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Rational Growth of a Class of Inverse Semigroups

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We give a sufficient condition for a finitely presented Rees quotient of a free inverse semigroup to have rational growth. Using related techniques we give a new proof that nonmonogenic free inverse semigroups have irrational growth. A new criterion for polynomial growth is proved and is used to show that polynomial growth implies rational growth. However we give an example of such a semigroup which has rational and exponential growth. © 1998 Academic Press

1. INTRODUCTION

In [2] Easdown and Shneerson initiated the study of growth of finitely presented Rees quotients of free inverse semigroups. Growth was known to be polynomial or exponential for semigroups from this class and an algorithm was given to recognise which type of growth occurred [2, Section 3]. In Theorem (4.8) we give a sufficient condition for rational growth, which is used in Section 5 to show that polynomial growth implies rational growth. Other sufficient conditions for rational growth are given in Proposition (4.10) and Theorem (5.5). Brazil [1] showed that nonmonogenic free inverse semigroups have irrational growth, using techniques from functional analysis. We give a new proof of this result in Section 3.

2. PRELIMINARIES

We assume familiarity with the basic definitions and the elementary results from the theory of semigroups, which can be found in [4]. Let S be

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a semigroup generated by a finite subset X. Recall that the length $l(t)$ of an element $t \in S$ (*with respect to* X) is the least number of factors in all representations of t as a product of elements of X. Let

$$g_S(m) = |\{t \in S | l(t) \leq m\}|.$$

We say that S has *polynomial growth* if there exist natural numbers q and d such that

$$g_S(m) \leq qm^d,$$

for all natural numbers m, and *exponential growth* if there exists a real number $\alpha > 1$ such that

$$g_S(m) \geq \alpha^m,$$

for all sufficiently large m. It is clear that if S has polynomial [exponential] growth with respect to a given finite set of generators then it has polynomial [exponential] growth with respect to any finite set of generators. Clearly, if S contains a noncyclic free subsemigroup then S has exponential growth.

The *growth series* of S with respect to X is defined to be the usual generating function for the sequence $g_S(m)$,

$$g_S(z) = \sum_{m=0}^{\infty} g_S(m) z^m.$$

A semigroup S is said to have *rational growth* if $g_S(z)$ is a rational function of z for some choice of X. In this paper it is more convenient to work with a slightly different series. We define a sequence,

$$h_S(m) = |\{t \in S | l(t) = m\}|,$$

and let $h_S(z)$ be its generating function. Because clearly $h_S(m) = g_S(m) - g_S(m-1)$, we have

$$h_S(z) = (1 - z) g_S(z).$$

Therefore S has rational growth if and only if $h_S(z)$ is a rational function of z.

Throughout this paper we shall denote the subring of rational functions in $\mathbb{R}[[z]]$ by $\mathrm{Rat}[[z]]$, that is, $\mathrm{Rat}[[z]] = \mathbb{R}[[z]] \cap \mathbb{R}(z)$. An element in $\mathrm{Rat}[[z]]$ can be written as $p(z)/q(z)$ for some polynomials $p(z)$ and $q(z)$ such that $q(z)$ has nonzero constant term. It can be easily checked that $\mathrm{Rat}[[z]]^*$, the group of units of $\mathrm{Rat}[[z]]$, is precisely $\mathbb{R}[[z]]^* \cap \mathrm{Rat}[[z]]$, the elements in $\mathrm{Rat}[[z]]$ whose series expansions have nonzero constant term.

Let A be a finite alphabet. We denote the free semigroup and the free inverse semigroup over A by F_A and FI_A, respectively. Equality in F_A is denoted by $\overset{\circ}{=}$. Recall that a word w is *reduced* if w does not contain xx^{-1} as a subword for any letter $x \in A \cup A^{-1}$.

Recall that elements of FI_A may be regarded as birooted word trees, the terminology and theory of which are explained in [4, Chapter 2] (see also [2, Section 2]). Thus if $u \in F_{A \cup A^{-1}}$, then regarded as an element of FI_A, u may be represented as a birooted tree,

$$\varphi(u) = (T(u), \alpha(u), \beta(u)).$$

Recall that u is an idempotent of FI_A if and only if $\alpha(u) = \beta(u)$. Given a birooted word tree $B = (T, \alpha, \beta)$, we shall adopt the following notation:

(i) $e(B) = \max\{d(\alpha, v) | v \in T\}$,

(ii) for $m \in \mathbb{N}$, $B|_m = (T|_m, \alpha, \alpha)$, where $T|_m$ is the subtree of T obtained by deleting all vertices of distance greater than m away from α, and all edges incident on such vertices.

It is often convenient to adjoin an identity called 1 to FI_A and adopt the convention that $l(1) = 0$ and $\varphi(1) = (\circ, \circ, \circ)$, the *null* birooted tree consisting of one vertex.

Given two trees $B_1 = (T_1, \alpha_1, \beta_1)$ and $B_2 = (T_2, \alpha_2, \beta_2)$, define $B_1 \oplus B_2 = (T, \alpha_1, \beta_2)$ where T is the word tree obtained by "pasting" T_1 and T_2 together, identifying β_1 with α_2 and then further identifying any isomorphic paths from the common vertex $\beta_1 = \alpha_2$. Note that $B_1 \oplus B_2 = \varphi(\varphi^{-1}(B_1)\varphi^{-1}(B_2))$.

A birooted word tree (T, α, β) on A is said to be *planted* if it is null or if it is nonnull and α is a leaf of the tree (that is, α is adjacent to exactly one vertex of T). We shall refer to a birooted word tree which is planted simply as a *planted tree*.

Given a nonnull planted tree $P = (T, \alpha, \beta)$, we shall adopt the following notation:

(i) $\gamma(P)$ is the unique vertex of T adjacent to α,

(ii) the label of P, denoted by label(P), is the label of the edge $\alpha \to \gamma(P)$.

Let $P_1 = (T_1, \alpha_1, \alpha_1)$ and $P_2 = (T_2, \alpha_2, \alpha_2)$ be two planted trees representing idempotents, with $P_1 \neq \varphi(1)$. If $P_2 = \varphi(1)$, or label$(P_1) \neq$ label$(P_2)^{-1}$, then define $P_1 \odot P_2$ to be the planted word tree (T, α_1, α_1), where T is the word tree obtained by pasting T_1 and T_2 together, by identifying $\gamma(P_1)$ with α_2 and then by further identifying any isomorphic paths from the common vertex $\gamma(P_1) = \alpha_2$. The reader can easily verify that order is unimportant where the operation $\odot$ is iterated.

