Properties of hitting times for $G$-martingales and their applications

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Abstract

In this article, we consider the properties of hitting times for $G$-martingales and the stopped processes. We prove that the stopped processes for $G$-martingales are still $G$-martingales and that the hitting times for a class of $G$-martingales including one-dimensional $G$-Brownian motion are quasi-continuous. As an application, we improve the $G$-martingale representation theorems of [7].

Keywords: $G$-martingale; Stopping time; Stopped process

1. Introduction

Recently, [2,4] introduced the notion of sublinear expectation space, which is a generalization of probability space. One of the most important sublinear expectation spaces is the $G$-expectation space. As the counterparts of those for the Wiener space in the linear case, the notions of $G$-Brownian motion, the $G$-martingale, and the Itô integral w.r.t. $G$-Brownian motion were also introduced. These notions have very rich and interesting new structures which nontrivially generalize the classical ones.

As is well known, stopping times play a major role in classical stochastic analysis. However, it is difficult to apply the stopping time technique in sublinear expectation space since the stopped process may not belong to the class of processes which are meaningful in the present situation. For example, letting \( \{ M_t \}_{t \in [0,T]} \) be a $G$-martingale and \( \tau \) be an $\mathbb{F}$-stopping time, we do not know whether \( M^\tau_t \) has a quasi-continuous version for \( t \in [0,T] \).
In this article, we consider the properties of hitting times for $G$-martingales and the stopped processes. We prove that the stopped $G$-martingales are still $G$-martingales and that the hitting times for symmetric $G$-martingales with strictly increasing quadratic variation processes are quasi-continuous. We also consider the hitting times for the quadratic process $\{\langle B \rangle_t \}$ of the one-dimensional $G$-Brownian motion $\{B_t\}$. We prove that the hitting times for $\{\langle B \rangle_t \}$ are quasi-continuous. As applications, we obtain the following results. We prove that any symmetric random variable can be approximated by bounded random variables that are also symmetric. Moreover, we improve the results of [7] on the $G$-martingale representation by a stopping time technique. Finally, we give some concrete examples of elements in $H^\alpha_G(0, T)$, for $\alpha \geq 1$.

This article is organized as follows. In Section 2, we recall some basic notions and results for $G$-expectation and the related space of random variables. In Section 3, we give several preliminary lemmas. In Section 4, we prove that the stopped processes for $G$-martingales are still $G$-martingales and that the hitting times for a class of $G$-martingales including one-dimensional $G$-Brownian motion are quasi-continuous. We also prove that the hitting times for the quadratic process $\{\langle B \rangle_t \}$ of the one-dimensional $G$-Brownian motion $\{B_t\}$ are quasi-continuous. In Section 5, we give some applications of the stopping time technique. In the Appendix, we introduce the decomposition theorem and a regularity property for $G$-martingales that were obtained in [7].

2. Preliminary

We recall some basic notions and results for $G$-expectation and the related space of random variables. More relevant details can be found in [3].

2.1. $G$-expectation

**Definition 2.1.** Let $\Omega$ be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on $\Omega$ with $c \in \mathcal{H}$ for all constants $c$. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to R$ having the following properties. For all $X, Y \in \mathcal{H}$, we have:

(a) Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$.
(b) Constant preservation: $\hat{E}(c) = c$.
(c) Sub-additivity: $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$.
(d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X), \lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

**Definition 2.2.** Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, for all $\varphi \in C_{l, L^p}(R^n)$, where $C_{l, L^p}(R^n)$ is the space of real continuous functions defined on $R^n$ such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \quad \text{for all } x, y \in R^n,$$

where $k$ and $C$ depend only on $\varphi$.

**Definition 2.3.** In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random vector $Y = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}_i$, is said to be independent of another random vector $X = (X_1, \ldots, X_m), X_i \in \mathcal{H}_i$, under $\hat{E}(\cdot)$ if for each test function $\varphi \in C_{l, L^p}(R^m \times R^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}]$. 
Definition 2.4 (G-Normal Distribution). A $d$-dimensional random vector $X = (X_1, \ldots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called $G$-normally distributed if for all $a, b \in \mathbb{R}_+$, we have

$$aX + b\hat{X} \sim \sqrt{a^2 + b^2}X,$$

where $\hat{X}$ is an independent copy of $X$. Here the letter $G$ denotes the function

$$G(A) := \frac{1}{2} \hat{E}[(AX, X)]: S_d \to \mathbb{R},$$

where $S_d$ denotes the collection of $d \times d$ symmetric matrices.

The function $G(\cdot): S_d \to \mathbb{R}$ is a monotonic, sublinear mapping on $S_d$ and $G(A) = \frac{1}{2} \hat{E}[(AX, X)] \leq \frac{1}{2}|A|\hat{E}[|X|^2] =: \frac{1}{2}|A|\hat{\sigma}^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subseteq S^+_d$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{Tr}(\gamma A).$$

If there exists some $\beta > 0$ such that $G(A) - G(B) \geq \beta \text{Tr}(A - B)$ for any $A \geq B$, we call the $G$-normal distribution non-degenerate, which is the case that we consider throughout this article.

Definition 2.5. (i) Let $\Omega_T = C_0([0, T]; \mathbb{R}^d)$ be endowed with the supremum norm, and let $\mathcal{H}^0_T := \{\varphi(B_{t_1}, \ldots, B_{t_n})|n \geq 1, t_1, \ldots, t_n \in [0, T], \varphi \in C_{Lip}(R^{d \times n})\}$. The $G$-expectation is a sublinear expectation defined by

$$\hat{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})]$$

$$= \hat{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \ldots, \sqrt{t_m - t_{m-1}}\xi_m)],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}) \in \mathcal{H}^0_T$, where $\xi_1, \ldots, \xi_n$ are identically distributed $d$-dimensional $G$-normally distributed random vectors in a sublinear expectation space $(\hat{\Omega}, \hat{\mathcal{H}}, \hat{E})$ such that $\xi_{i+1}$ is independent of $(\xi_1, \ldots, \xi_i)$ for each $i = 1, \ldots, m$. $(\Omega_T, \mathcal{H}^0_T, \hat{E})$ is called a $G$-expectation space.

(ii) Let us define the conditional $G$-expectation $\hat{E}_t$ of $\xi \in \mathcal{H}^0_T$ knowing $\mathcal{H}^0_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}$ with $t = t_i$, for some $1 \leq i \leq n$, and we put

$$\hat{E}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})]$$

$$= \hat{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\hat{\varphi}(x_1, \ldots, x_i) = \hat{E}[\varphi(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_m} - B_{t_{m-1}})].$$

Let $\|\xi\|_{p,G} = [\hat{E}(\|\xi\|^p)]^{1/p}$, for $\xi \in \mathcal{H}^0_T$ and $p \geq 1$. For all $t \in [0, T]$, $\hat{E}_t(\cdot)$ is a continuous mapping on $\mathcal{H}^0_T$ endowed with the norm $\|\cdot\|_{1,G}$. Therefore, it can be extended continuously to the completion $L^G_{\mathcal{H}}(\Omega_T)$ of $\mathcal{H}^0_T$ under the norm $\|\cdot\|_{1,G}$.

