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# Signatures and Signed Switching Classes

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## 1. INTRODUCTION

As various authors have pointed out [1, 5], several properties of switching classes and two-graphs can be explained using ideas from homological algebra. In this paper an analogous theory of signed switching classes and signatures is developed using similar techniques. The point of departure from the usual formulation is the fixing of an underlying simple graph $G$, and the use of edge signings of $G$. Beyond that, the development relies on choosing certain sets of subgraphs $\mathscr{X}_2$ and $\mathscr{X}_3$. By taking $G$ to be a complete graph, $\mathscr{X}_2$ its $K_3$-subgraphs, and $\mathscr{X}_3$ its $K_4$-subgraphs, the known results for switching classes and two-graphs are recouped. One of the themes of this paper is that through judicious choices of $G$, $\mathscr{X}_2$, and $\mathscr{X}_3$, the behavior of switching classes and two-graphs can be mimicked accurately.

The two main applications of this theory concern the circuits of $G$ and the induced circuits of $G$. The first application, which relies heavily on a theorem of Zaslavsky's (6.1), implies that a collection $X$ of circuits is precisely the set of unbalanced circuits of some edge signing when $X$ has an even number of elements contained in each theta subgraph. The second application, which is derived from a theorem of Truemper's (6.3), implies that a set $Y$ of induced circuits arises as the unbalanced induced circuits of some edge signing if and only if even numbers of members of $Y$ lie in each induced subgraph of three particular types (described in Sect. 6). Other

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examples detailed in Section 7 illustrate the wide variety of possible choices for $G$, $\mathscr{X}_2$, and $\mathscr{X}_3$.

A graph-theoretic corollary (8.1) is obtained from the application to induced circuits. This corollary determines all 2-connected graphs with the property that each edge lies in exactly two induced circuits. These graphs turn out to be precisely the three types of graphs mentioned above. This paper limits itself to the results obtained by applying homological techniques. The reader is advised to consult Wells [7] and Zaslavsky [11] for other approaches to signed switching classes and signatures.

All the graphs considered here are finite and simple, that is, without loops or multiple edges. An *empty graph* is a graph with no edges. If $G$ is a graph and $v$ is one of its vertices, let $G_v$ be the graph obtained from $G$ by deleting the edges containing $v$ and adding all other possible edges containing $v$. If $X \subseteq V(G)$, say $X = \{u, v, \ldots, w\}$, let $G_X = (((G_u)_v) \cdots)_w$. (Notice that the symmetric difference $E(G_X) \Delta E(G)$ is the set of pairs $\{a, b\}$ intersecting $X$ exactly once. Thus the order in which $u, v, \ldots, w$ are given does not affect the result, and $G_X$ is well defined.) If a graph $H$ is of the form $G_X$ for some $X \subseteq V(G)$, then $H$ is termed *switching equivalent* to $G$. Now "switching equivalent" is an equivalence relation [4]. The equivalence class of $G$, denoted $[G]$, is $\{G_X : X \subseteq V(G)\}$. Any set of the form $[G]$ is called a *switching class*, or a *Seidel switching class* to emphasize that it is not a signed switching class. Given a graph $G$, let $\Omega(G)$ be the collection of 3-subsets $\{a, b, c\}$ of $V(G)$ such that an odd number of $\{a, b\}$, $\{b, c\}$, $\{a, c\}$ are edges of $G$. Any set of the form $\Omega(G)$ is called a *two-graph*.

The properties of switching classes and two-graphs which we shall seek to mimic are these (cf. 5.6):

(1.1) *If $X$ and $Y$ are subsets of $V(G)$, then $G_X = G_Y$ if and only $X = Y$ or $X = V(G) \setminus Y$.*

(1.2) *If $G$ and $H$ are graphs with the same vertex set, then $\Omega(G) = \Omega(H)$ if and only if $G$ and $H$ are switching equivalent.*

(1.3) *A collection $\Omega$ of 3-subsets of a finite set $X$ is a two-graph if and only if each 4-subset of $X$ contains an even number of elements of $\Omega$.*

## 2. Signed Switching Classes and Signatures

In this section the basic objects of our study are introduced. They are derived from signed graphs, which we now review.

For the remainder of this paper $G$ will be a nonempty graph. A *signing* of $G$ is any function $f: E(G) \to \{+, -\}$. The pair $(G, f)$ is called a *signed graph*. For any $X \subseteq V(G)$ let $f_X$ be the signing obtained from $f$ by reversing the sign of each edge having exactly one vertex in $X$. If $g=f_X$ for some $X\subseteq V(G)$ then $f$ and $g$ are called *switching equivalent*. Noting $f=f_{\varnothing}$, $f=(f_X)_X$, and $(f_X)_Y=f_{(X\Delta Y)}$, we see this determines an equivalence relation. The equivalence class $[f]$ of $f$ is $\{f_X: X\subseteq V(G)\}$. The equivalence classes are termed *signed switching classes* of $G$.

If $G$ is a complete graph then signings of $G$ are in one-to-one correspondence with graphs having vertex set $V(G)$ (the signing $f$ corresponds to the graph with edge set $f^{-1}(-)$.) Furthermore, Seidel switching of graphs corresponds to the switching operation that has just been described: loosely put, $(f_X)^{-1}(-)=(f^{-1}(-))_X$. Thus signed switching classes in general are analogous to Seidel switching classes, and the two concepts are equivalent when the underlying graph $G$ is complete.

Assuming still that $G$ is complete, let $f$ be any signing of $G$. Let $\operatorname{sig}(f)$ be the set of triangles $T$ in $G$ such that $f(a)f(b)f(c)=-$, where $E(T)=\{a, b, c\}$. We call $\operatorname{sig}(f)$ the *signature* associated with $f$ (relative to the triangles.) Clearly these signatures are equivalent to two-graphs.

To extend this notion to an arbitrary underlying graph $G$, though, is a subtle matter. For example, $G$ might not have any triangles. In addition, it is not immediately clear how to guarantee the correctness of properties analogous with (1.1) through (1.3).

Our solution to this difficulty is to avoid as long as possible specifying the substitutes for 3-subsets and 4-subsets. Instead, let $\mathscr{X}_2$ and $\mathscr{X}_3$ be arbitrary collections of subgraphs of $G$. The conditions that need to be placed on $\mathscr{X}_2$ and $\mathscr{X}_3$ to ensure results such as (1.1) through (1.3) will be found in Sections 4 and 5. Specific choices for $\mathscr{X}_2$ and $\mathscr{X}_3$ are not discussed until Sections 6 and 7.

Now we can present the general definition of a signature: for any subgraph $H$ of $G$, let $f(H)$ be the product $\prod_{e\in E(H)} f(e)$. Define the *signature* $\operatorname{sig}(f)$ of *$f$ relative to $\mathscr{X}_2$* as $\{H\in\mathscr{X}_2: f(H)=-\}$. Note this is consistent with the previous usage of "signature" in this section.