Let P be a planted tree representing an idempotent with label x. Then P can be uniquely expressed in the form

$$P = \varphi(x) \oplus P_1 \oplus \cdots \oplus P_s \oplus \varphi(x^{-1}),$$

where $s \geq 0$, $P_1, \dots, P_s$ are nonnull planted trees with distinct labels, and for each i, P_i represents an idempotent and label$(P_i) \neq x^{-1}$. We shall call $P_1, \dots, P_s$ and the null tree the *components* of P.

3. FREE INVERSE SEMIGROUPS

In this section we give a new proof of a result due to Brazil [1], that the growth series for $FI_{\{x_1, \dots, x_n\}}$ is irrational if and only if $n \geq 2$. While being an interesting result in its own right, the proof also illustrates some counting techniques employed in the next section.

Let $A = \{x_1, \dots, x_n\}$. As was mentioned earlier, elements of $S = FI_A^{\ 1}$ can be regarded as birooted word trees. We will "build" the series $h_S(z)$ from the growth series for certain types of idempotents.

For $x \in A \cup A^{-1}$, define the sets,

$$A(x) = \{e \in E(S) | \varphi(e) \text{ is a planted tree with label } x\} \cup \{1\},$$
$$B(x) = \{u \in S \setminus E(S) | \varphi(u) \text{ is a planted tree with label } x\}.$$

By symmetry, for any given natural number m, the integers,

$$a_m = |\{e \in A(x) | l(e) = m\}| \quad \text{and} \quad b_m = |\{u \in B(x) | l(u) = m\}|$$

are independent of x. Denote the generating functions of the corresponding sequences by $a(z)$ and $b(z)$, respectively.

Now clearly $a_0 = 1$ and $a_1 = 0$. Suppose $m \geq 2$. If $e \in A(x)$ and $l(e) = m$, then

$$\varphi(e) = \varphi(xx^{-1}) \odot_y \varphi(e_y),$$

where y ranges over $(A \cup A^{-1}) \setminus \{x^{-1}\}$, e_y is some element of $A(y)$ for each y, such that $\sum l(e_y) = m - 2$. Therefore,

$$a_m = \sum_{m_1 + \cdots + m_{2n-1} = m-2} a_{m_1} \cdots a_{m_{2n-1}} \quad \text{for } m \geq 2.$$

Translating into generating functions, this gives

$$a(z)^{2n-1} = \sum_{m\geq 0}\left(\sum_{m_1+\cdots+m_{2n-1}=m} a_{m_1}\cdots a_{m_{2n-1}}\right)z^m = \sum_{m\geq 0} a_{m+2}z^m,$$

and hence,

$$z^2 a(z)^{2n-1} = a(z) - 1. \tag{1}$$

Next we turn to deriving an identity for $b(z)$. Clearly $b_0 = 0$. Suppose $m \geq 1$, $u \in B(x)$ and $l(u) = m$. There are two possibilities. If $\beta(u) = \gamma(u)$, then

$$\varphi(u) = \varphi(x) \oplus_y \varphi(e_y)$$

where y ranges over $(A \cup A^{-1}) \setminus \{x^{-1}\}$, $e_y \in A(y)$ for each y, and $\sum l(e_y) = m - 1$. If $\beta(u) \neq \gamma(u)$, then there is an $x' \in (A \cup A^{-1}) \setminus \{x^{-1}\}$ such that

$$\varphi(u) = \big(\varphi(x) \oplus_y \varphi(e_y)\big) \oplus \varphi(u'),$$

where y ranges over $(A \cup A^{-1}) \setminus \{x', x^{-1}\}$, $e_y \in A(y)$ for each y, $u' \in B(x')$, and $l(u') + \sum l(e_y) = m - 1$. Because there are $2n - 1$ choices for x', we get

$$\begin{aligned} b_m &= \sum_{m_1+\cdots+m_{2n-1}=m-1} a_{m_1}\cdots a_{m_{2n-1}} + (2n-1)a_{m_1}\cdots a_{m_{2n-2}}b_{m_{2n-1}} \\ &= a_{m+1} + (2n-1)\sum_{m_1+\cdots+m_{2n-1}=m-1} a_{m_1}\cdots a_{m_{2n-2}}b_{m_{2n-1}}, \end{aligned}$$

and hence,

$$b(z) = (2n-1)za(z)^{2n-2}b(z) + \frac{1}{z}\big(a(z) - 1\big). \tag{2}$$

Finally suppose $u \in S$ and $l(u) = m$. Again we have two cases. If $\alpha(u) = \beta(u)$, then

$$\varphi(u) = \bigoplus\nolimits_{y\in A\cup A^{-1}} \varphi(e_y),$$

where $e_y \in A(y)$ for each y, and $\sum l(e_y) = m$. If $\alpha(u) \neq \beta(u)$, then there is an $x \in A \cup A^{-1}$ such that

$$\varphi(u) = \Big(\bigoplus\nolimits_y \varphi(e_y)\Big) \oplus \varphi(u'),$$

where y ranges over $(A \cup A^{-1}) \setminus \{x\}$, $e_y \in A(y)$ for each $y, u' \in B(x)$, and $l(u') + \sum l(e_y) = m$. Because there are $2n$ choices for x, we get

$$h_S(m) = \sum_{m_1 + \cdots + m_{2n} = m} a_{m_1} \cdots a_{m_{2n}} + 2n a_{m_1} \cdots a_{m_{2n-1}} b_{m_{2n}},$$

for all $m \geq 0$. Therefore,

$$h_S(z) = a(z)^{2n-1}(2nb(z) + a(z)). \tag{3}$$

We can now show that $h_S(z)$ is irrational if $n \geq 2$. The result depends on the following lemmas.

(3.1) LEMMA. *Suppose that $f(z) \in \mathbb{R}[[z]]$ and $f(z) - 1 = z^s f(z)^t$ for some integers $s \geq 1$, $t \geq 2$. Then $f(z)$ is irrational.*

Proof. Suppose $f(z) = p(z)/q(z)$ for some polynomials $p(z)$ and $q(z)$. We can assume that $p(z)$ and $q(z)$ have no nonconstant common factors in $\mathbb{R}[z]$. Then

$$z^s p(z)^t = q(z)^{t-1}(p(z) - q(z)).$$

Thus $q(z)$ divides $z^s p(z)^t$ in $\mathbb{R}[z]$. Because $q(z)$ has a nonzero constant term (see comments following the definition of Rat[[z]]), it has no common factor with z^s either. Therefore $q(z)$ is a constant and $f(z)$ is a polynomial. Considering the degrees of the polynomials on both sides of $f(z) - 1 = z^s f(z)^t$, this is easily seen to be impossible. ∎

(3.2) LEMMA. *Suppose that $f(z) \in \mathbb{R}[[z]]$ and for some natural number t there exist $\alpha(z), \beta(z) \in$ Rat[[z]] such that*

$$z\alpha(z)f(z)^t = f(z) + \beta(z).$$

Suppose also that for some natural number $k \geq 1$, there exist $g_i(z) \in$ Rat[[z]] such that

$$f(z)^k = \sum_{i=0}^{k-1} g_i(z) f(z)^i.$$

Then $f(z) \in$ Rat[[z]].