Theorem 2.6 ([1]). There exists a tight subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$, the collection of all probabilities on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that

$$\hat{E}(\xi) = \max_{P \in \mathcal{P}} E_P(\xi) \quad \text{for all } \xi \in \mathcal{H}^0_T,$$

$\mathcal{P}$ is called a set that represents $\hat{E}$. 

Remark 2.7.  
(i) Let $\mathcal{A}$ denote the collection of all sets $\mathcal{P}$ representing $\hat{E}$ in the sense of Theorem 2.6. The set $\mathcal{P}^* = \{ P \in \mathcal{M}_1(\Omega_T) | E_P(\xi) \leq \hat{E}(\xi), \text{ for all } \xi \in \mathcal{H}^0_T \}$, is obviously the maximal one in $\mathcal{A}$; it is convex and weakly compact. All capacities induced by weakly compact sets of probabilities in $\mathcal{A}$ are the same, i.e. $c_P := \sup_{P \in \mathcal{P}} P = \sup_{P \in \mathcal{P}'} P =: c_{P'}$ for any weakly compact sets $\mathcal{P}, \mathcal{P}' \in \mathcal{A}$.

(ii) Let $(\Omega^0, \mathcal{F}^0 = \{ \mathcal{F}_t^0 \}, \mathcal{P}^0)$ be a filtered probability space, and $\{W_t\}$ be a $d$-dimensional Brownian motion. However, all results in the following sections of this article also hold for $\{ B_t^0(\omega) = \omega_t \}$ for $\omega \in \Omega^0_T$ is a martingale.

Definition 2.8.  
(i) Let $c$ be the capacity induced by $\hat{E}$. A map $X$ on $\Omega_T$ with values in a topological space is said to be quasi-continuous w.r.t. $c$ if for any $\varepsilon > 0$, there exists an open set $O$ with $c(O) < \varepsilon$ such that $X|_O$ is continuous.

(ii) We say that $X : \Omega_T \to R$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega_T \to R$ such that $X = Y$, $c$-q.s.  

Let $\| \psi \|_{p,G} = [\hat{E}(|\psi|^p)]^{1/p}$ for $\psi \in C_b(\Omega_T)$. The completions of $C_b(\Omega_T)$, $\mathcal{H}^0_T$ and $L_{ip}(\Omega_T)$ under $\| \cdot \|_{p,G}$ are the same and we denote them by $L^p_G(\Omega_T)$. Here

$$L_{ip}(\Omega_T) := \{ \varphi(B_{t_1}, \ldots, B_{t_n}) | n \geq 1, t_1, \ldots, t_n \in [0, T], \varphi \in C_{b,L_{ip}}(R^{d \times n}) \},$$

where $C_{b,L_{ip}}(R^{d \times n})$ denotes the set of bounded Lipschitz functions on $R^{d \times n}$.

Theorem 2.9 ([11]). For $p \geq 1$, the completion $L^p_G(\Omega_T)$ of $C_b(\Omega_T)$ is

$$L^p_G(\Omega_T) = \{ X \in L^0 : X \text{ has a q.c. version, } \lim_{n \to \infty} \hat{E}[|X|^p|1_{|X|>n}] = 0 \},$$

where $L^0$ denotes the space of all real valued measurable functions on $\Omega_T$.

2.2. Basic notions on stochastic calculus in a sublinear expectation space

For brevity we only give the definition of the Itô integral with respect to the one-dimensional $G$-Brownian motion. However, all results in the following sections of this article also hold for the $d$-dimensional case unless otherwise stated.

Let $H^0_G(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \ldots, t_N\} = \pi_T$ of $[0, T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_{j+1}(\omega) 1_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_t), \ i = 0, 1, 2, \ldots, N - 1$. For each $\eta \in H^0_G(0, T)$, let $\| \eta \|_{H^p_G} = \{ \hat{E}(\int_0^T |\eta_s|^2 ds)^{p/2} \}^{1/p}$ and let us denote by $H^p_G(0, T)$ the completion of $H^0_G(0, T)$ under the norm $\| \cdot \|_{H^p_G}$. 
Definition 2.10. For each \( \eta \in H^0_G(0, T) \) in the form
\[
\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)1_{[t_j, t_{j+1})}(t),
\]
we define
\[
I(\eta) = \int_0^T \eta(s) \, dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).
\]

By the B–D–G inequality under \( P \), for all \( P \in \mathcal{P}_M \), the mapping \( I : H^0_G(0, T) \to L^p_G(\Omega_T) \) is continuous under \( \| \cdot \|_{H^p_G} \) and, thus, it can be continuously extended to \( H^0_G(0, T) \).

Definition 2.11. A process \( \{M_t\} \) with values in \( L^1_G(\Omega_T) \) is called a \( G \)-martingale if \( \hat{E}_s(M_t) = M_s \) for any \( s \leq t \). If \( \{M_t\} \) and \( \{-M_t\} \) are both \( G \)-martingales, we call \( \{M_t\} \) a symmetric \( G \)-martingale.

Definition 2.12. For two processes \( \{X_t\}, \{Y_t\} \) with values in \( L^1_G(\Omega_T) \), we say that \( \{X_t\} \) is a version of \( \{Y_t\} \) if
\[
X_t = Y_t, \quad \text{q.s., for all } t \in [0, T].
\]

3. Some lemmas

Definition 3.1. We say that a process \( \{M_t\} \) with values in \( L^1_G(\Omega_T) \) is quasi-continuous if:
\[\text{for all } \varepsilon > 0, \text{ there exists an open set } G \text{ with } c(G) < \varepsilon \text{ such that } M.(\cdot) \text{ is continuous on } G^c \times [0, T].\]

By Theorem A.6, we know that any \( G \)-martingale \( \{M_t\} \) has a quasi-continuous version. So we shall only consider quasi-continuous \( G \)-martingales in what follows. The following lemma is the counterpart of Doob’s uniform integrability lemma, and the proof is adapted from [8].

Suppose that \( B_t = \sigma(B_s | s \leq t), F_t = \cap_{r > t} B_r \) and \( \mathbb{F} = \{F_t \}_{t \in [0, T]} \). The mapping \( \tau : \Omega_T \to [0, T] \) is called an \( \mathbb{F} \) stopping time if \( [\tau \leq t] \in F_t, \forall t \in [0, T] \).

Lemma 3.2. Let \( \{M_t\} \) be a symmetric or negative \( G \)-martingale with \( M_T \in L^p_G(\Omega_T) \) for \( p \geq 1 \). Then \( \{|M_{\sigma_i}|^p\}_{i \in I} \) is uniformly integrable under \( \hat{E} \) in the following sense:
\[
\sup_{i \in I} \hat{E} [\{|M_{\sigma_i}|^p 1_{\{|M_{\sigma_i}| > n\}}] \to 0, \quad \text{as } n \to \infty.
\]
Here \( \{\sigma_i | i \in I\} \) is an arbitrary family of stopping times w.r.t. \( \mathbb{F} \).

