Inseparability in recursive copies

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Abstract

In [7] and [8], it is established that given any abstract countable structure \( S \) and a relation \( R \) on \( S \), then as long as \( S \) has a recursive copy satisfying extra decidability conditions, \( R \) will be \( \Sigma^0_n \) on every recursive copy of \( S \) if \( R \) is definable in \( \mathcal{L}'s \) by a special type of infinitary formula, a \( \Sigma^0_n(\bar{p}) \) formula. We generalize the type of constructions of these papers to produce conditions under which, given two disjoint relations \( R_1 \) and \( R_2 \) on \( S \), there is a recursive copy of \( S \) in which \( R_1 \) and \( R_2 \) are \( \Delta^0_n \) inseparable. We then apply these theorems to specific everyday structures such as linear orderings, boolean algebras and vector spaces.

1. Introduction

Recursive structures were first studied seriously from the recursion-theoretic viewpoint by Russian mathematicians such as Goncharov and Nurtazin (see [11] for example). They worked with the notion of a constructivization, which was a many-one map \( v: \mathbb{N} \rightarrow S \) from the natural numbers onto an abstract structure \( S \) for which the inverse images of the relations and functions of \( S \) are uniformly recursive.

More recent studies of structures from the recursion-theoretic viewpoint have been done by mathematicians such as C.J. Ash, E.J. Barker, and J.F. Knight. The basic notion they have used resembles the following.

Consider a countable model \( S = \langle A, \{R_i\}_{i \in I}, \{f_j\}_{j \in J}, \{c_k\}_{k \in K} \rangle \), with universe \( A \), relations \( \{R_i\}_{i \in I} \), functions \( \{f_j\}_{j \in J} \) and constants \( \{c_k\}_{k \in K} \). By representing constants as 0-place functions, and \( n \)-ary functions as \( (n+1) \)-ary relations, it suffices to consider structures with no functions or constants, i.e. structures of the form \( \langle A, \{R_i\}_{i \in I} \rangle \).

Label the elements of \( A \) with some subset \( A^* \) of \( \mathbb{N} \). This then induces relations on \( A^* \), inherited from the corresponding relations and functions on \( A \). We say \( \langle A^*, \{R^*_i\}_{i \in I} \rangle \) is a copy of \( S \). Moreover, we say it is a recursive copy of \( S \) if \( A^* \) is a recursive subset of \( \mathbb{N} \), and the \( \{R^*_i\}_{i \in I} \) are recursive relations on \( A^* \).

In general, we are interested in investigating the way recursion-theoretic properties of relations may differ when considered over different recursive copies of a fixed structure.
Let us take an example. Consider any countably infinite graph $G$. We can label its vertices by the natural numbers. This then defines a symmetric binary relation $C$ on $\mathbb{N}$ such that for $x, y \in \mathbb{N}$, $C(x, y)$ iff $x$ and $y$ are labels of vertices of $G$ that are connected by an edge. Then $(\mathbb{N}, C)$ is a copy of the graph $G$.

There will be many different copies of the graph $G$, corresponding to different labellings of the vertices of $G$. We have that such a copy $(\mathbb{N}, C)$ is recursive iff the relation corresponding to $C$ is a recursive relation on $\mathbb{N}$. From now, we will be interested only in recursive copies of the graph $G$.

Consider a graph $G$ that looks as displayed in Diagram 1(a). This graph is then just the disjoint union of infinitely many pairs of vertices with a single edge connecting them, and infinitely many isolated vertices. We consider two specific recursive presentations of $G$. The first copy, $(\mathbb{N}, C_1)$, is that induced by the labelling shown in Diagram 1(b). Secondly, consider any r.e. nonrecursive set $K$. Let $f(n)$ be the $n$th element in some fixed enumeration of $K$. Take the copy $(\mathbb{N}, C_2)$ induced by the labelling shown in Diagram 1(c). Here, we use all the pairs of the form $(2i, 2f(i) + 1)$ to label the infinitely many connected pairs of vertices of $G$. As $K$ is infinite, this is possible. Moreover, as $K$ has infinite complement, there will be infinitely many odd numbers left over, that is odd numbers not of the form $2f(i) + 1$ for any $i$. These are used to label the isolated vertices of $G$.

Note that both $(\mathbb{N}, C_1)$ and $(\mathbb{N}, C_2)$ are recursive copies. To see this observe that for $x, y \in \mathbb{N}$

$$C_1(x, y) \text{ iff } x \equiv 1 \mod 3 \text{ and } y = x + 1 \text{ or } y \equiv 1 \mod 3 \text{ and } x = y + 1,$$

$$C_2(x, y) \text{ iff } x \text{ is even and } y = 2f(x/2) + 1 \text{ or } y \text{ is even and } x = 2f(y/2) + 1.$$
As $f$ is a recursive function, both $C_1$ and $C_2$ are recursive relations. Therefore, both $(\mathbb{N}, C_1)$ and $(\mathbb{N}, C_2)$ are recursive copies of $G$.

Consider now the unary relation $U$ of being an isolated point of $G$, i.e., the relation of being connected to no other points. In $(\mathbb{N}, C_1)$, this holds iff $x \equiv 0 \mod 3$. In $(\mathbb{N}, C_2)$, we have that this holds iff $x \neq 2f(i) + 1$ for all $i \in \mathbb{N}$, i.e., iff $x \neq 2z + 1$ for all $z \in K$.

So while the relation corresponding to $U$ is a recursive relation in $(\mathbb{N}, C_1)$, the relation corresponding to $U$ is not recursive in $(\mathbb{N}, C_2)$. For if $U$ was recursive in $(\mathbb{N}, C_2)$ then we would be able to determine membership in $K$ by saying $x \in K$ iff $\neg U(2x + 1)$.

So what we have is two recursive presentations of the graph $G$. In the first copy, $U$ is a recursive relation, in the second $U$ is not.

The moral then is that simple relations on simple structures may have different complexities in different recursive copies. This is one of the various types of phenomena which the area of recursive algebra tries to investigate.

Much work in this area has been done by Chris Ash. For instance, in [1] and [2], the question is asked as to when we have the situation in which all isomorphisms between any two recursive copies of a given structure are $\Delta^0_\alpha$ where $0 < \alpha < \omega^*_{CK}$.

An important technique in answering these sorts of questions is the use of $\alpha$-systems. These were invented by Chris Ash, and have appeared under many different disguises in earlier papers. In [1] and [2] they appeared under the name of "recursive labelling systems". It is the more polished version of the $\alpha$-systems, such as those appearing in [5], which we find of use.

In general, we will have some structure $S$ and some recursive copy $C$ of $S$. Assume we have some relation $R$ on $S$. Then we shall often (ambiguously) refer to the relation $R$ on $C$ as the relation between natural numbers which we inherit from the relation $R$ on $S$ when the elements of $S$ are labelled by natural numbers to form the copy $C$. Strictly speaking, a recursive copy should not only consist of $C$ but also an isomorphism $\nu: S \rightarrow C$, in which case the relation $R$ on $C$ simply refers to $\nu(R)$. Generally, however, we suppress the isomorphism $\nu$, for what we hope is a gain in clarity.

We will be interested specifically in the following pair of questions. Given some $\alpha: 0 < \alpha < \omega^*_{CK}$, a structure $S$ and a relation $U$ on $S$, can we find a recursive copy of $S$ in which the relation $U$ is not $\Sigma^0_\alpha$? Also given some $\alpha: 0 < \alpha < \omega^*_{CK}$, a structure $S$ and two disjoint relations $U_1$ and $U_2$ on it, can we find a recursive copy of $S$ in which the relations $U_1$ and $U_2$ are $\Delta^0_\alpha$ inseparable?

The first of these questions was investigated in [8]. In particular, it was demonstrated there that, given extra decidability conditions, if a relation is not definable by a particular type of infinitary formula (a $\Sigma^0_\alpha$ formula) then there is a recursive copy of the structure in which the relation is not a $\Sigma^0_\alpha$ set.

Barker's theorem will fall out as a corollary of our main theorem, Theorem 2.13, which helps us to answer our second question. Hence, part of our work can be regarded as a generalization of the work done in [8]. Theorem 2.13 is in turn also a generalization of Theorem 3.5 of [7].
We assume the reader is familiar with basic pure recursion theory. In particular, a knowledge of Kleene's Arithmetical Hierarchy will be assumed, together with an understanding of the constructive ordinals, ordinal notations, and the hyperarithmetical hierarchy. All this can be found in [12]. Also assumed is a basic knowledge of infinitary logic, that found in [10] is sufficient.

Part 2 will be devoted to the metatheorems giving the answers to these two questions. Section 2.1 will be an investigation of the questions for \( \alpha = 1 \). We will begin by reproving Theorem 3.5 of [7], as it is the ideas and method of this theorem which provide the basis for the remainder of the paper. This will then enable us to give good answers to both questions in the case \( \alpha = 1 \). Section 2.2 will then set up the machinery for answering our questions for all constructive \( \alpha \). In Section 2.3 the main theorems will then be proven, from which we shall deduce Barker's theorem and the theorem answering our second question for all \( \alpha \).

Part 3 is then concerned with applying the main theorems of Part 2 to specific structures. We shall be primarily concerned with Linear Orderings, Boolean Algebras and Vector Spaces. For instance, we shall demonstrate in Section 3.1 that there is a recursive copy of \( (\omega, \leq) \) in which the natural numbers \( \equiv 0 \mod 3 \) and the natural numbers \( \equiv 1 \mod 3 \) are recursively inseparable. We shall deduce similar examples of \( \Delta^0_2 \) inseparability for \( \alpha > 1 \). In Section 3.2, we shall demonstrate that there is a recursive copy of the Boolean algebra \( B(\omega) \) in which the ideal generated by the atoms and the filter generated by the co-atoms are recursively inseparable. Again, we formulate analogues of this statement for \( \alpha > 1 \). Finally, we shall show in Section 3.3 that two recursive subspaces of the vector space \( V_\alpha(F) \) can be made \( \Delta^0_2 \) inseparable under quite general conditions. It is hoped that this range of examples is such that the reader will be convinced that the extra assumptions required in the proof of theorems giving answers to both our questions are not unreasonable.

2. Getting to the main theorems

2.1. The case where \( \alpha = 1 \)

In this section, we consider the following two questions:

(Q1) Given a structure \( S \) and a relation \( U \) on \( S \), under what conditions can we find a recursive copy \( \mathcal{A} \) of \( S \) with the relation \( U \) not \( \Sigma^0_1 \), that is to say, not r.e.?

(Q2) Given a structure \( S \) and two disjoint relations \( U_1 \) and \( U_2 \) on \( S \), under what conditions can we find a recursive copy of \( S \) with the relations \( U_1 \) and \( U_2 \) recursively inseparable?

Question 1 was answered in [7], subject to an extra decidability condition, by Corollary 2.3 below. In [7] there also appears a more technical statement,
Theorem 2.2. We repeat here this work because we find that suitable modifications of Theorem 2.2 allow us to answer Question 2.

In order to answer these questions, let us consider possible situations where it might be impossible to find a recursive copy of a given structure \( S \) with a given unary relation not \( \Sigma^0_1 \). That is to say, the given unary relation is \( \Sigma^0_1 \) on every recursive copy of \( S \).

Take the example, similar to that at the beginning of Part 1, where \( S \) is a countable infinite graph. Call a vertex isolated if it is connected to no other vertex. Consider the language of graph theory, \( \{ - \} \) where if \( a, b \) are vertices of a graph, \( a \sim b \) iff \( a \) and \( b \) are connected by an edge. Then the relation of not being an isolated vertex is definable in the language of graph theory by an existential formula, in particular, \( x \) is not an isolated vertex iff \( \exists y \[ y \sim x \] \). Consider a fixed countable infinite graph \( S = \langle G, \sim \rangle \), and a recursive copy \( \langle T, C \rangle \) of \( S \). \( T \) a recursive set and \( C \) a recursive reflexive symmetric binary relation on \( T \). Let \( U(x) \) hold for \( x \) in \( T \) iff \( x \) is not an isolated vertex in \( \langle T, C \rangle \).

Then, as \( U(x) \) iff \( \exists y \[ y \sim x \] \), in the recursive copy \( \langle T, C \rangle \) of \( S \), we have \( U(x) \) iff \( \exists y \[ y E T & C(y, x) \] \). As \( C \) is a recursive relation, and \( T \) is a recursive set, \( U \) is then \( \Sigma^0_1 \) in \( \langle T, C \rangle \). So in every recursive copy of \( S \), the relation \( U \) is \( \Sigma^0_1 \). Thus, there is no recursive copy of \( S \) with \( U \) not \( \Sigma^0_1 \).

Consider then any countable structure \( S \). We define the recursive infinitary \( \Sigma_1 \) formulae, or \( \Sigma_1^* \) formulae, to be those of the form \( \bigwedge_{i \in I} \exists \tilde{y}_i \phi_i(\tilde{x}, \tilde{y}_i) \) for some r.e. set \( I \), and some set \( \{ \phi(x, \tilde{y}_i) \}_{i \in I} \) of finitary quantifier-free formulae of \( \mathcal{L}_S \) so that the map \( [i \mapsto \text{Gödel number of } \phi_i] \) is recursive, for some system of Gödel numbering finitary quantifier-free formulae of \( \mathcal{L}_S \) which we leave unspecified. So the \( \Sigma_1^* \) formulae are infinitary r.e. disjunctions of formulae of the form \( \exists \tilde{y} \phi(\tilde{x}, \tilde{y}) \), where \( \phi \) is a finitary quantifier-free sentence of \( \mathcal{L}_S \). That is to say, the \( \Sigma_1^* \) formulae will just be infinitary r.e. disjunctions of existential formulae in \( \mathcal{L}_S \). Often, we will allow finitely many parameters to appear in the formula. If these parameters are \( \tilde{p} \), we say then that the formula is \( \Sigma_1^*(\tilde{p}) \). We then have the following.

**Proposition 2.1.** Let \( S \) be a countable structure, and \( U \) a \( \Sigma_1^*(\tilde{p}) \) relation on \( S \) for some finite set of parameters \( \tilde{p} \) from \( S \). Then \( U \) is \( \Sigma_1^0 \) on every recursive copy of \( S \).

**Proof.** It suffices to show that every existentially definable relation on \( S \) is r.e. in every recursive copy of \( S \). From this, it will follow that an r.e. infinitary disjunction of existential formulae will just be the union of a uniformly r.e. sequence of r.e. sets, which is itself r.e., giving the desired conclusion. Consider then an existential formula \( \psi(\tilde{x}) \iff \exists \tilde{y} \phi(\tilde{y}, \tilde{x}) \), \( \phi \) a Boolean combination of atomic formulae in \( \mathcal{L}_S \) involving only the finitely many parameters \( \tilde{p} \). In a recursive copy \( \mathcal{A} \), all atomic terms become recursive relations and the finitely many parameters \( \tilde{p} \) from \( S \) get mapped to some finite set of parameters from \( \mathcal{A} \), so any Boolean combination
of atomic formulae gives us a recursive relation. It follows that, in any recursive copy of $S$, $\varphi$ is recursive relation. As $\varphi(\bar{x}) \leftrightarrow \exists \bar{y} \varphi(\bar{y}, \bar{x})$, we then have that $\psi$ is r.e. □

We aim to show that under certain conditions, the converse of this theorem is true—that if, on every recursive copy of $S$, $U$ is $\Sigma_1^0$, then $U$ must be $\Sigma_1^1(\bar{p})$ for some finite set of parameters $\bar{p}$. This will be a consequence of the next theorem, from [7].

For the proof, an extra condition is needed. Given a structure $S$, a relation $U$ on $S$, and a copy $\mathcal{A}$ of $S$, consider the following condition:

\begin{equation}
(*) \quad \text{There is a recursive procedure for determining, given an existential formula } \psi(\bar{x}, \bar{y}) \text{ of the form } \exists \bar{z} \varphi(\bar{z}, \bar{x}, \bar{y}), \text{ and elements } \bar{a} \text{ of the universe of } \mathcal{A}, \text{ whether } \mathcal{A} \vdash \psi(\bar{x}, \bar{a}) \rightarrow U(\bar{x}).
\end{equation}

The condition (*) just says that we can effectively recognize when any existential formula with parameters from $\mathcal{A}$ defines a subset of $U$. Note that this condition implies that $U$ is a recursive relation on $\mathcal{A}$. Now, let us state the theorem:

**Theorem 2.2 [7].** Let $S$ be a countable structure, $U$ a relation on $S$, and $\mathcal{A}$ a recursive copy of $S$ satisfying $(*)$. Then there is a recursive copy $\mathcal{B}$ of $S$ and some isomorphism $f: \mathcal{B} \rightarrow \mathcal{A}$ such that if $M$ is a $\Sigma_0^0$ relation on $\mathcal{B}$ with $M \subseteq f^{-1}(U)$, then there is a $\Sigma_1^1(\bar{p})$ relation $N$ on $\mathcal{B}$ with $M \subseteq N \subseteq f^{-1}(U)$, for some finite list $\bar{p}$ of parameters from $\mathcal{B}$.

This theorem is illustrated in Diagram 2.

Before we prove this theorem, let us see how it answers one of our main questions, (Q1).

**Corollary 2.3 [7].** Let $S$ be a structure and $U$ a relation on $S$ such that $S$ has a recursive copy with $U$ satisfying $(*)$. If $U$ is r.e. on every recursive copy of $S$, then $U$ is $\Sigma_1^1(\bar{p})$ for some finite list $\bar{p}$ of parameters from $\mathcal{B}$.

**Proof.** Let $\mathcal{B}$ be the recursive copy of $S$ given by Theorem 2.2. Then, we know $f^{-1}(U)$ is r.e. on $\mathcal{B}$, as $U$ is r.e. on every recursive copy of $S$. Set $M = f^{-1}(U)$. Then we know

![Diagram 2](image-url)
there is $N$ with $M \subseteq N \subseteq f^{-1}(U)$ such that $N$ is $\Sigma^*_1(\bar{p})$. As we must have $N = M = f^{-1}(U)$, we have that $f^{-1}(U)$ is $\Sigma^*_1(\bar{p})$, and thus $U$ is $\Sigma^*_1(\bar{p})$ for some finite list $\bar{p}$ of parameters, as desired. \qed

So, under certain assumptions, we will have shown that the converse of Proposition 2.1 is true. Hence, we know, under the extra condition given, that we can find a recursive copy of a structure with a given relation not $\Sigma^0_1$, as long as that relation is not $\Sigma^*_1(\bar{p})$ definable. This answers (Q1) as in [7].

We proceed now to a proof of Theorem 2.2, as it is the details of this proof which, when generalized, help to answer (Q2).

**Proof of Theorem 2.2.** For the sake of notational simplicity, we treat only the case where $U$ is a unary relation. For the case where $U$ has more than one argument, an analogous argument will yield the same result. Our construction of the recursive copy $\mathcal{B}$ and the isomorphism $f: \mathcal{B} \to \mathcal{A}$ will be by a finite injury priority argument. The universe of the given $\mathcal{A}$ is some recursive set. Fix a particular recursive enumeration 

\[
\{a_0, a_1, a_2, \ldots \}
\]

of this set. Also, fix a particular recursive enumeration of some other arbitrarily chosen infinite recursive set \{b_0, b_1, b_2, \ldots \}. This set is to become the universe of $\mathcal{B}$.

At any stage $s$ we will have some set $D_s$ of atomic sentences and negations of atomic sentences of $\mathcal{L}_\mathcal{A}$. This will be our approximation, at stage $s$, to the atomic diagram of $\mathcal{A}$. Consider then any recursive enumeration \{\varphi_0, \varphi_1, \ldots \} of the atomic sentences and negations of atomic sentences of $\mathcal{L}_\mathcal{A}$. Here, by $\mathcal{L}_\mathcal{B}$, we mean the language of the structure $\mathcal{B}$, that is the language containing all the relations of the structure $\mathcal{A}$ and having all elements of $a$ appearing as constant symbols.

We will ensure that, for $s < s'$, $D_s \subseteq D_{s'}$. Moreover, for all $i \in \mathbb{N}$, either $\varphi_i \in \bigcup_s D_s$ or $(\neg \varphi_i) \in \bigcup_s D_s$. We will ensure that $\bigcup_s D_s$ is in fact the atomic diagram of $\mathcal{B}$.

At any stage $s$ we will also have a function $f_s$ from some finite subset of $\mathcal{B}$ to some finite subset of $\mathcal{A}$. Although these functions will not form an increasing chain, they will, in some sense, be approximations of the final isomorphism $f: \mathcal{B} \to \mathcal{A}$, with only "finite error." We will in fact have that $f(b_i) = a_j$ iff $f_s(b_i) = a_j$ for all sufficiently large $s$.

We use the traditional notation \{$W_e$\}_{e < \omega} to denote some recursive enumeration of the r.e. sets. The requirements for our construction are as follows, for all $e \in \mathbb{N}$:

- **$P_e$:** $a_e \in \text{ran}(f)$ & $b_e \in \text{dom}(f)$.
- **$Q_e$:** $\varphi_e \in \bigcup_s D_s$ or $(\neg \varphi_e) \in \bigcup_s D_s$.
- **$R_e$:** There is $c_e \in W_e$ with $f(c_e) \notin U$.

Each of the requirements \{$P_e$\}_{e < \omega} and \{$Q_e$\}_{e < \omega} will be met. However, not all the requirements \{$R_e$\}_{e < \omega} need be eventually met.

The idea behind the requirements \{$R_e$\}_{e < \omega} is that we want $f^{-1}(U)$ to have as few $\Sigma^0_1$ subsets as possible. If a requirement $R_e$ is satisfied, we have $W_e \subseteq f^{-1}(U)$, as
desired. We say that \( R_e \) requires attention at some stage \( s \) if there is no \( c_e \in W_e^s \) with \( c_e \in \text{dom}(f_s) \) and \( f_s(c_e) \notin U \).

The construction is described inductively. At stage 0, let \( f_0 = \emptyset \) and \( D_0 = \emptyset \).

Say that \( f_s \) is \( D_s \) coherent if a sentence \( \psi(b_{i_1}, b_{i_2}, \ldots, b_{i_m}) \) being in \( D_s \) implies that \( b_{i_1}, b_{i_2}, \ldots, b_{i_m} \) are all in \( \text{dom}(f_s) \) and \( \mathcal{A} \models \psi(f_s(b_{i_1}), f_s(b_{i_2}), \ldots, f_s(b_{i_m})) \). This tells us that \( D_s \) accurately reports some finite part of the diagram of \( \mathcal{B} \) which any extension of \( f_s \) to an isomorphism from \( \mathcal{B} \) to \( \mathcal{A} \) would induce. It will be a characteristic of our construction that for all \( s, f_s \) will be \( D_s \) coherent, \( f_s \) will satisfy \( P_0, \ldots, P_s \) and \( D_s \) will satisfy \( Q_0, \ldots, Q_s \). In fact, given \( f_s \) and \( D_s \) satisfying these conditions, our aim will be to construct \( f_{s+1} \) and \( D_{s+1} \) satisfying these conditions, i.e., \( f_{s+1} \) is \( D_{s+1} \) coherent, \( f_{s+1} \) satisfies \( P_0, \ldots, P_{s+1} \) and \( D_{s+1} \) satisfies \( Q_0, \ldots, Q_{s+1} \). As we will have considerable freedom in creating such \( f_{s+1} \) and \( D_{s+1} \) from \( f_s \) and \( D_s \), we aim to choose \( f_{s+1} \) so that as many of the requirements \( \{R_e \}_{e < \omega} \) as possible are satisfied.

Assume we are at stage \( s + 1 \), for some natural number \( s \), and we have some \( f_s \) and \( D_s \) such that \( f_s \) is \( D_s \) coherent, \( P_0, \ldots, P_s \) and \( Q_0, Q_1, \ldots, Q_s \) are all satisfied by \( f_s \) and \( D_s \), respectively. Assume also that all requirements \( R_0, \ldots, R_{s+1} \) are satisfied by \( f_s \). Then, we can easily find an \( f_{s+1} \) such that \( f_{s+1} \) is a \( 1 \)-\( 1 \) map from some finite subset of \( B \) to some finite subset of \( A \). Next, let \( \mathcal{A} \models \psi(f_{s+1}(b_{i_1}), f_{s+1}(b_{i_2}), \ldots, f_{s+1}(b_{i_m})) \). If this is true, let \( D_{s+1} = D_s \cup \{ \varphi_{s+1} \} \). Otherwise let \( D_{s+1} = D_s \cup \{ -\varphi_{s+1} \} \). Then, \( D_s \subseteq D_{s+1} \), \( D_{s+1} \) satisfies \( Q_{s+1} \), and \( f_{s+1} \) is \( D_{s+1} \) coherent.

So \( P_1, \ldots, P_{s+1} \) are all satisfied by \( f_{s+1} \), \( Q_1, \ldots, Q_{s+1} \) are all satisfied by \( D_{s+1} \), and \( f_{s+1} \) is \( D_{s+1} \) coherent. We can now go to stage \( s + 2 \).

Assume then, in the previous case, that not every requirement \( R_1, \ldots, R_{s+1} \) was satisfied by \( f_s \). Choose some \( e \) such that \( 1 \leq e \leq s + 1 \) and \( R_e \) is not satisfied by \( f_s \). To begin with, choose the least such \( e \).

Let \( g_s \) be the least function such that \( g_s \subseteq f_s \), \( g_s \) satisfies \( P_1, \ldots, P_e \) and \( R_1, \ldots, R_{e-1} \). We try to find a \( g^*_s \) such that \( g_s \subseteq g^*_s \), \( g^*_s \) satisfies \( R_e \) and \( g^*_s \) is \( D_s \) coherent. Any such \( g^*_s \) will still satisfy all of \( P_1, \ldots, P_e \) and \( R_1, \ldots, R_{e-1} \), along with \( R_e \).

Let \( \bar{x}_s = \text{dom}(g_s) \). Let \( b_{i_1}, \ldots, b_{i_n} \) denote \( \text{dom}(f_s) \). Let \( \theta_s(\bar{x}_s, b_{i_1}, \ldots, b_{i_n}) \) be the conjunction of the formulae of \( D_s \), together with all equalities and inequalities that hold between \( \bar{x}_s \) and the elements \( b_{i_1}, \ldots, b_{i_n} \). Then, imagine there was some choice of \( a_{i_1}, \ldots, a_{i_n} \) from \( A \) such that \( \theta_s(f_s(\bar{x}_s), a_{i_1}, \ldots, a_{i_n}) \), and for some \( 1 \leq k \leq n, b_{i_k} \in W_e^{s+1} \), and \( a_{i_k} \notin U \). Then we could define \( g^*_s \) as follows:

\[
g^*_s(x) = \begin{cases} g_s(x) & \text{if } x \in \bar{x}_s, \\ a_{i_k} & \text{if } x = b_{i_k}, \quad 1 \leq k \leq n. \end{cases}
\]
Then, condition $R_s$ would be satisfied by $g^+_s$, since we have for some $k, b_k \in W_s$, and $g^+_s(b_k) \notin U$. Moreover, $g_s \subseteq g^+_s$, and by construction $g^+_s$ would be $D_s$ coherent. But how do we know if such $a_1, \ldots, a_n$ exist?