Proof. If $t \leq 1$ or $k = 1$ then one of the preceding equations will imply that $f(z) \in$ Rat[[z]]. So we can assume $k, t \geq 2$.

Using the fact that Rat[[z]] is a subring of $\mathbb{R}[[z]]$, it can easily be shown by induction that for all $m \geq k$,

$$f(z)^m = \sum_{i=0}^{k-1} f_i(z) f(z)^i,$$

for some $f_i(z) \in \mathrm{Rat}[[z]]$. In particular,

$$f(z)^{k+t-2} = \sum_{i=0}^{k-1} h_i(z) f(z)^i,$$

for some $h_i(z) \in \mathrm{Rat}[[z]]$. Therefore,

$$f(z)^{k-1} + \beta(z) f(z)^{k-2} = z\alpha(z) f(z)^{k+t-2} = \sum_{i=0}^{k-1} z\alpha(z) h_i(z) f(z)^i,$$

and, rearranging the terms,

$$(1 - z\alpha(z) h_{k-1}(z)) f(z)^{k-1} = \sum_{i=0}^{k-2} z\alpha(z) h_i(z) f(z)^i - \beta(z) f(z)^{k-2}.$$

Now $1 - z\alpha(z)h_{k-1}(z)$ has a nonzero constant term, and so is in $\mathrm{Rat}[[z]]^*$. Therefore,

$$f(z)^{k-1} = \sum_{i=0}^{k-2} g_i'(z) f(z)^i,$$

for some $g_i'(z) \in \mathrm{Rat}[[z]]$. We can repeat the process and decrement k each time until we get to the case $k = 1$. ∎

(3.3) THEOREM. *$FI_{\{x_1, \ldots, x_n\}}$ has irrational growth if and only if $n \geq 2$.*

Proof. Suppose $n \geq 2$. From (2) we have

$$b(z) z\left(1 - (2n-1) z a(z)^{2n-2}\right) = a(z) - 1.$$

Multiplying both sides by $a(z)$ and using (1), this becomes

$$b(z)(za(z) - (2n-1)(a(z) - 1)) = a(z)(a(z) - 1).$$

Putting this into (3) and using (1) again gives

$$\begin{aligned} h_S(z) z^2 (za(z) - (2n-1)(a(z)-1)) \\ &= (a(z)-1)a(z)(2n(a(z)-1) + za(z) \\ &\qquad -(2n-1)(a(z)-1)) \\ &= (a(z)-1)a(z)(a(z) - 1 + za(z)). \end{aligned}$$

Suppose that $h_S(z)$ is rational. Then by rearranging the previous equation, $(1+z)a(z)^3$ can be expressed in the form,

$$g_2(z) a(z)^2 + g_1(z) a(z) + g_0(z),$$

for some $g_i(z) \in \mathrm{Rat}[[z]]$. Because $1 + z \in \mathrm{Rat}[[z]]^*$, $a(z)^3$ can also be expressed in this form. By (1) and Lemma (3.2), $a(z) \in \mathrm{Rat}[[z]]$. But by (1) and Lemma (3.1), $a(z)$ is irrational, which yields a contradiction. Therefore $h_S(z)$ cannot be rational.

For $n = 1$, the Eqs. (1)–(3) still hold, and

$$a(z) = \frac{1}{1 - z^2} \quad \text{and} \quad h_S(z) = \frac{1}{(1 + z)(1 - z)^3}.$$

∎

We showed that, in the case of free inverse semigroups, $h_S(z)$ can be built out of the growth series $a(z)$ for certain idempotents, and the rationality of $h_S(z)$ depends on that of $a(z)$. This motivates the general construction in the next section.

4. RATIONAL GROWTH

In this section we give a sufficient graphical condition for a Rees quotient of a finitely generated free inverse semigroup by a finitely generated ideal to have rational growth series. Along the way we shall build $h_S(z)$ from some component series, and give an algorithm for doing so.

First we review some terminology from graph theory. A *cycle* in a directed graph is a path which starts and finishes at the same vertex. By a *loop at a vertex* v we mean a cycle which begins at v using no other vertex more than once. If there is a path from u to v, we say the vertex u is *reachable* from v. The *strongly connected components* of a directed graph are the equivalence classes of vertices under the "are mutually reachable" relation.

Let $\mathfrak{M}_{FI}$ denote the class of finitely presented inverse semigroups S with zero having a presentation of the form,

$$S = \langle A | c_i = 0 \text{ for } i = 1, \ldots, k \rangle,$$

where A is some finite alphabet, k is some nonnegative integer, $c_i \in F_{A \cup A^{-1}}$ for $i = 1$ to k. Then $\mathfrak{M}_{FI}$ is precisely the class of Rees quotients of finitely generated free inverse semigroups by finitely generated ideals. We shall write $A \cup A^{-1} = \{x_1, x_2, \ldots, x_{2n}\}$ and we shall put

$$d + 1 = \max\{3, |c_i| \,|\, i = 1, \ldots, k\},$$

where $|c_i|$ denotes the length of c_i with respect to $F_{A \cup A^{-1}}$. By our definition, $d \geq 2$. This is a slight modification of the convention in [2], where it is assumed $d \geq 1$. The reasons for this definition are given in the remarks following Theorem (4.8). Also we have the following useful criterion, which lies behind much of the proofs which follow and ensures the words we count are not zero in the semigroup.

(4.1) LEMMA. *Let* $w \in F_{A \cup A^{-1}}$. *Then* $w = 0$ *in* S *if and only if* $T(w)$ *contains* $T(c_i)$ *as a subtree for some* i.

We will now define a directed graph Γ'_S, depending on the presentation of S, called the *word tree graph of* S:

(i) the vertices of Γ'_S are planted trees $P = (T, \alpha, \alpha)$ on A satisfying the properties that $e(P) = d$ and $\varphi^{-1}(P)$ is nonzero in S;

(ii) there is a directed edge from vertex P to vertex Q if $Q|_{d-1}$ is a component of P and $\varphi^{-1}(P \odot Q)$ is nonzero in S.