Proof. First we recall that if \( M \) is a symmetric \( G \)-martingale (resp., \( G \)-martingale), it is a \( P \)-martingale (resp., \( P \)-supermartingale), for all \( P \in \mathcal{P}_M \). Then, for all \( \delta > 0, n \geq 1 \),
\[
E_P[|M_{\sigma_i}|^p 1_{\{|M_{\sigma_i}| > n\}}] \leq E_P[|M_T|^p 1_{\{|M_T| > \delta\}}]
\]
\[
\leq \delta^p P(|M_{\sigma_i}| > n) + E_P[|M_T|^p 1_{\{|M_T| > \delta\}}]
\]
\[
\leq \delta^p n^{-p} E_P(|M_{\sigma_i}|^p) + E_P[|M_T|^p 1_{\{|M_T| > \delta\}}]
\]
\[
\leq \delta^p n^{-p} E_P(|M_T|^p) + E_P[|M_T|^p 1_{\{|M_T| > \delta\}}].
\]
Let $E$ be a metric space and let a mapping $M$ given by

$$\tau = \inf \{ t \geq 0 | M_t \geq a \} \wedge T \text{ and } \tau_a = \inf \{ t \geq 0 | M_t > a \} \wedge T. $$

(i) $M_t \wedge \tau_a$ is continuous at any $\omega \in E$ with $M_t \wedge \tau_a(\omega) < a$ and $M_t \wedge \tau_a$ is continuous at any $\omega \in \hat{E}$ with $M_t \wedge \tau_a(\omega) = a$. Moreover, $-M_t \wedge \tau_a$, $M_t \wedge \tau_a$ are both lower semi-continuous.

(ii) $-\tau_a$ and $\tau_a$ are both lower semi-continuous.

Proof. (i) For $\omega$ with $M_t \wedge \tau_a(\omega) = a$, $M_t \wedge \tau_a(\cdot)$ is obviously continuous at $\omega$. Also, we claim that for $\omega$ with $M_t \wedge \tau_a(\omega) < a$, $M_t \wedge \tau_a(\cdot)$ is continuous at $\omega$. Otherwise, there exists a sequence $\{\omega_n\} \subset \Omega_T$ and a sequence $\{t_n\} \subset [0, T]$ such that $\omega_n \to \omega$ and $M_{t_n}(\omega_n) \geq a$. Assume that $t_n \to t' \in [0, T]$; then

$$|M_{t_n}(\omega_n) - M_{t'}(\omega)| \to 0.$$ 

So $M_t(\omega) \geq a$ and $M_t \wedge \tau_a(\omega) \geq a$, which contradicts the assumption.

For any $b \in R$, we claim that $[M_t \wedge \tau_a < b]$ and $[M_t \wedge \tau_a > b]$ are both open. If $b > a$, $[M_t \wedge \tau_a < b]$ is obviously open. Assume that $b \leq a$. For any $\omega \in [M_t \wedge \tau_a < b]$, there exists an open set $S$ such that $\omega \in \omega \subset [M_t \wedge \tau_a < b]$ since $M_t \wedge \tau_a$ is continuous at $\omega$. So $[M_t \wedge \tau_a < b]$ is open. Also, $[M_t \wedge \tau_a > b]$ is obviously open for $b \geq a$. Assume that $b < a$. If $M_t \wedge \tau_a(\omega) = a$, there exists an open set $S$ such that $\omega \in \omega \subset [M_t \wedge \tau_a > b]$ since $M_t \wedge \tau_a$ is continuous at $\omega$. For $b < M_t \wedge \tau_a(\omega) < a$, we have $b < M_t(\omega)$. Then there exists an open set $S$ such that $\omega \in \omega \subset [M_t > b] \subset [M_t \wedge \tau_a > b]$ since $M_t$ is continuous at $\omega$. So $[M_t \wedge \tau_a > b]$ is open.

(ii) For any $t \in [0, T]$, $[\tau_a < t]$ is obviously open. For any $t \in [0, T]$, $[\tau_a > t] = [M_t \wedge \tau_a < a]$ is open by (i). 

Lemma 3.4. For any closed set $F \subset \Omega_T$, we have

$$c(F) = \inf\{c(O) | F \subset O, O \text{ is open} \},$$

where $c$ is the capacity induced by $\hat{E}$.

Proof. It suffices to prove that for any closed set $F \subset \Omega_T$, $c(F) \geq \inf\{c(O) | F \subset O, O \text{ is open} \}$. In fact, for any closed set $F \subset \Omega_T$, there exists $\{\varphi_n\} \subset C_b(\Omega_T)$ such that $1 \geq \varphi_n \downarrow 1_F$. By Theorem 31 in [1], we have $c(F) = \lim_{n \to \infty} \hat{E}(\varphi_n)$. Let $O_n = [\varphi_n > 1 - 1/n]$. Then $O_n \supset F$ and $c(O_n) \leq \frac{1}{n} \hat{E}(\varphi_n) \to c(F)$. So $c(F) \geq \inf_n c(O_n) \geq \inf\{c(O) | F \subset O, O \text{ is open} \}$. 

Motivated by the comments from one of the referees, we add the following lemma.

Lemma 3.5. Let $\{M_t\}$ be a right continuous G-martingale (i.e., for any $\varepsilon > 0$, there exists an open set $O$ with $c(O) < \varepsilon$ such that for every $\omega \in O^c$, $\{M_t(\omega)\}_{t \in [0,T]}$ is right continuous) with $M_t$ quasi-continuous for all $t \in [0, T]$. Then $\{M_t\}$ is quasi-continuous.

Proof. Let $\{N_t\}$ be a quasi-continuous version of $\{M_t\}$. Fix $t \in [0, T]$. For any $\varepsilon > 0$, there exists an open set $O$ with $c(O) < \varepsilon / 2$ such that $M_t|_{O^c}$, $N_t|_{O^c}$ are both continuous since $M_t$, $N_t$ are both quasi-continuous. Noting that $O^c \cap [\{M_t - N_t\} > 0]$ is a polar set, $O^c \cap [\{M_t - N_t\} \geq 1/n]$
is a closed polar set for each \( n \in N \). By Lemma 3.4, there exists an open set \( G \) with \( c(G) < \varepsilon/2 \) such that \( O^c \cap \{ |M_t - N_t| > 0 \} \subset G \). So \( M_t = N_t \) on \( O^c \cap G^c \). By the right continuity of \( \{M_t\} \) and \( \{N_t\} \), for any \( \varepsilon > 0 \), there exists an open set \( O \) with \( c(O) < \varepsilon \) such that \( M_t(\omega) = N_t(\omega) \) on \( O^c \times [0, T] \). So \( \{M_t\} \) is quasi-continuous. \( \square \)

4. Hitting times for \( G \)-martingales

4.1. Hitting times for symmetric \( G \)-martingales

In this section, we try to define stopped processes for symmetric \( G \)-martingales.