Consider any value of $k$ such that $b_k \in W_{s+1}$. We then seek to ask if

$$(\exists a_1, \ldots, a_n)(\mathcal{A} \models \theta_s(f_s(j_s), a_1, \ldots, a_n \& a_k \notin U).$$

As $\mathcal{A}$ and $U$ satisfy $(\ast)$, this is decidable. We can ask this question for all $k$ such that $b_k \in W_{s+1}$, as $W_{s+1}$ is finite.

If the answer to any such question for any such choice of $b_k$ is “yes”, we can find a $D_s$ coherent $g_s^+$ as described above. Given this $g_s^+$, we can easily find a $D_s$ coherent $f_{s+1}$ with $g_s^+ \subseteq f_{s+1}$ and $f_{s+1}$ satisfying $P_1, \ldots, P_{s+1}$. The function $f_{s+1}$ will then also satisfy $R_1, \ldots, R_{e'}$. As before, we can extend $D_s$ to $D_{s+1}$ in such a way that $f_{s+1}$ is $D_s$ coherent, and $D_{s+1}$ satisfies $Q_1, \ldots, Q_{s+1}$, as we already know that $D_s$ satisfies $Q_1, \ldots, Q_{s+1}$. Having decided on $f_{s+1}$ and $D_{s+1}$, and noting that they satisfy the relevant requirements, we are now free to move to stage $s+2$. We say in this case, $R_e$ has been attacked. Note in this case that some $R_{e'}$ for $e' > e$ which may have previously been satisfied, may now not be satisfied. We say then that any such $R_{e'}$ has been injured. However, for all $R_{e'}$ with $e' < e$, $R_{e'}$ will remain satisfied if it was originally satisfied.

If, however, the answers to all the questions of the form $(\ast)$ for all applicable choices of $b_k$ is “no”, we consider next the least $e' > e$ such that $1 \leq e' \leq s + 1$, and $R_{e'}$ is not satisfied by $f_s$. Again, let $g_s$ be the least function such that $g_s \subseteq f_s$, $g_s$ satisfies $P_1, \ldots, P_{e'}$ and $g_s$ satisfies all those requirements $R_1, \ldots, R_{e'-1}$ that are already satisfied by $f_s$. Exactly as before, we try to find a $g_s^+$ extending $g_s$ which is $D_s$ coherent, and satisfies $R_{e'}$.

We obtain as before, a series of existential formulae whose truth we seek to decide. If any such formula is satisfiable, then as before we obtain $f_{s+1}$ satisfying $P_1, \ldots, P_{s+1}$, $f_{s+1}$ being $D_s$ coherent, $f_{s+1}$ satisfying $R_{e'}$, and all of the requirements $R_1, \ldots, R_{e'-1}$ which $f_s$ originally satisfied. From this, we can construct $D_{s+1}$ as before and move on to stage $s+2$.

If, however, the answers to all the questions for all applicable choices of $k$ is again “no”, we consider the next $e'' > e'$ such that $1 \leq e'' \leq s + 1$, and $R_{e''}$ is not satisfied, and repeat the same procedure. We repeat this entire process for $e'''', e''''', \ldots$. Eventually, we will either satisfy some previously unsatisfied requirement $R_l$ for some $l$ with $1 \leq l < s + 1$, and move on to stage $s + 2$, or all of the finitely many chances to do so will fail. In this final case, we simply expand $f_s$ to an $f_{s+1}$ such that $f_{s+1}$ contains in its domain all elements appearing in the formula $\varphi_{s+1}, f_{s+1}$ satisfies $P_{s+1}$, and such that $f_{s+1}$ is $D_s$ coherent. We can then expand $D_s$ to a suitable $D_{s+1}$ as before, with $D_{s+1}$ satisfying $Q_{s+1}$. We are then ready to move onto stage $s + 2$.

This completes the construction. It remains to prove that the construction has had the desired effect.

First, note that every $R_e$ is attacked only finitely often. This is true of $R_1$, as once $R_1$ is met it can never be injured. Assume that $R_1, R_2, \ldots, R_k$ are attacked only
finitely often. Let $s$ be a stage after which none of $R_1, R_2, \ldots, R_k$ is attacked. Then, if at some stage $s' > s$, $R_{k+1}$ is met, it can never thereafter be injured, as $R_{k+1}$ can only be injured by one of $R_1, R_2, \ldots, R_k$ being attacked. Thus, $R_{k+1}$ is attacked at most once after stage $s$. So $R_1, R_2, \ldots, R_k, R_{k+1}$ are all attacked only finitely often. By induction, it follows that all the $R_i$ for $i \in \mathbb{N}$ are attacked only finitely often. This is the "finite injury" argument, as it follows that each requirement is injured at most finitely often.

Define $f(b_i) = a_j$ iff $f_j(b_i) = a_j$ for all sufficiently large $s$. We show $f$ is an isomorphism from $\mathcal{B}$ to $\mathcal{A}$. First, we show $f$ is defined on all $b_i$. To see this, choose some particular $b_e$. At stage $e$, it gets included in the domain of $f_e$. (That is just to say that $P_e$ is met.) The only way we might have, for some $s' > s$, that $f_j(b_e) \neq f_r(b_e)$, is if some requirement $R_1, \ldots, R_{e-1}$ gets attacked. As each requirement is attacked only finitely often, there will be a natural number $t$ such that for all $t' > t$, $f_i(b_e) = f_r(b_e)$. Thus, $f$ is total, as desired.

It follows from the definition that $f$ is $1-1$. We prove $f$ is onto. This is essentially an identical argument. Fix some $a_e$. At stage $e$, it is included in the range of $f_e$. The only way its pre-image can change is if a requirement $R_1, \ldots, R_{e-1}$ is attacked. As this only happens finitely often, there is some stage after which the pre-image of $a_e$ stays the same. This gives us that $f$ is onto.

Thus, $f$ is a bijection from $\mathcal{B}$ to $\mathcal{A}$. Take any atomic $n$-ary relation $\varphi$, and elements $b_{i_1}, \ldots, b_{i_n}$ of $\mathcal{B}$. Let $D = \bigcup D_s$. Consider the sentence $\psi = \varphi(b_{i_1}, \ldots, b_{i_n})$. Then, as each $Q_i$ is met at stage $i$ (and cannot be injured), $\psi \in D$ or $\neg \psi \in D$. Let $t$ be a stage at which $f_i(b_{i_1}), \ldots, f_i(b_{i_n})$ have reached their final values. The formula $\varphi$ must appear as some $\varphi_e$ in the enumeration $\{ \varphi_{i_1} \}_{i < \omega}$ of atomic sentences and their negations in $\mathcal{L}_{\varphi}$. Let $s = \max(t, t')$. Then, at stage $s$, $\psi \in D_s$ or $\neg \psi \in D_s$, and as $f_i$ is $D_s$ coherent, $\psi \in D_s$ implies that $\mathcal{A} \models \varphi(f_i(b_{i_1}), \ldots, f_i(b_{i_n}))$, and $\neg \psi \in D_s$ implies that $\mathcal{A} \models \neg \varphi(f_i(b_{i_1}), \ldots, f_i(b_{i_n}))$. Thus, $D$ is just the atomic diagram of $\mathcal{B}$, which $\mathcal{B}$ inherits from $\mathcal{A}$ via $f$.

$\mathcal{B}$ is then a recursive copy. For, take any atomic $n$-ary relation $\varphi$, and elements $b_{i_1}, \ldots, b_{i_n}$. We must show there is a recursive procedure for deciding whether or not $\mathcal{B} \models \varphi(b_{i_1}, \ldots, b_{i_n})$. First, note that $D$ is r.e., as it is the union of a uniformly recursive chain of finite sets. So simply enumerate $D$, and eventually one of $\varphi(b_{i_1}, \ldots, b_{i_n})$ or $\neg \varphi(b_{i_1}, \ldots, b_{i_n})$ will appear. This gives us a decision procedure. So $\mathcal{B}$ has a recursive atomic diagram, and is hence a recursive copy of $S$. (A recursive copy is a copy with decidable atomic diagram.)

So we have $f: \mathcal{B} \simeq \mathcal{A}$ and that $\mathcal{B}$ is a recursive copy of $S$. It suffices now to show that $\mathcal{B}$ satisfies the conclusion of the theorem.

Take any $e \in \mathbb{N}$. Then consider a stage $s$ after which $R_1, \ldots, R_e$ are never attacked. As each is attacked at most finitely often, such a stage exists. Then let us consider separately the case where $R_e$ is met at this stage and the case where $R_e$ is not met at this stage.

So assume $R_e$ is met at this stage $s$. Then, there is $c_e \in W_e^e$ with $f_s(c_e) \notin U$, that is, $c_e \notin f_s^{-1}(U)$. But from this stage onwards, $R_e$ is never injured, so the image of $c_e$ under $f$ stays fixed from here on and so $c_e \notin f^{-1}(U)$. So $W_e \subseteq f^{-1}(U)$. 


Assume then that \( R_e \) is not met at this stage \( s \). Then, at all stages \( s' > s \), we will try to meet \( R_e \) by asking whether certain existential formulae are true, using condition (\( \ast \)), but it must always turn out that all such formulae are unsatisfiable. The question we will be asking at all stages \( t > s \) is if

\[
\exists a_{i_1} \ldots a_{i_n} (\mathcal{A} \models \theta_t(f_t(\bar{j}_t), a_{i_1}, \ldots, a_{i_n}) \& a_{i_k} \notin U)
\]

for any \( k \) such that \( b_{i_k} \in W_e^{s'} \), where \( \theta \) is the conjunction of the formulae of \( \mathcal{B}_s \) together with certain equalities and inequalities between \( \bar{j}_t \) and \( b_{i_1}, \ldots, b_{i_n} \), where \( \bar{j}_t \) is a specific finite sequence of elements. In particular, \( \bar{j}_t = \{ b_1, \ldots, b_{e'} \} \cup \{ f_t^{-1}(a_1), \ldots, f_t^{-1}(a_{e''}) \} \cup \{ c_1 | c_i \) witnesses that \( R_i \) is met for some \( i < e' \).

We know there will be some stage \( s' > s \) after which \( f(\bar{j}_t) \) and \( f(\bar{j}_t) \) have settled to some final value. Let the value \( f(\bar{j}_t) \) settle to be \( \bar{p} \). So, from this stage onwards we will then seek to know if, for some \( t > s' \),

\[
\exists a_{i_1} \ldots a_{i_n} (\mathcal{A} \models \theta_t(\bar{p}, a_{i_1}, \ldots, a_{i_n}) \& a_{i_k} \notin U)
\]

for any \( k \) with \( b_{i_k} \in W_e^{s'} \).

As we know that \( R_e \) is never satisfied after stage \( s \), we must have

\[
\exists a_{i_1} \ldots a_{i_n} (\mathcal{A} \models \theta_t(\bar{p}, a_{i_1}, \ldots, a_{i_n})) \text{ implies } a_{i_k} \in U
\]

for all \( k \) with \( b_{i_k} \in W_e^{s'} \), and \( t > s' \) where \( \theta_t \) is the formula earlier defined. Note, however, that from the fact that \( \bar{j}_t = f_t^{-1}(\bar{p}) \) the definition of \( \theta_t \), we have \( \mathcal{B} \models \theta_t(f_t^{-1}(\bar{p}), b_{i_1}, \ldots, b_{i_n}) \), so as \( f \) is an isomorphism

\[
(+) \quad \mathcal{A} \models \theta_t(\bar{p}, f(b_{i_1}), \ldots, f(b_{i_n})).
\]

So, for each \( b_{i_k} \in W_e \), we will certainly have some \( t \) such that

\[
\mathcal{A} \models \exists a_{i_1} \ldots a_{i_k} \ldots a_{i_n} \left( \theta_t(\bar{p}, a_{i_1}, \ldots, a_{i_k}, f(b_{i_k}), a_{i_k+1}, \ldots, a_{i_n}) \right).
\]

(Simply take \( t \) large enough that \( f(b_{i_k}) \) has settled by stage \( t \) and use (\( + \)).)

For each \( b_{i_k} \in W_e \) and natural number \( t > s' \), we get a formula \( \sigma_{b_{i_k},t}(x) \) on \( \mathcal{A} \) defined by

\[
\sigma_{b_{i_k},t}(x) \equiv \mathcal{A} \models \exists a_{i_1} \cdots a_{i_k} \cdots a_{i_n} (\theta_t(\bar{p}, a_{i_1}, \ldots, a_{i_n}, x, a_{i_k+1}, \ldots, a_{i_n})).
\]

We know that, for any \( b_{i_k} \in W_e \) and \( t > s' \), \( \sigma_{b_{i_k},t}(x) \rightarrow x \in U \). We also know that for any \( b_{i_k} \in W_e \) there is a \( t > s' \) such that \( \sigma_{b_{i_k},t}(f(b_{i_k})) \). So let \( M \) be the set of \( x \) in \( \mathcal{A} \) such that one of the formulae \( \sigma_{b_{i_k},t} \), is satisfied by \( x \). Then, \( f(W_e) \subseteq M \subseteq U \), so \( W_e \subseteq f^{-1}(M) \subseteq f^{-1}(U) \). But \( M \) is \( \Sigma^1_1(\bar{p}) \), as

\[
M(x) \leftrightarrow \bigvee_{b_{i_k} \in W_e, t > s'} \sigma_{b_{i_k},t}(x),
\]

and each \( \sigma_{b_{i_k},t}(x) \) is existential and involves only the finitely many parameters \( p \). Thus there is a \( \Sigma^1_1(\bar{p}) \) set \( M \) with \( W_e \subseteq M \subseteq f^{-1}(U) \) in \( \mathcal{B} \).
So if $R_e$ is eventually satisfied we have $W_e \subseteq f^{-1}(U)$, and if $R_e$ ends up unsatisfied $W_e \subseteq f^{-1}(U)$, but we can find a $\Sigma^0_1(\bar{p})$ set $M$ with $W_e \subseteq M \subseteq f^{-1}(U)$, as desired. Thus $\mathcal{B}$ has the desired property and the theorem is proven.

So we have a reasonable assumption, under which it turns out that if a specific relation is $\Sigma^0_1$ on every recursive copy of a structure, then that relation is definable by a $\Sigma^0_1(\bar{p})$ formula on that structure. This gives an answer to (Q1).

We now turn to our second question, (Q2) which is that of deciding, given two disjoint relations $U_1$ and $U_2$ on a structure $S$, if we can find a recursive copy of $S$ such that $U_1$ and $U_2$ are recursively inseparable in that copy.

To do this, we require a more general version of Theorem 2.2. The theorem we actually require is the following.

**Theorem 2.4.** Let $S$ be a structure, $U_1$ and $U_2$ relations on $S$, and $\mathcal{A}$ a recursive copy of $S$ such that both $U_1$ and $U_2$ satisfy ($*$) in $S$ (that is to say there is a recursive procedure for deciding sentences of the form $\psi(x,\bar{a}) \rightarrow U_1(x)$ for existential $\psi$, and another recursive procedure for deciding sentences of the form $\psi(x,\bar{a}) \rightarrow U_2(x)$ for existential $\psi$). Then there is a recursive copy $\mathcal{B}$ of $S$ and some isomorphism $f: \mathcal{B} \rightarrow \mathcal{A}$ such that if $M_1$ is a $\Sigma^0_1$ relation on $\mathcal{B}$ with $M_1 \subseteq f^{-1}(U_1)$, then there is a $\Sigma^0_1(\bar{p}_1)$ relation $N_1$ on $\mathcal{B}$ with $M_1 \subseteq N_1 \subseteq f^{-1}(U_1)$, and similarly if there is a $\Sigma^0_1$ relation $M_2$ with $M_2 \subseteq f^{-1}(U_2)$ then there is a $\Sigma^0_1(\bar{p}_2)$ relation $N_2$ on $\mathcal{B}$ with $M_2 \subseteq N_2 \subseteq f^{-1}(U_2)$.

What we have here is conditions for the conclusion of 2.2 to hold simultaneously for two relations $U_1$ and $U_2$.

**Proof of Theorem 2.4.** Again, we assume $U_1$ and $U_2$ are unary. As most of the details are as before, we indicate only the major changes. It is hoped that the many minor changes necessary will be apparent to the reader.

We now have the following list of requirements.

- $P_e$: $a_e \in \text{ran}(f) \& b_e \in \text{dom}(f)$.
- $Q_e$: $\forall \bar{x} \in \bigcup D_s \lor (-\exists \beta) \in \bigcup D_s$.
- $R_{2e}$: There is $c_e \in W_e$ with $f(c_e) \notin U_1$.
- $R_{2e+1}$: There is $c_e \in W_e$ with $f(c_e) \notin U_2$.

The strategy for attacking any $R_i$ is just the same. The same priority argument will work, and at the end we will have

If $R_{2e}$ is eventually met, $W_e \subseteq f^{-1}(U_1)$.
If $R_{2e+1}$ is eventually met, $W_e \subseteq f^{-1}(U_2)$.
If $R_{2e}$ is not eventually met, $W_e \subseteq f^{-1}(U_1)$ and there is a $\Sigma^0_1(\bar{p})$ set $M$ such that $W_e \subseteq M \subseteq f^{-1}(U_1)$. 


Diagram 3.

If \( R_{x+1} \) is not eventually met, \( W_c \subseteq f^{-1}(U_2) \) and there is a \( \Sigma_1(\bar{p}) \) set \( M \) such that \( W_c \subseteq M \subseteq f^{-1}(U_2) \).
Consequently we have the theorem. \( \square \)

Let us first note situations in which two disjoint relations of a structure might never be recursively inseparable in any recursive copy.

**Lemma 2.5.** Let \( S \) be a structure, let \( U_1 \) and \( U_2 \) be disjoint relations on \( S \) such that there are \( \Sigma_1(\bar{p}_1) \) and \( \Sigma_1(\bar{p}_2) \) sets \( F_1 \) and \( F_2 \) such that \( U_1 \subseteq F_1 \subseteq \bar{U}_2 \), \( U_2 \subseteq F_2 \subseteq \bar{U}_1 \) and \( F_1 \cup F_2 = S \) (see Diagram 3). Then, in every recursive copy of \( S \), there is a recursive set separating \( U_1 \) and \( U_2 \).

**Proof.** We assume, as usual, that \( U_1 \) and \( U_2 \) are unary. Take any recursive copy \( \mathcal{A} \) of \( S \). Then on \( \mathcal{A} \), \( F_1 \) and \( F_2 \) are r.e. We can define a recursive set \( R \) with \( U_1 \subseteq R \subseteq U_2 \) as follows: \( F_1 \) and \( F_2 \) are r.e., so assume \( F_1 = W_a \) and \( F_2 = W_b \) for some \( a, b \in \mathbb{N} \). Then, define \( x \in R \) if the least \( s \) such that \( x \in W_a \) (if it exists), is such that \( x \notin W_b \), and \( x \notin R \) if this condition fails or such an \( s \) does not exist. Then each element of \( F_1 \setminus F_2 \) is in \( R \), but no element of \( F_2 \setminus F_1 \) is in \( R \), so \( U_1 \subseteq R \subseteq U_2 \) as desired. Moreover, \( R \) is recursive. To see this, consider any \( x \in \mathcal{A} \). Enumerate \( W_a \) and \( W_b \) simultaneously. Then there will be a least \( s \) such that one of

1. \( x \in W_a \) and \( x \notin W_b \),
2. \( x \in W_a \) and \( x \in W_b \),
3. \( x \notin W_a \) and \( x \in W_b \)

is true. If it is 1, \( x \in R \). If it is 2 or 3, \( x \notin R \). So \( R \) is recursive as desired. \( \square \)

Using Theorem 2.4, we can prove a sort of converse to Lemma 2.5.

**Theorem 2.6.** Let \( S \) be a structure, \( U_1 \) and \( U_2 \) disjoint relations on \( S \), and \( \mathcal{A} \) a recursive copy of \( S \) such that \( (S, U_1) \) and \( (S, U_2) \) both satisfy \( (\ast) \) on \( \mathcal{A} \). Then, if in every recursive...
copy of $S$, $U_1$ and $U_2$ are recursively separable, there are $\Sigma_1^0(\bar{p})$ sets $F_1$ and $F_2$ such that $U_1 \subseteq F_1 \subseteq \bar{U}_2$, $U_2 \subseteq F_2 \subseteq \bar{U}_1$ and $F_1 \cup F_2 = S$.

**Proof.** Let $\mathcal{B}$ be the structure given by Theorem 2.4. Let $R$ be a recursive set separating $f^{-1}(U_1)$ and $f^{-1}(U_2)$ (see Diagram 4). We know then that $R$ is a recursive subset of $f^{-1}(\bar{U}_2)$. Thus, there is a $\Sigma_1^0(\bar{p})$ set $F_1$ with $R \subseteq F_1 \subseteq f^{-1}(\bar{U}_2)$. Likewise, $\bar{R}$ is a recursive subset of $f^{-1}(\bar{U}_1)$. Thus, there is a $\Sigma_1^0(\bar{p})$ set $F_2$ with $\bar{R} \subseteq F_2 \subseteq f^{-1}(\bar{U}_1)$. Consequently, $F_1 \cup F_2 = S$, so $F_1$ and $F_2$ are the sets desired. \[\square\]

Thus, under extra decidability conditions, we have necessary and sufficient conditions under which we can find a copy of the structure $S$ with two specified disjoint relations becoming recursively inseparable, namely that there are no two $\Sigma_1^0(\bar{p})$ relations $F_1$ and $F_2$ with $U_1 \subseteq F_1 \subseteq \bar{U}_2$, $U_2 \subseteq F_2 \subseteq \bar{U}_1$, and $F_1 \cup F_2 = S$. This answers (Q2).

This gives us reasonable answers to our main two questions for $\alpha = 1$, that of deciding when a given relation can be made not $\Sigma_1^0$ in a recursive copy, and when two given relations can be made recursively inseparable in a recursive copy.

Our ultimate aim is to give similar conditions under which a given relation can be made not $\Sigma_1^0$ in a recursive copy, and conditions for when two given relations can be made $\Sigma_1^0$ inseparable in a recursive copy, for $\alpha < \omega_1^{ck}$. Conditions answering the first question were originally given by Barker in [8]. The next section, Section 2.2, lays the groundwork and defines the notions necessary to prove the more general theorems giving such conditions which appear in Section 2.3.

2.2. The $\leq_\alpha$ relations and associated machinery

The primary aim of this section is to introduce and explain much of the machinery required to prove the metatheorems of the next section.

We have divided this section into five subsections, 2.2.1–5. We shall treat each section individually.
2.2.1. The recursive infinitary hierarchy

Our first aim is to define a hierarchy of formulae—the recursive infinitary formulae from [1]. In Proposition 2.1, it was demonstrated that any $\Sigma^*_1(\tilde{p})$ definable relation is necessarily r.e. on every recursive copy of $S$. We seek to find sufficient conditions for each $\alpha < \omega_1^{CK}$, such that any relation satisfying any such condition for $\alpha$ is $\Sigma^\alpha_\omega$ on all recursive copies of $S$.

In order to do this, we define the $\Sigma^*_\omega$ and $\Pi^*_\omega$ formulae on any structure $S$, for $\alpha < \omega_1^{CK}$. These were first defined and investigated in [1]. To begin with, define the $\Sigma^*_0$ and $\Pi^*_0$ formulae to be just the finitary quantifier-free formulae of $S$. We now proceed inductively. Given the $\Sigma^*_\lambda$ and $\Pi^*_\lambda$ formulae, define the $\Sigma^*_\lambda+1$ formulae to be those of the form $\bigwedge_{i \in I} \exists \bar{y}_i \varphi_i(\bar{y}_i, x)$ where $I$ is an r.e. set, and each $\varphi_i$ is a $\Pi^*_\lambda$ formula, so that the map $[i \rightarrow \text{the G"odel number of } \varphi_i]$ is recursive, for some system of G"odel numbering $\Pi^*_\lambda$ formulae which we leave unspecified. So the $\Sigma^*_\lambda+1$ formulae are just the r.e. disjunctions of existential quantifications of $\Pi^*_\lambda$ formulae. Likewise, we define the $\Pi^*_\lambda+1$ formulae to be those of the form $\bigwedge_{i \in I} \forall \bar{y}_i \varphi_i(\bar{y}_i, x)$, where $I$ is an r.e. set and each $\varphi_i$ is a $\Sigma^*_\lambda$ formula so that the map $[i \rightarrow \text{the G"odel number of } \varphi_i]$ is recursive, as above. The $\Pi^*_\lambda$ formulae are then just the r.e. conjunctions of universal quantifications of $\Sigma^*_\lambda$ formulae. If $\lambda$ is a limit ordinal, define the $\Sigma^*_\lambda$ formulae to be those of the form $\bigvee_{i \in I} \exists \bar{y}_i \varphi_i(\bar{y}_i, x)$, $I$ a r.e. set and each $\varphi_i$ a $\Sigma^*_\gamma$ formula for some $\gamma < \lambda$ so the map $[i \rightarrow \text{the G"odel number of } \varphi_i]$ is recursive. Likewise, the $\Pi^*_\lambda$ formulae are those of the form $\bigwedge_{i \in I} \forall \bar{y}_i \varphi_i(\bar{y}_i, x)$, $I$ a r.e. set and each $\varphi_i$ a $\Pi^*_\gamma$ formula for some $\gamma < \lambda$ so the map $[i \rightarrow \text{the G"odel number of } \varphi_i]$ is recursive. As before, we allow for finitely many parameters $\tilde{p}$. If a $\Sigma^*_\lambda$ formula involves the finitely many parameters $\tilde{p}$, we say it is $\Sigma^*_\lambda(\tilde{p})$. Likewise for the $\Pi^*_\lambda(\tilde{p})$ formulae. So a $\Sigma^*_\lambda$ or $\Pi^*_\lambda$ formulae is one without parameters.

Strictly speaking, in order to define the $\Sigma^*_\lambda$ formulae, we should proceed by recursive transfinite induction, simultaneously defining, for each $\beta < \alpha$, the G"odel numbers of the $\Pi^*_\beta$ relations and the $\Sigma^*_{\beta+1}$ formulae themselves, with suitable modifications at limit ordinals. In this and similar instances, we avoid this formality simply by referring to Kleene's method of recursive transfinite induction and hoping the omitted details are all apparent to the reader versed in this method. Further details can be found in [1].

This then defines the recursive infinitary hierarchy of formulae in the language of some structure $S$. We proved in Proposition 2.1 that the $\Sigma^*_1$ definable sets are r.e. in every recursive copy of $S$. We show now that the hierarchy we have just built allows us to make a more general statement.

**Lemma 2.7.** Let $S$ be a structure, and let $\alpha < \omega_1^{CK}$. Let $U$ be a $\Sigma^*_\alpha(\tilde{p})$ definable relation on $S$. Then $U$ is $\Sigma^\alpha_\omega$ on every recursive copy of $S$. Likewise, if $U$ is a $\Pi^*_\alpha(\tilde{p})$ definable relation on $S$, then $U$ is $\Pi^\alpha_\omega$ on every recursive copy of $S$.

**Proof.** We prove the claim using Kleene's method of recursive transfinite induction on $\alpha$. We know the result is true for $\alpha = 1$. Assume the result is true for some $\beta$. We
show it is true for $\beta + 1$. As mentioned before, we omit the many technical details about ordinal notations and Gödel numberings, favoring a more intuitive and less notationally cumbersome exposition.