Note that if $Q|_{d-1}$ is a component of P, then $\text{label}(P) \neq \text{label}(Q)^{-1}$, so $P \odot Q$ is defined. Furthermore $(P \odot Q)|_d = P$.

Denote the set of vertices of Γ'_S by $V(\Gamma'_S)$. For $P \in V(\Gamma'_S)$, define

$$V(P) = \{P' \in V(\Gamma'_S) | P \to P' \text{ is an edge in } \Gamma'_S\},$$

$$V_i(P) = \{P' \in V(P) | \text{label}(P') = x_i\}, \quad \text{for } i = 1, \ldots, 2n.$$

Here the label of a vertex in Γ'_S is simply its label as a planted tree.

For each vertex P in Γ'_S define $A(P)$ to be the set of $u \in FI_A$ such that u is nonzero in S, and $\varphi(u)$ is a planted word tree with $\varphi(u)|_d = P$. For $r, m \in \mathbb{N}$ put

$$A(P, r, m) = \{u \in A(P) | l(u) = m, d(\alpha(u), \beta(u)) = r\},$$

and

$$B(P, m) = \{u \in A(P) | l(u) = m, d(\alpha(u), \beta(u)) > 0\}.$$

Let $a(P, r, m) = |A(P, r, m)|$ and $b(P, m) = |B(P, m)|$. Denote their generating functions by $a(P, r)(z)$ and $b(P)(z)$, respectively.

Let P be a vertex of Γ'_S with label x_j. Then P can be uniquely expressed in the form,

$$P = \varphi(x_j) \oplus P_1 \oplus \cdots \oplus P_{2n} \oplus \varphi\left(x_j^{-1}\right),$$

where P_k is null for k such that $x_k = x_j^{-1}$ and, for each i, either P_i is null or P_i is a planted tree with label x_i. For $1 \leq i \leq 2n$, let $c(P, i, r, m)$ denote the number of planted trees $B = (T, \alpha, \beta)$ such that $(T, \alpha, \alpha) = P_i$, $d(\alpha, \beta) = r$, and $l(\varphi^{-1}(B)) = m$. Define the polynomials,

$$c(P, i, r)(z) = \sum_{m \geq 0} c(P, i, r, m) z^m.$$

Because $e(P_i) < d$ for each i, it is clear that $c(P,i,r)(z) = 0$ whenever $r \geq d$.

In the following propositions we shall construct $h_S(z)$ as an element of $\mathbb{Q}(z, a(P,0)(z) | P \in V(\Gamma'_S))$, the set of rational functions of z and the series $a(P,0)(z)(P \in V(\Gamma'_S))$. The proofs actually give an algorithm for doing so.

(4.2) PROPOSITION. *For each vertex P in Γ'_S,*

$$a(P,0)(z) = z^2 \prod_{i=1}^{2n} \left\{ c(P,i,0)(z) + \sum_{P' \in V_i(P)} a(P',0)(z) \right\},$$

$$a(P,1)(z) = z \prod_{i=1}^{2n} \left\{ c(P,i,0)(z) + \sum_{P' \in V_i(P)} a(P',0)(z) \right\},$$

and for all natural numbers $r \geq 2$,

$$a(P,r)(z) = z \sum_{j=1}^{2n} \left\{ c(P,j,r-1)(z) + \sum_{P' \in V_j(P)} a(P',r-1)(z) \right\}$$
$$\times \prod_{i \neq j} \left\{ c(P,i,0)(z) + \sum_{P' \in V_i(P)} a(P',0)(z) \right\}.$$

Proof. Let $P \in V(\Gamma'_S)$. Without loss of generality assume label$(P) = x_1$ and $x_1^{-1} = x_{2n}$. Thus,

$$P = \varphi(x_1) \oplus P_1 \oplus \cdots \oplus P_{2n-1} \oplus \varphi\left(x_1^{-1}\right),$$

where for each i, either P_i is null or P_i is a planted tree with label x_i.

We will first consider the case when $r \geq 2$, which is more complicated. Clearly $a(P,r,0) = a(P,r,1) = 0$. Consider an element u of $A(P,r,m)$ for $m \geq 2$. Because $\varphi(u)|_d = P$, for some $1 \leq j \leq 2n-1$,

$$\varphi(u) = \varphi(x_1) \oplus \varphi(e_1) \oplus \cdots \oplus \varphi(e_{j-1})$$
$$\oplus \varphi(e_{j+1}) \oplus \cdots \oplus \varphi(e_{2n-1}) \oplus \varphi(u'),$$

where $\sum l(e_i) + l(u') = m - 1$, and

(i) for each $i \neq j$, either $e_i \in A(P',0,l(e_i))$ for some $P' \in V_i(P)$ or $\varphi(e_i) = P_i$,

(ii) either $u' \in A(P', r-1, l(u'))$ for some $P' \in V_j(P)$ or $\varphi(u') = (T, \alpha, \beta)$ with $(T, \alpha, \alpha) = P_j$ and $d(\alpha, \beta) = r - 1$.

Therefore,

$$a(P,r,m) = \sum_{j=1}^{2n-1} \sum_{m_1+\cdots+m_{2n-1}=m-1} \Big\{c\big(P,j,r-1,m_j\big) + \sum_{P'\in V_j(P)} a\big(P',r-1,m_j\big)\Big\} \times \prod_{i\neq j}\Big\{c(P,i,0,m_i) + \sum_{P'\in V_i(P)} a(P',0,m_i)\Big\}.$$

Translating into generating functions gives the formula we want. (Note that, by our assumption at the beginning of this proof, $c(P,2n,0)(z) = 1$, $c(P,2n,i) = 0$ if $i \geq 1$, and $V_{2n}(P) = \emptyset$.)