We conclude with an elementary property of signed switching classes which will prove useful (cf, [10, 3.1]):

(2.1) *Let $F$ be a maximal forest of $G$, and $f$ any signing of $F$. For any signing $g$ of $G$ there is a unique signing of $G$ which is switching equivalent to $g$ and equal to $f$ on $F$.*

*Proof.* Existence. In every conected component of $F$ mark one vertex. While not all vertices of $F$ are marked do the following: choose an edge $\{v, w\}$ of $F$ such that $v$ is marked and $w$ is not marked; then mark $w$, and by inspection switch at $w$ as necessary to put $g$ in agreement with $f$ on $\{v, w\}$.

Uniqueness. Suppose $g_1$ and $g_2$ are switching equivalent and agree at every edge of $F$. Let $e\in E(G)\setminus E(F)$. Then $e$ forms a unique circuit $C$ with some of the elements of $F$. Now $g_1(C)=g_2(C)$ if and only if $g_1(e)=g_2(e)$. But $g_1(C)$ and $g_2(C)$ must be equal because $g_2$ can be obtained from $g_1$ by a sequence of single-vertex switchings. Thus $g_1=g_2$ on all edges in $E(G)\setminus E(F)$. ▮

## 3. The Associated Binary Vector Spaces

This section presents means of treating signings, signed switching classes, and signatures as elements of certain distinguished binary vector spaces. It also demonstrates that the operations of switching and forming signatures are linear transformations on these spaces. These observations will inform all of our study.

To give a unified presentation of the vector spaces, we make use of sets $\mathscr{X}_i$ of subgraphs of $G$, $-1\leqslant i\leqslant 3$. As before, $\mathscr{X}_2$ and $\mathscr{X}_3$ are allowed to be arbitrary collections of subgraphs. Let $\mathscr{X}_{-1}=\{\varnothing\}$, $\mathscr{X}_0=V(G)$, and let $\mathscr{X}_1$ be the set of single-edge subgraphs of $G$ (vertices included.) For $-1\leqslant i\leqslant 3$ we make the following definitions: let $V_i$ be the binary vector space of formal sums of elements of $\mathscr{X}_i$. Let $V^i$ be the dual of $V_i$, with distinguished dual basis $\mathscr{X}^i=\{x^*\colon x\in\mathscr{X}_i\}$. (Here we assume $F=\{0,1\}$ is the binary field with unit 1, and $x^*$ is the linear functional which is 1 on $x$ and 0 on all other elements of $\mathscr{X}_i$.)

If $0\leqslant i\leqslant 3$ define the boundary maps $\partial_i\colon V_i\to V_{i-1}$ by taking the linear extension of the function which maps each element of $\mathscr{X}_i$ to the sum of its subgraphs in $\mathscr{X}_{i-1}$. For example, the boundary of a sum of disjoint edges is the sum of their vertices. The boundary of that element in turn is $0\in V_{-1}$.

If $X$ is a subset of $\mathscr{X}_i$, where $-1\leqslant i\leqslant 3$, let $s_X$ in $V_i$ be the formal sum of the elements of $X$, as above and let $s_X^*$ be the sum of all $x^*\in\mathscr{X}^i$ for $x\in X$. This linear functional $s_X^*$ will be called (with some abuse of terminology) the *characteristic function* of $X$. For any $f\in V^i$ define the *support* of $f$ to be $\{x\in\mathscr{X}_i\colon f(x)=1\}$.

For $0\leqslant i\leqslant 3$, define $\delta^{i-1}\colon V^{i-1}\to V^i$ to be the dual of $\partial_i$. (That is, if $f\in V^{i-1}$, define $g=f\delta^{i-1}\colon V_i\to F$ such that $g$ evaluated at $x$ equals $f$ evaluated at $\partial_i x$. Then $g\in V^i$ and $\delta^{i-1}$ is linear.) The map $\delta^i$ is called the $i$th coboundary map. It is not hard to check that $(x^*)\,\delta^{i-1}$ is the characteristic function of the set of elements in $\mathscr{X}_i$ which contain $x$, assuming $x\in\mathscr{X}_{i-1}$. For example, $\varnothing\delta^{-1}$ is $s^*(V(G))$, and if $v\in V(G)$ then $v^*\delta^0$ is 1 on the edges containing $v$ and 0 all other edges.

These vector spaces and linear transformations represent the structures and operations described in Section 2: identify the sign $+$ with $0\in F$ and the sign $-$ with $1\in F$. Then the elements of $V^1$ correspond to signings of $G$. Now let $f\in V^1$. If $X\subseteq V(G)$ and $x=s_X^*\in V^0$, then $f_X=f+x\delta^0$. Clearly $[f]$ is the coset $f+(\operatorname{im}\delta^0)$ of $(\operatorname{im}\delta^0)$ in $V^1$. Furthermore $f\delta^1$ is 1 on the elements of sig($f$) and 0 on the remaining members of $\mathscr{X}_2$. Thus (im $\delta^1$) corresponds to the set of all possible signatures relative to $\mathscr{X}_2$.

Regarding $\delta^2$: $V^2 \to V^3$, note that any subset $Z$ of $\mathscr{X}_2$ has an even number of elements contained in each element of $X_3$ if and only if $(s_Z^*)\,\delta^2 = 0$.

We introduce the subspaces $B_i$, $Z_i$ in $V_i$ and $B^i$, $Z^i$ in $V^i$ as follows: $B_i = \text{im}\,\partial_{i+1}$, $-1 \leqslant i \leqslant 2$, is the space of *i-boundaries*; $Z_i = \ker \partial_i$, $0 \leqslant i \leqslant 3$, is the space of *i-cycles*; $B^i = \text{im}\,\delta^{i-1}$, $0 \leqslant i \leqslant 3$, is the space of *i-coboundaries*; and $Z^i = \ker \delta^i, -1 \leqslant i \leqslant 2$, is the space of *i-cocycles.*

To recapitulate the most important definitions: signings and signed switching classes are the elements of $V^1$ and $V^1/B^1$; signatures are the support of 2-coboundaries.

## 4. The Complex Condition

The sequence

$$V^{-1} \xrightarrow{\delta^{-1}} V^0 \xrightarrow{\delta^0} V^1 \xrightarrow{\delta^1} V^2 \xrightarrow{\delta^2} V^3 \tag{4.1}$$

is called a *complex* when the conditions $\delta^i\delta^{i+1} = 0$, $-1 \leqslant i \leqslant 1$, all hold. These conditions have the following significance vis a vis signatures and signed switching classes:

(4.2) *The following statements hold for any nonempty underlying graph $G$:*

(i) *The relation $\delta^{-1}\delta^0 = 0$ always holds. Thus $f_X = f_Y$ whenever $f \in V^1$, $X$, $Y \subseteq V(G)$, and $X = V(G) \setminus Y$.*

(ii) *If $\delta^0\delta^1 = 0$ then any two switching equivalent signings have the same signatures relative to $\mathscr{X}_2$.*

(iii) *If $\delta^1\delta^2 = 0$ then every signature has an even number of elements in each member of $\mathscr{X}_3$.*

*Proof.* (i) Notice that $\varnothing\delta^{-1} = \sum_{v \in V(G)} v^*$, so that $\varnothing\delta^{-1}\delta^0 = \sum_{v \in V(G)} v^*\delta^0$. Every $e^* \in \mathscr{X}^1$ occurs exactly twice in the latter sum, so $\delta^{-1}\delta^0 = 0$. Now let $X \subseteq V(G)$ and $Y = V(G) \setminus X$. Then $s_X^* = s_Y^* + \varnothing\delta^{-1}$. Thus for any signing $f \in V^1$, $f + s_X^*\delta^0 = f + s_Y^* + \varnothing\delta^{-1}\delta^0 = f + (s_Y^*)\,\delta^0$, i.e., $f_X = f_Y$.