Let

\[
\mathcal{Q}_T = \{(r, s) | T \geq r > s \geq 0, \ r, s \text{ are rational}\}
\]

and

\[
\mathcal{S}_a(M) = \{\omega \in \Omega_T | \text{there exists } (r, s) \in \mathcal{Q}_T \text{ s.t. } M_t(\omega) = a \text{ for all } t \in [s, r]\}.
\]

**Theorem 4.1.** Let \( \{M_t\}_{t \in [0, T]} \) be a quasi-continuous symmetric \( G \)-martingale. Then for all \( a > M_0 \) and \( \tau_a, \bar{\tau}_a \) defined in Lemma 3.3, we have the following conclusions:

(i) For all \( t \in [0, T] \), \( M_{t \wedge \tau_a} \) and \( M_{t \wedge \bar{\tau}_a} \) are both quasi-continuous. Consequently, \( \{M_{t \wedge \tau_a}\} \) and \( \{M_{t \wedge \bar{\tau}_a}\} \) are both symmetric \( G \)-martingales.

(ii) If in addition \( c(\mathcal{S}_a(M)) = 0 \), then \( \bar{\tau}_a, \tau_a \) are both quasi-continuous.

**Proof.** (i) Since \( \{M_t\}_{t \in [0, T]} \) is a symmetric \( G \)-martingale, it is a martingale under each \( P \in \mathcal{P}_M \).

Therefore, \( E_P(M_{t \wedge \tau_a}) = M_0 = E_P(M_{t \wedge \bar{\tau}_a}) \) for each \( P \in \mathcal{P}_M \). Consequently, \( \hat{E}(M_{t \wedge \tau_a} - M_{t \wedge \bar{\tau}_a}) = 0 \). Noting that \( M_{t \wedge \bar{\tau}_a} \geq M_{t \wedge \tau_a} \), we get \( M_{t \wedge \bar{\tau}_a} = M_{t \wedge \tau_a} \), q.s. Since \( \{M_t\} \) is quasi-continuous, for any \( c > 0 \), there exists an open set \( G \) with \( c(G) < \varepsilon/2 \) such that \( M(\cdot) \) is continuous on \( G^c \times [0, T] \). Let \( Q = \{(r, s)|r > s, \ r, s \text{ are rational}\} \). Noting that

\[
[M_{t \wedge \tau_a} > M_{t \wedge \bar{\tau}_a}] = \bigcup_{(r, s) \in Q} \{[M_{t \wedge \tau_a} \geq r, s \geq M_{t \wedge \bar{\tau}_a}]\},
\]

we have

\[
[M_{t \wedge \tau_a} > M_{t \wedge \bar{\tau}_a}] \subset G \bigcup \bigcup_{(r, s) \in Q} \{[M_{t \wedge \tau_a} \geq r, s \geq M_{t \wedge \bar{\tau}_a}] \cap G^c\}.
\]

By Lemma 3.3, \( [M_{t \wedge \tau_a} \geq r, s \geq M_{t \wedge \tau_a}] \cap G^c \) is closed for any \( (r, s) \in Q \). Since \( c([M_{t \wedge \tau_a} \geq r, s \geq M_{t \wedge \tau_a}] \cap G^c) = 0 \), by Lemma 3.4, there exists an open set \( O \) with \( c(O) < \varepsilon/2 \) such that

\[
\bigcup_{(r, s) \in Q} \{[M_{t \wedge \tau_a} \geq r, s \geq M_{t \wedge \tau_a}] \cap G^c\} \subset O.
\]

By Lemma 3.3, \( M_{t \wedge \tau_a} \) and \( M_{t \wedge \bar{\tau}_a} \) are both continuous on \( O^c \cap G^c \).

(ii) By the quasi-continuity of \( \{M_t\} \), for any \( \varepsilon > 0 \), there exists an open set \( G \) such that \( c(G) < \varepsilon/2 \) and \( M_t(\omega) \) is continuous on \( G^c \times [0, T] \). So

\[
G^c \cap [\tau_a > \bar{\tau}_a] \subset \mathcal{S}_a(M) \bigcup \bigcup_{r \in \mathbb{Q} \cap [0, T]} [M_{r \wedge \tau_a} < M_{r \wedge \bar{\tau}_a}],
\]

where \( \mathbb{Q} \) denotes the totality of rational numbers. Thus, from the assumption that \( C(\mathcal{S}_a(M)) = 0 \) combined with the proof of (i) of the present theorem we see that \( c(G^c \cap [\tau_a > \bar{\tau}_a]) = 0 \). Since

\[
G^c \cap [\tau_a > \bar{\tau}_a] = \bigcup_{(r, s) \in \mathcal{Q}_T} ([\tau_a \geq r, s \geq \tau_a] \cap G^c)
\]

and \( [\tau_a \geq r, s \geq \tau_a] \cap G^c \) is closed by Lemma 3.3, there exists an open set \( O \) such that \( c(O) < \varepsilon/2 \) and \( G^c \cap [\tau_a > \bar{\tau}_a] \subset O \). So on \( O^c \cap G^c \), \( \tau_a = \bar{\tau}_a \) are both continuous. \( \square \)
Remark 4.2. If the quadratic variation process of a symmetric $G$-martingale $\{M_t\}$ is strictly increasing except on a polar set, then $c(S_a(M)) = 0$ for any $a \in \mathbb{R}$.

Example 4.3. Let $\{B_t\}_{t \in [0,T]}$ be a one-dimensional $G$-Brownian motion. For $a > 0$, let $\tau_a = \inf\{t \geq 0| B_t \geq a\} \wedge T$ and $\bar{\tau}_a = \inf\{t \geq 0| B_t > a\} \wedge T$. Then we have:

(i) For any $t \in [0, T]$, $-B_t \wedge \bar{\tau}_a, B_t \wedge \bar{\tau}_a, -\tau_a$ and $\bar{\tau}_a$ are all lower semi-continuous.

(ii) For any $t \in [0, T]$, $B_t \wedge \bar{\tau}_a, B_t \wedge \tau_a, \bar{\tau}_a$ and $\tau_a$ are all quasi-continuous.

(iii) $\{B_t \wedge \bar{\tau}_a\}$ and $\{B_t \wedge \tau_a\}$ are both symmetric $G$-martingales.

4.2. Hitting times for possibly non-symmetric $G$-martingales

For each $P \in \mathcal{P}_M$ and $t \in [0, T]$, let $\mathcal{A}_{t,P} := \{Q \in \mathcal{P}_M|Q|\mathcal{F}_t = P|\mathcal{F}_t\}$. Theorem 2.3 in [5] implies the following result. For $t \in [0, T]$ and $\xi \in L^1_G(\Omega_T), \eta \in L^1_G(\Omega_t), \eta = \hat{E}_t(\xi)$ if and only if for each $P \in \mathcal{P}_M$,

$$\eta = \text{ess sup}_{Q \in \mathcal{A}_{t,P}}^P E_Q(\xi|\mathcal{F}_t), \quad P\text{-a.s.,}$$

where $\text{ess sup}_{Q \in \mathcal{A}_{t,P}}^P$ denotes the essential supremum under $P$.