We now consider the case when $r = 1$. Clearly $a(P,1,0) = 0$. Consider an element u of $A(P,1,m)$ for $m \geq 1$. Because $\varphi(u)|_d = P$,

$$\varphi(u) = \varphi(x_1) \oplus \varphi(e_1) \oplus \cdots \oplus \varphi(e_{2n-1}),$$

where $\sum l(e_i) = m-1$, and for each i, either $e_i \in A(P',0,l(e_i))$ for some $P' \in V_i(P)$ or $\varphi(e_i) = P_i$. Therefore,

$$a(P,1,m) = \sum_{m_1+\cdots+m_{2n-1}=m-1} \prod_{i=1}^{2n-1}\Big\{c(P,i,0,m_i) + \sum_{P'\in V_i(P)} a(P',0,m_i)\Big\}.$$

The formula for $r = 1$ now follows. The case for $r = 0$ is similar. ∎

(4.3) PROPOSITION. *For all $P \in V(\Gamma_S')$,*

$$b(P)(z) \in \mathbb{Q}\big(z, a(P',0)(z)\,|\,P' \in V(\Gamma_S')\big).$$

Proof. Write $V(\Gamma_S') = \{P_1, P_2, \ldots, P_t\}$. Because $c(P,i,r)(z) = 0$ whenever $r \geq d$, from the previous proposition,

$$a(P_i,r)(z) = \sum_{j=1}^{t} zf_{ij}(z)a\big(P_j,r-1\big)(z),$$

for all $r > d$, where $f_{ij}(z) \in \mathbb{Q}[z, a(P,0)(z)|P \in V(\Gamma_S')]$ are some series independent of r. Let M be the $t \times t$ matrix with entries $zf_{ij}(z)$. By the Hamilton–Cayley theorem,

$$M^t + f_{t-1}(z)M^{t-1} + \cdots + f_1(z)M + f_0(z)I = 0,$$

where $c_M(Y) = Y^t + f_{t-1}(z)Y^{t-1} + \cdots + f_0(z)$ is the characteristic polynomial of M, and I is the $t \times t$ identity matrix. Applying both sides to the column vector $[a(P_i, r)(z)]_{i=1}^t$, we get that

$$a(P, r+t)(z) + f_{t-1}(z)a(P, r+t-1)(z) + \cdots + f_0(z)a(P, r)(z) = 0,$$

for all $P \in V(\Gamma'_S)$, and for all $r \geq d$. Because $b(P)(z) = \sum_{r \geq 1} a(P, r)(z)$,

$$(1 + f_{t-1}(z) + \cdots + f_0(z))b(P)(z)$$

is a polynomial in $f_i(z)$ and $a(P', r)(z)$ for $P' \in V(\Gamma'_S)$ and $r \leq d + t$. Now each entry in the matrix M is an element of $\mathbb{Q}[z, a(P, 0)(z) | P \in V(\Gamma'_S)]$ with zero constant term, therefore each $f_i(z)$ is also. Thus,

$$1 + f_{t-1}(z) + \cdots + f_0(z) \in \mathbb{Q}\left[z, a(P, 0)(z) | P \in V(\Gamma'_S)\right],$$

and is invertible in $\mathbb{Q}[[z]]$. From the previous proposition, each $a(P', r)(z)$ is also in $\mathbb{Q}[z, a(P, 0)(z) | P \in V(\Gamma'_S)]$. The result now follows. ▮

(4.4) Proposition. $h_S(z) \in \mathbb{Q}(z, a(P, 0)(z) | P \in V(\Gamma'_S))$.

Proof. Let $\{P_1, \ldots, P_{2n}\}$ be a set of planted trees whose initial and terminal vertices coincide such that $\varphi^{-1}(P_1 \oplus \cdots \oplus P_{2n})$ is nonzero in S, and for each i, $e(P_i) \leq d$ and either P_i is null or label$(P_i) = x_i$. To control $h_S(z)$, it is enough to calculate for each such set of $\{P_1, \ldots, P_{2n}\}$ the number of $u \in FI_A$ which are nonzero in S and satisfy

$$\varphi(u)|_d = P_1 \oplus P_2 \oplus \cdots \oplus P_{2n}.$$

Consider such an element u. If $u \in E(S)$, then

$$\varphi(u) = \varphi(e_1) \oplus \cdots \oplus \varphi(e_{2n}),$$

where $\sum l(e_i) = l(u)$, and for each i, either $\varphi(e_i) = P_i$ or $e_i \in A(P_i, 0, l(e_i))$ (the latter can only occur if $P_i \in V(\Gamma'_S)$).

The contribution this makes to $h_S(z)$ is

$$a(P_1, 0)(z)a(P_2, 0)(z) \cdots a(P_{2n}, 0)(z),$$

where $a(P, 0)(z)$ has its usual meaning when $P \in V(\Gamma'_S)$, and $a(P_i, 0)(z) = z^{l(\varphi^{-1}(P_i))}$ otherwise. If $u \notin E(S)$, then for some $1 \leq j \leq 2n$,

$$\varphi(u) = \varphi(e_1) \oplus \cdots \oplus \varphi(e_{j-1}) \oplus \varphi(e_{j+1}) \oplus \cdots \oplus \varphi(e_{2n}) \oplus \varphi(u'),$$

where $\sum l(e_i) + l(u') = l(u)$, and

(i) for each $i \neq j$, either $\varphi(e_i) = P_i$ or $e_i \in A(P_i, 0, l(e_i))$ (the latter can only occur if $P_i \in V(\Gamma'_S)$),

(ii) either $\varphi(u') = (T, \alpha, \beta)$ with $(T, \alpha, \alpha) = P_j$ and $d(\alpha, \beta) > 0$, or $u' \in B(P_j, l(u'))$ (the latter can only occur if $P_j \in V(\Gamma'_S)$).

The contribution this makes to $h_S(z)$ is

$$b(P_j)(z) \prod_{i \neq j} a(P_i, 0)(z),$$

where $a(P, 0)(z)$ and $b(P)(z)$ have their usual meanings when $P \in V(\Gamma'_S)$, and are some polynomials in z otherwise.

Therefore $h_S(z)$ can be expressed as a polynomial in z, $a(P, 0)(z)$, and $b(P)(z)$, $P \in V(\Gamma'_S)$. The theorem now follows from the last proposition. ∎

Therefore, as an immediate corollary we have

(4.5) THEOREM. *Suppose $a(P, 0)(z)$ is rational for each $P \in V(\Gamma'_S)$. Then S has rational growth.*

Theorem (4.7) gives a condition on Γ'_S which guarantees the hypothesis of Theorem (4.5) is satisfied. First we prove a lemma.

(4.6) LEMMA. *Suppose $f(z) \in \mathrm{Rat}[[z]]$ has nonnegative coefficients. Then $f(z)$ has radius of convergence $R > 0$, and as $x \in \mathbb{R}$ increases to R, $f(x)$ increases to ∞.*

Proof. Suppose $f(z) = p(z)/q(z)$, where $p(z), q(z) \in \mathbb{R}[z]$ have no nonconstant common factors. Let $z_1, \ldots, z_s \in \mathbb{C}$ be the roots of $q(z)$. Then $f(z)$ has a positive radius of convergence $R = \min\{|z_1|, \ldots, |z_s|\}$. Let $R = |z_i|$, say.