Any $\Sigma^r_{\beta+1}$ relation is an r.e. disjunction of relations of the form $\exists \vec{y} \varphi(\vec{x}, \vec{y})$, with $\varphi$ a $\Pi^r_{\beta}$ relation. Take any recursive copy $\mathcal{A}$ of $S$. Then, by inductive hypothesis, $\varphi$ is $\Pi^0_0$ on $\mathcal{A}$, so the relation $\exists \vec{y} \varphi(\vec{x}, \vec{y})$ is $\Sigma^0_{\beta+1}$. Any $\Sigma^r_{\beta+1}$ relation is a uniformly r.e. disjunction of such relations, so any $\Sigma^r_{\beta+1}$ relation is thus the union of a uniformly r.e. sequence of $\Sigma^0_{\beta+1}$ sets. This is itself just a $\Sigma^0_{\beta+1}$ set, as desired.

So on any recursive copy, a $\Sigma^r_{\beta+1}$ relation defines a $\Sigma^0_{\beta+1}$ set. The result that a $\Pi^r_{\beta+1}$ relation defines a $\Pi^0_{\beta+1}$ set follows by taking negations.

For limit steps the induction is similar. Assume $\lim(\gamma)$. Then a $\Sigma^r_{\gamma}$ relation is a uniformly r.e. disjunction of relations of the form $\exists \vec{y} \varphi(\vec{x}, \vec{y})$, with $\varphi$ a $\Pi^r_{\gamma}$ relation for some $\gamma' < \gamma$. Each of these relations is just $\Sigma^0_{\gamma'+1}$, for $\gamma' < \gamma$. So a $\Sigma^r_{\gamma}$ relation is just the union of a uniformly r.e. sequence of $\Sigma^0_{\gamma}$ sets, which is $\Sigma^0_{\gamma}$ as desired. The proof for $\Pi^r_{\gamma}$ relations follows by taking negations. $\square$

We shall later show that given certain extra decidability conditions, the converse of this theorem is true, that is, that if in every recursive copy of $S$, a relation is $\Sigma^r_\alpha$, then that relation is $\Sigma^r_\alpha(\vec{p})$, for some finite set of parameters $\vec{p}$ from $S$. This result was due initially to Barker, and first appeared in [8].

2.2.2. The $\preceq_\alpha$ relations

We move on to the $\preceq_\alpha$ relations. There are several equivalent definitions available. We shall be mainly concerned with the following definition.

Fix a particular structure $\mathcal{A}$. Define, for two $n$-tuples $\vec{p}_1$ and $\vec{p}_2$ from $\mathcal{A}$, $\vec{p}_1 \preceq_1 \vec{p}_2$ iff for every finitary universal parameter-free formula $\varphi$ of $\mathcal{L}_\mathcal{A}$, $\mathcal{A} \not\models \varphi(\vec{p}_1)$ implies $\mathcal{A} \not\models \varphi(\vec{p}_2)$. Then, define inductively $\vec{p}_1 \preceq_{k+1} \vec{p}_2$ iff for each sequence $\vec{r}$ there is a sequence $\vec{q}$ of the same length such that $\vec{p}_2, \vec{r} \preceq_k \vec{p}_1, \vec{q}$. If $\lambda$ is a limit ordinal, define $\vec{p}_1 \preceq_\lambda \vec{p}_2$ iff $\vec{p}_1 \preceq_\gamma \vec{p}_2$ for all $\gamma < \lambda$. These relations have sometimes been referred to as the back and forth relations because of their relationship with back and forth arguments.

In fact, the $\preceq_\alpha$ relations have a quite elegant alternative characterization. Define the $\Sigma_0$ and $\Pi_0$ formulae to be the same as the $\Sigma^0_0$ and $\Pi^0_0$ formulae. Then define, for any $\alpha$, the $\Sigma_\alpha$ ($\Pi_\alpha$) formulae to be the infinitary, but not necessarily recursive or r.e., disjunction of existential (universal) quantifications over $\Pi^r_\beta$ ($\Sigma^r_\beta$) formulae for any $\beta < \alpha$. Then the $\Sigma_\alpha$ formulae are just the “recursive” $\Pi_\alpha$ formulae. A simple transfinite induction then shows that $\vec{p}_1 \preceq_\alpha \vec{p}_2$ iff every $\Pi_\alpha$ formulae satisfied by $\vec{p}_1$ is also satisfied by $\vec{p}_2$.

However, in the presence of an additional assumption we have this stronger result:

**Theorem 2.8.** Let $\mathcal{A}$ be a recursive copy of a structure $S$. Assume the existential diagram of $\mathcal{A}$ is decidable, and the relations $\{\preceq_\beta\}_{\beta < \alpha}$ are uniformly recursive in $\beta$. Then, for all $\vec{p}_1, \vec{p}_2$ of the same length from $\mathcal{A}$, $\vec{p}_1 \preceq_\alpha \vec{p}_2$ iff every $\Pi_\alpha$ sentence true of $\vec{p}_1$ is true of $\vec{p}_2$. 
So in a recursive copy with these extra decidability conditions, we have an alternative characterization of the $\leq_s$ relations. To prove the theorem, we use the following lemma.

**Lemma 2.9.** Suppose $\mathcal{A}$ is a recursive copy of a structure $S$ such that in $\mathcal{A}$, the relations $\{\leq_1, \leq_s\}$ are uniformly recursive, and the existential diagram of $\mathcal{A}$ is decidable. Then, for each $\gamma \leq \alpha$, we can effectively find, from any finite sequence $p_1$ from $\mathcal{A}$, a $\Pi^s_\gamma$ formula $\phi^s_\gamma$ such that for all $p_2 \in \mathcal{A}$ of the same length as $p_1$, $p_1 \leq_s p_2$ if $\mathcal{A} \models \phi^s_\gamma(p_2)$.

This was proved in [1]. We shall use this result to prove Theorem 2.8.

**Proof of Theorem 2.8.** Assume that every $\Pi^s_\gamma$ sentence true of $p_1$ is true of $p_2$, for some $\alpha < \omega_1^{CK}$. Let $\phi^s_\gamma$ be the formula given by Lemma 2.9, such that $\phi^s_\gamma(p_1)$ if $p_1 \leq_s p_2$. This formula can be taken to be $\Pi^s_\gamma$, so as every $\Pi^s_\gamma$ sentence true of $p_1$ is true of $p_2$, and obviously $\phi^s_\gamma(p_1)$, we must have $\phi^s_\gamma(p_2)$, and hence $p_1 \leq_s p_2$.

Conversely, assume $p_1 \leq_s p_2$ for some $\alpha < \omega_1^{CK}$. Then every $\Pi_\alpha$ formula true of $p_1$ is true of $p_2$. As the $\Pi^s_\gamma$ formulae form a subset of the $\Pi_\alpha$ formulae, it follows that every $\Pi^s_\gamma$ formula true of $p_1$ is true of $p_2$, as desired. \[\square\]

2.2.3. Coherence and the $\leq_s$ relations

In this section, we aim to look more closely at the notion of coherence. This notion was defined and used in Section 2.1. Here we define it again and look at its basic properties more carefully.

Imagine that we have a structure $S$, and a recursive copy $\mathcal{A}$ of $S$. Consider a recursive set $B = \{b_0, b_1, \ldots\}$ on which we might be trying to construct another recursive copy of $S$. Imagine a $1 \rightarrow 1$ function $f$ with finite domain and range from $\mathcal{B}$ into $\mathcal{A}$. Imagine also a finite set $D$ of atomic formulae and negations of atomic formulae of $\mathcal{L}_\mathcal{A}$. Assume also that every element of $\mathcal{B}$ appearing in a sentence of $D$ also appears in the domain of $f$.

We say then that $f$ is $D$-coherent, or that $f$ is coherent with respect to $D$, if for any sentence $\psi$ in $D$ of the form $\psi(b_1, \ldots, b_n)$, we have $b_1, \ldots, b_n$ are in $\text{dom}(f)$ and $\mathcal{A} \models \psi(f(b_1), \ldots, f(b_n))$. That is to say, $D$ is a set of sentences which accurately reports some finite part of the atomic diagram which $\mathcal{B}$ inherits from the partial isomorphism $f$ of $\mathcal{B}$ with $\mathcal{A}$. As the atomic diagram of $\mathcal{A}$ is recursive, we can recursively decide, given some finite function $f$ and set $D$ of atomic formulae and negations of atomic formulae in $\mathcal{L}_\mathcal{A}$, whether $f$ is $D$-coherent.

In the previous section, we used the following two lemmas freely. As they are needed later, we state and prove them once and for all. Throughout, we assume $S$ is some countable structure, $\mathcal{A}$ is some recursive copy of $S$, $\mathcal{B}$ is some recursive set on which we are trying to construct another recursive copy $\mathcal{B}$ of $S$, $f$ is a finite $1 \rightarrow 1$ function from $\mathcal{B}$ to $\mathcal{A}$, and $D$ is a set of atomic and negated atomic sentences in $\mathcal{L}_\mathcal{A}$. 


**Lemma 2.10.** If $f$ is coherent with respect to $D$, then for all $\bar{a} \in \mathcal{A}, \bar{b} \in \mathcal{B}$, there is a finite function $g$ with $f \subseteq g$, such that $g$ is coherent with respect to $D, \bar{b} \in \text{dom}(g)$ and $\bar{f} \in \text{ran}(g)$.

**Proof.** Select the images of $\bar{b}$ and the pre-image of $\bar{a}$ arbitrarily, making sure the function remains $1-1$ and extends $f$. As all elements appearing in a sentence of $D$ already appear in $\text{dom}(f)$, this will not interfere with coherence, and the resulting function will be coherent as desired. $\square$

**Lemma 2.11.** Let $f$ be coherent with respect to $D$. Let $\phi$ be an atomic formula or negation of an atomic formula of $\mathcal{L}_2$ asserted to hold of elements of $\mathcal{B}$. Assume these elements all appear in $\text{dom}(f)$. Then $f$ is coherent either with respect to $D \cup \{\phi\}$ or $D \cup \{\neg \phi\}$.

**Proof.** Let $\phi = \phi(b_{i_1}, \ldots, b_{i_n})$. If $\mathcal{A} \models \phi(f(b_{i_1}, \ldots, f(b_{i_n}))$, $f$ is coherent with respect to $D \cup \{\phi\}$. Otherwise, $\mathcal{A} \models \neg \phi(f(b_{i_1}, \ldots, f(b_{i_n}))$, and $f$ is coherent with respect to $D \cup \{\neg \phi\}$. $\square$

This proves these two simple but useful principles. For the proof of the main theorems, we will also need the following intricate lemma, known as the "weaving lemma."

Before we can state this theorem, we need to define what it might mean for one of the $\leq_{\mathcal{A}}$ relations to hold between two functions, as opposed to finite subsets of some structure. Let $\mathcal{A}$ and $\mathcal{B}$ be structures of the same type, and let $f$ and $g$ be partial functions from $\mathcal{B}$ to $\mathcal{A}$ with finite domain. Then, say $f \leq_{\mathcal{A}} g$ iff $\text{dom}(f) \subseteq \text{dom}(g)$, $\text{dom}(f) = b_1, b_2, \ldots, b_k$, and $f(b_1), f(b_2), \ldots, f(b_k) \leq_{\mathcal{A}} g(b_1), g(b_2), \ldots, g(b_k)$.

**The Weaving Lemma.** Assume $x_k > x_{k-1} > \cdots > x_2 > x_1 > x_0 \geq 1$, and that $f_k \leq_{\mathcal{A}} f_{k-1} \leq_{\mathcal{A}} \cdots \leq_{\mathcal{A}} f_1 \leq_{\mathcal{A}} f_0$, and $f_0$ is coherent w.r.t. some set $D$. Then there is $g \geq f_k$, such that $g$ is coherent w.r.t. $D$, and $f_0 \leq_{\mathcal{A}} g, f_1 \leq_{\mathcal{A}}, g, \ldots, f_{k-1} \leq_{\mathcal{A}} g$.

We refer the reader to [6] for a proof.

### 2.2.4. The $\mathcal{C}_{\mathcal{L}}$ relations

We move on to a study of the $\mathcal{C}_{\mathcal{L}}$ relations. These were first defined in [8]. We define them in terms of the $\leq_{\mathcal{A}}$ relations.

Consider a structure $\mathcal{S}$ and a recursive copy $\mathcal{A}$. Let $U$ be a unary relation on $\mathcal{A}$. Then, for each $\alpha < \omega_1^{\mathcal{A}}$, and each finite sequence $\bar{p}$ from $\mathcal{A}$, define $\mathcal{C}_{\mathcal{L}}(U, \bar{p})$ as follows.

If $\alpha$ is a successor, i.e. $\alpha = 1$, then $x \in \mathcal{C}_{\mathcal{L}}(U, \bar{p})$ iff for some $\bar{a}$, whenever we have $x'$ and $\bar{a}'$ such that $\bar{p}, x, \bar{a} \leq_{\mathcal{A}} \bar{p}, x', \bar{a}'$ then $x' \in U$.

If $\alpha$ is a limit, define $x \in \mathcal{C}_{\mathcal{L}}(U, \bar{p})$ iff $x \in \mathcal{C}_{\mathcal{L}}(U, \bar{p})$ for some $\beta < \alpha$.

The following theorem gives a nice equivalent definition of the $\mathcal{C}_{\mathcal{L}}$ relations.
Theorem 2.12. Let $S$ be a countable structure, $\mathcal{A}$ a recursive copy such that the existential diagram of $\mathcal{A}$ is recursive and the relations $\{ \leq_{\mathcal{A}} \}_{\mathcal{A}}$ are uniformly recursive in $\mathcal{A}$. Then, $\mathrm{Cl}_\alpha(U, \bar{p})$ is just the union of all $\Sigma^*_\alpha(p)$ definable subsets of $U$.

Proof. We prove the proposition first for $\alpha$ a successor. So let $\alpha = \beta + 1$. Assume $x$ is in some $\Sigma^*_\alpha(p)$ definable subset of $U$, but for all $\bar{a}$, there are $\bar{x}, \bar{a}'$ with $\bar{p}, x, \bar{a} \leq_{\mathcal{A}} \bar{p}, x', \bar{a}'$ and $x' \not\in U$. Let $\phi$ be a $\Sigma^*_\alpha(p)$ formula which defines a subset of $U$, and contains $x$. Then, $\forall z \exists \bar{y}_i \phi_i(\bar{y}_i, z)$, each $\phi_i$ a $\Pi^\mathcal{A}_\beta(p)$ formula. So at least one of the $\exists \bar{y}_i \phi_i(\bar{y}_i, z)$ must be true of $x$, i.e., $\exists \bar{y}_i \phi_i(\bar{y}_i, x)$ for some $i$. Let $\bar{a}$ be the relevant witness for $\bar{y}_i$. Then $\phi_i(\bar{a}, x)$. But $\phi_i$ is a $\Pi^\mathcal{A}_\beta(p)$ formula, and for all $\bar{a}$, there are $\bar{x}, \bar{a}'$ with $\bar{p}, x, \bar{a} \leq_{\mathcal{A}} \bar{p}, x', \bar{a}'$ and $x' \not\in U$. Thus, by Theorem 2.8 from $\phi_i(\bar{a}, x)$ we can conclude that $\phi_i(\bar{a}', x')$ for some $x' \not\in U$. Thus, $\exists \bar{y}_i \phi_i(\bar{y}_i, x')$, and so $\phi(x')$, contradicting the assumption that $\phi$ defines a subset of $U$. So, we must have that there is an $\bar{a}$ such that for all $x', \bar{a}'$ we have that $\bar{p}, x, \bar{a} \leq_{\mathcal{A}} \bar{p}, x', \bar{a}'$ implies $x' \in U$. Thus, $x \in \mathrm{Cl}_\alpha(U, \bar{p})$.

Assume then that $x \in \mathrm{Cl}_\alpha(U, \bar{p})$. We prove $x$ is in a $\Sigma^*_\alpha(p)$ definable subset of $U$. Now, $x \in \mathrm{Cl}_\alpha(U, \bar{p})$ implies that there must be an $\bar{a}$ such that $\forall x', \bar{a}'(\bar{p}, x, \bar{a} \leq_{\mathcal{A}} \bar{p}, x', \bar{a}' \rightarrow x' \in U)$. Let $\Phi^\mathcal{A}_{\beta, \alpha}(x, \bar{a})$ be the (parameter-free) $\Pi^\mathcal{A}_\beta$ formula asserted to exist by Lemma 2.9, such that $\Phi^\mathcal{A}_{\beta, \alpha}(x, \bar{a})$ iff $\bar{p}, x, \bar{a} \leq_{\mathcal{A}} \bar{w}$. Define $\psi(z) \Leftrightarrow \exists \bar{a}' \exists z' \Phi^\mathcal{A}_{\beta, \alpha}(z, \bar{a})$. Then $\psi(x)$, and $\psi(z) \rightarrow z \in U$. But $\psi$ is $\Sigma^*_\alpha(p)$, so $x$ is in a $\Sigma^*_\alpha(p)$ definable subset of $U$. So $\mathrm{Cl}_\alpha(U, \bar{p})$ is just the union of all the $\Sigma^*_\alpha(p)$ subsets of $U$, whenever $\alpha$ is a successor.

Assume then that $\alpha$ is a limit, and $x \in \mathrm{Cl}_\alpha(U, \bar{p})$. Then $x \in \mathrm{Cl}_\beta(U, \bar{p})$ for some $\beta < \alpha$. We can assume $\beta$ is a successor, so consequently $x$ is in some $\Sigma^*_\beta(p)$ subset of $U$. Therefore, $x$ is in a $\Sigma^*_\alpha(p)$ subset of $U$.

Finally, assume $x$ is in a $\Sigma^*_\alpha(p)$ subset of $U$ for a limit $\alpha$. Let $\phi$ define this subset, so $\phi(z) \Leftrightarrow \forall i \exists \bar{y}_i \phi_i(\bar{y}_i, z)$, where each $\phi_i$ is $\Pi^\mathcal{A}_\beta(p)$ for some $\beta < \alpha$. Then $\phi(x)$, so $\exists \bar{y}_i \phi_i(\bar{y}_i, x)$ for some $i \in I$, where $\phi_i$ is $\Pi^\mathcal{A}_\beta(p)$ for some $\beta < \alpha$. Thus, $\exists \bar{y}_i \phi_i(\bar{y}_i, z)$ defines a $\Sigma^*_{\beta+1}(p)$ subset of $U$ of which $x$ is a member. So $x$ is in $\mathrm{Cl}_{\beta+1}(U, \bar{p})$, and consequently $\mathrm{Cl}_\alpha(U, \bar{p})$.

So $\mathrm{Cl}_\alpha(U, \bar{p})$ is just the union of all the $\Sigma^*_\alpha(p)$ subsets of $U$, when $\alpha$ is a limit. This completes the theorem. □

2.2.5. $\alpha$-systems

Finally, we need to consider the powerful $\alpha$-systems invented by Chris Ash. These have undergone many changes, from [1] to [5]. For our purposes, the version from [4] seems simplest.

We define a tree to be a partially ordered set such that the set of predecessors of any element is well-ordered. Let the order type of the set of predecessors of an element be called the height of the element. We will be interested in trees with one node of height 0 such that all nodes have finite height, and each node has at least one successor. Call such a tree an $\omega$-tree.

Then let $(I, \leq)$ be a recursive $\omega$-tree. Let $U$ and $V$ denote the set of nodes of $T$ of even and odd height, so the unique node of height 0 is in $U$. Let $E$ be a
function that recursively assigns to each element of odd height, i.e. each \( v \in V \), an r.e. set \( E(v) \).

We can then play a game \( G \), in which player I will begin by playing the root node, player II will respond by playing a successor of the root node, player I will respond by playing a successor of the node just played by player II, and so on, the game determining an infinite branch in the tree \( T \), as each player chooses a successor of the previous move of the other player. Each game determines a path \( u_0, v_0, u_1, v_1, u_2, v_2, \ldots \) through the tree, with \( u_0 \) the root node, and \( u_i (v_i) \) the \((i + 1)\)st move of player I (II). We say that the game is winning for player II if \( \bigcup_{i<\omega} E(v_i) \) is r.e., otherwise that the game is winning for player I.

A strategy for player I is then a function \( s \) which assigns to each \( v \in V \) a successor \( s(v) \) of \( v \). A play \( u_0, v_0, u_1, v_1, \ldots \) is said to follow the player I strategy \( s \) if \( u_{i+1} = s(v_i) \) for all \( i \). Say that the strategy \( s \) is winning for player I if every play which follows \( s \) results in a player I win.

Take a recursive \( \omega \)-tree \( (T, \leq) \). Define an \( \alpha \)-system on \( T \) for \( \alpha < \omega_1^{CK} \) to be a uniformly r.e. set of relations \( \langle \leq, \gamma, <_{<\alpha} \rangle \) on \( V \) (the nodes of odd level of \( T \)) satisfying:

1. If \( u \leq_0 v' \), then \( E(u) \subseteq E(v') \).
2. If \( \gamma < \beta < \alpha \) and \( v \leq_\beta v' \), then \( v \leq_\gamma v' \).
3. Each \( \leq, \gamma \) is reflexive and transitive.
4. If \( \alpha > \gamma_1 > \gamma_2 > \cdots > \gamma_k, v_1 \leq_{\gamma_1} v_2 \leq_{\gamma_2} \cdots \leq_{\gamma_k} v_k \) and \( u \) is any successor of \( v_1 \), then there is a successor \( v \) of \( u \) with \( v_1 \leq_{\gamma_1} v \) and \( v_2 \leq_{\gamma_2} v \) and \( \cdots \) and \( v_k \leq_{\gamma_k} v \).

The following theorem is then from [4].

**The \( \alpha \)-system Theorem.** Let \( (T, \leq) \) be a recursive \( \omega \)-tree such that there is an \( \alpha \)-system \( \langle \leq, \gamma, <_{<\alpha} \rangle \) on \( T \). Then there is no \( \Delta_0^\alpha \) winning strategy for player I.

This theorem has been of much use in proving results in recursive model theory, from [1] through [5], and [6] and [8]. We do not prove it here, but use it to prove our main results in the next section. For a proof, see [4].

2.3. **The main theorems**

The main aim of this section is to prove the following generalization of Theorem 2.2, for every \( \alpha < \omega_1^{CK} \).

**Theorem 2.13.** Let \( S \) be a countable structure, \( \alpha \) an ordinal with \( 1 < \alpha < \omega_1^{CK} \), \( R \) a relation on \( S \) such that there is a recursive copy \( \mathcal{A} \) of \( S \) satisfying:

1. \( \mathcal{A} \) has decidable existential diagram.
2. \( R \) is recursive on \( \mathcal{A} \).
3. The relations \( \{ \leq_\beta \}_{\beta < \alpha} \) are uniformly recursive.
4. The relation \( x \in \text{Cl}_s(R, \bar{p}) \) is uniformly recursive in \( x \) and \( \bar{p} \).
Then, there is a recursive copy $\mathcal{B}$ of $\mathcal{A}$, and a bijection $f: \mathcal{B} \to \mathcal{A}$, such that for every $\Sigma^0_2$ set $S$ with $S \subseteq f^{-1}(R)$, there is a $\Sigma^0_2(\bar{p})$ set $F$ with $S \subseteq F \subseteq f^{-1}(R)$.

What this theorem is saying is that for any $\Sigma^0_2$ approximation $S$ of $f^{-1}(R)$ from inside, there is another $\Sigma^0_2(\bar{p})$ approximation from inside which is just as good, if not better (see Diagram 5). It is this proof which requires the use of an $\alpha$-system.

**Proof of Theorem 2.13.** Again, it suffices to assume $R$ is a unary relation. The proof presented here will be in some sense analogous to the proof of Theorem 2.2. We shall in fact begin by forming a similar list of requirements. As before, we will be constructing some isomorphism $f$ from $\mathcal{B}$ to $\mathcal{A}$, where $\mathcal{A}$ is the given recursive copy and $\mathcal{B}$ is the one being constructed. We can assume we have recursive enumerations $\{a_i\}_{i<\omega}$ of the universe of $\mathcal{A}$ and $\{b_i\}_{i<\omega}$ of the universe of $\mathcal{B}$. Also, let $\{\varphi_i\}_{i<\omega}$ be an enumeration of all atomic sentences and negations of atomic sentences in $\mathcal{L}_2$. To ensure that $\mathcal{B}$ is recursive, we require that its diagram is recursive. Let its diagram be $D$. Then $D$ will be a complete set of atomic sentences of $\mathcal{L}_2$. Assume also we have some enumeration $\{W^\varphi_e\}_{e<\omega}$ of the $\Sigma^0_2$ sets of natural numbers. Our requirements will then be as follows:

$P_e$: $a_e \in \text{ran}(f)$ and $b_e \in \text{dom}(f)$.

$Q_e$: $\varphi_e \in D$ or $\neg \varphi_e \in D$.

$R_e$: There is $c_e \in W^\varphi_e$ with $f(c_e) \notin R$.

Again, our wish that $f^{-1}(R)$ has as few $\Sigma^0_2$ subsets as possible leads us to formulate the requirements $\{R_e\}_{e<\omega}$. So to construct our $\alpha$-system, we begin by constructing our recursive tree $T$ with no terminal nodes. To do this, we first construct two sets $U'$ and $V'$. Every element of the tree will then be of the form $\langle u_0, v_0, u_1, v_1, \ldots, u_n \rangle$ or $\langle u_0, v_0, u_1, v_1, \ldots, u_n, v_n \rangle$ for some $u_0, u_1, \ldots, u_n \in U'$ and $v_0, v_1, \ldots, v_n \in V'$. The tree will then just be some set $T$ of ordered tuples of this form, ordered by initial segments.

To begin, $U'$ will just be the set of all ordered $n$-tuples of finite sets with $n \in \mathbb{N}$. So each element of $U'$ will be of the form $\langle s_1, s_2, \ldots, s_n \rangle$, $n \in \mathbb{N}$, each $s_i$ a finite set of natural numbers. The intended interpretation of such an element $u = \langle s_1, s_2, \ldots, s_n \rangle$
is as follows: let \( r \) be a node in the tree of the form \( \langle u_0, v_0, u_1, v_1, \ldots, u_m, v_m \rangle \) with \( u_m = u \). Then at node \( r \), for each \( i: 1 \leq i \leq m, s_i \) is the set of elements enumerated by \( W_i \) in the first \( m \) stages relative to some universal \( A^0_\alpha \) oracle. That is, \( s_i \) is our \( m \)th approximation to \( W_i \). So, as we move to higher levels of the tree, the \( i \)th co-ordinate of the rightmost \( u \) appearing in any particular node will enumerate \( W_i \) more and more.