**Theorem 4.4.** Let $\{M_t\}_{t \in [0,T]}$ be a quasi-continuous $G$-martingale. For all $a > \{|M_0|, M_t \wedge \bar{\sigma}_a$ and $M_t \wedge \sigma_a\}$ are both $G$-martingales, where $\sigma_a = \inf\{t \geq 0| M_t \leq -a\} \wedge T$ and $\bar{\sigma}_a = \inf\{t \geq 0| M_t < -a\} \wedge T$.

**Proof.** For each $P \in \mathcal{P}_M, \{M_t\}_{t \in [0,T]}$ is a supermartingale. Noting that $\sigma_a \leq \bar{\sigma}_a$, for each $t \in [0, T]$, we have

$$E_P(M_t \wedge \bar{\sigma}_a|\mathcal{F}_t \wedge \sigma_a) \leq M_t \wedge \sigma_a$$

by the Doob optimal stopping theorem. This implies that $E_P(M_t \wedge \bar{\sigma}_a - M_t \wedge \sigma_a) \geq 0$. On the other hand, it is obvious that $M_t \wedge \bar{\sigma}_a \leq M_t \wedge \sigma_a$. So $M_t \wedge \bar{\sigma}_a = M_t \wedge \sigma_a$ q.s. By the same arguments as in Theorem 4.1, for any $\epsilon > 0$, there exists an open set $O$ such that $c(O) < \epsilon$ and $M_t \wedge \bar{\sigma}_a = M_t \wedge \sigma_a$ are continuous on $O^c$. So $M_t \wedge \bar{\sigma}_a$ and $M_t \wedge \sigma_a$ are both quasi-continuous.

Suppose that $\sigma = \bar{\sigma}_a$ or $\sigma_a$.

For $P \in \mathcal{P}_M$, we recall that $M$ is a $Q$-supermartingale for all $Q \in \mathcal{A}_{s,P}$. Therefore, for $0 \leq s < t \leq T$,

$$\hat{E}_s(M_t \wedge \sigma) = \text{ess sup}_{Q \in \mathcal{A}_{s,P}}^P E_Q(M_t \wedge \sigma|\mathcal{F}_s) \leq M_s \wedge \sigma, \quad P\text{-a.s.}$$

On the other hand, for all $Q \in \mathcal{A}_{s,P}$,

$$E_Q(M_t \wedge \sigma|\mathcal{F}_s) = -a1_{[\sigma \leq s]} + E_Q(M_t \wedge \sigma|\mathcal{F}_s)1_{[\sigma > s]} \geq -a1_{[\sigma \leq s]} + E_Q[M_t|\mathcal{F}_s, \sigma > s]1_{[\sigma > s]} = -a1_{[\sigma \leq s]} + E_Q(M_t|\mathcal{F}_s)1_{[\sigma > s]} = -a1_{[\sigma \leq s]} + E_Q(M_t|\mathcal{F}_s)1_{[\sigma > s]}.$$
Lemma 3.5

Let \( \tau \), \( \xi \in L^1(\Omega_T) \) be symmetric if \( \hat{E}(\xi) + \hat{E}(-\xi) = 0 \).

Theorem 4.5. Let \( \tau_a = \inf\{t \geq 0| (B)_t \geq a\} \wedge T \) and \( \tau_a = \inf\{t \geq 0| (B)_t > a\} \wedge T \). Then \( \tau_a, \tau_a \) are both quasi-continuous.

Proof. If Theorem A.6, \( \langle B \rangle_t - \tilde{\sigma}^2 t \) has a quasi-continuous version. Consequently, \( \langle B \rangle_t \) has a quasi-continuous version. We assume that \( \langle B \rangle_t \) is quasi-continuous. For any \( \varepsilon > 0 \), there exists an open set \( G \) with \( c(G) < \varepsilon/2 \) such that \( \langle B \rangle_t(\omega) \) is continuous on \( G \times [0, T] \). By Lemma 3.3, \( -\tau_a \) and \( \tau_a \) are both lower semi-continuous on \( G \). Since \( \langle B \rangle_t \) is strictly increasing q.s., \( [\tau_a > \tau_a] \) is a polar set. Since

\[
[\tau_a > \tau_a] \subset G \bigcup_{(r,s) \in Q_T} ([\tau_a \geq r,s \geq \tau_a] \cap G^c)
\]

and \( [\tau_a \geq r,s \geq \tau_a] \cap G^c \) is a closed polar set for every \( (r,s) \in Q_T \), there exists an open set \( O \) with \( c(O) < \varepsilon/2 \) such that \( \bigcup_{(r,s) \in Q} ([\tau_a \geq r,s \geq \tau_a] \cap G^c) \subset O \). Therefore, \( \tau_a, \tau_a \) are both continuous on \( O^c \cap G^c \).

The following corollary is straightforward.

Corollary 4.6. For any \( a \geq 0 \), suppose that \( \tau_a = \tau_a \) or \( \tau_a \). \( B_{\tau_a} \) is quasi-continuous.

5. Applications

We call a random variable \( \xi \in L^1(\Omega_T) \) symmetric if \( \hat{E}(\xi) + \hat{E}(-\xi) = 0 \).

Theorem 5.1. Let \( \xi \in L^\beta(\Omega_T) \) for some \( \beta \geq 1 \) be symmetric. Then there exists a sequence \( \{\xi^n\} \subset L^1(\Omega_T) \) of elements which are bounded and symmetric such that \( \hat{E}(||\xi - \xi^n||^\beta) \to 0 \).

Proof. Let \( M \) be a quasi-continuous version of \( \langle \hat{E}_t(\xi) \rangle_{t \in [0,T]} \). For each \( n \in N \), let \( \sigma_n = \inf\{t \geq 0| M_t > n\} \wedge T \), and \( \tau_n = \inf\{t \geq 0| M_t > n\} \wedge T \). By Theorem 4.1, \( \{M_{\tau_n}^\xi\}_{t \in [0,T]} \) is a symmetric G-martingale with \( M_{\tau_n}^\xi \) quasi-continuous for each \( t \in [0,T] \). So \( \{M_{\tau_n}^\xi\}_{t \in [0,T]} \) is quasi-continuous by Lemma 3.5. Let \( s_n = \inf\{t \geq 0| M_{\tau_n}^\xi > n\} \wedge T \). By the same arguments, \( \{M_{\tau_n}^\xi \} \) is a bounded symmetric G-martingale. So \( M_{\tau_n}^\xi = M_{\tau_n}^\xi \wedge \bar{s}_n \) is a bounded symmetric G-martingale.

\[
|M_{\sigma_n} - M_T|^\beta \leq 2^{\beta-1}(M_{\sigma_n} - (M_T \wedge n) \vee (-n))|^\beta + (M_T \wedge n) \vee (-n) - M_T)^\beta
\]

\[
\leq 2^{\beta-1}|M_{\sigma_n}|^\beta 1_{|M_{\sigma_n}| \geq n} + 2^{\beta-1}|M_T|^\beta 1_{|M_T| > n}.
\]
Hence, by Lemma 3.2,
\[ \hat{E}(|M_{\sigma_n} - M_T|^\beta) \leq 2^{2^\beta - 1} \sup_i \hat{E}(|M_{\sigma_i}|^\beta 1_{|M_{\sigma_i}| \geq n}) + 2^{\beta - 1} \hat{E}(|M_T|^\beta 1_{|M_T| > n}) \to 0. \]

The following corollary improves Corollary A.3.