Suppose $f(R)$ converges. Because $f(z)$ has nonnegative coefficients, for each z with $|z| \leq R$, $f(z)$ converges and $|f(z)| \leq f(R)$. But if $|z| < R$ and z approaches z_i, $|p(z)|$ is bounded below by a positive real number and $q(z) \to 0$, so $|f(z)|$ is not bounded. This contradiction shows that $f(R)$ must diverge. ∎

(4.7) THEOREM. *$a(P, 0)(z) \in \mathrm{Rat}[[z]]$ for each $P \in V(\Gamma'_S)$ if and only if Γ'_S does not contain a vertex P such that P is in two cycles,*

$$P \to P_1 \to \cdots \to P_s \to P \quad \textit{and} \quad P \to P'_1 \to \cdots \to P'_t \to P,$$

with $\mathrm{label}(P_1) \neq \mathrm{label}(P'_1)$.

Proof. Let $V_1, \ldots, V_t$ be the strongly connected components of Γ'_S, ordered so that the vertices in V_i are reachable from the vertices in V_j only if $i \leq j$.

We shall prove the "if" part by induction. Suppose that the condition on Γ'_S holds. Suppose $a(P, 0)(z) \in \mathrm{Rat}[[z]]$ for all vertices $P \in V_1 \cup V_2 \cup \cdots \cup V_k$. (So $k = 0$ initially.)

Let $V_{k+1} = \{P_1, \ldots, P_s\}$. Because the vertices in V_{k+1} are mutually reachable, the condition on Γ'_S implies that if $P' \in V_i(P)$ and $P'' \in V_j(P)$

for some P, P', and P'' in V_{k+1}, then label(P') = label(P'') and so $i = j$. By the formula for $a(P,0)(z)$ in Proposition (4.2) and the inductive hypothesis, there are functions $g_i(z), f_{ij}(z) \in \mathrm{Rat}[[z]]$ such that

$$a(P_i,0)(z) = z^2\left(g_i(z) + \sum_{j=1}^{s} f_{ij}(z)a(P_j,0)(z)\right),$$

for each i. Let M be the matrix with entries,

$$M_{ij} = \begin{cases} 1 - z^2 f_{ij}(z), & \text{if } i = j; \text{ and} \\ -z^2 f_{ij}(z), & \text{if } i \neq j. \end{cases}$$

Then

$$M\left[a(P_i,0)(z)\right]_{i=1}^{s} = \left[z^2 g_i(z)\right]_{i=1}^{s}.$$

Because $\det M$ clearly has constant term 1, $\det M \in \mathrm{Rat}[[z]]^*$ and M is invertible in the ring of $s \times s$ matrices over $\mathrm{Rat}[[z]]$. Therefore $a(P,0)(z) \in \mathrm{Rat}[[z]]$ for all $P \in V_{k+1}$. By induction on k, $a(P,0)(z) \in \mathrm{Rat}[[z]]$ for all $P \in V(\Gamma'_S)$.

Conversely, suppose $P \in V(\Gamma'_S)$ is in two cycles,

$$P \to P_1 \to \cdots \to P_s \to P \quad \text{and} \quad P \to P'_1 \to \cdots \to P'_t \to P,$$

with label$(P_1) \neq$ label(P'_1). It is easy to check, by the definition of Γ'_S, that if $u \in A(P,0,l(u))$ and $u' \in A(P,0,l(u'))$, then

$$\varphi^{-1}\big(P\odot\big(P_1\odot\big(P_2\odot\cdots\odot\big(P_s\odot\varphi(u)\big)\cdots\big)\big)$$
$$\odot\big(P'_1\odot\big(P'_2\odot\cdots\odot\big(P'_t\odot\varphi(u')\big)\cdots\big)\big)\big)$$

is in $A(P,0,l(u)+l(u')+l)$, for some positive integer l which is independent of u and u'. Thus for $m \geq l$,

$$a(P,0,m) \geq \sum_{i+j=m-l} a(P,0,i)a(P,0,j).$$

Therefore,

$$a(P,0)(z) - z^l a(P,0)(z)^2 = a(P,0)(z)\big(1 - z^l a(P,0)(z)\big)$$

is a power series with nonnegative coefficients.

Suppose $a(P,0)(z) \in \mathrm{Rat}[[z]]$ and has radius of convergence R. Because $a(P,0)(z)$ has nonnegative coefficients, as x increases to R, $a(P,0)(x) \to \infty$ by Lemma (4.7), contradicting that $1 - x^l a(P,0)(x) \geq 0$. Therefore $a(P,0)(z) \notin \mathrm{Rat}[[z]]$. ∎

Combining this with Theorem (4.5) gives a sufficient graphical condition for rational growth.

(4.8) THEOREM. *Suppose* Γ'_S *does not contain a vertex* P *in two cycles,*

$$P \to P_1 \to \cdots \to P_s \to P \quad \textit{and} \quad P \to P'_1 \to \cdots \to P'_t \to P,$$

with $\text{label}(P_1) \neq \text{label}(P'_1)$. *Then* S *has rational growth.*

It is an open problem whether the converse of the theorem holds also.

We conclude this section with a few remarks on the definitions of d and Γ'_S. The fact that

$$d + 1 \leq \max\{|c_i| \mid i = 1, \ldots, k\}$$

ensures that if $P \to P'$ is an edge in Γ'_S and $u \in A(P')$, then $\varphi^{-1}(P \odot \varphi(u))$ is nonzero in S. The fact that $d \geq 2$ ensures that if $P', P'' \in V(P)$ have different labels, then $\varphi^{-1}((P \odot P') \odot P'')$ is nonzero in S. These two observations, easily seen using Lemma (4.1), lie behind many of the proofs.

For each natural number $i \geq 1$ we now define a directed graph $\Gamma'_S(i)$ as follows:

(i) the vertices of $\Gamma'_S(i)$ are planted trees $P = (T, \alpha, \alpha)$ on A satisfying the properties that $e(P) = i$ and $\varphi^{-1}(P)$ is nonzero in S;

(ii) there is a directed edge from vertex P to vertex Q if $Q|_{i-1}$ is a component of P and $\varphi^{-1}(P \odot Q)$ is nonzero in S.

Obviously $\Gamma'_S = \Gamma'_S(d)$.

(4.9) PROPOSITION. *Suppose* $i \geq d$. *Then* $\Gamma'_S(i + 1)$ *does not contain a vertex* P *in two cycles,*

$$P \to P_1 \to \cdots \to P_s \to P \quad \textit{and} \quad P \to P'_1 \to \cdots \to P'_t \to P,$$

with $\text{label}(P_1) \neq \text{label}(P'_1)$ *if and only if* $\Gamma'_S(i)$ *does not.*

Proof. The proof in one direction is obvious given the observation that if

$$P \to P_1 \to \cdots \to P_s \to P$$

is a cycle in $\Gamma'_S(i + 1)$, then

$$P|_i \to P_1|_i \to \cdots \to P_s|_i \to P|_i$$

is a cycle in $\Gamma'_S(i)$. (This does not require $i \geq d$.)