(ii) Two signings of $G$ are switching equivalent if and only if they differ by an element of im $\delta^0$. They have the same signatures if and only if they differ by an element of $\ker \delta^1$. Since $\delta^0\delta^1 = 0$ means im $\delta^0 \subseteq \ker \delta^1$, statement (ii) holds.

(iii) Recall that signatures are supports of elements of im $\delta^1$. Now $\delta^1\delta^2=0$ means im $\delta^1 \subseteq \ker \delta^2$. In light of the remark concerning $\delta^2$ made in the last section, (iii) must hold. ∎

*Note.* The converses of (ii) and (iii) also hold, by the same argument essentially. Thus (4.1) *must* be a complex if the analogs of properties (1.1) through (1.3) are to hold for signatures and signed switching classes relative to $G$, $\mathscr{X}_2$, and $\mathscr{X}_3$.

We digress briefly to discuss dual (or transpose) linear transformations: for any vector space $V$ over a field $K$, let $V^*$ be its dual. For any subspace $U$ of $V$, let the *annihilator* of $U$ be the space $\operatorname{Ann} U=\{\phi \in V^*: \phi u=0\ \forall u \in U\}$. If $T: V \to W$ is a linear transformation, its *dual* is the linear map $T^*: W^* \to V^*$ such that $w^*T^*: V \to K: (w^*T^*)\,v=w^*(Tv)$, for any $w^* \in W^*$. Then a simple argument shows

(4.3) *Let $T: V \to W$ be a linear transformation and $T^*: W^* \to V^*$ its dual. Then* $\operatorname{Ann}(\ker T)=\operatorname{im} T^*$, *and* $\operatorname{Ann}(\operatorname{im} T)=\ker T^*$.

Recalling that $\delta^{i-1}$ is the dual of $\partial_i$, we see that im $\delta^{i-1}=\operatorname{Ann}(\ker \partial_i)$ and $\ker \delta^i=\operatorname{Ann}(\operatorname{im} \partial_{i+1})$. Thus $\delta^{i-1}\delta^i=0$ if and only if im ${}^{i-1} \subseteq \ker \delta^i$ if and only if $\operatorname{Ann}(\ker \partial_i) \subseteq \operatorname{Ann}(\operatorname{im} \partial_{i+1})$ if and only if im $\partial_{i+1} \subseteq \ker \partial_i$ if and only if $\partial_{i+1}\partial_i=0$. Which shows

(4.4) *For each $i$ in the range $0 \leqslant i \leqslant 2$, $\delta^{i-1}\delta^i=0$ if and only if $\partial_i\partial_{i+1}=0$.*

Therefore the sequence

$$V_{-1} \xleftarrow{\partial_0} V_0 \xleftarrow{\partial_1} V_1 \xleftarrow{\partial_2} V_2 \xleftarrow{\partial_3} V_3 \tag{4.5}$$

is a complex if and only if (4.1) is.

Whether $\delta^0\delta^1$ is 0 hinges on the choice of $\mathscr{X}_2$, so we shall say that $(G; \mathscr{X}_2)$ is *complex* when $\delta^0\delta^1=0$. Similarly $\delta^1\delta^2=0$ depends on the choices of $\mathscr{X}_2$ and $\mathscr{X}_3$. We say that $(G; \mathscr{X}_2, \mathscr{X}_3)$ is complex when both $\delta^0\delta^1=0$ and $\delta^1\delta^2=0$. In addition, a graph $H$ (possibly empty) is called *even* if the degree of each vertex is even. We can now state and prove these results:

(4.6) *$(G; \mathscr{X}_2)$ is complex if and only if every element of $X_2$ is an even graph.*

(4.7) *The composition $\delta^1\delta^2$ is zero if and only if for each graph $H$ in $\mathscr{X}_3$ and each edge $e$ in $H$, the number of graphs in $\mathscr{X}_2$ containing $e$ and contained in $H$ is even.*

*Proof of* (4.6). Assume $\delta^0\delta^1=0$. Then $\partial_1\partial_2=0$ by (4.4). Thus $\partial_1\partial_2 H=0$

for all $H \in \mathscr{X}_2$. But $\partial_1 \partial_2 H$ is the sum of the vertices of $H$ of odd degree. This forces the members of $\mathscr{X}_2$ to be even graphs. The converse can be proved by reversing this argument. ▮

*Proof of* (4.7). First assume $\delta^1 \delta^2 = 0$. Let $H \in \mathscr{X}_3$ and $e \in E(H)$. Then $e^* \in V^1$ and $e^* \delta^1$ is the characteristic function of a signature. By (4.2)(ii) the support of $e^* \delta^1$ has an even number of elements contained in $H$. Thus there is an even number of members of $\mathscr{X}_2$ containing $e$ and contained in $H$. This holds for every $H \in \mathscr{X}_3$ and $e \in E(H)$. Again, the converse follows by reversing the argument. ▮

## 5. Exactness

We have seen that requiring (4.1) to be an algebraic complex has fruitful implications for signatures and signed switching classes. We shall now show that (4.1) is an exact sequence precisely when all the appropriate analogs of (1.1) through (1.3) are valid (cf. 5.6).

If $(G; \mathscr{X}_2, \mathscr{X}_3)$ is complex, then define the binary vector spaces $H^i$ to be $H^i = (\ker \delta^i)/(\operatorname{im} \delta^{i-1})$, $0 \leqslant i \leqslant 2$. Due to (4.4) it is also possible to make the definition $H_i = (\ker \partial_i)/(\operatorname{im} \partial_{i+1})$, $0 \leqslant i \leqslant 2$. Furthermore the remarks preceding (4.5) can be used to show $\dim H^i = \dim H_i$, $0 \leqslant i \leqslant 2$. Notice also that $H^0$ and $H_0$ depend only on $G$; $H^1$ and $H_1$ depend on $G$ and $\mathscr{X}_2$; and $H^2$ and $H_2$ depend on $G$, $\mathscr{X}_2$, and $\mathscr{X}_3$. Define $(G; \mathscr{X}_2)$ to be *exact* when $H^0 = H^1 = 0$. Similarly call $(G; \mathscr{X}_2, \mathscr{X}_3)$ exact when all three of $H^0$, $H^1$, and $H^2$ vanish. (The complex (4.1) is said to be exact precisely when $(G; \mathscr{X}_2, \mathscr{X}_3)$ is exact.) Recall that a subgraph $G'$ of a graph $G$ is an induced subgraph when $E(G') = \{e \in E(G) : e \subseteq V(G')\}$.

