in \mathbb{F}$ and $P_x^\alpha \in B$. Since $(X, \mathbb{F})$ is fuzzy regular and has σ-fuzzy locally finite base $\mathbb{A} = \bigcup \{\mathbb{A}^n : n = 1, 2, \ldots\}$ there exist $A_j^m \in \mathbb{A}^m$, $A_i^n \in \mathbb{A}^n$ such that

$$P_x^\alpha \in \overline{A_j^m} \subset A_i^n \subset B.$$

It follows that

$$P_x^\alpha \in \overline{A_j^m} \subset \bar{A}_i \subset O_{i,d} \subset A_i^n \subset B,$$

for every $d \in D$. If $P_y^\beta \notin B$, then $f_i(P_y^\beta) = 1$, hence $e_{m,n}(P_x^\alpha, P_y^\beta) = 1$. This shows that, if $e_{m,n}(P_x^\alpha, P_y^\beta) < 1$, then $P_y^\beta \in B$ and so $B_1(P_x^\alpha) \subset B$. Therefore $B \in \mathbb{F}_e$ and so $\mathbb{F} \subset \mathbb{F}_e$.

THEOREM 8.4. *If fts $(X, \mathbb{F})$ is a fuzzy T_3 space and has σ-fuzzy locally finite base, then it is a fuzzy metric space.*

Proof. Since fuzzy T_3 space is fuzzy regular, by Theorem 8.3, the fts $(X, \mathbb{F})$ is a fuzzy pseudo-metric space. Hence we need only show the corresponding fuzzy pseudo-metric e satisfying axiom (M4).

For any point $P_x^\alpha \in \mathbb{P}$ by Theorem 4.4 on fuzzy T_1 space, we have

$$P_x^\alpha = \bigcap \{B_\varepsilon(P_x^\alpha) : \varepsilon > 0\}.$$

Therefore for any point $P_y^\beta \not\subset P_x^\alpha$ there is $B_\varepsilon(P_x^\alpha)$ such that $P_y^\beta \not\subset B_\varepsilon(P_x^\alpha)$ and so $e(P_x^\alpha, P_y^\beta) > 0$. For any $P_y^0 \in \mathbb{P}_*$, $y \neq x$, consider $(P_y^1)'$, it is an open fuzzy set and $P_x^\alpha \in (P_y^1)'$. Thus there is $\varepsilon > 0$ such that $B_\varepsilon(P_x^\alpha) \subset (P_y^1)'$, and it follows $e(P_x^\alpha, P_y^0) \geq \varepsilon > 0$. Similarly, we can prove $e(P_x^0, P_y^\beta) > 0$ for any P_x^0, $x \in X$, and $P_y^\beta \in \mathbb{P}_*$ $P_y^\beta \neq P_x^0$. This shows that e satisfies axiom (M4).

Corresponding to S-neighborhood, we have introduced the concept of S-cover and S-compact [6].

A family $\mathbb{C}$ of fuzzy sets in $(X, \mathbb{F})$ is called an S-cover of fuzzy set A iff every $P(x, A(x)) \in \mathbb{P}_*$ has a neighborhood germ N_U^α and there is a $C \in \mathbb{C}$ such that

$$P(x, A(x)) \in N_U^\alpha \subset C.$$

A fuzzy set A is said to be S-compact iff each of its S-covers has finite S-subcover. A fts $(X, \mathbb{F})$ is said to be S-compact if the fuzzy set 1_X is S-compact.

A family $\mathbb{D}$ of fuzzy sets is called a refinement of an S-cover $\mathbb{C}$ iff $\mathbb{D}$ is also S-cover and for any $D \in \mathbb{D}$ there exists $C \in \mathbb{C}$ such that $D \subset C$.

DEFINITION 8.2. A fts $(X, \mathbb{F})$ is called S-paracompact iff its every S-cover has a fuzzy locally finite refinement.

THEOREM 8.5. *If a fts $(X, \mathbb{F})$ is S-paracompact then its initial topological space $(X, \iota(\mathbb{F}))$ is paracompact.*

Proof. Suppose $\mathscr{U} = \{U : U \in \iota(\mathbb{F})\}$ is an open cover of $(X, \iota(\mathbb{F}))$, then $\{N_U^1 : U \in \mathscr{U}\}$ is an S-cover of $(X, \mathbb{F})$ and it has a fuzzy locally finite refinement $\{A_\lambda : \lambda \in \Lambda\}$. Let $W_\lambda = \sigma_1(A_\lambda)$ then $\{W_\lambda : \lambda \in \Lambda\}$ is a locally finite refinement of $\mathscr{U}$.

A fts $(X, \mathbb{F})$ is called a product-induced S-paracompact T_3 space if it is a product-induced space $(X, \mathbb{F}_{\mathscr{T} \times \theta_r})$ and its initial topological space $(X, \iota(\mathbb{F}))$ is a paracompact T_3 space.