**Corollary 5.2.** Suppose that \( \xi \in L_G^\beta(\Omega_T) \) for some \( \beta > 1 \) with \( \hat{E}(\xi) + \hat{E}(-\xi) = 0 \). Then there exists \( \{Z_t\}_{t \in [0,T]} \in H_G^\beta(0, T) \) such that
\[ \xi = \hat{E}(\xi) + \int_0^T Z_s dB_s, \quad \text{q.s.} \]

**Proof.** By Theorem 5.1, there exists a sequence \( \{\xi^n\} \subset L^1(\Omega_T) \) of elements which are bounded and symmetric such that \( \hat{E}(|\xi^n|^\beta) \to 0 \). By Corollary A.3, there exists \( \{Z^n_t\}_{t \in [0,T]} \in H_G^\beta(0, T) \) such that
\[ \xi^n = \hat{E}(\xi^n) + \int_0^T Z^n_s dB_s. \]

By the B–D–G and Doob’s maximal inequalities, \( \{Z^n_t\}_{t \in [0,T]} \) is a Cauchy sequence in \( H_G^\beta(0, T) \). So there exists \( \{Z_t\} \in H_G^\beta(0, T) \) such that \( \|Z^n - Z\|_{H_G^\beta} \to 0 \). Then
\[ \xi = \lim_{L_G^n, n \to \infty} \xi^n = \lim_{L_G^n, n \to \infty} \left[ \hat{E}(\xi^n) + \int_0^T Z^n_s dB_s \right] = \hat{E}(\xi) + \int_0^T Z_s dB_s. \]

**Theorem 5.3.** Let \( \xi \in L_G^2(\Omega_T) \) be bounded above. Then \( M_t = \hat{E}_t(\xi), t \in [0, T] \) has the following representation:
\[ M_t = M_0 + \int_0^t Z_s dB_s - K_t, \quad (5.0.1) \]
where \( \{Z_t\} \in M_G^2(\Omega_T), \{K_t\} \) is a quasi-continuous increasing process with \( K_0 = 0 \) and \( \{-K_t\}_{t \in [0,T]} \) a G-martingale.

**Proof.** Without loss of generality we assume that \( \xi \leq 0 \). Let \( \{M_t\} \) be a quasi-continuous version of \( \{\hat{E}(\xi)\} \). For \( n \in N \), suppose that \( \sigma_n = \sigma_{n+1} \), defined in Theorem 4.4. Then \( M_{\sigma_n} \) is bounded. By Theorems 4.4 and A.2, we have the following representation:
\[ M_t^{\sigma_n} = \hat{E}(M_T) + \int_0^t Z^n_s dB_s - K_t^n =: n^n_t - K_t^n, \quad \text{q.s.,} \]
where \( \{Z^n_t\}_{t \in [0,T]} \in H_G^2(0, T) \) and \( \{K_t^n\}_{t \in [0,T]} \) is a continuous increasing process with \( K_0^n = 0 \) and \( \{-K_t^n\}_{t \in [0,T]} \) a G-martingale. For \( m > n \), by the uniqueness of the decomposition of a semimartingale under \( P \), for all \( P \in \mathcal{P}_M, N_t^n = (N^n_t)^{\sigma_n}_t \) and \( K_t^n = (K^n_t)^{\sigma_n}_t \). So \( \tilde{K}_t := K_t^m - K_t^n \) is a continuous increasing process. Let
\[ \tilde{M}_t := M_t^m - M_t^{\sigma_n} = (N_t^m - N_t^n) - (K_t^m - K_t^n) =: \tilde{N}_t - \tilde{K}_t. \]
By Itô’s formula under \( P \), for all \( P \in \mathcal{P}_M \), we have
\[
\hat{E}(\hat{N}_T^2) \leq \hat{E}(\hat{M}_T^2) + 2\hat{E}\left( \int_0^T \hat{M}_s^* d\hat{K}_s \right).
\]
Noting that \( \hat{M}_s^* \leq n \), we have
\[
\hat{E}(\hat{N}_T^2) \leq \hat{E}(\hat{M}_T^2) + 2n\hat{E}(\hat{K}_T).
\]
We also have
\[
\hat{E}(\hat{M}_T^2) \leq 2[\hat{E}((M_T - M_{\sigma_n})^2) + \hat{E}((M_T - M_{\sigma_m})^2)]
\]
and
\[
2n\hat{E}(\hat{K}_T) = 2n[\hat{E}(M_{\sigma_n} - M_{\sigma_m}) + \hat{E}(M_{\sigma_m} - M_{\sigma_n})]
\leq 4n[\hat{E}(|M_{\sigma_n} - M_T|) + \hat{E}(|M_{\sigma_m} - M_T|)].
\]
By the same arguments as in Theorem 5.1, we can show that \( \hat{E}(\hat{M}_T^2) \to 0 \) as \( m, n \to \infty \). Now we show that \( 2n\hat{E}(\hat{K}_T) \to 0 \) as \( m, n \to \infty \). Indeed,
\[
|M_{\sigma_n} - M_T| \leq |M_{\sigma_n} - M_T \vee (-n)| + |M_T \vee (-n) - M_T|
\leq 2|M_{\sigma_n}|1[|M_{\sigma_n}| \geq n] + |M_T|1[|M_T| > n].
\]
So
\[
2n\hat{E}(\hat{K}_T) \leq 8n\hat{E}(M_{\sigma_n}1[|M_{\sigma_n}| \geq n]) + 4n\hat{E}(M_T1[|M_T| > n])
+ 8m\hat{E}(M_{\sigma_m}1[|M_{\sigma_m}| \geq m]) + 4m\hat{E}(M_T1[|M_T| > m])
\leq 8\hat{E}(M_{\sigma_n}^21[|M_{\sigma_n}| \geq n]) + 4\hat{E}(M_T^21[|M_T| > n])
+ 8\hat{E}(M_{\sigma_m}^21[|M_{\sigma_m}| \geq m]) + 4\hat{E}(M_T^21[|M_T| > m]).
\]
So \( 2n\hat{E}(\hat{K}_T) \to 0 \) as \( m, n \to \infty \) by Lemma 3.2. Consequently, we conclude that \( \hat{E}(\hat{N}_T^2) \to 0 \)
and \( \hat{E}(\hat{K}_T^2) \to 0 \) as \( m, n \to \infty \).