The spaces $H_0$, $H_1$, and $H_2$ we have introduced are generalizations of homology over the binary field (see [3] for the topological definitions.) In fact, $H_0$ is the reduced 0-homology (mod 2) of the graph, as a 1-dimensional complex. If every element of $\mathscr{X}_2$ is an induced circuit, then adjoin discs to form a 2-dimensional cell complex; $H_1$ is the 1-homology of this complex. Similarly, if each element of $\mathscr{X}_3$ induces (in this complex) the 2-skeleton of a 3-cell, adding further patches produces a 3-dimensional cell complex whose second homology is our $H_2$.

Now we state six results, which are subsequently proved. Their purpose is to illuminate the significance of exactness *vis a vis* signatures and signed switching classes, and to describe the graph-theoretic conditions which ensure exactness:

(5.1) *The dimension of $H^0$ is one less than the number of connected components of $G$.*

(5.2) *The following three statements are equivalent*:

(i) $\dim H^0(G) = 0$;

(ii) *G is connected*;

(iii) *For every signing f of G and all sets X,* $Y \subseteq V(G)$, $f_X = f_Y$ *if and only if* $X = Y$ *or* $X = V(G) \setminus Y$.

We need to precede (5.4) with a lemma:

(5.3) *Let* $\mathscr{C}$ *be the set of circuits of G and* $\mathscr{I}$ *the set of induced circuits. Then* $\dim H^1(G; \mathscr{C}) = 0$ *and* $\dim H^1(G; \mathscr{I}) = 0$.

(5.4) *Assume* $(G; \mathscr{X}_2)$ *is complex. Then the following are equivalent*:

(i) $\dim H^1(G; \mathscr{X}_2) = 0$;

(ii) *Whenever two signings of G have the same signature relative to* $\mathscr{X}_2$ *they are switching equivalent*;

(iii) $\mathscr{X}_2$ *spans the cycle space of G*;

(iv) *For every circuit C of G, there is a set* $\{G_1, ..., G_t\}$ *of graphs in* $\mathscr{X}_2$ *such that* $E(C) = E(G_1) \Delta \cdots \Delta E(G_t)$.

(v) *For every induced circuit C of G the conclusion of* (iv) *holds.*

(5.5) *Assume* $(G; \mathscr{X}_2, \mathscr{X}_3)$ *is complex. Then the following statements are equivalent*:

(i) $\dim H^2(G; \mathscr{X}_2, \mathscr{X}_3) = 0$;

(ii) *Every subsets S of* $\mathscr{X}_2$ *such that, for all* $H \in \mathscr{X}_3$ *an even number of elements of S are contained in H, is a signature relative to* $\mathscr{X}_2$.

The constraints put on $\mathscr{X}_2$ and $\mathscr{X}_3$ by the condition $\dim H^2(G; \mathscr{X}_2, \mathscr{X}_3) = 0$ are difficult to describe unless $\mathscr{X}_2$ is specified. See (6.5) and (6.6) for the constraints when $\mathscr{X}_2$ is the set of all circuits in $G$, or just the set of induced circuits.

Finally we state the summary theorem:

(5.6) $(G; \mathscr{X}_2, \mathscr{X}_3)$ *is exact if and only if all three of the following statements hold for all X,* $Y \subseteq V(G)$, *all* $f, f_1, f_2 \in V^1$, *and all* $S \subseteq \mathscr{X}_2$:

(i) $f_X = f_Y$ *if and only if* $X = Y$ *or* $X = V(G) \setminus Y$;

(ii) $f_1$ *and* $f_2$ *are switching equivalent if and only if they have the same signature relative to* $\mathscr{X}_2$;

(iii) *A subset S of* $\mathscr{X}_2$ *is a signature if and only if each member of* $\mathscr{X}_3$ *has an even number of elements of S as subgraphs.*

*Proof of* (5.1). By (2.1), any switching class contains $2^{|E(F)|}$ signings, where $F$ is a maximal forest of $G$. Thus $\dim(\operatorname{im} \delta^0) = |E(F)|$, and dim(ker $\delta^0$) therefore equals dim $V^0 - |E(F)| = |V(G)| - |E(F)|$. But this last term is the number of connected components of $G$. Since dim(im $\delta^{-1}$) = 1 and $H^0 = (\ker \delta^0)/(\operatorname{im} \delta^{-1})$, the result holds. ∎

*Proof of* (5.2). Because of (5.1) we need only show dim $H^0 = 0$ is equivalent to (iii). First assume dim $H^0 = 0$ and let $f \in V^1$; $X, Y \subseteq V(G)$ with $f_X = f_Y$. Then $f = (f_X)_X = (f_Y)_X = f_{Y \Delta X} = f + s^*_{Y \Delta X}\delta^0$. Thus $s^*_{Y \Delta X} \in \ker \delta^0 = \{0, s^*_{V(G)}\}$. So either $X = Y$ or $X = V(G) \setminus Y$.

Now suppose dim $H^0 \geqslant 1$ and let $C$ be a connected component of $G$. Then $f_{V(C)} = f$ for any $f \in V^1$, so (iii) does not hold. ∎

*Proof of* (5.3). First consider $(G; \mathscr{C})$. We need only show $Z^1 \subseteq B^1$, i.e., that any signing $f$ with empty signature is switching equivalent to the all + signing. Let $F$ be a maximal forest of $G$. By (2.1), $f$ can be switched so that, without loss of generality, $f$ is + on every edge of $F$. Now let $e \in E(G) \setminus E(F)$. Together with $F$, $e$ forms a circuit $C$. By assumption $f(C) = +$, so $f(e) = +$. Thus $f \in B^1$.

Now consider $(G; \mathscr{I})$. The argument proceeds just as above, except that $f(C) = +$ is not necessarily known. But $E(C)$ is the symmetric difference of the edge sets of some of the circuits in $\mathscr{I}$. Because $f$ is + on all the induced circuits, it follows that $f$ is + on $C$. So $f(e) = +$ and $f \in B^1$. ∎

*Proof of* (5.4). (i), (ii) Let $f, g \in V^1$. Then sig($f$) = sig($g$) if and only if $f\delta^1 = g\delta^1$ if and only if $(f-g)\,\delta^1 = )$ if and only if $(f-g) \in \ker \delta^1$. Also $[f] = [g]$ if and only if $f + (\operatorname{im} \delta^0) = g + (\operatorname{im} \delta^0)$ if and ony if $(f-g) \in \operatorname{im} \delta^0$. Thus (ii) holds if and only if im $\delta^0 = \ker \delta^1$, i.e., if and only if dim $H^1(G; x_2) = 0$.