THEOREM 8.6. *A fts $(X, \mathbb{F})$ is a product-induced S-paracompact T_3 space iff it is an S-paracompact and ST_3 space.*

Proof. Sufficiency. We need only to prove that for any fundamental fuzzy set N_U^α and point $P_{x_0}^{\alpha_0} \in N_U^\alpha$ there exists an open fuzzy set $B \in \mathbb{F}$ such that $P_{x_0}^{\alpha_0} \in B \subset B_U^\alpha$, i.e., $N_U^\alpha \in \mathbb{F}$, because all N_U^α forms a base of $\mathbb{F}_{\mathscr{T} \times \theta_r}$.

Since $(X, \mathbb{F})$ is an ST_3 space, there are open fuzzy sets D and G such that

$\alpha_0 < D(x_0) < \alpha$, $P_{x_0}^{\alpha_0} \in G \subset \bar{G} \subset D$ and D is an S-neighborhood of $\bar{G}$. Given any real number γ satisfying inequality $G\ (x_0) < \gamma < D(x_0)$. Since $\bar{G}$ is an upper semicontinuous function in $(X, \iota(\mathbb{F}))$, there is a neighborhood W of x_0 such that $G(x) < \gamma$ for every $x \in W$. Let $V = U \cap W$, then the points $P_{x_0}^1$ and P_x^1, $x \in X \sim V$, are strong quasi-discoincident. Using a similar way of proof as for Theorem 4.9, we can prove that there exist quasi-discoincident open fuzzy sets B_x and A_x such that $P_{x_0}^1 \in N_{V_x}^1 \subset A_x$ and $P_x^1 \in N_{U_x}^1 \subset B_x$, where $N_{V_x}^1$ and $N_{U_x}^1$ are the neighborhood germs of $P_{x_0}^1$ and P_x^1, respectively. Since open fuzzy sets B_x and A_x are not quasi-coincident, for any $x \in X \sim V$ we have $\bar{B}_x(x_0) = 0$. Obviously, the family $\mathbb{B} = \{B_x : x \in X \sim V\}$ is an S-cover of fuzzy set $1_{X \sim V}$. Since $(X, \mathbb{F})$ is s-paracompact and $X \sim V$ is a closed subset of X, we can prove that the S-cover $\mathbb{B}$ of $1_{X \sim V}$ has fuzzy locally finite refinement $\{A_\lambda : \lambda \in \Lambda\}$. By Theorem 8.1, we have

$$\overline{\bigcup_{\lambda \in \Lambda} A_\lambda} = \bigcup_{\lambda \in \Lambda} \bar{A}_\lambda.$$

Since for any A_λ there exists B_γ such that $A_\lambda \subset B_\gamma$ it follows that $\bar{A}_\lambda(x) = 0$ and so

$$\overline{[\bigcup_{\lambda \in \Lambda} A_\lambda]}(x_0) = \bigcup_{\lambda \in \Lambda} \bar{A}_\lambda(\chi_0) = 0.$$

Let $A = (\overline{\bigcup_{\lambda \in \Lambda} A_\lambda})'$, then $\sigma_0(A) \subset V$ and $A(x_0) = 1$.

Let $B = A \cap G$, then we have $P_{x_0}^{\alpha_0} \in B \subset N_U^\alpha$, and this shows $\mathbb{F} = \mathbb{F}_{\mathcal{T} \times \theta_I}$, $\mathcal{T} = \iota(\mathbb{F})$. By Theorem 8.5, the fts $(X, \mathbb{F})$ is a product-induced S-paracompact T_3 space $(X, \mathbb{F}_{\mathcal{T} \times \theta_I})$.

Necessity. If $(X, \mathbb{F})$ is a product-induced S-paracompact T_3 space, then $(X, \iota(\mathbb{F}))$ is a paracompact T_3 space. By Theorems 4.13 and 8.5, $(X, \mathbb{F})$ is an ST_3 and S-paracompact space.

THEOREM 8.7. *A fts $(X, \mathbb{F})$ is a product-induced fuzzy metric space $(X, \mathbb{F}_{\mathcal{T}_d \times \theta_I})$ iff it is an ST_3, S-paracompact fts and the initial topology $\iota(\mathbb{F}) = \mathcal{T}_d$ has σ-locally finite base.*

Proof. Sufficiency. Since fts $(X, \mathbb{F})$ is S-paracompact and ST_3, by Theorem 8.6, it is a product-induced S-paracompact space. From $(X, \mathbb{F})$ being an ST_3 space it follows that the $(X, \iota(\mathbb{F}))$ is a T_3 space. Therefore $(X, \iota(\mathbb{F}))$ is a metric space and so $(X, \mathbb{F})$ is a product-induces fuzzy metric space $(X, \mathbb{F}_{\mathcal{T}_d \times \theta_I})$, $\mathcal{T}_d = \iota(\mathbb{F})$.

Necessity. Obvious.

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