So there exists \( \{Z_t\} \in H_G^2(0, T) \) and \( \{K_t\} \) with values in \( L_G^2(\Omega_T) \) such that
\[
\hat{E}\left[ \int_0^T |Z_s^n - Z_s|^2 ds \right] \to 0
\]
and
\[
\hat{E}[\sup_{t \in [0, T]} |K_t^n - K_t|^2] \to 0
\]
as \( n \) goes to infinity.

Then
\[
M_t = \lim_{L^2_G, n \to \infty} M_t^n = \lim_{L^2_G, n \to \infty} \int_0^t Z_s^n dB_s - \lim_{L^2_G, n \to \infty} K_t^n = \int_0^t Z_s dB_s - K_t. \quad \square
\]

In this theorem, for \( \xi \in L^2_G(\Omega_T) \) bounded above, we have \( K_T \in L^2_G(\Omega_T) \). This result improves Theorem A.2.
For \( \eta \in H^0_{\alpha}(0, T) \), [3] proved that \( M_t(\eta) := \int_0^t \eta_s \, dB_s - 2 \int_0^t G(\eta_s) \, ds \) is a non-increasing \( G \)-martingale. Let

\[
A^0 = \left\{ \int_0^T \eta_s \, dB_s - 2 \int_0^T G(\eta_s) \, ds \mid \eta \in H^0_G(0, T) \right\}
\]

and

\[
A^p = \{ M_T \in L^p_G(\Omega_T) \mid \{ M_t \} \text{ is a non-increasing } G \text{-martingale with } M_0 = 0 \}.
\]

**Theorem 5.4.** For any \( p \geq 1 \) and \( \xi \in A^p \), there exists \( \{ \eta^n \} \subset H^0_G(0, T) \) such that

\[
\hat{E}[ \sup_{t \in [0, T]} |M_t(\eta^n) - M_t|^p ] \rightarrow 0,
\]

where \( \{ M_t \} \) is a quasi-continuous version of \( \{ \hat{E}_t(\xi) \} \) and \( M_t(\eta^n) = \int_0^t \eta^n_s \, dB_s - 2 \int_0^t G(\eta^n_s) \, ds \).

**Proof.** For \( \xi \in A^p \), let \( \{ M_t \} \) be a quasi-continuous version of \( \{ \hat{E}_t(\xi) \} \). Suppose that \( \sigma_n = \overline{\sigma}_n \). By Theorem 4.4, \( \{ M^\sigma_n \} \) is a bounded non-increasing \( G \)-martingale. So \( M_{\sigma_n} \in A^\beta \) for any \( \beta \geq 1 \).

\[
\sup_{t \in [0, T]} |M^\sigma_n_t - M_t|^p = |M_{\sigma_n} - M_T|^p \\
\leq 2^{p-1}(|M_{\sigma_n} - M_T \lor (-n)|^p + |M_T \lor (-n) - M_T|^p) \\
\leq 2^{p-1} |M_{\sigma_n}|^p 1_{|M_{\sigma_n}| \geq n} + 2^{p-1} |M_T|^p 1_{|M_T| > n}.
\]

Hence, by Lemma 3.2,

\[
\hat{E}( \sup_{t \in [0, T]} |M^\sigma_n_t - M_t|^p ) \leq 2^{2p-1} \sup_i \hat{E}[|M_{\sigma_i}|^p 1_{|M_{\sigma_i}| \geq n}] + 2^{p-1} \hat{E}[|M_T|^p 1_{|M_T| > n}] ightarrow 0.
\]

On the other hand, by the proof to Theorem A2 in [7], for each \( n \), there exists \( \{ \eta^n_k \} \subset H^0_G(0, T) \) such that

\[
\| \sup_{t \in [0, T]} |M_t(\eta^n_k) - M^\sigma_n_t| \|_{L^p_G} \rightarrow 0
\]

as \( k \) goes to infinity. So we get the desired result. \( \square \)

**Theorem 5.5.** Let \( \{ B_t \}_{t \in [0, T]} \) be a one-dimensional \( G \)-Brownian motion and let \( \tau \) be an \( \mathcal{F} \) stopping time. If \( B_t \) belongs to \( L^1_G(\Omega_T) \), then \( \{ 1_{[0, \tau]}(s) \}_{s \in [0, T]} \in H^0_G(0, T) \) for any \( \alpha > 1 \).

**Proof.** \( B_t \in L^1_G(\Omega_T) \) implies that \( B_t \) belongs to \( L^\alpha_G(\Omega_T) \) for any \( \alpha > 1 \) by Doob’s maximal inequality under \( P \), for all \( P \in \mathcal{P}_M \). By Corollary 5.2, there exists \( \{ Z_t \}_{t \in [0, T]} \in H^0_G(0, T) \) for all \( \alpha > 1 \) such that

\[
B_\tau = \int_0^\tau Z_s \, dB_s, \quad \text{q.s.}
\]

On the other hand, under every \( P \in \mathcal{P}_M \),

\[
B_\tau = \int_0^\tau 1_{[0, \tau]}(s) \, dB_s, \quad P - \text{a.s.}
\]
By the B–D–G inequality under \( P \), we have
\[
E_P \left[ \left( \int_0^T |Z_s - 1_{[0, \tau]}(s)|^2 \, ds \right)^{\alpha/2} \right] = 0.
\]
So \( \hat{E} \left[ \left( \int_0^T |Z_s - 1_{[0, \tau]}(s)|^2 \, ds \right)^{\alpha/2} \right] = 0 \) and consequently \( 1_{[0, \tau]}(s) \in H_G^\alpha(0, T) \). \( \square \)

**Corollary 5.6.** Let \( \{B_t\}_{t \in [0, T]} \) be a one-dimensional \( G \)-Brownian motion.

(i) \( \{1_{[\sup_{s \leq T} B_s \leq a]} \} \in H_G^\alpha(0, T) \) for all \( \alpha > 1 \). Similarly, \( \{1_{[\sup_{s \leq T} B_s \leq a]} \} \in H_G^\alpha(0, T) \) for all \( \alpha > 1 \).

(ii) Let \( \{M_t\} \) be a symmetric \( G \)-martingale. If \( c(S_a(M)) = 0 \), then \( \{1_{[\sup_{s \leq T} M_s \leq a]} \} \in H_G^\alpha(0, T) \) for all \( \alpha > 1 \).

(iii) \( 1_{[\{B_s\}_{s \leq T} \leq a]} \) \in \( H_G^\alpha(0, T) \) for all \( \alpha > 1 \).

**Proof.** (i) Let \( \tau = \tau_a \) be defined as in Example 4.3. Then we have \( B_\tau \in L_G^1(\Omega_T) \) by Example 4.3, so \( \{1_{[\sup_{s \leq T} B_s \leq a]} \} = \{1_{[\tau, T]}(s)\} \in H_G^\alpha(0, T) \).

(ii) Let \( \tau = \tau_a \) be defined as in Theorem 4.1. Since \( \tau \) is quasi-continuous, \( B_\tau \) is quasi-continuous and consequently belongs to \( L_G^1(\Omega_T) \). So \( \{1_{[\sup_{s \leq T} M_s \leq a]} \} = \{1_{[0, \tau]}(s)\} \in H_G^\alpha(0, T) \).