(iii), (iv), and (v) are plainly equivalent, so it only remains to show that (i)$\Leftrightarrow$(iv). Recall dim $H^1 = \dim H_1$, so (i) holds if and only if dim $H_1(G; \mathscr{X}_2) = 0$ if and only if $Z_1 \subseteq B_1$. Now (5.3) dualized reveals that $Z_1$ is spanned by the boundaries of circuits. Thus $Z_1 \subseteq B_1$ is equivalent to (iii). ∎

*Proof of* (5.5). The signatures are the supports of elements of $B^2$. The subsets of $\mathscr{X}_2$ with an even number of elements in each member of $\mathscr{X}_3$ are the supports of functionals in $Z^2$. Thus (ii) is equivalent to $Z^2 \subseteq B^2$, which is equivalent to $Z^2 = B^2$, given that $(G; \mathscr{X}_2, \mathscr{X}_3)$ is a complex. ∎

*Proof of* (5.6). This simply summarizes parts of (4.2), (4.6), (4.7), (5.2), (5.4), and (5.5). ∎

## 6. Two General Examples

This section describes two general methods for endowing any connected graph $G$ with sets $\mathscr{X}_2$ and $\mathscr{X}_3$ so that $(G; \mathscr{X}_2, \mathscr{X}_3)$ is exact. The chief distinction between these two methods is that in one, $\mathscr{X}_2$ and $\mathscr{X}_3$ are collections of ordinary subgraphs of $G$, and is the other, they are sets of induced subgraphs of $G$. Recall that subdividing an edge $e$ of a graph $H$ consists of placing a new vertex upon $e$, so that $e$ becomes two edges. A subdivision of $H$ is $H$ itself or any graph obtained from $H$ by a sequence of edge subdividions.

We shall need the following notation and terminology in presenting the first construction: let $G$ be a simple nonempty graph, and $\mathscr{C}$ the set of circuits of $G$. Let $K_4'$ be $K_4$ with an edge deleted. A graph $H$ is called a theta-graph if $H$ is isomorphic to a subdivision of $K_4'$. Let $\Theta$ be the collection of all theta subgaphs of $G$. A subset $\Omega'$ of $\mathscr{C}$ is called ○ *additive* if, whenever $C_1, C_2 \in \mathscr{C}$, $C_1 \cup C_2 \in \Theta$, and both or neither of $C_1$, $C_2$ lie in $\Omega'$, then the symmetric difference $C_1 \Delta C_2$ lies in $\Omega'$. Also, for any edge signing $f$ of $G$, the *balanced* circuits of $G$ relative to $f$ are those circuits $C$ such that $\prod_{e \in E(C)} f(e) = +$.

We first state a theorem of Zaslavsky's [9, Theorem 6].

(6.1) *Let $G$ be a graph and $\Omega'$ a set of circuits of $G$. Then there is a signing $f$ of $G$ such that $\Omega'$ is the collection of balanced circuits of $G$ relative to $f$, if and only if $\Omega'$ is circle additive.*

Of course $\Omega'$ is the set of balanced circuits of $f$ if and only if $\Omega = \mathrm{sig}(f)$ relative to $\mathscr{C}$, where $\Omega = \mathscr{C} \setminus \Omega'$. Thus $\Omega'$ is circle additive if and only if $\Omega$ is a signature. We now show that (6.1) quickly leads to

(6.2) *If $G$ is a nonempty simple connected graph, $\mathscr{C}$ the set of circuits of $G$, and $\Theta$ the set of theta subgraphs of $G$, then $(G; \mathscr{C}, \Theta)$ is exact.*

*Proof.* Note that each theta graph contains three circuits; that the symmetric difference of any two of these is the third; and that each edge lies in exactly two of the circuits. It is easy to verify now that $(G; \mathscr{C}, \Theta)$ is complex. Also (5.2) and (5.3) apply, so that it only remains to show $\dim H^2 = 0$. But the following statements are equivalent to each other in turn: $\Omega \subseteq \mathscr{C}$ is a signature; $\Omega' = \mathscr{C} \setminus \Omega$ is circle additive; each theta subgraph contain an odd number of members of $\Omega'$; each theta subgraph contains an even number of members of $\Omega$. Thus (ii) of (5.5) holds, where $\mathscr{X}_2 = \mathscr{C}$ and $\mathscr{X}_3 = \Theta$. Therefore (5.5)(i) is true, i.e., $\dim H^2 = 0$. ∎

The second method for endowing any simple connected graph with appropriate sets $\mathscr{X}_2$ and $\mathscr{X}_3$ makes use of a result of K. Truemper [6, Corollary 1.1]. To describe his theorem, we need to define certain graphs: *A wheel graph $W_n$, $n \geqslant 3$,* is formed by adding one vertex $v$ *to a circuit $C$ of length $n$, and joining $v$ to all vertices of $C$. The rim of $W_n$ is $C$. The triangular prism $P$* is the graph formed by the six vertices and nine edges of a solid triangular prism; the *long edges* of $P$ are the ones not contained in a triangle. Let $P'$ be the graph obtained from $P$ by deleting one of the edges in a triangle. If $\mathscr{X}_2$ is a set of subgraphs of $G$, $X \subseteq \mathscr{X}_2$, and if $H$ is a subgraph of $G$, let $X|_H$ be the set of members of $X$ which are subgraphs of $H$. Let $\mathscr{I}$ be the set of induced circuits of $G$ and $\mathscr{X}_3$ the set of induced subgraphs of $G$ of one of these three types:

($I_0$) $P$ or $P'$ or one of their subdividions along the long edges;

($I_1$) the complete bipartite graph $K_{2,3}$ or one of its subdivisions;

($I_2$) a wheel graph or one of its subdivisions along the rim.

Truemper's result can now be stated:

(6.3) *Let $\Omega \subseteq \mathscr{I}$. Then $\Omega$ is a signature of $G$ relative to $\mathscr{I}$ if and only if, for every $H \in \mathscr{X}_3$, $\Omega|_H$ is a signature of $H$ relative to $\mathscr{I}|_H$.*

Our second method is given in (6.4).

(6.4) *If $G$ is a nonempty connected graph, $\mathscr{I}$ is the set of induced circuits of $G$, and $\mathscr{X}_3$ is the set of induced subgraphs of the form $I_0$, $I_1$, or $I_2$, then $(G; \mathscr{I}\ \mathscr{X}_3)$ is exact.*

The proof will require two lemmas:

(6.5) *Let $H$ be a 2-connected nonempty graph such that each edge of $H$ lies in exactly two induced circuits. Let $\mathscr{X}_2$ be the set of induced circuits of $H$. Then $(H; \mathscr{X}_2, \{H\})$ is exact.*

(6.6) *Any graph of type $I_0$, $I_1$, or $I_2$ satisfies the hypothesis of* (6.5).

*Proof of* (6.5) By (4.6), (4.7), (5.2), and (5.3), we need only show $\dim H^2(H; \mathscr{X}_2, \{H\}) = 0$, i.e., $\dim H_2(H; \mathscr{X}_2, \{H\}) = 0$, i.e., $Z_2 \subseteq B_2$. Now $B_2 = \{0, \partial_3 H\} = \{0, s_2\}$ where $s_2$ is the $V_2$-sum of all elements of $\mathscr{X}_2$. Thus we must show any element $s_X$ of $Z_2$, $X \subseteq \mathscr{X}_2$, is either 0 or $s_2$. So let $s_X \in Z_2$, $X \subseteq \mathscr{X}_2$, $X \neq \varnothing$. We need only show $X = \mathscr{X}_2$.