The set \( V' \) will consist of finite sequences of the form \( \langle D, f_1, f_2, \ldots, f_n \rangle \). The set \( D \) will be some finite subset of the set of atomic sentences and negations of atomic sentences in \( \mathcal{L}_\mathcal{A} \). \( f_1 \), \( f_2 \), \ldots, \( f_n \) will be a chain of partial functions from \( \mathcal{A} \) to \( \mathcal{A} \), such that \( f_n \) is \( D \) coherent. Also, for \( 1 \leq i \leq n, b_i \in \text{dom}(f_i) \), \( a_i \in \text{ran}(f_i) \), and \( \varphi_i \in D \) or \( \neg \varphi_i \in D \). The intended interpretation of such an element \( v = \langle D, f_1, f_2, \ldots, f_n \rangle \) is as follows: let \( r \) be a node of the tree of the form \( \langle u_0, v_0, u_1, v_1, \ldots, u_m, v_m \rangle \) with \( v_m = v \). Then at node \( r, D \) is our \( m \)th finite approximation to the diagram of the model \( \mathcal{B} \) being constructed. Also, the final co-ordinate of \( v \), in this case \( f_n \), gives the \( m \)th finite approximation to the isomorphism \( f \) we hope to construct from \( \mathcal{B} \to \mathcal{A} \). As we move higher in the tree, our information about the diagram of \( \mathcal{B} \) will increase, and \( f \) will become more complex. The function \( f_n \) is called the principal function of \( v \).

We construct the tree inductively as follows. Given a node of the form \( \tau = \langle u_0, v_0, u_1, v_1, \ldots, u_n \rangle \), we recursively construct a subset \( V_i \) of \( V' \), such that the set of successors of the node in question are the nodes of the form \( \langle u_0, v_0, u_1, v_1, \ldots, u_m, v_m \rangle \), with \( v_m \in V_i \). Analogously, for each node of the form \( \tau = \langle u_0, v_0, u_1, v_1, \ldots, u_n, v_n \rangle \), we construct a subset \( U_i \) of \( U' \), such that the set of successors of the node in question are nodes of the form \( \langle u_0, v_0, u_1, v_1, \ldots, u_n, v_n, u_{n+1} \rangle \), with \( u_{n+1} \in U_i \). We also specify the root node \( \langle u_0 \rangle \), and the set \( V_0 \) such that the successors of the root node are just the nodes of the form \( \langle u_0, v_0 \rangle, v_0 \in V_0 \). Once this is done, we will have inductively constructed our recursive tree \( T \) with no terminal nodes.

To start, the root node will be the singleton \( \langle u_0 \rangle \), where \( u_0 = \emptyset \). This indicates that, at stage 0 of enumeration, all the sets \( W_0^* \) are empty. The only successor of \( \langle u_0 \rangle \) will be \( \langle u_0, v_0 \rangle \), where \( v_0 = \langle \emptyset, \langle \emptyset, \emptyset \rangle \rangle \). This indicates that, at such a node, the diagram of the structure \( \mathcal{A} \) we are trying to construct is empty, and our approximation to the isomorphism \( f: \mathcal{B} \to \mathcal{A} \) is just the empty function.

Consider a node of the form \( \tau = \langle u_0, v_0, u_1, u_2, \ldots, u_n, v_n \rangle \). Let \( u_n = \langle s_1, s_2, \ldots, s_m \rangle \). Then, let \( U_r = \{ \langle s'_1, s'_2, \ldots, s'_r \rangle | r > m \} \). Then, let \( U_r = \{ \langle s'_1, s'_2, \ldots, s'_r \rangle | r > m \} \). Then, the set of successors of \( t \) will just be nodes of the form \( \langle u_0, v_0, u_1, \ldots, u_n, v_n, u_{n+1} \rangle \), with \( u_{n+1} \in U_r \). All this requirement gives us is the obvious condition that, as the number of stages increases, our approximations of the r.e. sets (relative to the universal \( A^0_\alpha \) oracle) should form an increasing chain of finite sets. That is, our approximations of any particular \( W_i^* \) should form an increasing chain of finite sets.

Assume then we have a node of the form \( \tau = \langle u_0, v_0, u_1, v_1, \ldots, v_{n-1}, u_n \rangle \). Let \( u_n = \langle s_1, s_2, \ldots, s_m \rangle \). We seek to determine the least \( e \) such that requirement \( R_e \) is not met by \( f_e \). Each requirement \( R_e \) is of the form: there is \( c_e \in W_e^* \) with \( f(c_e) \notin R \). In order to determine the least such \( e \), we replace each \( W_e^* \)
by its approximation, \( s_n \), at the node \( u_n \). Then, to see if requirement \( R_e \) is met, we ask if there is \( c \in s_n \) with \( f_n(c) \notin R \). As \( s_n \) is finite, \( f_n \) is finite, and \( R \) recursive, we can answer this question. And so we can determine the least \( e \) such that requirement \( R_e \) is not met, relative to our current approximation of the \( \Sigma^0_2 \) sets at \( u_n \).

At this stage we wish to attack requirement \( R_e \). Loosely speaking, our idea is as follows. In attacking requirement \( R_e \), we do not want to injure any of the requirements \( R_i \) for any \( i < e \). However, in the spirit of the finite injury arguments, we will quite happily injure any requirement \( R_i \) for any \( i > e \) if it enables us to meet \( R_e \). Now, consider any element \( v \) of \( V'_1 \) of the form \( \langle D, f_1, \ldots, f_n \rangle \). We know \( f_1 \subseteq f_2 \subseteq \cdots \subseteq f_n \), and that \( f_n \) is \( D \)-coherent. Whenever we make a deliberate effort to satisfy some \( R_j \), we make sure the corresponding \( c_j \) that satisfies \( c_j \in \mathcal{W}^p_1 \) and \( f(c_j) \notin R \) is placed in the domain of \( f_j \). So, if we have made a special effort to meet \( R_j \), and have done so successfully, the witness \( c_j \) can be found in \( f_j \). This avoids giving witnesses special names. That is not to say that if \( f_n \) satisfies \( R_j \), \( f_j \) will satisfy \( R_j \), but if \( f_n \) satisfies \( R_j \) was met in some deliberate attempt at satisfying \( R_j \), then \( f_j \) will satisfy \( R_j \).

So when we attack \( R_e \), we will move from some node \( \tau \) ending in \( \langle D, f_1, \ldots, f_n \rangle \) to some node ending in \( \langle D', f_1, \ldots, f_{e-1}, g^+ \rangle \), where \( g^+ \) satisfies \( R_e \). Now assume we have established that \( R_e \) is the least such requirement not met. Fix \( \bar{d} = \text{ran}(f_{e-1}) \). We want to make sure that \( g^+ \) agrees with \( f_n \) on the pre-images of \( \bar{d} \). The plan of attack is as follows. Assume we have \( x \in W^p_2 \), with \( x \in \text{dom}(f_n) \). We try to find a \( g^+ \) as described with \( g^+(x) \notin R \). Assume that \( f_n(x) \in \mathcal{C}_h(R, \bar{d}) \). Then it will turn out that we cannot easily find such a \( g^+ \) satisfying \( R_e \), as the fact that \( f_n(x) \in \mathcal{C}_h(R, \bar{d}) \), that is, the fact that \( f_n(x) \) is in some \( \Sigma^0_2(\bar{d}) \) definable subset of \( R \), spoils any simple attempt at finding such a \( g^+ \) with \( g^+(x) \notin R \). However, if \( f_n(x) \notin \mathcal{C}_h(R, \bar{d}) \), we will be able to construct such a \( g^+ \) quite straightforwardly. Let us examine the details.

Once we have determined the least \( e \) so that \( R_e \) is not satisfied, fix \( \bar{d} = \text{ran}(f_{e-1}) \). Then we ask if there is \( x \in W^p_2 \) with \( x \in \text{dom}(f_n) \), and \( f_n(x) \notin \mathcal{C}_h(R, \bar{d}) \). To do this, we simply test all \( x \in s_e \), as \( s_e \) is our current approximation to \( W^p_2 \), and use condition (3) in the statement of the lemma.

Assume such an \( x \in s_e \) is found with \( x \in \text{dom}(f_n) \) and \( f_n(x) \notin \mathcal{C}_h(R, \bar{d}) \). Then the set of successors of \( \tau \) will be the set of nodes of the form \( \langle u_0, v_0, u_1, v_1, \ldots, u_n, v_n \rangle \) with \( v_n = \langle D', f_1, f_2, \ldots, f_{e-1}, g^+ \rangle \), such that \( f_n \leq g^+ \) for some \( \beta < \alpha \) if \( \alpha \) is a limit, otherwise for the \( \beta \) such that \( \beta + 1 - \alpha, D \subseteq D', \varphi_n \in D' \) or \( \neg \varphi_n \in D', f_{e-1} \equiv g^+, y^+ \) is a \( D' \)-coherent partial function from \( \mathcal{P} \) to \( \mathcal{A} \), \( b = \text{dom}(g^+) \), \( a \in \text{ran}(g^+) \), and \( g^+(x) \notin R \). In this case we gladly sacrifice all information contained in \( f_1, \ldots, f_n \) in order to find a \( g^+ \) satisfying \( R_e \) so that in this case, \( R_e \) is met.

Note that in this case the set of possible such \( v_n \) is non-empty. To see this, note that from \( f_n(x) \notin \mathcal{C}_h(R, \bar{d}) \), we may conclude that for all \( \bar{d} \), there are \( z', \bar{a}' \) such that we have \( \bar{d}, f_n(x), \bar{a} \leq \bar{d}, z', \bar{a}' \) with \( z' \notin R \) for some (in fact all) \( \beta < \alpha \) if \( \alpha \) is a limit, otherwise for the \( \beta \) such that \( \beta + 1 = \alpha \). So fix \( \bar{a} = \text{ran}(f_n) - \{ \bar{d}, f_n(x) \} \). Then we know we can find \( z \) and \( \bar{a}' \) with \( \bar{d}, f_n(x), \bar{a} \leq \bar{d}, z, \bar{a}' \) and \( z \notin R \). From this, we can define a function \( g \) by

\[
g(f_n^{-1}(\bar{d})) = \bar{d}, \quad g(x) = z \quad \text{and} \quad g(f_n^{-1}(\bar{a})) = \bar{a}'.
\]
Then $f_n \preceq g$. Expand $g$ to a $g^+$ such that $b_e \in \text{dom}(g^+), a_e \in \text{ran}(g^+)$. Then $f_n \preceq g^+$ and $g^+$ is $D$-coherent. Expand $D$ to $D'$ such that $g^+$ is $D'$-coherent, and $\varphi_n \in D'$. Then, $g^+(\varphi_n) \not\in R$. Thus, we have constructed a possible successor of $\tau$. So, as desired, the set of possible such $v_n$ is non-empty.

Assume then that there was no $x \in s_\tau$ found with $x \in \text{dom}(f_n)$ and $f_n(x) \not\in \text{Cl}_{\tau}(R, \overline{d})$. As in the proof of Theorem 2.2, we let $e'$ be the least number such that $e < e' \leq n$ and requirement $R_{e'}$ is not satisfied. We then fix $\overline{d} = \text{ran}(f_{e'-1})$ and repeat essentially the same procedure, that is, ask if there is $x \in s_{e'}$ with $x \in \text{dom}(f_n)$ and $f_n(x) \not\in \text{Cl}_{\tau}(R, \overline{d})$. If there is, a set of possible successors of $\tau$ is defined as before. Else, we go to the next least $e'' < n$ with $R_{e''}$ not satisfied, and proceed analogously. Repeating this process, we get finitely many chances to attack some $R_i, i \leq n$.

If all such attempts fail, then let the set of successors of $\tau$ be the set of nodes of the form $\langle u_0, v_0, u_1, v_1, \ldots, u_n, v_n \rangle$ with $v_n = \langle D', f_1, f_2, \ldots, f_n, f_{n+1} \rangle$, such that $D \subseteq D', \varphi_n \in D'$ or $\neg \varphi_n \in D', f_n \subseteq f_{n+1}, f_{n+1}$ is a $D'$-coherent partial function from $\emptyset$ to $\mathcal{A}$, $b_{n+1} \in \text{dom}(f_{n+1})$, and $a_{n+1} \in \text{ran}(f_{n+1})$. The principal function of $v_n$ will then be $f_{n+1}$, and (for all we know) all previously unsatisfied requirements $R_e$ are left unsatisfied.

This completes the construction of the tree $T$.

We have defined our recursive tree $T$ with no terminal nodes. We define the function $E$ as follows: let $v$ be a node at an odd level, so $v = \langle u_0, v_0, \ldots, u_n, v_n \rangle$ and $v_n = \{D, f_1, \ldots, f_k\}$. Then define $E(v) = D$.

We seek now to apply the $\alpha$ system theorem. To do this, we need to define relations $\langle \leq \rangle_{\tau < \alpha}$ on the nodes of odd levels. We do this as follows: let $n_1$ and $n_2$ be nodes of odd levels, and let $f_{n_1}$ and $f_{n_2}$ be the principal functions of the rightmost elements in each of these nodes. Then write $n_1 \leq \tau n_2$ iff $n_1$ occurs below $n_2$ in the tree ordering and $f_{n_1} \preceq \tau f_{n_2}$.

We seek now to verify conditions (1), (2), (3) and (4),

1. If $v \preceq_\tau v'$, then $E(v) \preceq E(v')$.
2. If $\gamma' \leq \beta \leq \alpha$ and $v \preceq_\gamma v'$, then $v \preceq_\gamma v'$.
3. Each $\tau$ is reflexive and transitive.
4. If $\alpha > \gamma_1 > \gamma_2 > \cdots > \gamma_k$, and $v_1 \preceq_{\gamma_k} v_2 \preceq_{\gamma_2} \cdots \preceq_{\gamma_k} v_k$, and $u$ is any successor of $v_1$, then there is a successor $v$ of $u$ such that $v \preceq_{\gamma_i} v$ for each $i = 1, \ldots, k$.

(1) follows from the fact that if $v \preceq_\alpha v'$, then $v'$ lies above $v$ in the tree ordering, and hence $E(v) \preceq E(v')$. Conditions (2) and (3) follow simply from basic facts about the $\preceq_\gamma$ relations. We shall now demonstrate that (4) holds. To do this, we fix some $v_1$, and a successor $u$ of $v_1$. Assume, for some $v_1, \ldots, v_k$ we have $v_1 \preceq_{\gamma_1} v_2 \preceq_{\gamma_2} \cdots \preceq_{\gamma_k} v_k$, so that $f_{v_1} \preceq_{\gamma_1} f_{v_2} \preceq_{\gamma_2} \cdots \preceq_{\gamma_k} f_{v_k}$, for some $\alpha > \gamma_1 > \gamma_2 > \cdots > \gamma_k$ where $f_{v_i}$ is the principal function of $v_i$.

There are two cases to consider. First, the nodes above $u$ were chosen in the case where there was no $x \in W^*_e$ with $x \in \text{ran}(f_n)$ and $f_n(x) \not\in \text{Cl}_{\tau}(R, \overline{d})$ for any selected $e$. In this case, the successors of $u$ are nodes of the form $\langle \ldots, v_1, u, v \rangle$, where if $v_1 = \langle D, f_1, \ldots, f_n \rangle$, then $v$ is required to be of the form $\langle D', f_1, \ldots, f_n, f_{n+1} \rangle$ such
that $D \subseteq D', \varphi_n \in D'$ or $\neg \varphi_n \in D'$, $f_n \subseteq f_{n+1}$, $f_{n+1}$ is a $D'$-coherent partial function from $\mathcal{A}$ to $\mathcal{A}$. $b_{n+1} \in \text{dom}(f_{n+1})$, and $a_{n+1} \in \text{ran}(f_{n+1})$. Note that, as $f_n$ is the principal function of $v_1$, we can write $f_{v_1}$ instead of $f_n$ whenever our notation requires it, so $f_{v_1} = f_n$. To start with, we know $f_{v_1}$ is $D$-coherent. As $f_{v_1}$ is the principal function of a node above $v_1$, it follows that $f_{v_1}$ is also $D$-coherent. Applying the Weaving Lemma, we obtain a function $h$ such that for each $i = 1, \ldots, k$ we have $f_{v_i} \leq h$, $f_{v_i} \leq h$, and $h$ is $D$-coherent. So extend $h$ to a function $h^+$ which is $D$-coherent, and $b_{n+1} \in \text{dom}(h^+)$, $a_{n+1} \in \text{ran}(h^+)$. Then extend $D$ to $D'$ such that $h^+$ is $D'$-coherent, and $\varphi_n \in D'$ or $\neg \varphi_n \in D'$. Then $v = \langle D', f_1, \ldots, f_m, h^+ \rangle$ gives a suitable successor of $u$ satisfying condition (4).

Secondly, consider the case where there was $x \in W^\alpha$ with $x \in \text{dom}(f_n)$ and $f_n(x) \notin \text{Cl}_\alpha(R, d)$ for some selected $e$. In this case, the successors of $u$ are nodes of the form $\langle \ldots, v_1, u, v \rangle$, where if $v_1 = \langle D, f_1, \ldots, f_n \rangle$, $v$ is required to be of the form $\langle D', f_1, f_2, \ldots, f_{e-1}, g^+ \rangle$. $D \subseteq D'$, $\varphi_n \in D'$ or $\neg \varphi_n \in D'$ or $\neg \varphi_n \in D'$, $f_{v_1} \leq g^+$ for some $\beta < \alpha$ if $\alpha$ is a limit, otherwise for the $\beta$ such that $\beta + 1 = \alpha$, $f_{e-1} \leq g^+$, $g^+$ is $D'$-coherent function, $b_{n+1} \in \text{dom}(g^+)$, $a_{n+1} \in \text{ran}(g^+)$, and $g^+(x) \notin \mathcal{R}$.

As before, take $h$ to be the function given by the Weaving Lemma, so $f_{v_1} \subseteq h$. Put $\text{ran}(f_{e-1}) = \delta, a = \text{ran}(h) - \{d, f_n(x)\}$. Since $f_n(x) \notin \text{Cl}_\alpha(R, d)$, then we can find $z, \tilde{a}$ such that $\delta, f_n(x), \tilde{a} \leq z, \tilde{a}$, and $z \notin \mathcal{R}$ where $\delta = \gamma_1$ if $\alpha$ is a limit, otherwise $\delta$ is the predecessor of $\alpha$. From this, we can define a function $g$ such that $h \leq g$ and $g(x) \notin \mathcal{R}$. Extending $g$ to contain $b_{n+1}$ in its domain and $a_{n+1}$ in its range, we obtain some $g^+$ which is $D$-coherent. $D$ may then be extended to $D'$ to include $\varphi_n$ or $\neg \varphi_n$. Then $v = \langle D', f_1, \ldots, f_{e-1}, g^+ \rangle$ gives a suitable successor of $u$ satisfying condition (4).

Thus, in both cases, condition (4) holds, and we have an $\alpha$-system.

Consider the game then that gets played on this tree. This consists of player I playing a node from $U$, and player II playing a node from $V$ in such a way that this process, when repeated, yields a branch of the tree $T$. So in effect, player I gives information about the $\Sigma^0_\alpha$ sets as player II attempts to construct $\mathcal{B}$ and an isomorphism $f: \mathcal{A} \rightarrow \mathcal{A}$. As player I gives more and more information about the $\Sigma^0_\alpha$ sets, player II modifies his construction in an attempt to meet all the $P_e, Q_e$ and $R_e$ requirements.

We know then that from the $\alpha$-system theorem, for any $\Delta^0_\alpha$ I-strategy, there is a play which follows it and is winning for II. Consider then the following I-strategy. Fix any $\Delta^0_\alpha$ set $O$ such that all the $\Sigma^0_\alpha$ sets are just r.e. in $O$, that is, $O$ is a universal $\Delta^0_\alpha$ set. Then, located at any node $\langle u_0, v_1, \ldots, u_n, v_n \rangle$, player I plays $\langle u_0, v_1, \ldots, u_n, v_n, u_{n+1} \rangle$, where $u_{n+1} = \langle s_1, s_2, \ldots, s_n, s_{n+1} \rangle$, and for $1 \leq i \leq n + 1$, $s_i$ is just the elements enumerated by $W_i$ relative to $O$ in the first $n + 1$ stages of enumeration. Thus, player I provides all the “true” information about the $\Sigma^0_\alpha$ sets. We know then that by the $\alpha$-system theorem there is a play which follows it and is winning for II. This play defines a path $(u_0, (u_0, v_0), (u_0, v_0, u_1), \ldots)$ which is winning for II. Let $S$ be the infinite sequence $(v_0, v_1, v_2, \ldots)$. Each $v_i$ will be of the form $\langle D_i, f'^1_i, f^1_i, \ldots, f^k_i \rangle$ for some $k$ (a function of $v_i$). Moreover, $D_0 \subseteq D_1 \subseteq \cdots$ and $D = \bigcup_{i < \alpha} D_i$ will provide an r.e. set of atomic formulae and negations of atomic formulae such that, for all
quantifier-free $\varphi \in \mathcal{L}_\varphi$ we have $\varphi \in D$ or $\neg \varphi \in D$. That $D$ is r.e. follows from the fact that the play is winning for II. From the fact that $D$ is r.e., it follows that in fact $D$ is recursive. Thus $D$ defines a recursive structure $\mathcal{A}$. We now construct an isomorphism $f: \mathcal{A} \cong \mathcal{A}$ from the sequence $S$. First, note that once all requirements \{R_1, R_2, \ldots, R_e\} are met for some $e$, they can never be injured. So, by a standard finite injury argument, each $R_e$ is attacked at most finitely often. Consequently, we have $\lim_{i \to \infty} f_i^\alpha$ exists for each $i$, as the only way a $f_i^\alpha$ can change is if requirement $R_j$ is attacked where $j < i$. Let, for each $i$, $\lim_{n \to \infty} f_i^\alpha$ be $f_i$. Then the $f_i$ will form a chain, $f_1 \subseteq f_2 \subseteq f_3 \cdots$. Let $f = \bigcup_{i<\omega} f_i$. Each $f_i$ is a partial function from $\mathcal{A}$ to $\mathcal{A}$, with $b_i \in \text{dom}(f_i)$ and $a_i \in \text{ran}(f_i)$. So $f$ is a bijection from the set $\{b_i\}_{i<\omega}$ to $\{a_i\}_{i<\omega}$. Moreover, using the fact that each $f_i^\alpha$ is $D_i$ coherent, it follows as in the proof of Theorem 2.2 that $D$ is in fact just the atomic diagram of the presentation $f^{-1}(\mathcal{A})$, which we call $\mathcal{A}$.

Having constructed the recursive presentation $\mathcal{A}$, it remains to verify that $\mathcal{A}$ satisfies the conditions of the theorem.

Fix a particular $e$. Assume requirement $R_e$ is met at some stage $s$ by player II and is never thereafter injured. Then we know there is $c_e \in W_\varphi$ with $f(c_e) \notin R$. Consequently $f(c_e) \notin R$. So $W_\varphi \models \neg \varphi$.

Assume then that requirement $R_e$ is not eventually met by player II. Consider some stage $s$ at which $f_1, f_2, \ldots, f_{e-1}$ have settled to their final values, so $d = \text{ran}(f_{e-1})$ is fixed. Then it must be the case that, for all $x \in W_\varphi$, $f(x) \in \mathcal{C}_1(R, d)$. So $x \in W_\varphi \Rightarrow f(x) \in \mathcal{C}_1(R, d)$. Now, $r \in \mathcal{C}_1(R, d)$ iff there is some $a$, such that whenever, $\bar{d}, r, a \leq \bar{d}, r, a$ for all $\beta < \alpha$, we must have $r' \in R$. So given any choice of $r \in \mathcal{C}_1(R, d)$, there is some corresponding choice of $\bar{a}$ which makes the above sentence true. Moreover, the choice of $\bar{a}$ can be found by a $\mathcal{D}_\varphi$ process in $r$, given the recursiveness of the $\leq_\beta$ relations for all $\beta \leq \alpha$. Denote this choice of $\bar{a}$ by $\bar{a}(r)$.

Define a predicate $\varphi^{\bar{d}, r, \bar{a}}$ as follows:

$$\varphi^{\bar{d}, r, \bar{a}}(r^*, \bar{a}^*) \text{ iff } \bar{d}, r, \bar{a} \leq_\beta \bar{d}, r^*, \bar{a}^*.$$

This predicate is $\Pi_\beta(\bar{d})$ by Lemma 2.9.

First, we consider the case where $\alpha$ is a successor ordinal. Assume $\alpha = \beta + 1$. Consider the relation $Q$ on $\mathcal{A}$ such that

$$Q(x) \iff \bigvee_{(r, \bar{a}(r))} \exists \bar{z} \varphi^{\bar{d}, r, \bar{a}(r)}(x, \bar{z})$$

that is,

$$Q(x) \iff \bigvee_{(r, \bar{a}(r))} \exists \bar{z} [\bar{d}, r, \bar{a}(r) \leq_\beta \bar{d}, x, \bar{z}]$$

where $\varphi^{\bar{d}, r, \bar{a}}$ is $\Pi_\beta(\bar{d})$. So $Q$ is definable as a uniformly $\Sigma^0_\beta$ disjunction of formulae of the form $\exists \bar{z} \varphi(\bar{z})$, $\varphi$ a $\Pi_\beta(d)$ formulae, $\beta \geq 1$. We show how any such formula can be
redefined as a uniformly r.e. disjunction of formulae of the same form, from which we will be able to conclude that \( Q \) is a \( \Sigma^a_1(d) \) relation.

Assume \( I \) is a \( \Sigma^0_1 \) set, and we have a formula \( \bigvee_{i \in I} \varphi_i \) where each \( \varphi_i \) is of the form \( \exists \bar{z} \psi_i(\bar{z}), \psi_i \) a \( \Pi^0_1(d) \) formula, \( \beta \geq 1 \). Now, \( i \in I \) iff \( \exists j P(i,j) \) for some \( \Pi^0_1 \) relation \( P \). Then let \( \theta_{ij} \) be some \( \Pi^1_1 \) formula recursively obtainable from \( i, j \) which is a tautology if \( P(i,j) \) and a contradiction if \( \neg P(i,j) \) (such \( \theta_{ij} \) are easily constructed). Then

\[
\bigvee_{i \in I} \varphi_i \equiv \bigvee_{i, j} (P(i,j) \& (\varphi_i)) \equiv \bigvee_{i, j} (\theta_{ij} \& \varphi_i).
\]

Now, \( \theta_{ij} \& \varphi_i \) is of the form \( [\Pi^1_1 \& \exists \Pi^0_1(d)] \). Thus \( \bigvee_{i \in I} \varphi_i \) is logically equivalent to a uniformly recursive disjunction of existential quantifications over \( \Pi^0_1(d) \) formulae, which is \( \Sigma^a_1(d) \), as desired. In particular, our relation \( Q \) is \( \Sigma^a_1(d) \).

From the definition of \( \bar{a}(r) \), we know that for all \( x \in \mathcal{A}, Q(x) \rightarrow x \in R \). So the set of \( x \) satisfying \( Q(x) \) is a subset of \( R \). Moreover, for any \( x \) in \( \mathcal{B} \), \( x \in W^*_x \rightarrow f(x) \in \text{Cl}_x(R, d) \rightarrow Q(f(x)) \). So \( f(W^*_x) \subseteq Q \). We have \( f(W^*_x) \subseteq Q \subseteq R \), so \( W^*_x \subseteq f^{-1}(Q) \subseteq f^{-1}(R) \). But \( Q \) is \( \Sigma^a_1(d) \) on \( \mathcal{A} \), hence \( Q \) is \( \Sigma^a_1(d) \) on \( \mathcal{B} \) with \( \bar{d} = f^{-1}(d) \). Thus there is a \( \Sigma^a_1(d) \) definable set separating \( W^*_x \) and \( f^{-1}(R) \) as desired.