For the converse we first need to observe that if $P_1 \to P_2 \to P_3$ in $\Gamma'_S(i)$, then $P_1 \odot P_2 \to P_2 \odot P_3$ is an edge in $\Gamma'_S(i + 1)$. This is because

$$\varphi^{-1}((P_1 \odot P_2) \odot (P_2 \odot P_3)) = \varphi^{-1}(P_1 \odot (P_2 \odot P_3))$$

is nonzero in S, given that $i \geq d$.

With this observation, we see that if

$$P \to P_1 \to \cdots \to P_s \to P$$

is a cycle in $\Gamma'_S(i)$, and $P'_1 \in V(P)$ with label$(P_1) \neq$ label(P'_1), then

$$(P \odot P_1) \odot P'_1 \to P_1 \odot P_2 \to \cdots \to P_s \odot P \to (P \odot P_1) \odot P'_1$$

is a cycle in $\Gamma'_S(i+1)$. The proof of the converse is now obvious. ∎

This proposition means that it is at least plausible that the converse of Theorem (4.8) holds, because the condition on Γ'_S is "stable" with respect to d. As previously remarked, only half of the proof of Proposition (4.9) requires $i \geq d$. So we have

(4.10) PROPOSITION. *If* $\Gamma'_S(i)$ *does not contain a vertex P in two cycles,*

$$P \to P_1 \to \cdots \to P_s \to P \quad \textit{and} \quad P \to P'_1 \to \cdots \to P'_t \to P,$$

with label$(P_1) \neq$ label(P'_1), *then* $\Gamma'_S(i+1)$ *does not either.*

This gives a possible shortcut for showing S has rational growth: for Γ'_S to satisfy the condition in Theorem (4.8), it is enough to show that $\Gamma'_S(i)$ satisfies the same condition for some $i \leq d$.

5. THE WORD TREE GRAPH AND POLYNOMIAL GROWTH

In this section we apply Theorem (4.8) to prove that if a semigroup S from our class $\mathfrak{M}_{FI}$ has polynomial growth then it has rational growth. Along the way we develop a new graphical criterion for polynomial growth.

Easdown and Shneerson introduced in [2] a directed graph Γ_S, the *Ufnarovsky graph of S* (depending on the presentation of S). Vertices of Γ_S are defined to be reduced words of length d which are nonzero in S. If v_1 and v_2 are vertices then a directed edge from v_1 to v_2 is defined in Γ_S if there exist letters $g, h \in A \cup A^{-1}$ such that $v_1 g$ is a reduced word which is nonzero in S and $v_1 g \doteq h v_2$. We regard the letter g as a label for this edge. Paths in Γ_S may then be labelled by reduced words which are nonzero in S. The pair (w, x) is an *adjacent pair* if w is a reduced word which labels a loop in Γ_S at a vertex v and x is a letter labelling an edge which emanates from v and terminates outside the loop. The following is Theorem 1 of [2].

(5.1) THEOREM. *Let* $S = \langle A | c_i = 0 \ \ for \ \ i = 1, \dots, k \rangle$ *be an inverse semigroup from the class* $\mathfrak{M}_{FI}$. *Then the following conditions are equivalent*:

(a) *S has polynomial growth.*

(b) *S does not contain any noncyclic free subsemigroup.*

(c) (i) Γ_S *has no vertex contained in different cycles*; *and*

(ii) *If* (w, x) *is an adjacent pair in* Γ_S *then*

$$w^{d+1} x x^{-1} w^{d+1} = 0 \quad in\ S.$$

Using the word tree graph we can give yet another characterisation of polynomial growth. This condition is slightly neater than Theorem (5.1), but generally the word tree graph is much more complicated than the Ufnarovsky graph. Other criteria for polynomial growth can be found in [3]. First we prove a technical lemma.

(5.2) LEMMA. *Let P be a vertex in* Γ'_S, *and* $P_1, \dots, P_s$ *be distinct elements of* $V(P)$. *Then there are constants* $n_1, \dots, n_s \in \mathbb{N}$ *such that for m sufficiently large,*

$$a(P, 0, m) \geq a(P_1, 0, m - n_1) + \cdots + a(P_s, 0, m - n_s).$$

Proof. Let $u_i \in A(P_i, 0, l(u_i))$ for each i. Let $v_i = \varphi^{-1}(P \odot \varphi(u_i))$. Then $v_i \in (P, 0, l(v_i))$, and $l(v_i) = n_i + l(u_i)$ for some constant n_i independent of the choice of u_i. Moreover if $i \neq j$ then $v_i \neq v_j$ because

$$\varphi(v_i)\big|_{d+1} = P \odot P_i \neq P \odot P_j = \varphi(v_j)\big|_{d+1}.$$

The result now follows. ∎

(5.3) THEOREM. *Let* $S = \langle A | c_i = 0 \ \ for \ \ i = 1, \dots, k \rangle$ *be an inverse semigroup from the class* $\mathfrak{M}_{FI}$. *Then S has polynomial growth if and only if* Γ'_S *has no vertex contained in different cycles.*

Proof. Suppose the vertex P is in two different cycles. Applying the previous lemma to the vertices in the two cycles we find that for m sufficiently large,

$$a(P, 0, m) \geq a(P, 0, m - n_1) + a(P, 0, m - n_2),$$

for some constants $n_1, n_2 \in \mathbb{N}$. Let $n = n_1 n_2$. Then for m large,

$$\begin{aligned} a(P, 0, m - n_1) &\geq a(P, 0, m - 2n_1) + a(P, 0, m - n_1 - n_2) \\ &\geq a(P, 0, m - 2n_1). \end{aligned}$$

So repeating we get

$$a(P, 0, m - n_1) \geq a(P, 0, m - 2n_1) \geq \cdots \geq a(P, 0, m - n).$$

Similarly $a(P, 0, m - n_2) \geq a(P, 0, m - n)$. Therefore $a(P, 0, m) \geq 2a(P, 0, m - n)$. Now $a(P, 0, s) > 0$ for infinitely many $s \in \mathbb{N}$ because we have a cycle at the vertex P. Fix an s sufficiently large. Then for all $t \in \mathbb{N}$, $a(P, 0, s + tn) \geq 2^t$. Therefore S has exponential growth.