Let $F$ be the set of all edges $f$ in $E(H)$ such that both induced circuits of $H$ containing $f$ lie in $X$. Plainly the proof will be finished if $F$ equals $E(H)$.

First note that, assuming $e$ is an edge and $C_1$, $C_2$ are the two induced circuits of $H$ in which $e$ lies, if one of $C_1$, $C_2$ lies in $X$ then so does the other, and consequently $e \in F$. The reason for this is that $s_X \in Z_2$, so $\partial_2 s_X = 0$ and $X$ has an even number of members containing $e$. But there are only two possible members of $X$ containing $e$—$C_1$ and $C_2$—and $X$ already contains one. So $X$ must contain both $C_1$ and $C_2$, and by definition $e \in F$. In particular, $F \neq \varnothing$ because $X \neq \varnothing$.

Suppose there were an edge $e \in E(H) \setminus F$. For any such $e$ let the *height* $h(e)$ of $e$ be the minimum number of edges among all circuits $D$, if any, such that $D$ contains $e$ and at least one edge of $F$. Such a $D$ must exist for each $e$ because $H$ is 2-connected: for example, let $f \in F$ and choose $D$ to be a circuit of $H$ containing both $f$ and $e$. Thus $h(e)$ is defined for all $e \in E(H) \setminus F$.

Let $k$ be the minimum value of $h(e)$ over all $e$ in $E(H) \setminus F$. Let $e \in E(H) \setminus F$, let $D$ be a circuit of $H$, and let $f \in F$ with $e, f \in E(D)$ and $|E(D)| = k$.

If $D$ were an induced circuit it would lie in $X$ because $f \in F$ and $f \in E(D)$. But $D \in X$ and $e \in E(D)$ would force $e \in F$, a contradiction.

Thus the circuit $D$ is not induced, and there is an edge $e_0 \in E(H) \setminus E(D)$ with $e_0 \subseteq V(D)$. Together with $e_0$, $D$ forms a theta graph. Call the three circuits of this theta $D$, $C_1$, and $C_2$. Then $e_0$ lies in $C_1$ and $C_2$, and without loss of generality $f \in E(C_1)$. Note furthermore that $|E(C_1)|, |E(C_2)| < k$. Choose $i$ so that $e \in E(C_i)$.

Now $f \in F$; $e_0, f \in E(C_1)$; and $|E(C_1)| < k$. The minimality of $k$ forces $e_0 \in F$. But then $e_0, e \in E(C_i)$ and $|E(C_i) < k$, so the minimality of $k$ forces $e \in F$, a contradiction. Thus $F$ equals $E(H)$. ∎

LEMMA (6.6) *is true by inspection.*

*Proof of* (6.4). First (5.3) and (6.6) show $(G; \mathscr{I}, \mathscr{X}_3)$ is complex and $(G; \mathscr{I})$ is exact. Thus we need only show dim $H^2(G; \mathscr{I}, \mathscr{X}_3) = 0$. Choose any $\Omega \subseteq \mathscr{X}_2$, and let $H \in \mathscr{X}_3$. (6.3) shows $\Omega\,|_H$ is a signature of $H$ relative to $\mathscr{I}\,|_H$ if and only if $H$ contains an even number of elements of $\Omega\,|_H$, i.e., of $\Omega$. Thus Truemper's result implies $\Omega$ is a signature of $G$ relative to $\mathscr{I}$ if and only if each member of $\mathscr{X}_3$ contains an even number of elements of $\Omega$. (5.5) concludes the proof. ∎

Two simple corollaries are these:

(6.7) *Assume $G$ is a nonempty connected graph and $\mathscr{X}_3$ is any set of subgraphs of $G$. Then $(G; \mathscr{C}, \mathscr{X}_3)$ is exact if and only if both of these conditions hold*:

(i) *For every $H \in \mathscr{X}_3$ and $e \in E(H)$, there is an even number of circuits containing $e$ and contained in $H$.*

(ii) *For every theta subgraph $\theta$ of $G$, there is a subset $\{G_1, ..., G_t\}$ of $\mathscr{X}_3$ such that every circuit of $G$ lies in an even number of $\theta, G_1, ..., G_t$.*

(6.8) *Let $G$ be a nonempty connected graph and $\mathscr{X}_3$ any collection of subgraphs of $G$. Then $(G; \mathscr{I}, \mathscr{X}_3)$ is exact if and only if both of the following conditions hold*:

(i) *For every $H \in \mathscr{X}_3$ and $e \in E(H)$, there is an even number of induced circuits containing $e$ and contained in $H$.*

(ii) *For every induced subgraph $I$ of one of the three types $I_0$, $I_1$, or $I_2$, there is a subset $\{G_1,..., G_t\}$ of $\mathscr{X}_3$ such that every induced circuit of $G$ lies in an even number of $I, G_1,..., G_t$.*

Proofs of (6.7) and (6.8). In each case condition (i) holds iff $\delta^1\delta^2=0$, and condition (ii) holds iff $Z_2 \subseteq B_2$.

## 7. Additional Examples

We simply state a variety of examples now and defer their proofs until the end of the section.

(7.1) *Let $G$ be complete, and $\mathscr{X}_i$ the set of subgraphs isomorphic to $K_{i+1}$, $i=2, 3$. Then $(G; \mathscr{X}_2, \mathscr{X}_3)$ is exact.*

(Of course, this implies (1.1) through (1.3).)

(7.2) *Let $G$ be a complete bipartite graph, $\mathscr{X}_2$ the set of subgraphs isomorphic to $K_{2,2}$, and $\mathscr{X}_3$ the set of subgraphs isomorphic to $K_{2,3}$. Then $(G; \mathscr{X}_2, \mathscr{X}_3)$ is exact.*

(7.3) *Let $G$ have the property that every induced circuit of $G$ is contained in the neighborhood of a vertex. Then $H^1(G; \mathscr{T})=0$, where $\mathscr{T}$ is the set of triangles of $G$.*

(7.4) (Zaslavsky). *Let $G$ be a complete tripartite graph, and $\mathscr{X}_2$ the set of circuits of length 3. If $\mathscr{X}_3$ is the collection of subgraphs isomorphic to $K_{2,2,2}$, then $(G; \mathscr{X}_2, \mathscr{X}_3)$ is exact.*

(7.5) *Let $G$ be a complete multipartite graph with at least four parts. If $\mathscr{X}_i$ is the set of complete subgraphs on $i+1$ vertices, $i=2, 3$, then $(G; \mathscr{X}_2, \mathscr{X}_3)$ is exact.*

Let $N_1,..., N_r$ be disjoint nonempty sets. The complete circular multipartite graph $C_r[N_1,..., N_r]$ has vertex set $\bigcup_i N_i$, and edge set $\{\{u, v\}: u \in N_i, v \in N_j, j \equiv i+1 \pmod r\}$. Then we have

(7.6) (Zaslavsky). *Let $G$ be the complete circular multipartite graph $C_r[N_1,..., N_r]$ with $r \geqslant 4$. Let $\mathscr{X}_2$ be the collection of circuits of length $r$ of the form $x_1 \cdots x_r$, where $x_i \in N_i$, $i=1,..., r$. Let $\mathscr{X}_3$ be the collection of subgraphs of $G$ of the form $C_r[M_1,..., M_r]$, where $M_i \subseteq N_i$, $1 \leqslant i \leqslant r$, two of the $M_i$ have cardinality 2, all remaining $M_i$ are singletons, and the two $M_i$ of size 2 do not have subscripts which are consecutive modulo $r$. Then $(G; \mathscr{X}_2, \mathscr{X}_3)$ is exact.*

The $n$-cube $Q_n$, $n \geqslant 2$, is the graph whose vertices are the elements of $F^n$, the $n$-tuple vector space over $F$, with $u$ adjacent to $v$ when $u$ and $v$ differ in only one coordinate.