(iii) Let \( \tau = \tau_a \) be defined as in Theorem 4.5. By just the same arguments as in (ii), we have \( B_\tau \in L_G^1(\Omega_T) \). So \( \{1_{[\{B_s\}_{s \leq T} \leq a]} \} = \{1_{[0, \tau]}(s)\} \in H_G^\alpha(0, T) \). \( \square \)

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**Appendix**

In this Appendix, we mainly introduce the decomposition theorem and a regularity property for \( G \)-martingales, which were obtained in [7].

[3] conjectured that for any \( \xi \) \in \( L_G^1(\Omega_T) \), we have the following decomposition:
\[
X_t := \hat{E}_t(\xi) = \hat{E}(\xi) + \int_0^t Z_s \, dB_s - K_t, \quad t \in [0, T],
\]
where \( K_0 = 0 \) and \( \{K_t\} \) is an increasing process. [3] proved the conjecture for cylindrical functions in \( L_{ip}(\Omega_T) \) by Itô’s formula in the setting of \( G \)-expectation space.

[5] defined a norm
\[\|\xi\|_{L_2^0} = [\hat{E}[\sup_{t \in [0, T]} \hat{E}_t(|\xi|^2)]]^{1/2}\]

on \( L_{ip}(\Omega_T) \) and generalized the above result to the completion \( L_2^0(\Omega_T) \) of \( L_{ip}(\Omega_T) \) under the norm \( \| \cdot \|_{L_2^0} \).

However, [5] fell short of establishing the relations between the two norms \( \| \cdot \|_{L_2} \) and \( \| \cdot \|_{L_2^0} \). The space \( L_2^0 \) is just an abstract completion and we know nothing from [5] about the space \( L_2^0 \) except the fact that \( L_{ip}(\Omega_T) \subset L_2^0 \subset L_G^2(\Omega_T) \).
For \( \xi \in \mathcal{H}^0_T \), let \( \mathcal{E}(\xi) = \hat{E}\left[\sup_{u \in [0,T]} \hat{E}_u(\xi)\right] \), where \( \hat{E} \) is the \( G \)-expectation. For \( \alpha \geq 1 \) and \( \xi \in \mathcal{H}^0_T \), define \( \|\xi\|_{\alpha,G} = [\mathcal{E}(\|\xi\|)]^{1/\alpha} \) and denote as \( L_\alpha^G(\Omega_T) \) the completion of \( \mathcal{H}^0_T \) under \( \| \cdot \|_{\alpha,G} \).

**Theorem A.1** ([7]). For any \( \alpha \geq 1 \) and \( \delta > 0 \), we have \( L_\alpha^{G+\delta}(\Omega_T) \subset L_\alpha^G(\Omega_T) \). More precisely, for any \( 1 < \gamma < \beta :=(\alpha+\delta)/\alpha \), \( \gamma \leq 2 \), we have

\[
\|\xi\|_{\alpha,G} \leq \gamma^*\{\|\xi\|_{\alpha+\delta,G} + 14^{1/\gamma} C_{\beta/\gamma} \|\xi\|_{\alpha+\delta,G}^{(\alpha+\delta)/\gamma}\}, \quad \text{for all } \xi \in L_{ip}(\Omega_T),
\]

where \( C_{\beta/\gamma} = \sum_{i=1}^{\infty} i^{-\beta/\gamma} \), \( \gamma^* = \gamma/(\gamma - 1) \).

After the completion of [7] and its submission to arxiv, we became aware of a new version [6] of [5], which was submitted to arxiv a few days later than my submission. In [6], they proved that \( L_p^G(\Omega_T) \subset L_{ip}^G(\Omega_T) \) for any \( p > 2 \), which is a particular case of Theorem A.1.

By the estimates given in Theorem A.1, [7] obtained the following decomposition theorem for \( G \)-martingales.

**Theorem A.2** ([7]). For \( \xi \in L_\alpha^G(\Omega_T) \) with some \( \beta > 1 \), \( X_t = \hat{E}_t(\xi), \ t \in [0,T] \) has the following decomposition:

\[
X_t = X_0 + \int_0^t Z_s \, dB_s - K_t, \quad \text{q.s.,}
\]

where \( \{Z_t\} \in H_1^1(0,T) \) and \( \{K_t\} \) is a continuous increasing process with \( K_0 = 0 \) and \( \{-K_t\}_{t \in [0,T]} \) a \( G \)-martingale. Furthermore, the above decomposition is unique and \( \{Z_t\} \in H_1^1(0,T), K_T \in L_\alpha^G(\Omega_T) \) for any \( 1 \leq \alpha < \beta \).

[7] presented, as corollaries of the above decomposition theorem, the following representation theorem for symmetric \( G \)-martingales.

**Corollary A.3** ([7]). Suppose that \( \xi \in L_\alpha^G(\Omega_T) \) for some \( \beta > 1 \) with \( \hat{E}(\xi) + \hat{E}(-\xi) = 0 \). Then there exists \( \{Z_t\}_{t \in [0,T]} \in H_1^1(0,T) \) such that

\[
\xi = \hat{E}(\xi) + \int_0^T Z_s \, dB_s.
\]

Furthermore, the above representation is unique and \( \{Z_t\} \in H_1^1(0,T) \) for any \( 1 \leq \alpha < \beta \).

**Remark A.4.** In Corollary A.3, for \( \xi \in L_\alpha^G(\Omega_T) \), we only know that the related \( \{Z_t\}_{t \in [0,T]} \in H_\alpha^1(0,T) \) for any \( 1 \leq \alpha < \beta \). But for the case \( \beta = 2 \), the following theorem shows that the related \( \{Z_t\}_{t \in [0,T]} \in H_\alpha^2(0,T) \). Fortunately, by a stopping time technique, Corollary 5.2 in this article shows that \( \{Z_t\}_{t \in [0,T]} \) does belong to \( H_\alpha^2(0,T) \) for \( \xi \in L_\beta^G(\Omega_T) \).

**Theorem A.5** ([7]). Suppose that \( \xi \in L_\alpha^2(\Omega_T) \) with \( \hat{E}(\xi) + \hat{E}(-\xi) = 0 \). Then there exists \( \{Z_t\}_{t \in [0,T]} \in M_{H_\alpha^2}(0,T) \) such that

\[
\xi = \hat{E}(\xi) + \int_0^T Z_s \, dB_s.
\]
[7] proved, as a corollary of Theorem A.1, that any $G$-martingale $\{M_t\}$ with $M_T \in L^\beta_G(\Omega_T)$ for some $\beta > 1$ has a quasi-continuous version. Using a more elaborate proof, [7] obtained the following theorem.

**Theorem A.6 ([7]).** Any $G$-martingale $\{M_t\}$ has a quasi-continuous version.

This theorem turns out to be quite important for studying the properties of the stopped processes for $G$-martingales in this article.

**References**