To prove the converse, we use condition (c) of Theorem (5.1). It is clear that mapping a vertex w of Γ_S to $\varphi(ww^{-1})$ embeds Γ_S in Γ'_S. Therefore if no vertex of Γ'_S is contained in different cycles then no vertex of Γ_S is contained in different cycles.

Now suppose $w^{d+1}xx^{-1}w^{d+1}$ is nonzero in S for some adjacent pair (w, x) in Γ_S. We shall show that this gives a vertex in two different cycles in Γ'_S. Let s be the least multiple of $l(w)$ such that $s \geq d$. For $0 \leq i \leq s$, denote the prefix of w^{d+1} of length i by v_i, and the suffix of $w^{d+1}v_i$ of length d by w_i. Let $P_i = \varphi(w_i w_i^{-1})$, which is planted because w is cyclically reduced. Then

$$P_0 \to P_1 \to \cdots \to P_{s-1} \to P_s = P_0$$

is a cycle at P_0 in Γ'_S. For $1 \leq i \leq d - 1$, denote the suffix of w^{d+1} of length $d - i$ by u_i, and let $P'_i = \varphi(u_i xx^{-1} v_i (u_i xx^{-1} v_i)^{-1})$. Then

$$P_0 \to P'_1 \to P'_2 \to \cdots \to P'_{d-1} \to P_d \to P_{d+1} \to \cdots \to P_s = P_0$$

is a different cycle at P_0. ∎

Clearly if Γ'_S has no vertex in two cycles then the hypothesis of Theorem (4.8) is satisfied. So we have

(5.4) THEOREM. *If S has polynomial growth, then it has rational growth.*

It is of course interesting that a simple condition on Γ'_S gives a criterion for polynomial growth. But in fact Theorem (5.4) could have been deduced easily from Theorem (5.1) and the following result, which gives yet another possible shortcut for showing S has rational growth.

(5.5) THEOREM. *The following conditions are equivalent*:

(a) *Γ_S has no vertex contained in different cycles.*

(b) *If a vertex P in Γ'_S is contained in two cycles,*

$$P \to P_1 \to \cdots \to P_{s-1} \to P_s = P \quad \textit{and} \quad P \to P'_1 \to \cdots \to P'_{t-1} \to P'_t = P,$$

then $\operatorname{label}(P_i) = \operatorname{label}(P'_i)$ *for each* $i = 1, \ldots, \min\{s, t\}$.

Proof. First we make the following observation: if $P_1 \to P_2 \to \cdots \to P_d$ is a path in Γ'_S, and $a_i = \operatorname{label}(P_i)$ for each i, then $\varphi(a_1 a_2 \cdots a_d)$ is a subtree of $P_1 \odot (P_2 \odot \cdots \odot (P_{d-1} \odot P_d) \cdots)$. Therefore $a_1 a_2 \cdots a_d$ is a vertex of Γ_S.

To show (a) implies (b), suppose (b) does not hold. Let

$$P \to P_1 \to \cdots \to P_s \to P_{s+1} = P \quad \text{and} \quad P \to P_1' \to \cdots \to P_t' \to P_{t+1}' = P$$

be two cycles in Γ_S'. Let $a = \text{label}(P)$, $a_i = \text{label}(P_i)$, and $b_i = \text{label}(P_i')$ for each i. Suppose $a_i \neq b_i$ for some $1 \leq i \leq \min\{s+1, t+1\}$.

By replacing $P \to P_1 \to \cdots \to P_s \to P$ by

$$P \to P_1 \to \cdots \to P_s \to P \to P_1 \to \cdots \to P_s \to P,$$

and so on if necessary, we can assume $s \geq d$. Applying the foregoing observation to each subpath of length d of

$$P \to P_1 \to \cdots \to P_s \to P \to P_1 \to \cdots \to P_{d-1},$$

we see that

$$\begin{aligned} aa_1 \cdots a_{d-1} &\to a_1 a_2 \cdots a_d \to \cdots \to a_{s-d+2} \cdots a_s a \\ &\to a_{s-d+3} \cdots a_s a a_1 \to \cdots \to a_s a a_1 \cdots a_{d-2} \to a a_1 \cdots a_{d-1} \end{aligned}$$

is a cycle in Γ_S. But applying the observation to

$$P \to P_1 \to \cdots \to P_s \to P \to P_1' \to \cdots \to P_t' \to P \to P_1 \to \cdots \to P_{d-1},$$

shows that

$$\begin{aligned} aa_1 \cdots a_{d-1} &\to a_1 a_2 \cdots a_d \to \cdots \to a_{s-d+2} \cdots a_s a \\ &\to a_{s-d+3} \cdots a_s a b_1 \to \cdots \to b_t a a_1 \cdots a_{d-2} \to a a_1 \cdots a_{d-1} \end{aligned}$$

is another cycle in Γ_S. Because $a_i \neq b_i$ for some i, $aa_1 \cdots a_{d-1}$ is in two different cycles in Γ_S.

Now suppose (b) holds. Given two cycles,

$$w \to w_1 \to \cdots \to w_s \to w_{s+1} = w \quad \text{and}$$
$$w \to w_1' \to \cdots \to w_t' \to w_{t+1}' = w,$$

in Γ_S, we need to show that they are the same cycle. We can assume that the cycles are actually loops.

Let $P = \varphi(ww^{-1})$, $P_i = \varphi(w_i w_i^{-1})$, $P_i' = \varphi(w_i w_i^{-1})$. By considering Munn trees, it is clear that w_i is the concatenation of the labels of the first d vertices in the sequence,

$$P_i \to P_{i+1} \to \cdots \to P_s \to P \to P_1 \to \cdots.$$

Therefore condition (b) applied to the cycles,

$$P \to P_1 \to \cdots \to P_s \to P \to P_1 \to \cdots \quad \text{and}$$
$$P \to P_1' \to \cdots \to P_t' \to P \to P_1' \to \cdots,$$

actually shows that $w_i = w_i'$ for each i. So the original cycles are the same. ∎

If the equivalent conditions in Theorem (5.5) are satisfied by a semigroup S from the class $\mathfrak{M}_{FI}$, then clearly S has rational growth by Theorem (4.8). However the hypothesis in Theorem (4.8) is weaker than the conditions stated in Theorem (5.5). One example is $S = \langle x, y | x^2 = y^2 = 0 \rangle$. In this example $d = 2$, and Γ_S is isomorphic to Γ'_S. By Theorems (4.8) and (5.3), S also serves as an example of a semigroup in the class $\mathfrak{M}_{FI}$ which has rational and exponential growth.

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