(7.7) *Let $G$ be the n-cube $Q_n$, $n \geqslant 2$, and $\mathscr{X}_i$ the set of subgraphs isomorphic to $Q_i$, $i = 2, 3$. Then $(G; \mathscr{X}_2, \mathscr{X}_3)$ is exact.*

A well-known strongly regular graph is $L_2(n)$, $n \geqslant 2$, which has vertex set of the form $X \times X$ for some $n$-set $X$. Two vertices $(x_1 x_2)$ and $(y_1, y_2)$ are adjacent if and only if $x_1 = y_1$ or $x_2 = y_2$. For strongly regular graphs, one would naturally focus on circuits of length 3 and 4, and this technique succeeds for $L_2(n)$:

(7.8) *Let $G = L_2(n)$, $n \geqslant 4$, and set $\mathscr{X}_2$ to be the collection of all induced circuits of $G$ of length 3 or 4. Let $\mathscr{X}_3$ be the set of complete subgraphs on 4 vertices together with the induced subgraphs isomorphic to the triangular prism. Then $(G; \mathscr{X}_2, \mathscr{X}_3)$ is exact.*

*Note.* If $\mathscr{X}_2$ contains only the induced circuits of length 3, then $\dim H^1 = (n-1)^2$.

The approach taken for $L_2(n)$ will not work for all strongly regular graphs, however; there are some for which the circuits of length 3 and 4 do not span the cycle space. For example, let $F_n$ be the folded $n$-cube, $n \geqslant 5$, which is obtained from $Q_n$ by identifying vertices which are unequal in every coordinate. Then $F_5$ is strongly regular, none of the $F_n$ have any triangles, and the following result holds:

(7.9) *Let $G = F_n$, $n \geqslant 5$. If $\mathscr{X}_2$ is the set of subgraph of $G$ isomorphic to $Q_2$ then* $\dim H^1(G; \mathscr{X}_2) = 1$.

We can generalize (7.9) to other quotient graphs of $Q_n$, $n \geqslant 5$, as follows: let $C \subseteq V(Q_n)$ be any binary linear code of length $n$ and minimum weight 5 or greater. We define a graph $G = Q_n/C$ by identifying vertices of $Q_n$ if their difference is in $C$. (When $C$ is the span of the all-$l$ vector, then $G$ is $F_n$.) The minimum weight condition ensures that 4-cycles in $G$ all come from 4-cycles in $Q_n$. We claim that $C$ can be recovered from $G$:

(7.10) *If $G = Q_n/C$ and $\mathscr{X}_2$ is the set of 4-cycles of $G$, then $H_1(G; \mathscr{X}_2) \simeq C$.*

Finally, there is a general method for spanning the cycle space of a graph with circuits of limited size, which is due to Lemmens [2]: given a system $\sum$ of 1-dimensional subspaces of $\mathbb{R}^n$, form a graph $G$ with vertex set $\sum$, two vertices being adjacent if they are not orthogonal. If $\sum$ gives rise to $G$ in this way, call $\sum$ an *Eucidean realization* of $G$. An argument involving Gram matrices can be used to show that every graph has an Euclidean realization. Lemmens showed that the girth of $G$ is at most $n+2$, and that the circuit space of $G$ is generated by the set $X_2$ of circuits of length at most $n$. Thus we have

(7.11) *Let n be the minimum dimension of the Euclidean realizations of G, and let $\mathscr{X}_2$ be the sets of circuits of G of length at most n. If G is connected, then $(G; \mathscr{X}_2)$ is exact.*

In each case it is relatively easy to verify $(G; \mathscr{X}_2, \mathscr{X}_3)$ is complex. All the underlying graphs are connected, so the following sketchy proofs only establish that the first and second cohomology vanish.

*Proof of* (7.1). This follows from the vanishing cohomology of the $(n-1)$-simplex [3], and it is also an immediate corollary of (6.2). Direct proofs can be found in [4, 5]. ∎

*Proof of* (7.2). This is also an immediate corollary of (6.2), although an easy direct proof can also be given. ∎

*Proof of* (7.3). Under these assumption, every induced circuit is a symmetric difference of triangles. Thus (5.4)(v) is satisfied, and $H^1(G; T) = 0$. ∎

*Proof of* (7.4). The induced circuits of $G$ are of two types: triangles with each vertex in a different vertex part, and $K_{2,2}$ subgraphs, with 2 vertices in each of 2 parts. Induced circuits of both types are contained in the neighborhood of a vertex. By (7.3), $H^1$ vanishes. Finally [11, Theorem 2] shows that $\dim H^2 = 0$. ∎

*Proof of* (7.5). As before, the hypothesis of (7.3) is satisfied, so $\dim H^1 = 0$. For proofs that $H^2$ vanishes, see [7, p. 11] or [11, Theorem 2]. ∎

*Proof of* (7.6) is due to Zaslavsky [11, Theorem 4]. ∎

*Proof of* (7.7). To show $\dim H^1 = 0$, it is enough to show that every signing with empty signature relative to $\mathscr{X}_2$ is switching equivalent to the all + signing. So let $f \in V^1$ with $\operatorname{sig}(f) = \varnothing$. We proceed by induction on $n$: if $n = 2$, choose a spanning tree $T_2$ for $Q_2$. Switch $f$ so that it is + on the edges of $T_2$. Then plainly $f$ is the all + signing. Assuming the correctness of the result now for $Q_n$, let $T_n$ be a spanning tree for $Q_n$. The graph $Q_{n+1}$ can be formed by taking two copies of $Q_n$, and joining corresponding vertices. Take the corresponding two copies of $T_n$, join them with an edge $e_{n+1}$ of $Q_{n+1}$, and call the resulting graph $T_{n+1}$. Clearly $T_{n+1}$ is a spanning tree for $Q_{n+1}$. Switch $f$ so that it is all + on the edges of $T_{n+1}$. By the inductive hypothesis, $f$ is + on all edges of both copies of $Q_n$. But $f(e_{n+1}) = +$ also, and it is not hard to see that $f$ must be + on all remaining edges of $Q_{n+1}$.