Next consider the case where \( x \) is a limit. Assume \( r \in \text{Cl}_x(R, d) \). Then \( r \) is in some \( \Sigma^a_1(d) \) subset of \( R \). Consequently, \( r \) must be in some \( \Sigma^a_1(d) \) subset of \( R \), \( \beta_r < \omega \), and so \( r \in \text{Cl}_{\beta_r}(R, d) \). So there is some \( \bar{a} \) such that whenever \( d, r, \bar{a} \leq_{\beta_r} d, r', \bar{a}' \), we have \( r' \in R \).

Let this \( \bar{a} \) be \( \bar{a}(r) \) as before. Then, let

\[
Q(x) \leftrightarrow \bigvee_{(r, \bar{a}(r)) \in \text{Cl}_x(R, d)} \exists \bar{z} \varphi^x_{\bar{a}(r)}(x, \bar{z}).
\]

We have that \( Q \) is \( \Sigma^a_1(d) \), and by a similar argument, \( f(W^*_x) \subseteq Q \subseteq R \).

So if \( R_x \) is eventually met, \( W^*_x \subseteq f^{-1}(R) \). However, if \( R_x \) is not eventually met, \( W^*_x \subseteq f^{-1}(R) \), but we have a \( \Sigma^a_1(d) \) set \( Q \) between \( W^*_x \) and \( f^{-1}(R) \). Thus the recursive copy \( \mathcal{B} \) satisfies the desired conditions and the theorem is proven.

As with Theorem 2.4, it is also straightforward to do the argument presented on two relations \( R_1 \) and \( R_2 \) at once. All this requires is a doubling up of requirements as before, so we have:

\[
\begin{align*}
P_e: & \quad a_e \in \text{ran}(f) \& b_e \in \text{dom}(f). \\
Q_e: & \quad \varphi_e \in D \text{ or } (\neg \varphi_e) \in D. \\
R_{2e}: & \quad \text{There is } c_e \in W^*_x \text{ with } f(c_e) \notin R_1. \\
R_{2e+1}: & \quad \text{There is } c_e \in W^*_x \text{ with } f(c_e) \notin R_2.
\end{align*}
\]

The theorem that can thus be proved is

**Theorem 2.14.** Let \( S \) be a countable structure, \( R_1 \) and \( R_2 \) relations on \( S \) such that there is a recursive copy \( \mathcal{A} \) of \( S \) satisfying:

1. \( \mathcal{A} \) has decidable existential diagram.
2. \( R_1 \) and \( R_2 \) are both recursive on \( \mathcal{A} \).
(3) The relations \( \{ \leq \}_{\beta \in \omega} \) are uniformly recursive.

(4) The relations \( x \in \text{Cl}_\alpha (R_1, \bar{p}) \) and \( x \in \text{Cl}_\alpha (R_2, \bar{p}) \) are uniformly recursive in \( x \) and \( \bar{p} \).

Then there is a recursive copy \( \mathcal{B} \) of \( \mathcal{A} \), and a bijection \( f: \mathcal{B} \rightarrow \mathcal{A} \), such that if \( S \subseteq f^{-1}(R_1) \) for some \( \Sigma^0_\alpha \) set \( S \), there is a \( \Sigma^*(\bar{p}) \) set \( F_1 \) with \( S \subseteq F_1 \subseteq f^{-1}(R_1) \). Likewise, if \( S \subseteq f^{-1}(R_2) \) for some \( \Sigma^0_\alpha \) set \( S \), there is a \( \Sigma^*(\bar{p}) \) set \( F_2 \) with \( S \subseteq F_2 \subseteq f^{-1}(R_2) \).

Using these theorems, and taking the assumptions (1) to (4) in each of these theorems for granted, we can then answer the two questions of when we can find a recursive copy of a structure with a given relation not \( \Sigma^0_\alpha \) on that copy for some \( \alpha < \omega^1 \), and when we can find a recursive copy of a structure with two given disjoint relations \( \Delta^0_\alpha \) inseparable in that copy. An answer to the first question had actually already been obtained by Barker in [8], but also falls out as a corollary of our Theorem 2.13.

Consider the first question. We know already that if a relation is expressible by a \( \Sigma^0_\alpha \) formula, then it is \( \Sigma^0_\alpha \) on every recursive copy of the structure in question. In Section 2.1, we found a converse to this statement for \( \alpha = 1 \). Using Theorem 2.13, we have the converse for all \( \alpha < \omega^1 \) from [8].

**Corollary 2.15** [8]. Let \( S \) be a countable structure, \( R \) a relation on \( S \) such that there is a recursive copy \( \mathcal{A} \) of \( S \) satisfying (1)-(4) of Theorem 2.13. Then, if \( R \) is \( \Sigma^0_\alpha \) on every recursive copy of \( S \), \( R \) is \( \Sigma^*(\bar{p}) \).

**Proof.** Let \( \mathcal{B} \) be the recursive copy asserted to exist by Theorem 2.13. Then, \( f^{-1}(R) \) is \( \Sigma^0_\alpha \) by assumption, so \( f^{-1}(R) \) is itself a \( \Sigma^0_\alpha \) subset of \( f^{-1}(R) \), thus \( f^{-1}(R) \) is \( \Sigma^*(\bar{p}) \) for some finite set \( \bar{p} \) of parameters, by the conclusion of Theorem 2.13. Thus \( R \) is \( \Sigma^*(\bar{p}) \), as desired. \( \square \)

So we know, given the additional assumptions (1)-(4), that we can find a copy of a structure with a given relation not \( \Sigma^0_\alpha \) iff that relation is not \( \Sigma^*(\bar{p}) \) for any \( \bar{p} \). This answers our first question. Consider then the second question, that of when we can find a recursive copy of a given structure with two relations \( \Delta^0_\alpha \) inseparable in that copy.

We have the following analogue of Lemma 2.5.

**Lemma 2.16.** Let \( S \) be a structure, let \( U_1 \) and \( U_2 \) be disjoint relations on \( S \) such that, for some finite lists \( \bar{p}_1 \) and \( \bar{p}_2 \) of parameters, there are \( \Sigma^*(\bar{p}_1) \) and \( \Sigma^*(\bar{p}_2) \) sets \( F_1 \) and \( F_2 \) such that \( U_1 \subseteq F_1 \subseteq \bar{U}_2 \), \( U_2 \subseteq F_2 \subseteq \bar{U}_1 \) and \( F_1 \cup F_2 = S \) (see Diagram 6). Then, in every recursive copy of \( S \), there is a \( \Delta^0_\alpha \) set separating \( U_1 \) and \( U_2 \).

**Proof.** Take any recursive copy \( \mathcal{A} \) of \( S \). Then, on \( \mathcal{A} \), \( F_1 \) and \( F_2 \) will be \( \Sigma^0_\alpha \). In particular, \( F_1 \) and \( F_2 \) are r.e. in some \( \Delta^0_\alpha \) set. So, relativizing the proof of Lemma 2.5, we have a set recursive in a \( \Delta^0_\alpha \) set separating \( F_1 \) and \( F_2 \), that is, a \( \Delta^0_\alpha \) set separating \( U_1 \) and \( U_2 \). \( \square \)
Using Theorem 2.14 we can obtain, under extra conditions, the converse to this theorem.

**Theorem 2.17.** Let $\mathcal{S}$ be a structure, $R_1$ and $R_2$ disjoint relations on $\mathcal{S}$, and $\mathcal{A}$ a recursive copy of $\mathcal{S}$ satisfying:

1. $\mathcal{A}$ has decidable existential diagram.
2. $R_1$ and $R_2$ are both recursive on $\mathcal{A}$.
3. The relations $\{ \preceq \}_{\beta \leq \gamma}$ are uniformly recursive.
4. The relations $x \in \text{Cl}_A(R_1, \vec{p})$ and $x \in \text{Cl}_A(R_2, \vec{p})$ are uniformly recursive in $x$ and $\vec{p}$.

Then, if in every recursive copy $R_1$ and $R_2$ are $\Delta_0^0$ separable, there are $\Sigma^0_1(\vec{p})$ sets $R_1$ and $R_2$ such that $R_1 \subseteq F_1 \subseteq \bar{\bar{R}}_1, R_2 \subseteq F_2 \subseteq \bar{\bar{R}}_2$, and $F_1 \cup F_2 = \mathcal{S}$.

**Proof.** Let $\mathcal{B}$ be the structure given by Theorem 2.14. Then let $P$ be a $\Delta_0^0$ set separating $R_1$ and $R_2$ (see Diagram 7).

We know $P$ is a $\Delta_0^0$ subset of $f^{-1}(\bar{\bar{R}}_2)$. Thus, there is a $\Sigma^0_1(\vec{p})$ set $F_1$ with $P \subseteq F_1 \subseteq f^{-1}(\bar{\bar{R}}_2)$. Likewise, $\bar{P}$ is a $\Delta_0^0$ subset of $f^{-1}(\bar{\bar{R}}_1)$. Thus, there is a $\Sigma^0_1(\vec{p})$ set $F_2$ with $\bar{P} \subseteq F_2 \subseteq f^{-1}(\bar{\bar{R}}_1)$. Consequently, $F_1 \cup F_2 = \mathcal{A}$, and so $F_1$ and $F_2$ are the desired sets. $\square$
So, given the additional assumptions (1)–(4), we can find a copy of a structure with a pair of given disjoint unary relations $\Delta^0_0$ inseparable as long as there are no two $\Sigma^0_1$ sets $F_1$ and $F_2$ with the relevant inclusions holding. This answers the second question.

We have answered our main questions, under additional assumptions. In the next section, we will use the theorems of this and the previous sections on specific everyday structures $S$. It will be seen then that conditions (1)–(4) of the various theorems presented are met under a large variety of circumstances, and so our answers to our main questions are good ones.

3. Applications

3.1. Linear orderings

In this and the remaining two sections, we seek to apply the theorems of the previous section to specific structures. In this section, we shall be interested in linear orderings. In particular, we shall be interested in linear orderings of the type $(\alpha, <)$, where $\alpha$ is a constructive ordinal, and $<$ is the natural ordering on the set of ordinals less than $\alpha$.

The simplest infinite linear ordering is just $(\omega, <)$, isomorphic to the usual ordering on natural numbers.

Theorem 3.1. Let $S_1 = \{x \in \mathbb{N} : x \equiv 1 \text{ mod } 3\}$, and $S_2 = \{x \in \mathbb{N} : x \equiv 2 \text{ mod } 3\}$. Then there is a recursive copy of $(\omega, <)$ with $S_1$ and $S_2$ recursively inseparable.

In fact, more generally,

Theorem 3.2. Let $S_1$ and $S_2$ be infinite disjoint recursive sets of natural numbers, such that $\mathbb{N} - (S_1 \cup S_2)$ is infinite. Then, there is a recursive copy $\mathscr{B}$ of $(\omega, <)$ with $S_1$ and $S_2$ recursively inseparable.

Before we prove Theorem 3.2, let us first note that given such sets $S_1$ and $S_2$, we cannot hope to find a recursive copy $\mathscr{B}$ of $(\omega, <)$ in which $S_1$ and $S_2$ are $\Delta^0_0$ inseparable. For take any $n$ in $(\omega, <)$. Then, define $\varphi_n(z)$ to hold iff $\exists x_1 \cdots x_n \forall y [x_1 < x_2 < \cdots < x_n < z \land y < z \rightarrow (y = x_1 \lor y = x_2 \lor \cdots \lor y = x_n)]$. So $\varphi_n(z)$ holds iff $z = n$. So, let $S$ be any recursive set of natural numbers. Then we can define $x \in S \leftrightarrow \bigwedge_{n \in S} \varphi_n(x)$. So $S$ is $\Sigma^0_2$ (with no parameters). But $S$ is also $\Sigma^0_2$, hence $S$ is $\Delta^0_2$ in every recursive copy, and so $\Delta^0_2$ separable from any disjoint set. But $S_1$ and $S_2$ are recursive, so $S_1$ and $S_2$ are $\Delta^0_2$ separable in every recursive copy of $(\omega, <)$.

Thus, Theorem 3.2 gives us the best "separability bounds" for $S_1$ and $S_2$. We proceed now with the proof.
Proof of Theorem 3.2. We seek to apply Theorem 2.6. First, we need to find a recursive copy $\mathcal{A}$ of $(\omega, <)$ satisfying extra decidability conditions. Fortunately, there is a very natural recursive copy of $(\omega, <)$. Label the $n$th least element of the structure $(\omega, <)$ with the natural number $n - 1$. This defines a recursive copy $\mathcal{A}$. We shall verify that the extra decidability conditions hold for $\mathcal{A}$, namely that both $S_1$ and $S_2$ satisfy $(\ast)$. Let $\overline{S}_1 = R$. We must verify then that given any existential formula $\phi(x, \bar{y})$ and parameters $\bar{a}$ from $\mathcal{A}$, we can decide whether or not $\phi(x, \bar{a}) \rightarrow R(x)$ is true in $\mathcal{A}$. That is to say, we must decide whether or not $\{x : \phi(x, \bar{a})\} \subseteq R$. Consider the following claim:

Claim. Let $\phi(x, \bar{y})$ be any existential formula. Let $\bar{a} \in \mathcal{A}$. Then there is some finite set $F$ such that $\{x : \phi(x, \bar{a})\} = F$ or $\mathbb{N} - F$. We can recursively compute which case holds, and the set $F$. Furthermore, if $\{x : \phi(x, \bar{a})\}$ is finite, $\{x : \phi(x, \bar{a})\}$ is bounded above by $\max(\bar{a})$.

Let us see how the whole theorem follows from this claim. Note that as $R$ is both infinite and has infinite complement, the only case where $\{x : \phi(x, \bar{a})\} \subseteq R$ is when $\phi$ defines a finite subset of $R$. As $R$ in $\mathcal{A}$ is $\Delta^0_1$, we can decide for a given $\phi$ whether or not this is the case: if $\phi$ defines an infinite set (which we can decide by the claim) then we know $\{x : \phi(x, \bar{a})\}$ is co-finite and hence not a subset of $R$. If $\{x : \phi(x, \bar{a})\}$ defines a finite set $F$, then we can recursively find $F$, and test to see if $F \subseteq R$, as $R$ is recursive in $\mathcal{A}$. Thus, $R$ satisfies $(\ast)$, that is, $\overline{S}_1$ satisfies $(\ast)$. Obviously, $\overline{S}_2$ also satisfies $(\ast)$, and thus we have verified that $\mathcal{A}$ satisfies all extra decidability conditions.

It suffices now to verify that there are no two $\Sigma^1_1(\bar{p})$ sets $F_1$ and $F_2$ with $S_1 \subseteq F_1 \subseteq \overline{S}_2, S_2 \subseteq F_2 \subseteq \overline{S}_1$, and $F_1 \cup F_2 = \omega$. Consider any such $F_1$. Then, we have $x \in F_1$ iff $\bigcup_{i \in I} \phi_i(x, \bar{p})$, for some r.e. set $I$ and some uniformly recursive set of existential formula $\{\phi_i(x, \bar{p})\}_{i \in I}$.

Let $T_i = \{x : \phi_i(x, \bar{p})\}$. Then $F_1 = \bigcup_i T_i$. But each $T_i$ is finite or co-finite by the claim. If there is an $i$ with $T_i$ co-finite, then $F_1$ is co-finite. But $F_1 \subseteq \overline{S}_2$, so $F_1$ must have an infinite set in its complement. Thus, each $T_i$ is finite. By the claim, each $T_i$ is bounded above by $\max(\bar{p})$, and thus $F_1$ is bounded above by $\max(\bar{p})$. But $S_1 \subseteq F_1$, so $F_1$ is required to be infinite, contradiction. No such $F_1$ exists, and the theorem is established. It remains now to verify the claim.

Verification of Claim. Let $\phi(x, \bar{a})$ be an existential sentence, $\phi(x, \bar{a}) \leftrightarrow \exists \bar{z} \psi(\bar{z}, x, \bar{a})$. Assume $\psi$ is in disjunctive normal form. Then we can write $\exists \bar{z} \psi(\bar{z}, x, \bar{a})$ as $\bigwedge_{i = 1, \ldots, n} \exists \bar{z} \psi_i(\bar{z}, x, \bar{a})$, for some natural number $n$, where each $\psi_i$ is a conjunction of atomic sentences and negations of atomic sentences.

Call a set of natural numbers good if it is finite and bounded above by $\max(\bar{a})$, or if it is co-finite. We must show that $\{x : \exists \bar{z} \psi(\bar{z}, x, \bar{a})\}$ is good. As the union of finitely many good sets is good, it suffices to show for each $i$ that $\{x : \exists \bar{z} \psi_i(\bar{z}, x, \bar{a})\}$ is good.
The formula $\psi_i$ can assert one of two things, first a contradiction (for instance $x > x$) or second, that a particular (consistent) order relation holds between $\tilde{z}, x, \tilde{a}$ (for instance $z_1 > x$ and $x < a_3$). If $\psi_i$ asserts a contradiction, then $\{x : \exists \tilde{z} \varphi_i(\tilde{z}, x, \tilde{a})\}$ is the empty set, which is good. So assume from here on that $\psi_i$ is not of this form.

Note that there are only finitely many ways in which $\tilde{z}, x, \tilde{a}$ can be ordered, and corresponding to each ordering, there is a formula $\theta_j(\tilde{z}, x, \tilde{a})$ which just asserts that $\tilde{z}, x$ and $\tilde{a}$ are ordered in this way.

One can recursively list the possible orderings of $\tilde{z}, x$ and $\tilde{a}$ consistent with $\psi_i(\tilde{z}, x, \tilde{a})$. Consequently, we will be able to express $\psi_i(\tilde{z}, x, \tilde{a})$ as a disjunction of finitely many $\theta_j(\tilde{z}, x, \tilde{a})$, i.e., for some finite set $J, \psi_i(\tilde{z}, x, \tilde{a}) \leftrightarrow \bigvee_{j \in J} \theta_j(\tilde{z}, x, \tilde{a})$. Again, as the union of finitely many good sets is good, it suffices to show that for any $j$, $\{x : \exists \tilde{z} \theta_j(\tilde{z}, x, \tilde{a})\}$ is good.

Assume $\tilde{a} = \{a_1, a_2, \ldots, a_k\}, a_1 < a_2 < \cdots < a_k$. Without loss of generality, we may assume $a_1 = 0$, the least element of $\omega$. Then, for each $j$, $\theta_j(\tilde{z}, x, \tilde{a})$ implies $a_1 \leq x \leq a_i + 1$ for some $1 \leq i \leq k$, or $\theta_j(\tilde{z}, x, \tilde{a})$ implies $x \geq a_k$. We consider two cases:

Case 1: $\theta_j(\tilde{z}, x, \tilde{a}) \rightarrow a_i \leq x \leq a_i + 1$ for some $i$. Note that $\{x : \exists \tilde{z} \theta_j(\tilde{z}, x, \tilde{a})\}$ is bounded above by $a_i + 1$, and is therefore good as desired. We must show how to recursively find $\{x : \exists \tilde{z} \theta_j(\tilde{z}, x, \tilde{a})\}$. The formula $\theta_j$ just tells us that for some $m$ there are $z_1, \ldots, z_m$ such that

$$a_i < z_1 < z_2 < \cdots < z_n < x < z_{n+1} < \cdots < z_m < a_i + 1.$$  

(Note first that we can ignore all variables asserted to exist outside the interval $[a_i, a_i + 1]$. Secondly, some of these inequalities may in fact not be strict. We assume for notational simplicity that they are and hope that the minor changes required when they are not will be apparent to the reader.)

Let $\Phi(x)$ hold iff

$$\exists z_1 \cdots z_m[a_i < z_1 < \cdots < z_n < x < z_{n+1} < \cdots < z_m < a_i + 1].$$

Then $\Phi(x) \leftrightarrow \exists \tilde{z} \theta_j(\tilde{z}, x, \tilde{a})$. Also,

$$\Phi(x) \leftrightarrow [(a_i + n) < x < a_{i+1} - (m - n)].$$

So

$$\{x : \exists \tilde{z} \theta_j(\tilde{z}, x, \tilde{a})\} = \{x : a_i + n < x < a_{i+1} - (m - n)\},$$

and Case 1 is done.

Case 2. $\theta_j(\tilde{z}, x, \tilde{a}) \rightarrow x \geq a_k$. Then $\theta_j(\tilde{z}, x, \tilde{a})$ just tells us that for some $m$ there are $z_1, \ldots, z_m$ such that $a_i < z_1 < \cdots < z_m < x$. (Note again that variables outside $[a_i, x]$ are irrelevant and that we are assuming all inequalities are strict.) As before, we then have

$$\{x : \exists \tilde{z} \theta_j(\tilde{z}, x, \tilde{a})\} = \{x : x > a_k + m\}.$$  

This set is good, and so we are finished with Case 2.

This completes the verification of the claim, and gives us the theorem. □
Theorem 3.2 then gives surprisingly general conditions under which we can make two sets of natural numbers recursively inseparable in a recursive copy of \((\omega, <)\). We seek to generalize Theorems 3.1 and 3.2, to find natural examples of \(\mathcal{A}\) inseparability in a recursive copy of \((\omega, <)\). This will require the use of Theorem 2.17. Consequently, we will need an understanding of the behaviour of the \(\leq_{\beta}\) relations on linear orderings. In particular, we are interested in the way the \(\leq_{\beta}\) relations behave on linear orderings of type \((\gamma, <)\), where \(\gamma\) is a constructive ordinal.

The following two theorems provide us with all the data we shall require on the \(\leq_{\beta}\) relations and constructive ordinals. Both are due to Ash, and are taken from [5]. First, we require a slight generalization of the \(\leq_{\beta}\) relations. Recall that, given a structure \(\mathcal{A}\) and \(n\)-tuples \(\bar{c}\) and \(\bar{d}\) from \(\mathcal{A}\), we say \(\bar{c} \leq_{\beta} \bar{d}\) iff every \(\Pi_{2}^{n}\) formula true of \(\bar{c}\) is true of \(\bar{d}\). Now, let \(\mathcal{A}\) and \(\mathcal{B}\) be structures of the same type, let \(\bar{c}\) be an \(n\)-tuple from \(\mathcal{A}\), and let \(\bar{d}\) be an \(n\)-tuple from \(\mathcal{B}\). Then, say \((\mathcal{A}, \bar{c}) \leq_{\beta}(\mathcal{B}, \bar{d})\) iff for every \(\Pi_{2}^{n}\) formula \(\psi, \mathcal{A} \models \psi(\bar{c}) \rightarrow \mathcal{B} \models \psi(\bar{d})\). In our old notation, \(\bar{c} \leq_{\beta} \bar{d}\) iff \(\mathcal{A} \models \psi(\bar{c}) \rightarrow \mathcal{B} \models \psi(\bar{d})\). Write \(\mathcal{A} \leq_{\alpha} \mathcal{B}\) iff \((\mathcal{A}, \emptyset) \leq_{\alpha}(\mathcal{B}, \emptyset)\), where \(\emptyset\) is the unique \(0\)-tuple, that is to say \(\mathcal{A} \leq_{\alpha} \mathcal{B}\) iff for all \(\Pi_{2}^{n}\) sentences \(\psi, \mathcal{A} \models \psi \rightarrow \mathcal{B} \models \psi\).

Given any linear ordering \(\mathcal{A}\), any finite \(n\)-tuple determines a partition of \(\mathcal{A}\) into sections \(\langle \mathcal{A}_{0}, \mathcal{A}_{1}, \ldots, \mathcal{A}_{n} \rangle\) such that \(\mathcal{A} = \mathcal{A}_{0} + 1 + \mathcal{A}_{1} + 1 + \cdots + 1 + \mathcal{A}_{n}\). Let \(\bar{a} = \langle a_{1}, a_{2}, \ldots, a_{n} \rangle\) be any \(n\)-tuple, with \(a_{1} < a_{2} < \cdots < a_{n}\).

Defining
\[
\mathcal{A}_{0} = \{x : x \in \mathcal{A} \land x < a_{1}\}, \quad \mathcal{A}_{n} = \{x : x \in \mathcal{A} \land x > a_{n}\}
\]
and
\[
\mathcal{A}_{i} = \{x : x \in \mathcal{A} \land a_{i} < x < a_{i+1}\} \quad \text{for } 1 \leq i < n
\]
gives the desired partition.

Consider two linear orderings \(\mathcal{A}\) and \(\mathcal{B}\), and two strictly increasing \(n\)-tuples \(\bar{c}\) and \(\bar{d}\) from \(\mathcal{A}\) and \(\mathcal{B}\). The \(n\)-tuple \(\bar{c}\) breaks \(\mathcal{A}\) into sections \(\{\mathcal{A}_{i}\}_{1 \leq i \leq n}\) as described above. Likewise, \(\mathcal{B}\) is broken into sections \(\{\mathcal{B}_{i}\}_{1 \leq i \leq n}\) by \(\bar{d}\). We say that the pairs \(\langle \mathcal{A}_{i}, \mathcal{B}_{i} \rangle\) are “corresponding” sections, for \(i \leq n\). We have our first theorem:

**Theorem 3.3.** Assume \(\mathcal{A}\) and \(\mathcal{B}\) are linear orderings, \(\bar{c}\) is a strictly increasing \(n\)-tuple from \(\mathcal{A}\) and \(\bar{d}\) is a strictly increasing \(n\)-tuple from \(\mathcal{B}\). Then, \((\mathcal{A}, \bar{c}) \leq_{\alpha}(\mathcal{B}, \bar{d})\) iff \(\mathcal{A}_{i} \leq_{\alpha} \mathcal{B}_{i}\) for each pair \(\langle \mathcal{A}_{i}, \mathcal{B}_{i} \rangle\) of corresponding sections.

We refer the reader to [5] for a proof.

This theorem enables us to break down the \(\leq_{\alpha}\) relations between a pair of structures with parameters into a set of \(\leq_{\alpha}\) relations holding between certain pairs of structures with no added parameters.

So if we are interested in the case where \(\mathcal{A}\) and \(\mathcal{B}\) are linear orderings isomorphic to some recursive ordinal, this simplifies the \(\leq_{\alpha}\) relations between \(n\)-tuples from \(\mathcal{A}\) and
to the \( \leq_\alpha \) relations between pairs of structures with no parameters. However, we need to be able to answer the question “When is \( \gamma \leq_\alpha \gamma' \) for \( \gamma, \gamma' < \omega_1^{CK} \)” in order for this to be of any use. The full answer to this question is given by the following. First note that we write \( \alpha < \delta \beta \) to mean \( \alpha = \delta + \beta \) and not \( \alpha < \beta \).

**Theorem 3.4.** Let \( \alpha, \beta, \alpha', \beta', \gamma \) and \( \rho \) be ordinals, where \( \rho < \omega^\omega \), and let \( m, n < \omega \). Then facts (i)-(iii) below, together with the fact that \( \alpha < \delta \beta \to \alpha < \delta' \beta' \) for \( \delta < \delta' \), yields all true statements of the form \( \alpha < \delta \beta \), for \( \delta < \omega_1^{CK} \).