To show $H^2$ vanishes we consider any subset $S$ of $\mathscr{X}_2$ which has an even number of elements in each $Q_3$-subgraph of $G$. We must show $S$ is a signature, i.e., that $S$ arises from an edge signing. An inductive argument similar to the previous one can be given, and we omit the details. ∎

*Proof of* (7.8). The graph $L_2(n)$ has two sets of $n$ disjoint cliques each, say $A_1,..., A_n$ and $C_1,..., C_n$. Every edges lies in exactly one of these $2n$ cliques. Choose a spanning tree for $C_1$ and each $A_i$, $1 \leqslant i \leqslant n$. Their union is a spanning tree $T$ for $L_2(n)$. First suppose $f \in V^1$ with $\operatorname{sig}(f) = \varnothing$. Switch $f$ so that $f(e) = +$ for every $e \in E(T)$. By (7.1), $f$ is $+$ on all edges of $C_1, A_1,..., A_n$. The only remaining edges lie in circuits of length 4 with three such edges. Thus $f$ is the all $+$ signing of $G$, and $\dim H^1 = 0$.

Now assume we have an element $\xi$ of $Z^2$; $\xi$ is a signing of induced 3-cycles and 4-cycles, and we wish to show $\xi \in B^2$, i.e., that $\xi$ is the coboundary of some edge signing. We define a signing $f$ in the following way: note that $\xi$ restricted to the triangles of $C_1$ is a 2-cocycle relative to the set of $K_4$'s in $C_1$. By (7.1), $\xi$ (within $C_1$) is the coboundary of a signing $f_1$ of $C_1$. Thus, for edges in $C_1$, define $f$ to be equal to $f_1$. Similarly we may define $f$ on the edges of $A_1,..., A_n$ so that the coboundary of $f$ restricted to the triangles of $C_1, A_1,..., A_n$ agrees with $\xi$.

Consider any other edge $e$ of $L_2(n)$. It lies in one of the $C_i$, $2 \leqslant i \leqslant n$, and joins vertices $a_i \in V(A_i)$ and $a_j \in V(A_j)$. This edge forms a unique 4-cycle $D$ with edges $e_1$, $e_2$, $e_3$ in $A_i$, $A_j$, and $C_1$. Let $f(e) = f(e_1) f(e_2) f(e_3)\, \xi(D)$. This completes the definition of $f$ on all edges of $L_2(n)$. Furthermore $f\delta^1$ agrees with $\xi$ on all triangles in $C_1, A_1,..., A_n$ and on all induced 4-cycles in $L_2(n)$ containing one edge in $C_1$.

We must show that $f\delta^1 = \xi$ on all of $\mathscr{X}_2$. First consider any triangle $T_1$ not in $C_1, A_1,..., A_n$. It lies entirely in one of the $C_i$, $2 \leqslant i \leqslant n$. There is a unique triangle in $C_1$ which forms an induced triangular prism $P$ with $T_1$. But now it is easy to check that

$$(f\delta^1)(T_1) = \prod_{e \in T_1} f(e) = \prod_{\Gamma} \prod_{e \in F} f(e) = \prod_{\Gamma} \xi(\Gamma) = \xi(T_1),$$

where $\Gamma$ ranges over the four induced circuits of $P$ other than $T_1$.

Finally consider any induced 4-cycle $Q$ with no edges in $C_1$. It lies in a unique induced triangular prism with one edge in $C_1$. By the same argument as for $T_1$ above, $(f\delta^1)(Q) = \xi(Q)$. Thus $\xi = f\delta^1$ ∎

*Proof of* (7.9). This is an immediate consequence of (7.10) if the code $C$ is chosen to be the span of the all-$l$ vector. ∎

*Proof of* (7.10). Recall that $F$ represents the binary field. Each edge of $G$ is associated with a vector of weight 1 in $F^n$ (take the difference of appropriate coset representatives of the two vertices.) Hence there is a linear map $T^0\colon V_1 \to F^n$. Let $T$ denote its restriction to $Z_1$. Since $T$ annihilates 4-cycles, we have $B_1 \subseteq \ker T$. Conversely, if $f \in \ker T$, then we can add edges with multiplicity 2 to find a closed walk in $G$ corresponding to $f$. By a sequence of operations consisting of either

(i) replacing successive edges of type $i, j$ by successive edges of type $j, i$, or

(ii) deleting pairs of successive edges of the same type,

we can transform $f$ to 0 by adding to it a sum of 4-cycles, i.e., an element of $B_1$. So $B_1 = \ker T$. Thus $H_1 = Z_1/B_1 \simeq \operatorname{im} T$, which is isomorphic to $C$, since a closed walk in $G$ lifts to a walk in $Q_n$ from 0 to an element of $C$ and vice versa. ∎

*Proof of* (7.11). This follows from results of Lemmens, as described before. ∎

## 8. Three Applications

We present three graph-theoretical results which follow from the preceding work.

(8.1) *Let $G$ be a nonempty 2-connected graph such that every edge lies in exactly two induced circuits. Then $G$ is a graph of type $I_0$, $I_1$, or $I_2$ as defined in Section* 6.

*Proof.* Result (6.5) states that $(G; \mathscr{I}, \{G\})$ is exact, where $\mathscr{I}$ is the collection of induced circuits of $G$. By (6.8), statement (6.8)(ii) must hold. (6.8)(ii) implies either

(a) $G$ has no induced subgraphs of types $I_0$, $I_1$, or $I_2$; or

(b) there is an induced subgraph $H$ of type $I_0$, $I_1$, or $I_2$ such that every induced circuit of $G$ lies in neither or both of $H$ and $G$; or

(c) there is an induced subgraph $H$ of type $I_0$, $I_1$, or $I_2$ such that every induced circuit of $G$ does not lie in $H$.

Now (c) is clearly impossible because $H$ would contain some induced circuits. If (b) holds, let $e$ be any edge of $G$. By 2-connectedness, $e$ lies in a circuit of $G$, and therefore in some induced circuit $C$ of $G$. The relation $C \subseteq G$ forces $C \subseteq H$, so $e \in E(H)$ and thus $H = G$, establishing the desired conclusion. Finally (a) is impossible for these reasons: $(G; \mathscr{I}, \{G\})$ is exact, so $Z_2 = Z_2(G; \mathscr{I})$ is spanned by the boundary of $G$, and is clearly nonzero. But $(G; \mathscr{I}, \mathscr{X}_3)$ is also exact, where $\mathscr{X}_3$ is the set of induced subgraphs of type $I_0$, $I_1$, or $I_2$. Now $Z_2$ is also spanned by the boundaries of members of $\mathscr{X}_3$, and because $Z_2 \neq 0$, we must have $\mathscr{X}_3 \neq \varnothing$. ∎

We can apply (6.2) and (6.4) to immediately obtain these two corollaries:

(8.2) *Let G be a nonempty graph, and S a set of circuits of G. Then S the set of unbalanced circuits of G under some signing if and only if each theta subgraph of G contains an even number of members of S.*

(8.3) *Let G be a nonempty graph, and S a collection of induced circuits of G. Then S is the set of unbalanced induced circuits of some signing of G if and only if each induced subgraph of G of type $I_0$, $I_1$, or $I_2$ (as described in Section 6) contains an even number of members of S.*

*Note.* (8.2) is essentially a restatement of [9, Theorem 6]. (8.3) is a consequence of [6, Corollary 1.1], i.e., (6.3), as Zaslavsky and Seymour first discovered [12].

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