(i) If \( \alpha = \omega^\rho \cdot m + \rho, \beta = \omega^\rho \cdot n + \rho, \) and \( m > n \geq 1 \), then \( \alpha < \beta + 1 \).

(ii) If \( \alpha = \omega^\rho + \gamma \cdot m + \rho, \beta = \omega^\rho \cdot n + \rho, \gamma > 1 \) and \( n > 1 \), then \( \alpha < \beta + 1 \).

(iii) If \( \alpha = \omega^\rho + \gamma \cdot m + \rho, \beta = \omega^\rho + \gamma \cdot n + \rho, \gamma' > 2 \) and \( m > n \), then \( \alpha < \beta + 1 \).

Again, we refer the reader to [5] for a proof.

Note that every ordinal \( \gamma \) less than \( \omega_1^{CK} \) can be written in the form

\[ \gamma = \omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \cdot n_2 + \cdots + \omega^{\alpha_m} \cdot n_m, \]

for some natural numbers \( m, n_1, n_2, \ldots, n_m \), and some finite sequence of constructive ordinals \( \alpha_1 > \alpha_2 > \cdots > \alpha_m \). This is known as Cantor Normal Form. Let \( a_1, a_2, \ldots, a_m \) be ordinal notations for \( \alpha_1, \alpha_2, \ldots, \alpha_m \). Then encode \( a_1, a_2, \ldots, a_m, n_1, \ldots, n_m \) into a single natural number \( d \). We call \( d \) the Cantor Code of \( \gamma \). The Cantor code gives us more information about an ordinal than the ordinal notation from which the Cantor code is derived.

Fix any ordinal \( \gamma \). Then, consider a function \( d \) on \( \gamma \times \gamma \) defined as follows: for \( \alpha_1 \) and \( \alpha_2 \) in \( \gamma \) with \( \alpha_1 < \alpha_2 \), \( \{ x : \alpha_1 < x < \alpha_2 \} \) is a well-ordered set of some order type \( \delta \). Define \( d(\alpha_1, \alpha_2) \) to be a Cantor code for this order type. Call \( d \) the “distance function” on \( \gamma \). We then have the following lemma.

**Lemma 3.5.** Let \( \mathcal{A} \) be a recursive copy of the linear ordering \( (\omega^\omega, <) \) for some \( \gamma < \omega_1^{CK} \). Then, if the distance function is recursive on \( \mathcal{A} \), the existential diagram of \( \mathcal{A} \) is decidable and the relations \( \{ \leq_\gamma \} \) are uniformly recursive.

**Proof.** We show first that under the conditions of the theorem, the existential diagram of \( \mathcal{A} \) is decidable. Consider an existential sentence of the form \( \exists \bar{z} \psi(\bar{z}, \bar{a}) \). As in the proof of the claim in Theorem 3.2, we can assume \( \psi \) is in disjunctive normal form, write \( \exists \bar{z} \psi(\bar{z}, \bar{a}) \leftrightarrow \bigwedge_{i=1}^{\gamma} \exists \bar{z} \psi_i(\bar{z}, \bar{a}) \) where each \( \psi_i \) is a conjunction of atomic relations and negations of atomic relations, and conclude that it suffices to give a decision procedure for each sentence \( \exists \bar{z} \psi_i(\bar{z}, \bar{a}) \).

\( \psi_i \) will then just assert that some ordering relation holds between the \( \bar{z} \) and \( \bar{a} \). If this ordering relation is inconsistent, the sentence \( \exists \bar{z} \psi_i(\bar{z}, \bar{a}) \) is false. Otherwise, \( \psi_i(\bar{z}, \bar{a}) \) can just be written as \( \bigwedge_{j \in J} \theta_j(\bar{z}, \bar{a}) \) for some finite set \( J \), where each \( \theta_j \) describes completely
some particular (unique) ordering between $\vec{z}$ and $\vec{a}$, as in the verification of the claim in Theorem 3.2. So

$$\exists \vec{z} \psi(\vec{z}, \vec{a}) \iff \exists \vec{z} \bigwedge_{j \in J} \theta_j(\vec{z}, \vec{a}) \iff \bigwedge_{j \in J} \exists \vec{z} \theta_j(\vec{z}, \vec{a}),$$

and consequently it suffices to demonstrate a decision procedure for each sentence of the form $\exists \vec{z} \theta_j(\vec{z}, \vec{a})$. Assume $\vec{a} = \{a_1, a_2, \ldots, a_k\}$. Without loss of generality, we may assume $a_1 = 0$. Then, $\theta_j(\vec{z}, \vec{a})$ just asserts

"there are at least $f_1$ elements between $a_1$ and $a_2$, and there are at least $f_2$ elements between $a_2$ and $a_3$, 

\ldots

there are at least $f_{k-1}$ elements between $a_{k-1}$ and $a_k$, and there are at least $f_k$ elements greater than $a_k"$

for some finite sequence $f_1, f_2, \ldots, f_k$ of natural numbers, recursively computable from $\theta_j$ as in Theorem 3.2. Then, $\exists \vec{z} \theta_j(\vec{z}, \vec{a})$ iff $\forall i (1 \leq i \leq k, d(a_i, a_{i+1}) \geq f_i)$. As the distance function is recursive, this is decidable. So the existential diagram of $\mathcal{A}$ is decidable.

We now show that the relations $\{ \leq_{\beta} \}_{\beta < 2^{\gamma} + 1}$ are uniformly recursive on $\mathcal{A}$. Fix $\delta \leq 2^{\gamma} + 1$, and let $\vec{c}, \vec{a}$ be finite $n$-tuples from $\mathcal{A}$. We must give a procedure for deciding if $\vec{c} \leq_{\delta} \vec{a}$. Using the recursiveness of the distance function and Lemma 3.3, we can recursively obtain the Cantor codes for two sequences of ordinals $\{\alpha_i\}_{i<n}$ and $\{\beta_i\}_{i<n}$ with each $\alpha_i$ and $\beta_i$ less than $\omega^\gamma$, such that $\vec{c} \leq_{\delta} \vec{a}$ iff for all $i < n, \alpha_i \leq_{\delta} \beta_i$. However, we can decide if $\alpha_i \leq_{\delta} \beta_i$ by using Theorem 3.4 and the information contained in Cantor codes. Thus, the relations $\{ \leq_{\beta} \}_{\beta < 2^{\gamma} + 1}$ are uniformly recursive.

Given these preliminaries, we are now ready to generalize Theorem 3.2 for arbitrary successor ordinals.

**Theorem 3.6.** Let $\gamma + 1 < \omega_1^\text{CK}$, and let $\mathcal{A}$ be a recursive copy of the linear ordering $(\omega^\gamma + 1, <)$ in which the distance function is recursive. Consider the set $S = \{\omega^\gamma \cdot n, n \in \mathbb{N}\}$ in $\mathcal{A}$. Let $S_1$ and $S_2$ be two infinite disjoint recursive subsets of $S$. Then, there is a recursive copy $\mathcal{B}$ of $(\omega^\gamma + 1, <)$ in which $S_1$ and $S_2$ are $\Delta_2^{\omega^\gamma + 1}$ inseparable.

**Proof.** We aim, of course, to apply Theorem 2.17. Using 3.5, the only condition (1)–(4) requiring verification is that the relations $\text{Cl}_{\omega^\gamma + 1}(S_1, \vec{p})$ and $\text{Cl}_{\omega^\gamma + 1}(S_2, \vec{p})$ are uniformly recursive in $\vec{p}$. Let us defer a proof of this for a paragraph.

Assuming then that conditions (1)–(4) have been met, it suffices now to show that there is no pair $F_1$ and $F_2$ of $\Sigma_2^{\omega^\gamma + 1}(\vec{p})$ sets with $S_1 \subset F_1 \subset S_2, S_2 \subset F_2 \subset S_1$ and $F_1 \cup F_2 = \mathcal{A}$. Assume there was such $F_1$ and $F_2$. Let $x$ and $y$ be elements of the form $\omega^\gamma \cdot n$ such that $x > y > \max(\vec{p})$. We aim to show that $\vec{p}, x \leq_{2^{\gamma} + 1} \vec{p}, y$. By
Theorem 3.3, it suffices to show that if \( m > n \geq 1 \), then \( \omega^y \cdot m \leq 2^{y+1} \omega^y \cdot n \). This follows from Theorem 3.4(i). So \( \bar{p}, x \leq \bar{S}_2, \bar{p}, y \). Hence, any \( \Sigma^+_{2^{y+1}}(\bar{p}) \) formula satisfied by \( y \) is also satisfied by \( x \). Therefore, if \( F_1 \) contains an \( x \) such that \( x = \omega^y \cdot n \) for some \( x \in \mathbb{N} \) with \( x > \max(\bar{p}) \), we must have that \( F_1 \) contains all \( x \) of the form \( \omega^y \cdot m \) with \( m > n \). But \( F_1 \subseteq \bar{S}_2 \), so this is not possible. Thus, \( F_1 \) contains no \( x \) with \( x = \omega^y \cdot n \) and \( x > \max(\bar{p}) \). But \( S_1 \subseteq F_1 \), so we have a contradiction. Therefore no such sets \( F_1 \) and \( F_2 \) exist, so by Theorem 2.17 there is a recursive copy of \((\omega^{y+1}, <)\) in which \( S_1 \) and \( S_2 \) are \( \Delta^0_{2y+1} \) inseparable.

It suffices then to show that the relations \( x \in \text{Cl}^{2y+1}_y(S_1, \bar{p}) \) and \( x \in \text{Cl}^{2y+1}_y(S_2, \bar{p}) \) are uniformly recursive in \( x \) and \( \bar{p} \). Let \( R = \bar{S}_1 \). We show \( \text{Cl}^{2y+1}_y(R, \bar{p}) \) has the required property. The proof for \( R = \bar{S}_2 \) will then be identical.

Note first that \( R \) is just the union of the set of ordinals not of the form \( \omega^y \cdot n \) in some infinite subset of \( \mathbb{N} \). We show first that the set of ordinals not of the form \( \omega^y \cdot n \) for any \( n \) is a subset of \( \text{Cl}^{2y+1}_y(R, \bar{p}) \), for any \( \bar{p} \). Recall that \( \text{Cl}^{2y+1}_y(R, \bar{p}) \) is the union of the \( \Sigma^+_{2^{y+1}}(\bar{p}) \) definable subsets of \( R \). It suffices to show that we can define the complement of \( \{ \omega^y \cdot n, n \in \mathbb{N} \} \) in a \( \Sigma^+_{2^{y+1}} \) way, without parameters. Note that \( \{ \omega^y \cdot n, n \in \mathbb{N} \} \) is just the set of \( y \)th limit points of \( \Pi^1_y \). A simple induction shows that the \( y \)th limit points of any ordinal is \( \Pi^1_y \) definable without parameters. Thus, the complement of \( \{ \omega^y \cdot n, n \in \mathbb{N} \} \) is \( \Sigma^+_{2^{y+1}} \), definable, as desired.

It remains then to ask, given any element \( x \in \omega^y \cdot n \) in \( R \), whether or not \( x \in \text{Cl}^{2y+1}_y(R, \bar{p}) \). Enumerate \( \bar{p} \) as \( p_1 \prec p_2 \prec \cdots \prec p_k \). Then, we will have one of

\begin{enumerate}
  \item \( x \leq p_1 \),
  \item \( p_i \leq x \leq p_{i+1} \) for some \( 1 \leq i < k \),
  \item \( p_k \leq x \).
\end{enumerate}

Moreover, we can recursively decide which case holds. We show in each case how to decide if \( x \in \text{Cl}^{2y+1}_y(R, \bar{p}) \).

Case 1. Assume there are \( m \) points of the form \( \omega^y \cdot i \) with \( \omega^y \cdot i \prec p_1 \), so \( \omega^y \), \( \omega^y \cdot 2 \), \ldots , \( \omega^y \cdot m \) are all less than \( p_1 \) and there is no \( y \)th limit point between \( \omega^y \cdot m \) and \( p_1 \). Then, each \( \omega^y \cdot i, 1 \leq i \leq m \), is \( \Sigma^+_{2^{y+1}} \) definable in the parameter \( p_1 \) as follows: let \( \phi_i(x) \leftrightarrow \exists z_1 \cdots \exists z_{i-1} \exists z_{i+1} \cdots \exists z_m[z_1, \ldots , z_{i-1}, z_{i+1}, \ldots , z_m \text{ and } x \text{ are all } y \text{th limit ordinals, and } z_1 < \cdots < z_{i-1} < x < z_{i+1} < \cdots < z_m < p_1] \). Then \( \phi_i(x) \leftrightarrow x = \omega^y \cdot i \).

So each element of \( R \) of the form \( \omega^y \cdot i \) with \( \omega^y \cdot i \prec p_1 \) is in \( \text{Cl}^{2y+1}_y(R, \bar{p}) \) of course, if \( \omega^y \cdot i \prec p_1 \) then \( \omega^y \cdot i \) is in \( \text{Cl}^{2y+1}_y(R, \bar{p}) \) as well. So in Case 1, there is nothing to check.

Case 2. The details here are analogous to Case 1. Again, it turns out that if \( \omega^y \cdot j \) is an element of \( R \) and \( p_i \leq \omega^y \cdot j \leq p_{i+1} \), then \( \omega^y \cdot j \) is an element of \( \text{Cl}^{2y+1}_y(R, \bar{p}) \), and so there is nothing to check.

Case 3. Assume \( p_k \leq x \), where \( x = \omega^y \cdot n \). Assume \( x \in \text{Cl}^{2y+1}_y(R, \bar{p}) \). Let \( S \) be a \( \Sigma^+_{2^{y+1}}(\bar{p}) \) subset of \( R \) containing \( x \). Then, as long as \( p_k \neq x \), we must have that \( \omega^y \cdot n \in S \) for all \( n' > n \), using Lemma 3.4(i). So \( \{ x^y \cdot n' \mid n' > n \} \subseteq \text{Cl}^{2y+1}_y(R, \bar{p}) \subseteq R \), contradiction. So \( \{ x^y \cdot n \mid x^y \cdot n \geq p_k \} \cap \text{Cl}^{2y+1}_y(R, \bar{p}) = \emptyset \). However, if \( p_k = \omega^y \cdot n \) obviously the singleton \( \omega^y \cdot n \) is \( \Sigma^+_{2^{y+1}}(p_k) \) definable, and \( \omega^y \cdot n \in \text{Cl}^{2y+1}_y(R, \bar{p}) \). This gives us a clear algorithm for deciding if \( \omega^y \cdot n \in \text{Cl}^{2y+1}_y(R, \bar{p}) \) in Case 3.
Thus, as we can decide which of the cases 1, 2, 3 holds, $Cl_{2\gamma+1}(R, \bar{p})$ is recursive and the theorem is proven. □

We can then immediately give a generalization of Theorem 3.1, as a corollary of Theorem 3.6.

**Theorem 3.7.** Let $\gamma < \omega_1^{CK}$. Let $S_1 = \{\omega^{\gamma} \cdot n : n \equiv 1 \mod 2\}$ and $S_2 = \{\omega^{\gamma} \cdot n : n \equiv 0 \mod 2\}$. Then there is a recursive copy of $(\omega^{\gamma+1}, <)$ in which $S_1$ and $S_2$ are $A^0_{2\gamma+1}$ inseparable.

**Proof.** This follows from Theorem 3.7 and the existence of a recursive copy of $(\omega^{\gamma+1}, <)$ in which $S_1, S_2$ and the distance function are also recursive. □

It is natural now to ask what can be done with a recursive copy of $(\omega^{\gamma}, <), \gamma < \omega_1^{CK}$, where $\gamma$ is a limit ordinal. To begin with, consider $(\omega^n, <)$. Theorem 3.6 suggests that we might try and make some pair of disjoint, cofinal subsets of $(\omega^n, <)$ $A^0_n$ inseparable. The following theorem shows that this will not be possible.

**Lemma 3.8.** Every singleton in $(\omega^n, <)$ is $\Sigma^r_n$ definable (without parameters) for some $n < \omega$.

**Proof.** Each singleton is expressible in the form

$$\omega^{k_1} \cdot n_1 + \omega^{k_2} \cdot n_2 + \cdots + \omega^{k_j} \cdot n_j,$$

for some decreasing sequence $k_1 > k_2 > \cdots > k_j$ of natural numbers, and some $\{n_1, n_2, \ldots, n_j\} \subseteq \mathbb{N}$. Such a singleton is the unique element $z$ having the property that

"there are $n_1$ $k_1$th limit ordinals $l^1_{n_1}, l^2_{n_1}, \ldots, l^{k_1}_{n_1}$, and $n_2$ $k_2$th limit ordinals $l^3_{n_2}, l^2_{n_2}, \ldots, l^{k_2}_{n_2}$

\[\ldots\]

and $n_j$ $k_j$th limit ordinals $l^{k_j}_{n_j}, l^n_{n_j}, \ldots, l^{k_j}_{n_j}$, such that

$$l^1_{n_1} < \cdots < l^{k_1}_{n_1} < l^2_{n_2} < \cdots < l^{k_2}_{n_2} < \cdots < l^1_{n_j} < \cdots < l^{k_j}_{n_j} < z$$

and for all $(l^1_{n_1})^*, (l^2_{n_2})^*, \ldots, (l^{k_1}_{n_1})^*, (l^{k_2}_{n_2})^*, \ldots, (l^{k_j}_{n_j})^*$ ordered in the same way with $(l^{k_j}_{n_j})^* < z$ and

$(l^1_{n_1})^*, (l^1_{n_1})^*, \ldots, (l^{k_1}_{n_1})^*$ all $k_1$th limit ordinals

\[\ldots\]

$(l^{k_j}_{n_1})^*, (l^{k_j}_{n_1})^*, \ldots, (l^{k_j}_{n_j})^*$ all $k_j$th limit ordinals,

we have $l^1_{n_1} = (l^1_{n_1})^*, \ldots, l^{k_j}_{n_j} = (l^{k_j}_{n_j})^*$.

This can be seen to be $\Sigma^r_n$ (with no parameters) for some $n < \omega$. □
Theorem 3.9. Let $S$ be a set which is recursive in some recursive copy of $(\omega^\omega, <)$. Then, $S$ is $\Sigma^\omega_1$ definable.

Proof. Let $\mathcal{A}$ be a copy in which $S$ is a recursive set. For each $\gamma \in \omega^\omega$ let $\phi_\gamma(x)$ be the formula which holds of the single variable $x$ iff $x = \gamma$, given by the previous theorem. Then $x \in S \iff \bigvee_{\gamma \in S} \phi_\gamma(x)$ in $\mathcal{A}$. So $S$ is $\Sigma^\omega_1$. □

Thus, given two disjoint subsets $S_1$ and $S_2$ of $(\omega^\omega, <)$, which are recursive in some copy of $(\omega^\omega, <)$, we cannot hope to find a recursive copy of $(\omega^\omega, <)$ with $S_1$ and $S_2$ $\Delta^\omega_1$ inseparable, as there will always be $\Sigma^\omega_1$ sets $F_1$ and $F_2$ with $S_1 \subset F_1 \subset \bar{S}_2, S_2 \subset F_2 \subset \bar{S}_1, F_1 \cup F_2 = \omega^\omega$, by simply letting $F_1 = S_1$ and $F_2 = S_2$.

Moreover, the same comments apply to any $\omega^\gamma, \gamma < \omega_1^\alpha$ and $\lim(\gamma)$. For any such $\gamma$, all singletons in $\omega^\gamma$ will be $\Sigma^\gamma_1$ definable without parameters, and so we will obtain a result analogous to Theorem 3.9. It seems then that the inseparability results we have in Theorem 3.6 are the best we can achieve in this direction.

3.2. Boolean algebras

We turn our attention now to Boolean Algebras. There is a very close link between countable boolean algebras and countable linear orderings. Given any linear ordering $L$, one can consider the set of finite unions of semi-open intervals of $L$, that is, the set of finite unions of intervals of the form $[a, b)$ with $a \in L, b \in L$ or $b = \infty$. This set is then a boolean algebra under union, intersection and complementation. So given any linear ordering $L$, we have a corresponding boolean algebra $B(L)$. Moreover, any countable boolean algebra is isomorphic to $B(L)$ for some countable linear ordering $L$.

We will be particularly interested in boolean algebras generated by recursive well-orderings, that is, boolean algebras of the form $B(\alpha), \alpha < \omega_1^{CK}$. These boolean algebras are known as superatomic boolean algebras, as they have no atomless subalgebras.

Associated with any boolean algebra, we have a set of atoms and a set of co-atoms. The atoms of a boolean algebra $B$ are just the set of elements $x$ of $B$ such that there is no $z$ with $0 < z < x$. The dual notion of the set of atoms is the set of co-atoms.

The set of atoms generate an ideal. This ideal simply contains the unions of finite sets of atoms, and is known as the ideal generated by the atoms. Dually, we obtain the filter generated by the co-atoms. We begin with the following theorem.

Theorem 3.10. Let $\alpha$ be an infinite ordinal less than $\omega_1^{CK}$. Then there is a recursive copy of $B(\alpha)$ in which the ideal generated by the atoms and the filter generated by the co-atoms are recursively inseparable.

Proof. We seek (of course) to apply Theorem 2.6. First, we must verify that there is a recursive copy of $B(\alpha)$ satisfying the extra assumptions of Theorem 2.6. Let $I$ be the
ideal generated by the atoms of $B(a)$ and $F$ be the filter generated by the co-atoms of $B(a)$. We must produce a recursive copy $\mathcal{A}$ of $B(a)$ in which $\bar{I}$ and $\bar{F}$ satisfy condition $(\ast)$. So, given any existential formula $\psi(x, \bar{a})$ with $\bar{a} \in \mathcal{A}$, we must be able to decide if $\psi(x, \bar{a}) \rightarrow x \in \bar{I}$ and if $\psi(x, \bar{a}) \rightarrow x \in \bar{F}$. Let us fix a $\psi$ and an $\bar{a}$, and concentrate on deciding if $\psi(x, \bar{a}) \rightarrow x \in \bar{I}$.

We know that each element $x$ of $B(a)$ may be represented as

$$[x_1, x_2) \cup [x_3, x_4) \cup \cdots \cup [x_{2n-1}, x_{2n})$$

for $x_1 < x_2 < \cdots < x_{2n}$ all less than or equal to $a$. Let $z_1, z_2, \ldots, z_{2n}$ be Cantor codes for $x_1, x_2, \ldots, x_{2n}$. Code $z_1, z_2, \ldots, z_{2n}$ into a single natural number $z$. Then, say a copy of $B(a)$ has a recursive representation function iff the function $r$ which maps an element $x$ of the boolean algebra to the value $z$ described above is a recursive function. We shall demonstrate that if $B(a)$ has a recursive representation function, then we can decide whether or not $\psi(x, a) \rightarrow x \in I$.

Enumerate $\bar{a}$ as $\{a_1, \ldots, a_k\}$. Then consider the set $\bar{b}$ of elements of $\mathcal{A}$ of the form $b_1 \cap b_2 \cap \cdots \cap b_k$, where $b_1 = a_1$ or $a_1', b_2 = a_2$ or $a_2', \ldots, b_k = a_k$ or $a_k'$. Then, distinct elements of $\bar{b}$ are disjoint, and each element of $\bar{a}$ can be represented as the union of some finite number of elements of $\bar{b}$. Hence, we can express the formulae $\psi(x, \bar{a})$ by some equivalent formula $\psi^*(x, \bar{b})$ involving the parameters $\bar{b}$. Moreover, each element of $\bar{b}$ can be represented as the union of finitely many half-open intervals of $a$, that is $\bar{b} = c_1 \cup c_2 \cup \cdots \cup c_k$ for some $k$ where each $c_i$ is just a half-open interval of $a$. We can thus express the formula $\psi^*(x, \bar{b})$ by some equivalent formula $\psi^{**}(x, \bar{c})$ involving only parameters $\bar{c}$ which are disjoint, half-open intervals. Thus, it suffices to restrict our attention to formulae involving only sets of disjoint half-open intervals as parameters. We may also assume that $\bigcup_{i<s} I_i$ is just $a$, by adding dummy parameters if necessary. Consequently, we shall from now on assume that $\bar{a}$ consists solely of disjoint half-open intervals. Enumerate this set of intervals $\{I_{i,s}\}_{i<s}$ for some $r \in \mathbb{N}$.

We seek to decide whether or not we can have $\psi(x, \bar{a})$ with $x$ an atom. Assume $\psi(x, \bar{a}) \leftrightarrow \exists \bar{z} \phi(z, x, \bar{a})$, and let $x, \bar{z}$ be such that $\phi(z, x, \bar{a})$. Pick any interval $I$ from $\{I_{i,s}\}_{i<s}$. Assume $z_1, \ldots, z_t$. Then consider the set of subintervals of $I$ of the form $y_1 \cap \cdots \cap y_t \cap w \cap I$ where $y_i = z_i$ or $z_i'$ and $w = x$ or $x'$. This gives us a set $\{J_{i,s}\}_{i<s}$ of disjoint subintervals of $I$, such that each element from $x, \bar{z}$ may be expressed as the union of some elements of $\{J_{i,s}\}_{i<s}$. We are now free to increase or decrease the size of these subintervals $\{J_{i,s}\}_{i<s}$ to produce new intervals $\{J^*_i\}_{i<s}$ of $I$, as long as the $J^*_i$ are all pairwise disjoint subintervals of $I$, $|J_i| \neq 0$ iff $|J^*_i| \neq 0$, and $|I - \bigcup_{i<s} J_i| \neq 0$ iff $|I - \bigcup_{i<s} J^*_i| \neq 0$. From these new intervals $\{J^*_i\}_{i<s}$ we can obtain new values for $x \cap I$ and each $z_i \cap I$, call these $\hat{x}$ and $\hat{z}_i$. Then, let $x^* = (x - I) \cup \hat{x}$ and $z_i^* = (z_i - I) \cup \hat{z}_i$. No atomic sentence with parameters $\bar{a}$ will be able to discern between $z, x$ and $z^*, x^*$ so we will therefore have $\phi(z^*, x^*, \bar{a})$.

Choose $J$ from $\{I_{i,s}\}_{i<s}$ and let $\{J_{i,s}\}_{i<s}$ be the set of subintervals of $I$ which $\bar{z}$ and $x$ define. Let $\{J_{i,s}\}_{i<s}$ be that subset of $\{J_{i,s}\}_{i<s}$ such that $J_{i,s} \cup \cdots \cup J_{i,s} = x \cap I$. Assume $I - x$ is non-empty. Then we may decrease each of $J_{i,s}$, $J_{i,s}$ to finite size, while increasing the size of $I - x$. That $I - x$ is non-empty is important, as otherwise
we would have that \( \{ J_1, \ldots, J_p \} = \{ J_1, \ldots, J_s \} \), and that \( I - \bigcup_{i<s} J_i \) was being increased in size from an empty interval to a non-empty interval.

We can repeat this process on all intervals \( \{ I_i \}_{i<s} \) such that \( I_i - x \) is non-empty, obtaining a final \( x^* \) and \( \varepsilon^* \) with \( \varphi(\varepsilon^*, x^*, \bar{a}) \). Moreover, for any interval \( I_i \), if \( x^* \) does not contain \( I_i \) then \( x^* \cap I_i \) is finite, by construction. Also, if an interval \( I_i \) is finite, we will obviously have that \( x^* \cap I_i \) is finite. So as long as there is no \( i \) such that \( I_i \) is infinite and \( I_i \subseteq x \), we will have that for all \( i, x^* \cap I_i \) is finite.

Hence, if we begin this process with an \( x \) such that we do not have \( I_i \subseteq x \) for any infinite interval \( I_i \), the \( x^* \) we end up with will be an atom. However, if for all \( x \) satisfying \( \exists \bar{a} \varphi(\varepsilon, x, \bar{a}) \) we have that \( I_i \subseteq x \) for some infinite interval \( I_i \), then there is clearly no atom \( x \) satisfying \( \exists \varepsilon \varphi(\varepsilon, x, \bar{a}) \). Thus, it suffices to decide whether or not there is an \( x \) such that we do not have \( I_i \subseteq x \) for any infinite interval \( I_i \), and \( \exists \varepsilon \varphi(\varepsilon, x, \bar{a}) \).

Enumerate the infinite intervals \( \{ I_{i_1}, \ldots, I_{i_p} \} \) of \( \{ I_i \}_{i<s} \). We must decide if \( \exists \varepsilon \varphi(\varepsilon, x, \bar{a}) \rightarrow (I_{i_1} \subseteq x \lor \cdots \lor I_{i_p} \subseteq x) \). Equivalently, as each interval \( I_{i_1}, \ldots, I_{i_p} \) corresponds to some parameter \( a_{i_1}, \ldots, a_{i_p} \), it suffices to decide whether or not \( \exists \varepsilon \varphi(\varepsilon, x, \bar{a}) \rightarrow (a_{i_1} \cup x = x \lor \cdots \lor a_{i_p} \cup x = x) \). We have reduced our decision procedure to deciding whether or not a certain \( \Pi^0_1(\bar{p}) \) sentence is true or not, with no mention of atoms. Hence, if we can show that the existential diagram of \( \mathcal{A} \) is decidable, we will be done. We are interested in the following claim.

**Claim.** If \( \mathcal{A} \) has a recursive representation function, then the existential diagram of \( \mathcal{A} \) is recursive.

For the sake of continuity, a proof of this claim is deferred until the end of the proof. For the moment we assume it is true. As a consequence, we have that any recursive copy with recursive representation function will satisfy the requirements of Theorem 2.6. Applying the conclusion of this theorem, we know that if we can prove that there are no two \( \Sigma^0_1(\bar{p}) \) sets \( S_1 \) and \( S_2 \) with \( I \subset S_1 \subset \bar{F}, F \subset S_2 \subset \bar{I} \) and \( S_1 \cup S_2 = B(\alpha) \), then there is a recursive copy \( \mathcal{B} \) of \( B(\alpha) \) in which \( I \) and \( F \) are recursively inseparable.

So assume there are two such sets, \( S_1 \) and \( S_2 \). The parameters \( \bar{p} \) define (as before) a partition of \( x \) into finitely many disjoint intervals, \( \{ I_i \}_{i<s} \). Let \( \{ I_{i_1}, \ldots, I_{i_p} \} \) again be a list of the infinite intervals of this partition. Let \( T \) be the set of \( x \) such that \( I_{i_1} \subseteq x, \ldots, I_{i_p} \subseteq x \) are false. We know that any existential formula with parameters \( \bar{p} \) satisfied by an element of \( T \) is satisfied by an atom. By an analogous argument, let \( T^* \) be the set of \( x \) such that \( x \cap I_{i_1} = 0, \ldots, x \cap I_{i_p} = 0 \) are all false. Then, any existential formula with parameters \( \bar{p} \) satisfied by an element of \( T^* \) is satisfied by a co-atom. However, we can easily construct \( x \in T \cap T^* \). Moreover, \( x \in S_1 \) or \( x \in S_2 \). Assume \( x \in S_1 \). Then \( S_1 \) is just a union of an r.e. sequence of existentially definable sets in the parameters \( \bar{p} \). So \( x \) is an element of some existentially definable set \( S \) in the parameters \( \bar{p} \). Therefore, \( S \) contains a co-atom, contradicting \( S_1 \subseteq \bar{F} \). If \( x \in S_2 \), we contradict \( S_2 \subseteq \bar{I} \) similarly. Thus, no such sets exist and the theorem is proven.

It remains now to verify the claim made earlier.
Verification of Claim. Let \( \mathcal{A} \) be a recursive copy with recursive representation function. We must show that the existential diagram of \( \mathcal{A} \) is decidable. Take any formula \( \exists \bar{z} \varphi(z, x, \bar{a}) \), involving the parameters \( \bar{a} \). We must give a procedure to decide if \( \exists \bar{z} \varphi(z, x, \bar{a}) \) is true. Fix any value of \( x \). Once \( x \) is fixed, we may treat it as a parameter. Hence, from here onwards we avoid explicitly mentioning \( x \), assuming it is now an element of the sequence of parameters \( \bar{a} \). So we must produce a decision procedure for sentences of the form \( \exists \bar{z} \varphi(z, \bar{a}) \). As before, we may assume \( \bar{a} \) consists of disjoint elements whose union is \( z \). Also by writing \( \varphi(z, \bar{a}) \) as \( \bigwedge_{i=1, \ldots, n} \varphi_i(z, \bar{a}) \) where each \( \varphi_i \) is just the conjunction of atomic sentences and negations of atomic sentences, we can see that it suffices to produce a decision procedure for each sentence \( \exists \bar{z} \varphi_i(z, \bar{a}) \).

Each \( \varphi_i \) is just a conjunction of statements at the atomic level about \( z, \bar{a} \). However, up to logical equivalence, there are only finitely many such statements possible. To see this, assume \( |\bar{a}| = m \), and let \( B_m \) be the free boolean algebra on \( m \) generators. Then there is a subset \( S \) of \( B_m \) such that \( |S| = m \), and exactly the same atomic relations hold amongst \( S \) as hold amongst \( z, \bar{a} \). So each \( z, \bar{a} \) is "copied" by a subset of \( B_m \). Pick any subset \( S \) of \( B_m \) with \( |S| = m \). From this we construct a formula \( \theta \), which is just the conjunction of all the atomic relations and negated atomic relations which hold amongst \( S \). As there are only finitely many such sets \( S \) (because \( B_m \) is finite), we can then enumerate the set of all such possible sentences \( \{\theta_i\}_{i<\tau} \).

For some values of \( j, \varphi_j(z, \bar{a}) \) will be consistent with \( \theta_j(z, \bar{a}) \). Let \( J \) be the set of such values of \( j \). Then in fact \( \varphi_j(z, \bar{a}) \) is equivalent to \( \bigwedge_{j \in J} \varphi_i(z, \bar{a}) \). Moreover, if we know the sentence \( \varphi_j \), then we can recursively compute the finite set \( J \). If \( J \) is empty, then there is no \( z \) with \( \varphi_j(z, \bar{a}) \), so the sentence is false and the decision procedure ends. Otherwise we can write \( \exists \bar{z} \varphi(z, x, \bar{a}) \leftrightarrow \bigwedge_{j \in J} \exists \bar{z} \varphi_j(z, \bar{a}) \). It thus suffices to give a procedure for deciding for any \( j \in J \), whether or not \( \exists \bar{z} \varphi_j(z, \bar{a}) \).

Recall that \( \bar{a} \) just consists of half-open intervals, whose union is \( z \). Assume \( \bar{a} = \{a_1, \ldots, a_k\} \). The sentence \( \exists \bar{z} \theta_j(z, \bar{a}) \) is certainly consistent with the theory of boolean algebras, as it is satisfied in \( B_m \). Now, \( B_m \) may also be viewed as the boolean algebra generated by the \( 2^m \)-element linear ordering. Let \( \bar{z}^*, \bar{a}^* \) be elements of \( B_m \) satisfying \( \theta_j \). Again \( \bar{a}^* \) partitions the \( 2^m \)-element linear ordering into finitely many disjoint finite pieces. Take any \( a_t^* \). Then \( a_t^* \) will just be some finite subinterval of the \( 2^m \)-element interval. The elements of \( \bar{z}^* \) will induce a partition inside the interval \( a_t^* \), in the usual way. Assume the elements of \( \bar{z}^* \) partition \( a_t^* \) into \( t \) non-empty pieces. Then, recalling that \( a_t \) is an element of \( B(z) \), as long as \( |a_t| \geq t \), we will be able to partition \( a_t \) into \( t \) non-empty pieces, obtaining \( \bar{z}_i \) in \( B(z) \) such that \( \bar{z}_i \) satisfies precisely the same atomic formulae in \( B(z) \) as \( \bar{z}^* \cap a_t^* \) satisfies in \( B_m = B(2^m) \). For each interval \( a_t \), we can recursively find such a natural number \( t_i \), and conclude that if \( |a_t| \geq t_i \), then there is \( \bar{z}_i \) in \( B(z) \) with \( \bar{z}_i \subseteq a_t \), such that \( \bar{z}_i, x_i \) and \( \bar{z}^*_i \cap a_t^* \) satisfy the same atomic formula. Moreover, if \( |a_t| < t_i \), then such \( \bar{z}_i \) and \( x_i \) will fail to exist.

So if for some \( i, |a_i| < t_i \), then there is no \( z \) satisfying \( \theta_j(z, \bar{a}) \), and the decision procedure ends. However, if for all \( i, |a_i| \geq t_i \), we obtain \( \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_k \) as detailed above. Let \( z = z_1 \cup z_2 \cup \cdots \cup z_k \). Then \( \theta_j(z, \bar{a}) \). Thus, \( \exists z \theta_j(z, \bar{a}) \) is true iff \( |a_1| \geq t_1, \ldots, |a_k| \geq t_k \). As \( \mathcal{A} \) has recursive representation function, we can decide if
Thus we have a complete decision procedure. This completes the verification of the claim, and with it the theorem. □

The next theorem tells us that recursive inseparability is the best we can hope for in this direction.

**Theorem 3.11.** Let \( \alpha \) be an ordinal less than \( \omega_1^{CK} \). Then in all recursive copies of \( B(\alpha) \), the ideal generated by the atoms, \( I \), and the filter generated by the co-atoms, \( F \), are \( \mathcal{A}_2^0 \) separable.

**Proof.** If \( \alpha \) is finite, then the theorem is trivial. Assume \( \alpha \) is infinite. Let \( p = [0, \omega) \), so \( p \in B(\alpha) \). Define \( x \in S_1 \) iff \( x \cap p \) is finite, \( x \in S_2 \) iff \( x \cap p \) is co-finite. We shall show that \( S_1 \) and \( S_2 \) are \( \Sigma_2 \) separable. From this, we know \( I \subseteq S_1 \subseteq \overline{F}, F \subseteq S_2 \subseteq I \) and \( S_1 \cup S_2 \) is equal to \( B(\alpha) \). Hence \( I \) and \( F \) are \( \mathcal{A}_2^0 \) separable in every recursive copy of \( B(\alpha) \).

Now, \( x \in S_1 \iff \exists x_1 \cdots x_n [x_1 \lor \cdots \lor x_n = x \land p \) and \( \forall z [z < x_1 \) or \( \cdots \) or \( z < x_n \rightarrow z = 0] \), which is \( \Sigma_2(p) \).

Also, \( x \in S_2 \iff \exists x_1 \cdots x_n [p - (x_1 \lor \cdots \lor x_n) = x \land p \) and \( \forall z [z < x_1 \) or \( \cdots \) or \( z < x_n \rightarrow z = 0] \), which is also \( \Sigma_2(p) \). □

We seek now to generalize Theorem 3.10. We do this by generalizing the notion of the ideal generated by the atoms and the filter generated by the co-atoms.

Take any boolean algebra of the form \( B(\alpha), \alpha < \omega_1^{CK} \). Let \( I_1 \) be the ideal generated by the atoms of \( B(\alpha) \). Take any \( x \in B(\alpha) \). Then \( x \) is just some subset of \( \alpha \). As a subset of \( \alpha \), it has an order type, \( \tau(x) \). So \( I_1 \) is just \( \{x : x \in B(\alpha) \land \tau(x) < \omega\} \). Let \( I_2 = \{x : x \in B(\alpha) \land \tau(x) < \omega^2\} \). Then it is easily seen that \( I_2 \) is also an ideal of \( B(\alpha) \). We call \( I_2 \) the second ideal of atoms.

In general, for all \( \gamma < \omega_1^{CK} \), we can define \( I_\gamma = \{x : x \in B(\alpha) \land \tau(x) < \omega^\gamma\} \). Then \( I_\gamma \) is an ideal of \( B(\alpha) \), known as the \( \gamma \)th ideal of atoms. The definition we have given relies on the fact that we begin with a superatomic boolean algebra. However, there is an alternative algebraic definition of the \( \gamma \)th ideal of atoms, equivalent to our definition in the superatomic case, that applies to all boolean algebras, including ones generated by linear orderings which are not well-orderings and uncountable boolean algebras which are not necessarily generated by any linear ordering at all. However, as we do not require the extra generality such an alternative definition would give us, we do not present it here, but remain content with the less general definition presented.

Dually, we also can define the \( \gamma \)th filter of co-atoms as just the complement of the \( \gamma \)th ideal of atoms. Our aim is to use these generalized ideals and filters to provide examples of \( \mathcal{A}_2^0 \) inseparability for \( \alpha < \omega_1^{CK} \) with \( \alpha > 1 \). In order to apply Theorem 2.17, we need an understanding of the way the \( \leq_\beta \) relations behave on boolean algebras. We seek analogous of Theorems 3.3 and 3.4 of the previous section to boolean algebras.
Consider any boolean algebra of the form $B(L)$ for some linear ordering $L$. Fix some natural number $n$, and let $\vec{c}$ and $\vec{d}$ be $n$-tuples from $B(L)$. We know that $\vec{c}$ determines a partition of $L$, as follows. Let $\vec{c} = c_1, \ldots, c_k$. Then, the set of elements of $B(L)$ of the form $g_1 \cap g_2 \cap \cdots \cap g_k$, where each $g_i$ is $c_i$ or $c_i'$, is a set of disjoint subsets of $L$. Enumerate these subsets as $\{\hat{c}_i\}_{i \in 2^k}$. Then let $e = L - \bigcup_{i \in 2^k} \hat{c}_i$. The elements $\{\hat{c}_i\}_{i \in 2^k}$ together with $e$ form a partition of $L$. Likewise, beginning with the $n$-tuple $\vec{d}$, we construct elements $\{\hat{d}_i\}_{i \in 2^k}$ and $f$ which form a partition of $L$.

Every $\hat{c}_i$ is of the form $g_1 \cap g_2 \cap \cdots \cap g_k, g_i = c_i$ or $c_i'$. Likewise, every $\hat{d}_i$ is of the form $h_1 \cap h_2 \cap \cdots \cap h_k, h_i = d_i$ or $d_i'$. Say that $\hat{c}_i$ and $\hat{d}_i$ are corresponding segments of $L$ iff $g_i = c_i$ precisely when $h_i = d_i$. Likewise, we say $e$ and $f$ are corresponding segments of $L$. Each of $\hat{c}_i, \hat{d}_i, e$ and $f$ may then be viewed as a linear ordering. Recall that, given two structures of the same type, say $\mathcal{A}$ and $\mathcal{B}$, we said $\mathcal{A} \models \mathcal{B}$ iff for all sentences $\phi$ in the common language of $\mathcal{A}$ and $\mathcal{B}$, we have $\mathcal{A} \models \phi$. We now present our analogue to Theorem 3.3 of the preceding section.

**Theorem 3.12.** Let $B$ be a boolean algebra of the form $B(L)$ for some countable linear ordering $L$. Let $\vec{c}$ and $\vec{d}$ be $n$-tuples from $B$. Then $\vec{c} \leq \vec{d}$ iff, for all pairs $\hat{c}^*, \hat{d}^*$ of corresponding segments, we have $B(\hat{c}^*) \leq B(\hat{d}^*)$.

We refer the reader to [3] for details.

We are primarily interested in the case where $L$ is a well-ordering of type $\omega_1^{CK}$. In this case, $\hat{c}^*$ and $\hat{d}^*$ will also be well-orderings. So to understand when two $n$-tuples $\vec{c}$ and $\vec{d}$ from any $B(\gamma)$ are such that $\vec{c} \leq \vec{d}$ ($\gamma, \alpha < \omega_1^{CK}$), it suffices to understand when we have $B(\gamma) \leq B(\gamma')$, for $\gamma, \gamma'$ and $\alpha$ all less than $\omega_1^{CK}$.

The full details of this are contained in the following theorem, an analogue of Theorem 3.4 of the previous section. First, for any ordinal $\delta$, let $\delta'$ be the greatest ordinal less than $\delta$ of the form $\omega^\rho \cdot m$, for some ordinal $\rho$ and nonzero $m \in \mathbb{N}$. Then it is straightforward to see that $B(\delta') \models B(\delta)$. Thus it suffices to consider only boolean algebras of the form $B(\omega^\rho \cdot m), \rho$ an ordinal, $m \in \mathbb{N}$.

**Theorem 3.13.** Let $\alpha, \beta$ be ordinals $< \omega_1^{CK}, \delta$ a limit ordinal $< \omega_1^{CK}, m, n$ and $k$ natural numbers. Then

(a) $B(\omega^\rho \cdot m) \leq \delta + 2k + 1 B(\omega^\rho \cdot n)$ iff one of

(i) $\alpha = \beta, \beta < \delta + k$ and $m = n$,
(ii) $\alpha = \beta, \beta = \delta + k$ and $m > n$,
(iii) $\alpha \geq \delta + k + 1$ and $\beta \geq \delta + k$.

(b) $B(\omega^\rho \cdot m) \leq \delta + 2k B(\omega^\rho \cdot n)$ iff one of

(i) $\alpha = \beta, \beta < \delta + k$ and $m = n$,
(ii) $\alpha \geq \delta + k, \beta \geq \delta + k$.

Again, the interested reader is referred to [3] for details. Theorems 3.12 and 3.13 then give us all the information we need about recursive superatomic boolean algebras. Importantly, we have the following lemma.
Lemma 3.14. Let $\mathcal{A}$ be a recursive copy of the boolean algebra of the form $B(\gamma), \gamma < \omega_1^{CK}$. Then, if $\mathcal{A}$ has a recursive representation function, the existential diagram of $\mathcal{A}$ is decidable, and the relations $\{\leq_\beta\}_{\beta \leq 2\gamma + 1}$ are uniformly recursive.

Proof. We know already that the existential diagram of any such $\mathcal{A}$ must be decidable, by the claim in Theorem 3.10. So let $\bar{c}$ and $\bar{d}$ be $n$-tuples from $B(\gamma)$. Using the recursiveness of the representation function, we can determine the order type of corresponding segments from $\bar{c}$ and $\bar{d}$, and then use Theorem 3.13 to determine if $\bar{c} \leq_\alpha \bar{d}$ for any particular $\alpha \leq 2\gamma + 1$ using information from the Cantor codes.  

We are now ready to generalize Theorem 3.10.

Theorem 3.15. Let $\gamma$ be an ordinal less than $\omega_1^{CK}$, and let $\alpha$ be an ordinal $\geq \omega^{\gamma+1}$. Then there is a recursive copy of $B(\alpha)$ in which $I_{\gamma+1}$ and $F_{\gamma+1}$ are $\Delta_0^{\omega^{\gamma+1}}$ inseparable.

Proof. We seek to apply Theorem 2.17. Fix a recursive copy $\mathcal{A}$ of $B(\alpha)$ in which the representation function is recursive. As $I_{\gamma+1}$ is just the set of elements of $B(\alpha)$ of total length less than $\omega^{\gamma+1}, I_{\gamma+1}$ will be recursive in any such copy. We also know that in any such copy, the existential diagram of $\mathcal{A}$ will be recursive and the relations $\{\leq_\beta\}_{\beta \leq 2\gamma + 1}$ will be uniformly recursive. It suffices only to verify that the relations $\text{Cl}_{2\gamma+1}(I_{\gamma+1}, \bar{\rho})$ and $\text{Cl}_{2\gamma+1}(F_{\gamma+1}, \bar{\rho})$ are recursive. Let us defer a proof of this fact for a moment, and continue with the remainder of the proof.

Assuming that conditions (1)–(4) of Theorem 2.17 hold, it suffices to show that there are no two $\Sigma_{2\gamma+1}(\bar{\rho})$ sets $S_1$ and $S_2$ such that $I_{\gamma+1} \subseteq S_1 \subseteq F_{\gamma+1}, F_{\gamma+1} \subseteq S_2 \subseteq I_{\gamma+1}$ and $S_1 \cup S_2 = B(\alpha)$. We may assume, as explained earlier, that $\bar{\rho}$ just consists of finitely many half-open intervals whose union is $\alpha$. Enumerate these intervals, $\{I_i\}_{i < \tau}$.

Consider any formula $\psi(x)$ equivalent to $\exists \bar{z} \varphi(\bar{z}, x)$ for some $\Pi_{2\gamma}(\bar{\rho})$ relation $\varphi$. Let $\{I_{i_1}, \ldots, I_{i_q}\}$ be a list of the intervals of this partition which have length $\geq \omega^{\gamma+1}$. Let $T$ be the set of $x$ such that for each $j$, the length of $I_{i_j} - x$ is $\geq \omega^\gamma$. We shall show that if $\psi$ is satisfied by an element of $T$, then $\psi$ is satisfied by an element of $I_{\gamma+1}$. It follows (by taking duals), that if we let $T^*$ be the set of $x$ such that for each $j$, the length of $I_{i_j} \cap x$ is $\geq \omega^\gamma$, then if $\psi$ is satisfied by an element of $T^*$, $\psi$ is satisfied by an element of $F_{\gamma+1}$.

Assume that a sentence of the form $\exists \bar{z} \varphi(\bar{z}, x)$ with $\varphi$ a $\Pi_{2\gamma}(\bar{\rho})$ relation is satisfied by some $x \in T$. Fix $\bar{z}$ to be the sequence of elements making $\varphi(\bar{z}, x)$ true.

Consider any interval $I$ from amongst $\{I_{i_1}, \ldots, I_{i_q}\}$. Then $\bar{z}, x$ partitions $I$ into further subintervals $\{J_{i_1}, \ldots, J_{i_q}\}$ such that $J_{i_1} \cup \cdots \cup J_{i_q} = I$, and each element from $x, \bar{z}$ may be expressed as the union of finitely many $J$'s. By assumption, $I$ has length $\geq \omega^{\gamma+1}$, and $I - x$ and has length $\geq \omega^\gamma$. But $I \cap x$ is the union of finitely many elements from $\{J_{i_1}, \ldots, J_{i_q}\}$. So let $J^*$ be the subset of elements of $J$ of $\{J_{i_1}, \ldots, J_{i_q}\}$ such that $J \subseteq x$ and the length of $J$ is $\geq \omega^\gamma$. If $J^*$ is empty, $I \cap x \in I_{\gamma}$. Assume the set $J^*$ is non-empty. Also, let $\tilde{J}$ be any element of $\{J_{i_1}, \ldots, J_{i_q}\}$ which is disjoint from $x$ and has length $\geq \omega^\gamma$. Take any element of $J^*$. Then replace $J^*$ by a subinterval of size $\omega^\gamma$, and re-allocate the remains of $J^*$ to $\tilde{J}$. In this process we have decreased an
interval of size $\geq \omega^\beta$ to an interval of size $\omega^\beta$ and increased an interval of size $\omega^\beta$ to an even larger size. Let $x$ and $\beta$ be ordinals $\geq \omega^\beta$. Then, $B(x) \leq \omega_2(B(\omega^\beta))$ and $B(x) \leq \omega_2, B(\beta)$ for all such $x$ and $\beta$ by Theorem 3.13(b). Repeat this process for all elements of $J^*$, decreasing the size of these elements while increasing the size of $\hat{J}$. At the end of this procedure, we will then have new $x^*$ and $\hat{z}^*$ such that $x^* \cap I$ has length $< \omega^{\alpha+1}$ and such that $x, \hat{z}, \hat{p} \leq x^*, \hat{z}^*, \hat{p}$. Repeat this process for the finitely many choices for $I$. In the end, an $x^*$ and $\hat{z}^*$ are obtained such that $x^* \in I_{\gamma+1}$, and $x, \hat{z}, \hat{p} \leq x^*, \hat{z}^*, \hat{p}$. Hence, as $\phi(\hat{z}, x)$ is true and $\phi$ is $\Pi_2(\hat{p})$, we must have $\phi(\hat{z}^*, x^*)$. Hence the formula $\exists \hat{z} \phi(\hat{z}, x)$ is satisfied by an element of $I_{\gamma+1}$, as desired.

As before, let $S_1$ and $S_2$ be $\Sigma_2^{\gamma+1}(\hat{p})$ sets with $I_{\gamma+1} \subseteq S_1 \supseteq F_{\gamma+1} \subseteq S_2 \subseteq \bar{I}_{\gamma+1}$ and $S_1 \cup S_2 = B(\alpha)$. We can easily construct a nonzero $x$ in $T \cap T^*$. But we have $x \in S_1$ or $x \in S_2$. Assume $x \in S_1$. Now $x \in S_1$ iff $\bigwedge_{i \in I} \exists \hat{z} \phi(\hat{z}, x)$ for some $\Sigma_2^{\gamma+1}$ set $I$ and some set $\{\phi(\hat{z}, x)\}_{i \in I}$ of $\Pi_2(\hat{p})$ formulae. So for some $i \in I$, $\phi(\hat{z}, x)$. As $x \in T^*, \exists \hat{z} \phi(\hat{z}, x^*)$, for some $x^* \in F_{\gamma+1}$. So $S_1$ contains an element of $F_{\gamma+1}$, contradiction. Likewise, if $x \in S_2$, $S_2$ contains an element of $I_{\gamma+1}$, contradiction. Hence the desired sets $S_1$ and $S_2$ do not exist, giving us that there is a recursive copy of $B(\alpha)$ in which $I_{\gamma+1}$ and $F_{\gamma+1}$ are $\Delta_2^{\gamma+1}$ inseparable.

The only ingredient missing from our proof is a demonstration that in $\mathcal{A}$, the relations $x \in C_{\gamma+1}(\bar{I}_{\gamma+1}, \hat{p})$ and $x \in C_{\gamma+1}(\bar{I}_{\gamma+1}, \hat{p})$ are recursive. We demonstrate that the relation $x \in C_{\gamma+1}(\bar{I}_{\gamma+1}, \hat{p})$ is recursive, a proof that the later relation is recursive being analogous.

As before, we can assume $\hat{p}$ partitions $x$ into intervals $\{I_i\}_{i<\gamma}$. Let $\{I_{i_1}, \ldots, I_{i_q}\}$ be a list of the intervals of this partition which have length $\geq \omega^{\gamma+1}$. Let $T$ be the set of $x$ such that for each $j$, the length of $I_{i_j} - x$ is $\geq \omega^\gamma$. We know that if any $\Sigma_2^{\gamma+1}(\hat{p})$ relation is satisfied by an element of $T$, then it is satisfied by an element of $\bar{I}_{\gamma+1}$. So $C_{\gamma+1}(\bar{I}_{\gamma+1}, \hat{p})$ is disjoint from $T$. Therefore $C_{\gamma+1}(\bar{I}_{\gamma+1}, \hat{p}) \subseteq \bar{T}$. We show that $\bar{T}$ is a $\Sigma_2^{\gamma+1}(\hat{p})$ subset of $\bar{I}_{\gamma+1}$, from which we conclude $C_{\gamma+1}(\bar{I}_{\gamma+1}, \hat{p}) = \bar{T}$.

Now, $x$ is in $\bar{T}$ iff for some $j$, the length of $I_{i_j} - x$ is $< \omega^\gamma$. Certainly no element of $\bar{T}$ can be in $I_{\gamma+1}$, so $T \subseteq I_{\gamma+1}$. But $x \in T \iff \bigwedge_{j=1, \ldots, q} [(I_{i_j} - x) \in I_{\gamma+1}]$. A straightforward induction shows that $I_{\gamma}$ is $\Sigma_2$ definable in any superatomic boolean algebra. Hence, the relation $x \in T$ is $\Sigma_2\gamma(p)$, and hence $\bar{T}$ is certainly a $\Sigma_2^{\gamma+1}(\hat{p})$ subset of $\bar{I}_{\gamma+1}$ as desired. Therefore $C_{\gamma+1}(\bar{I}_{\gamma+1}, \hat{p}) = \bar{T}$. However, given the recursiveness of the representation function in $\mathcal{A}$, it follows that the set $T$ is recursive and so the relation $x \in C_{\gamma+1}(\bar{I}_{\gamma+1}, \hat{p})$ is recursive as desired. This completes a proof of the recursiveness of the $\mathcal{C}$ relations, and with it the theorem. □

In analogy with Theorem 3.11, it may be demonstrated that $I_{\gamma+1}$ and $F_{\gamma+1}$ are always $\Delta_2^{\gamma+2}$ separable, so our theorem does give the best possible separability bounds.

3.3. Recursive vector spaces

In this final section, we shall consider the recursion-theoretic structure of vector spaces. Much literature has appeared on recursive vector spaces, in which there has
been much confusion of terminology. We shall try to employ what seems to be the simplest terminology consistent with the terminology already presented herein.

Let $F$ be an infinite recursive field. From this, we can construct an $\mathbb{N}_0$-dimensional vector space, with scalar field $F$. Call this structure $V_\omega(F)$. To begin with, the language of vector spaces will be given by a binary function for addition, and an indexed set \( \{ c \cdot ( ) \}_{c \in F} \) of infinitely many unary functions. For every $c \in F$, the function $c \cdot ( )$ represents scalar multiplication by $c$.

Every copy of $V_\omega(F)$ is therefore of the form $\mathcal{A} = \langle A, +, \{ c \cdot ( ) \}_{c \in F} \rangle$, where $A$ is the underlying set of the copy. To say that a copy of $V_\omega(F)$ is recursive is then to say that $A$ is a recursive set, $+$ a recursive operation, and the functions $\{ c \cdot ( ) \}_{c \in F}$ are uniformly recursive. Note that this implies the field $F$ is recursive.

Given any two subspaces $S_1$ and $S_2$ of $V_\omega(F)$ such that $S_1 \cap S_2 = \{0\}$, and $S_1 \oplus S_2$ has infinite co-dimension in $V_\omega(F)$, that is, $\dim(V_\omega(F)/(S_1 \oplus S_2)) = \infty$, we seek to find conditions under which there is a recursive copy of $V_\omega(F)$ with $S_1 \setminus \{0\}$ and $S_2 \setminus \{0\}$ recursively inseparable. First, we will want that there is no pair $F_1$ and $F_2$ of $\Sigma_1^0$ sets such that $S_1 \setminus \{0\} \subseteq F_1 \subseteq S_2 \setminus \{0\}$ and $S_2 \setminus \{0\} \subseteq F_2 \subseteq S_1 \setminus \{0\}$ and $F_1 \cup F_2 = V_\omega(F)$. The following lemma in fact tells us substantially more.

**Lemma 3.16.** Let $F$ be a field with a recursive copy, and $S$ be a subspace of $V_\omega(F)$ such that $S$ has infinite dimension. For any $\alpha < \omega_1^{CK}$ let $F$ be a $\Sigma_1^\alpha(p)$ subset of $V_\omega(F)$ such that $S \subset F$. Then, $F$ contains the (set-theoretic) complement of a finite-dimensional subspace of $V_\omega(F)$.

**Proof.** Let $F$ be such a $\Sigma_1^\alpha(p)$ set. Let $x_1$ and $x_2$ be elements of $V_\omega(F)$ both of which lie outside the subspace generated by $\bar{p}$, that is, let $x_1, x_2 \notin \text{Span}(\bar{p})$. Then there is an automorphism $\pi$ of $V_\omega(F)$ with $\pi(x_1) = x_2$ and $\pi(\bar{p}) = \bar{p}$. Consequently, if $F$ contains any element $\notin \text{Span}(\bar{p})$, then $F$ contains all elements not in $\text{Span}(\bar{p})$. As $S \subset F$, and $S$ is infinite-dimensional, $F$ must contain an element not in $\text{Span}(\bar{p})$, thus $F$ contains the complement of $\text{Span}(\bar{p})$, that is, $F$ contains the complement of a finite-dimensional subspace of $V_\omega(F)$. □

Before we apply our main theorem, we need to understand the $\leq_{\gamma}$ relations on $V_\omega(F)$. This is simpler than in linear orderings or Boolean algebras, as the following theorem shows.

**Theorem 3.17.** Let $F$ be a recursive field, $1 \leq \gamma < \omega_1^{CK}$. Let $\bar{x} = \{x_1, \ldots, x_n\}$, $\bar{y} = \{y_1, \ldots, y_n\}$ be a pair of $n$-tuples from $V_\omega(F)$. Then $\bar{x} \leq_{\gamma} \bar{y}$ iff for all $c_1, \ldots, c_n$ from $F$,

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0 \iff c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0.$$

**Proof.** Consider first $\gamma = 1$. If there are $c_1, \ldots, c_n$ from $F$ with exactly one of $\sum_{i \leq n} c_i x_i = 0$ or $\sum_{i \leq n} c_i y_i = 0$ holding, then here is a parameter and quantifier-free
sentence true of $\bar{x}$, and false of $\bar{y}$. However, if for all $c_1, \ldots, c_n$ from $F$, 
$\sum_{i \leq n} c_i x_i = 0 \iff \sum_{i \leq n} c_i y_i = 0$, then there is an automorphism $\pi$ of $V_{\omega}(F)$ with 
$\pi(x_i) = y_i, i \leq n$. Thus, any finitary parameter-free universal sentence true of $\bar{x}$ is true 
of $\bar{y}$. This completes the case $\gamma = 1$.

Assume $\bar{x} \leq 2 \bar{y}$. It follows that $\bar{x} \leq 1 \bar{y}$, and hence $\sum_{i \leq n} c_i y_i = 0 \implies \sum_{i \leq n} c_i y_i = 0$, for 
all $c_1, \ldots, c_n$ from $F$. Assume $\sum_{i \leq n} c_i x_i = 0 \iff \sum_{i \leq n} c_i y_i = 0$ for all $c_1, \ldots, c_n$ from $F$, 
but that there is $\bar{r}$ such that for all $\bar{q}$ we have $\bar{y}, \bar{r} \leq 1 \bar{x}, \bar{q}$ is false. Let $\pi$ be an 
arithmetic of $V_{\omega}(F)$ taking $\bar{y}$ pointwise onto $\bar{x}$. Then let $\bar{q} = \pi(\bar{r})$. Enumerate 
$\bar{y}, \bar{r} - y_1, \ldots, y_1$ and $\bar{x}, \bar{q} = x_1, \ldots, x_1$. Then $\sum_{i \leq 1} c_i x_i = 0 \iff \sum_{i \leq 1} c_i y_i = 0$, so 
$\bar{y}, \bar{r} \leq 1 \bar{x}, \bar{q}$ by assumption, contradiction. So for all $\bar{r}$ there is $\bar{q}$ with $\bar{y}, \bar{r} \leq 1 \bar{x}, \bar{q}$. Thus 
$\bar{x} \leq 2 \bar{y}$. This completes $\gamma = 2$ and moreover shows that $\bar{x} \leq 1 \bar{y} \iff \bar{x} \leq 2 \bar{y}$. However, it 
then follows by normal transfinite induction using the definition of $\leq$, that for all 
$\gamma < \omega_1^{CK}$, $\bar{x} \leq 1 \bar{y} \iff \bar{x} \leq 2 \bar{y}$, from which the theorem follows. $\square$

From this, we can deduce:

**Theorem 3.18.** Let $F$ be a field with a recursive copy. Assume there is a recursive 
layer $A$ of $V_{\omega}(F)$ such that in $A$ there is an algorithm to decide whether or not 
any finite set of vectors is linearly dependent. Then, for all $\alpha > 1$ with $\alpha < \omega_1^{CK}$ there is 
a recursive copy $B$ with two subspaces $T_1$ and $T_2$ such that $T_1 \setminus \{0\}$ and $T_2 \setminus \{0\}$ are 
$A_0$ inseparable.

**Proof.** Let $S_1$ and $S_2$ be subspaces of $A$ such that  
1. $S_1 \cap S_2 = \{0\}$,
2. both $S_1$ and $S_2$ are recursive as subsets of $A$.
3. $\dim(S_1) = \dim(S_2) = \infty$, and $\dim(V/(S_1 \oplus S_2)) = \infty$.

First we verify that $A$ satisfies the assumptions (1)-(4) of Theorem 2.17. We know 
already that $S_1 \setminus \{0\}$ and $S_2 \setminus \{0\}$ are recursive in $A$. We give a decision procedure for the 
extistental diagram of any copy of $V_{\omega}(F)$.

Let $\phi$ be an existential sentence in the finitely many parameters $\bar{p}$ in the copy $A$. 
We give a procedure to determine if $\phi$ is true or false.

Assume $\phi$ is of the form $\exists \bar{x} \psi(\bar{x}, \bar{p})$, where $\psi(\bar{x}, \bar{p})$ is in disjunctive normal form. Then 
$\phi$ can be expressed in the form $\bigwedge_{i=1}^{n} \exists \bar{x} \psi_i(\bar{x}, \bar{p})$, where each $\psi_i$ is a conjunction of 
atomic sentences and negations of atomic sentences, for $i$ with $1 \leq i \leq n$ for some 
$n \in \mathbb{N}$. It suffices to give a decision procedure for each sentence $\exists \bar{x} \psi_i(\bar{x}, \bar{p})$. We can 
assume $\psi_i$ is a conjunction of sentences of the form 
$t(\bar{x}, \bar{p}) = 0$ and $\neg t(\bar{x}, \bar{p}) \neq 0$, $t$ some term of the language of vector spaces.

Consider first any sentence of the form $t(\bar{x}, \bar{p}) = 0$. For definiteness, assume $\bar{x}$ is an 
m-tuple, that is, $\bar{x} = x_1, \ldots, x_m$, some particular variable $x_j$ actually appears in 
t(\bar{x}, \bar{p}) and no variable $x_1, \ldots, x_{j-1}$ appears in t(\bar{x}, \bar{p}). We can then write t(\bar{x}, \bar{p}) in the 
form $x_j = (\bar{x} \setminus \{x_j\}, \bar{p})$, simply by transposition and field operations. Then, substitute
the expression \( t(\bar{x}\setminus\{x_j\}, \bar{p}) \) for all instances of the variable \( x_j \) in \( \psi_i(\bar{x}, \bar{p}) \). Thus, the sentence

\[
\exists x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m [t(\bar{x}, \bar{p}) \& A(\bar{x}, \bar{p})]
\]

reduces to an equivalent

\[
\exists x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m [A^*(\bar{x}, \bar{p})]
\]

where \( A^* \) is obtained from \( A \) by the substitution indicated.

We may continue this process on all formulae \( t(\bar{x}, \bar{p}) = 0 \) (in which some \( x_i \) appears) appearing in \( \psi_i(\bar{x}, \bar{p}) \). Eventually, the sentence \( \exists \bar{x} \psi_i(\bar{x}, \bar{p}) \) will be reduced to an equivalent sentence of the form \( \exists \bar{z} \psi^*_i(\bar{z}, \bar{p}) \), where \( \bar{z} \subseteq \bar{x} \). \( \psi^*_i \) is a conjunction of formulae of the form \( "t(\bar{x}, \bar{p}) = 0" \) and \( "t(\bar{x}, \bar{p}) \neq 0" \), such that each formulae \( "t(\bar{x}, \bar{p}) = 0" \) involves only the parameters \( \bar{p} \).

Each formula \( "t(\bar{x}, \bar{p}) = 0" \) will then be either true for all substitutions of \( \bar{x} \) or false for all substitutions of \( \bar{x} \) as \( t \) only actually involves the parameters \( \bar{p} \). Moreover, this can be decided given the recursiveness of the field \( F \). If \( t(\bar{x}, \bar{p}) = 0 \) is true for all substitutions \( \bar{x} \), we can simply omit the conjunct \( t(\bar{x}, \bar{p}) = 0 \) from the sentence \( \exists \bar{z} \psi^*_i(\bar{z}, \bar{p}) \). Otherwise, we know immediately that the sentence \( \exists \bar{z} \psi^*_i(\bar{z}, \bar{p}) \) is false.

Hence, we can assume \( \psi^*_i \) is a conjunction of formulae of the form \( "t(\bar{x}, \bar{p}) \neq 0" \). Assume \( \bar{z} \) is an \( r \)-tuple, \( z_1, \ldots, z_r \). Then imagine \( z_1, \ldots, z_r \) such that the set \( \{z_1, \ldots, z_r, \bar{p}\} \) is linearly independent. Such a set exists, as \( V_c(F) \) is infinite-dimensional. Moreover, for such a choice of \( z_1, \ldots, z_r \), \( "t(\bar{x}, \bar{p}) \neq 0" \) is true for any term \( t \) in our agreed language of vector spaces. Thus, the sentence \( \exists \bar{z} \psi^*_i(\bar{z}, \bar{p}) \) is true.

We thus have a decision procedure for each \( \exists \bar{z} \psi^*_i(\bar{z}, \bar{p}) \). This gives us a decision procedure for each \( \exists \bar{x} \psi_i(\bar{x}, \bar{p}) \). Hence we can decide if \( \varphi \) is true or false, as

\[
\varphi(\bar{x}) \iff \bigwedge_{i=1, \ldots, n} \exists \bar{x} \psi_i(\bar{x}, \bar{p}).
\]

So the existential diagram of \( \mathcal{A} \) is decidable. To show the relations \( \{\leq_x\}_{x \in \mathcal{A}} \) are recursive, it suffices to show that we can decide, given \( x_1, \ldots, x_n, y_1, \ldots, y_n \), whether or not for all \( c_1, \ldots, c_n \) from \( F \), \( \sum_{i=1}^n c_i x_i = 0 \leftrightarrow \sum_{i=1}^n c_i y_i = 0 \). To do this, find a maximal independent subset of \( x_1, \ldots, x_n \) and express the others as linear combinations from this. Then, we simply see whether the corresponding \( y \)'s are linearly independent and whether the remaining \( y \)'s can be expressed as linear combinations of the independent \( y \)'s in the same way the corresponding \( x \)'s are expressible as linear combinations of the independent \( x \)'s. This can be done, as we have a way of deciding dependence in \( \mathcal{A} \), and hence the \( \{\leq_x\}_{x \in \mathcal{A}} \) relations are recursive. We seek now to verify that the relations \( x \in \text{Cl}_a(S_1 \setminus \{0\}, \bar{p}) \) and \( x \in \text{Cl}_a(S_2 \setminus \{0\}, \bar{p}) \) are recursive.

Consider the relation \( x \in \text{Cl}_a(S_1 \setminus \{0\}, \bar{p}) \). \( \text{Cl}_a(S_1 \setminus \{0\}, \bar{p}) \) is just the union of all \( \Sigma^*_x(\bar{p}) \) subsets of \( S_1 \cup \{0\} \) containing \( \bar{p} \). Let \( F \) be a \( \Sigma^*_x(\bar{p}) \) subset of \( S_1 \cup \{0\} \). Then we know that if \( F \) contains an element not in \( \text{Span}(\bar{p}) \), \( F \) contains all elements not in \( \text{Span}(\bar{p}) \),
so \( F \) contains the complement of a finite-dimensional subspace. But \( S_1 \) is infinite-dimensional, and \( F \subseteq \overline{S_1} \cup \{0\} \), so therefore \( F \) does not contain any elements not in \( \text{Span}(\bar{p}) \). So \( F \subseteq \text{Span}(\bar{p}) \cap \overline{(S_1 \cup \{0\})} \subseteq \text{Cl}_\alpha(S_1 \setminus \{0\}, \bar{p}) \). Clearly, every element of \( \text{Span}(\bar{p}) \cap \overline{(S_1 \cup \{0\})} \) is in \( \text{Cl}_\alpha(S_1 \setminus \{0\}, \bar{p}) \) for any \( \alpha \geq 1 \), thus \( \text{Cl}_\alpha(S_1 \setminus \{0\}, \bar{p}) = \text{Span}(\bar{p}) \cap \overline{(S_1 \cup \{0\})} \). \( \text{Span}(\bar{p}) \) is recursive, as the dependence relation on \( \mathcal{A} \) is recursive, and so \( \text{Cl}_\alpha(S_1 \setminus \{0\}, \bar{p}) \) is recursive as desired.

Given the satisfaction of conditions (1)--(4), we are now able to apply Theorem 2.17, from which we can conclude that if there are no two \( \Sigma\bar{p}(\bar{p}) \) sets \( F_1 \) and \( F_2 \) with \( S_1 \setminus \{0\} \subseteq F_1 \subseteq S_2 \setminus \{0\}, S_2 \setminus \{0\} \subseteq F_2 \subseteq S_1 \setminus \{0\} \) and \( F_1 \cup F_2 = V_\alpha(F) \), then there is a recursive copy of \( V_\alpha(F) \) satisfying the conclusion of our theorem.

Consider any \( \Sigma\bar{p}(\bar{p}) \) set \( F_1 \) with \( S_1 \setminus \{0\} \subseteq F_1 \subseteq S_2 \setminus \{0\} \). As \( S_1 \) is infinite-dimensional, \( F_1 \) contains an element not in \( \text{Span}(\bar{p}) \). Thus, \( F_1 \) contains all elements not in \( \text{Span}(\bar{p}) \). So \( F_1 \) contains a subspace of finite co-dimension. However, \( S_2 \) has infinite dimension and \( F_1 \subseteq S_2 \setminus \{0\} \), contradiction. Hence no such \( F_1 \) exists.

We therefore conclude that there is a recursive copy \( \mathcal{B} \) of \( V_\alpha(F) \) in which \( S_1 \setminus \{0\} \) and \( S_2 \setminus \{0\} \) are \( \mathcal{A}_0 \) inseparable, as desired. Letting \( T_1 = f^{-1}(S_1) \) and \( T_2 = f^{-1}(S_2) \) gives us the theorem. \( \square \)

We thus know that under very general conditions we can make two subspaces \( \mathcal{A}_0 \) inseparable in some recursive copy \( \mathcal{B} \) of \( V_\alpha(F) \).

Consider for some fixed \( n \), the \( n \)-ary dependence relation on \( V_\alpha(F) \), that is the \( n \)-ary relation \( \text{Dep}_n \) defined by "\( \text{Dep}_n(x_1, \ldots, x_n) \) iff \( x_1, \ldots, x_n \) are linearly dependent in \( V_\alpha(F) \)". We can then add these relations to the language of vector spaces, creating the expanded language \( \{+, \{c(\cdot)\}_{c \in F}, \{\text{Dep}_n\}_{n \in \mathbb{N}}\} \).

Given a vector space \( V_\alpha(F) \), we reserve the term recursive copy for a copy in which \( + \) and \( c(\cdot) \) \( c \in F \) are (uniformly) recursive. We call a recursive copy fully effective if the relations \( \{\text{Dep}_n\}_{n \in \mathbb{N}} \) are also uniformly recursive. Thus, a fully effective copy is just a recursive copy in the expanded language \( \{+, \{c(\cdot)\}_{c \in F}, \{\text{Dep}_n\}_{n \in \mathbb{N}}\} \). So the set of fully effective copies is just the subset of recursive copies in which there is a dependence algorithm.

The following lemma is of importance in the study of fully effective copies.

**Lemma 3.19.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be fully effective presentations of \( V_\alpha(F) \), for some recursive field \( F \). Then there is a \( \mathcal{A}_0 \) isomorphism \( f: \mathcal{A} \cong \mathcal{B} \).

**Proof.** By a recursive back and forth argument. Enumerate \( \mathcal{A} \) as \( \{a_1, a_2, \ldots\} \) and \( \mathcal{B} \) as \( \{b_1, b_2, \ldots\} \). We then construct a chain of functions \( f_0 \subset f_1 \subset f_2 \subset \cdots \) such that \( a_i \in \text{dom}(f_i), b_i \in \text{ran}(f_{i+1}), \bigcup_{i \in \mathbb{N}} f_i \) is an isomorphism from \( \mathcal{A} \) to \( \mathcal{B} \).

Begin by letting \( f_0 = \emptyset \). We proceed inductively. Assume we have constructed \( f_i \) such that \( \text{dom}(f_i) \) is finite. We seek to construct \( f_{i+1} \). Assume \( i + 1 = 2j \) for some \( j \in \mathbb{N} \). Thus we only need make sure that \( a_j \in \text{dom}(f_{i+1}) \). Let \( \bar{a} = \text{dom}(f_j) \), and \( \bar{b} = \text{ran}(f_j) \). If \( a_j \) is already in \( \bar{a} \), then we are done. Otherwise, begin by finding
a maximal linearly independent subset \( \tilde{a}^* \) of \( \tilde{a} \). This is possible, as the dependence relations are recursive in \( \mathcal{A} \). There are two cases.

**Case 1:** \( \{a_j, \tilde{a}^*\} \) is a linearly independent set. Then, pick any \( z \) in \( \mathcal{B} \) such that \( z \notin \text{Span}(b) \). We can find such a \( z \) recursively using the dependence algorithms. Then let

\[
f_{i+1}(x) = \begin{cases} f_i(x) & \text{if } x \in \tilde{a}, \\ z & \text{if } x = a_j. \end{cases}
\]

**Case 2:** \( \{a_j, \tilde{a}^*\} \) is not a linearly independent set. As \( \tilde{a}^* \) is a linearly independent set, we must then have that \( a_j \) is some linear combination of the \( \tilde{a}^* \); that is \( a_j = t(\tilde{a}^*) \) for some term \( t \). By enumerating \( \{t(\tilde{a}^*): t \text{ is a term in } \{+, \cdot, (\ )_{c \in \mathcal{F}}\}\} \), we can recursively find the term \( t \) such that \( a_j = t(\tilde{a}^*) \). Then, let \( z = t(f_i(\tilde{a}^*)) \); and let

\[
f_{i+1}(x) = \begin{cases} f_i(x) & \text{if } x \in \tilde{a}, \\ z & \text{if } x = a_j. \end{cases}
\]

This completes the construction of \( f_{i+1} \). We assumed \( i + 1 \) was even. If \( i + 1 \) is odd, we must make sure that some \( b_j \) is in the range of \( f_{i+1} \). The details are analogous.

So \( f = \bigcup_{n<\omega} f_n \) is 1–1 and onto. It is clear from the construction that \( f \) preserves the structure of the vector space, and is hence an isomorphism. Moreover, \( f \) is recursive, as it is the union of a uniformly recursive chain of finite functions.

So all fully effective presentations are \( \Delta^0_1 \) isomorphic. As \( \Delta^0_1 \) isomorphisms preserve the recursion-theoretic structure in which most recursion theorists are interested, it is convenient to phrase Lemma 3.19 as “up to recursive isomorphism, there is exactly one recursive copy of \( V_\infty(F) \).”

We seek then to generalize Theorem 3.18 to find conditions under which we can find a fully effective copy of a vector space with two subspaces \( \Delta^0_1 \), inseparable. As can be seen by the following theorem, the conditions are more strict.

**Theorem 3.20.** Let \( F \) be a countable field. Assume there is a fully effective presentation \( \mathcal{A} \) of \( V_\infty(F) \) with decidable existential diagram.

Then, for all \( \alpha > 1 \) with \( \alpha < \omega^K \), there is a fully effective copy \( \mathcal{B} \) with two subspaces \( T_1 \) and \( T_2 \) such that \( T_1 \setminus \{0\} \) and \( T_2 \setminus \{0\} \) are \( \Delta^0_\alpha \) inseparable.

**Proof.** Let \( S_1 \) and \( S_2 \) be subspaces of \( V_\infty(F) \) in \( \mathcal{A} \) such that

1. \( S_1 \cap S_2 = \{0\} \),
2. both \( S_1 \) and \( S_2 \) are recursive as subsets of \( \mathcal{A} \),
3. \( \dim(S_1) = \dim(S_2) = \infty \), and \( \dim(V/(S_1 \oplus S_2)) = \infty \).

By assumption, \( \mathcal{A} \) has decidable existential diagram and \( S_1 \) and \( S_2 \) are both recursive in \( \mathcal{A} \). We still have that any \( \Sigma^0_1(f) \) (in the new expanded language) subset of a vector space which contains an infinite-dimensional subspace also contains a subspace of
finite co-dimension. Thus, using the same argument as in the proof of Theorems 3.17 and 3.18, the \( \{ \preceq \_ p \} \) relations are recursive in \( \mathcal{A} \) (in fact, the \( \preceq \) relations work out to be the same as those given by Theorem 3.17), and \( \text{Cl}_d(S_1 \setminus \{0\}, \overline{p}) = (S_1 \cup \{0\}) \cap \text{Span}(\overline{p}) \).

By the same argument that appears in the proof of Theorem 3.18, we also know that there are no two \( \Sigma^*_d \) sets \( F_1 \) and \( F_2 \) with \( S_1 \setminus \{0\} \subseteq F_1 \subseteq S_2 \setminus \{0\} \) and 
\[
S_2 \setminus \{0\} \subseteq F_2 \subseteq S_1 \setminus \{0\}, |F_1 \cup F_2| = \mathcal{A}.
\]

So we know there is \( f: \mathcal{B} \to \mathcal{A} \) in which \( f^{-1}(S_1 \setminus \{0\}) \) and \( f^{-1}(S_2 \setminus \{0\}) \) are \( \Delta^0_k \) inseparable. So letting \( T_1 = f^{-1}(S_1) \) and \( T_2 = f^{-1}(S_2) \) we have the theorem. \( \square \)

The main difference between the extra assumptions in Theorems 3.18 and 3.20 is that in Theorem 3.20 we specifically require that \( V_\omega(F) \) has a fully effective copy with decidable existential diagram in the stronger language of vector spaces. In Theorem 3.18 we were able to deduce that any recursive copy of \( V_0(F) \) had decidable existential diagram in the stronger language and so did not need the extra assumption. It turns out that not every fully effective copy will have decidable existential diagram in the stronger language. In particular, if the field \( F \) has a recursive copy which does not have a splitting algorithm, this may be the case. See [9] for this result. (A field is said to have a splitting algorithm if there is a procedure to decide when any polynomial in \( F[x] \) is reducible.) So we cannot expect to be able to show a general decision procedure for existential sentences in the stronger language of vector spaces without additional assumptions. Hence, we cannot expect to avoid stronger conditions in Theorem 3.20.

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