On a result of G. Brosch

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Abstract

We prove a uniqueness theorem for non-constant meromorphic functions \( f, g \) which share three values \( 0, 1, \infty \) and \( f - a, g - b \) share the value 0 for \( a, b \notin \{0, 1, \infty\} \). Our theorem improves a result of G. Brosch.

Keywords: Meromorphic functions; Weighted sharing; Uniqueness

1. Introduction, definitions and results

Let \( f \) and \( g \) be two non-constant meromorphic functions defined in the open complex plane \( \mathbb{C} \). For \( a \in \mathbb{C} \cup \{\infty\} \) we say that \( f, g \) share the value \( a \) CM (counting multiplicities) if \( f, g \) have the same \( a \)-points with the same multiplicity and we say that \( f, g \) share the value \( a \) IM (ignoring multiplicities) if we do not consider the multiplicities.

Following result is known as Nevanlinna’s four points uniqueness theorem.

Theorem A. (Cf. [14], [15, p. 218].) Let \( f \) and \( g \) be distinct non-constant meromorphic functions sharing four values CM. Then \( f \) is a bilinear transformation of \( g \).

G.G. Gundersen [4] improved Theorem A as follows:

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Theorem B. [4] Let \( f \) and \( g \) be two distinct non-constant meromorphic functions sharing two values \( \text{CM} \) and two other values \( \text{IM} \). Then \( f \) and \( g \) share all the four values \( \text{CM} \) and the conclusion of Theorem A holds.

In 1989 G. Brosch (cf. [2], [15, p. 329]) improved Theorem A in another direction and proved the following result.

Theorem C. ([2], [15, p. 329]) Let \( f \) and \( g \) be two distinct non-constant meromorphic functions sharing 0, 1, \( \infty \) \( \text{CM} \). Let \( a, b \) be two complex numbers such that \( a, b \notin \{0, 1, \infty\} \). If \( f - a \) and \( g - b \) share 0 \( \text{IM} \) then \( f \) is a bilinear transformation of \( g \).

So far the authors know the problem of finding the relationship of two non-constant meromorphic functions sharing three values \( \text{IM} \) and one value \( \text{CM} \) has still not been completely solved. In this direction the following result of G.G. Gundersen [5] is worth mentioning.

Theorem D. ([5], [15, p. 248]) Let \( f \) and \( g \) be non-constant meromorphic functions sharing three values \( a_1, a_2, a_3 \) \( \text{IM} \) and a fourth value \( a_4 \) \( \text{CM} \). Suppose that there exist some real constant \( \lambda > \frac{4}{3} \) and some set \( I \subset (0, \infty) \) of infinite linear measure such that \( N(r, a_4; f) \geq \lambda T(r, f) \) for all \( r \in I \). Then \( f \) and \( g \) share all the four values \( \text{CM} \) and the conclusion of Theorem A holds.

We now state the result of Alzahary and Yi [1].

Example 1.1. [3] Let \( f(z) = \frac{e^{z+1}}{(e^{z}-1)^2} \) and \( g(z) = \frac{(e^{z+1})^2}{8(e^{z}-1)} \). Then \( f \) and \( g \) share 0, 1, \( \infty \) \( \text{IM} \). Since \( f(z) + \frac{1}{2} = \frac{e^{2z}+3}{2(e^{z}-1)^2} \) and \( g(z) - \frac{1}{4} = \frac{e^{2z}+3}{8(e^{z}-1)} \), we see that \( f + \frac{1}{2} \) and \( g - \frac{1}{4} \) share 0 \( \text{CM} \). Also clearly \( f \) is not a bilinear transformation of \( g \).

Recently T.C. Alzahary and H.X. Yi [1] used the idea of weighted sharing, introduced in [7,8], to improve Theorem C. We now explain in the following definition the notion of weighted sharing which measures how close a shared value is to being shared \( \text{IM} \) or to being shared \( \text{CM} \).

Definition 1.1. [7,8] Let \( k \) be a nonnegative integer or infinity. For \( a \in \mathbb{C} \cup \{\infty\} \) we denote by \( E_k(a; f) \) the set of all \( a \)-points of \( f \) where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). If \( E_k(a; f) = E_k(a; g) \), we say that \( f \), \( g \) share the value \( a \) with weight \( k \).

The definition implies that if \( f \), \( g \) share a value \( a \) with weight \( k \) then \( z_0 \) is a zero of \( f - a \) with multiplicity \( m \) \( (\leq k) \) if and only if it is a zero of \( g - a \) with multiplicity \( m \) \( (\leq k) \) and \( z_0 \) is a zero of \( f - a \) with multiplicity \( m \) \( (> k) \) if and only if it is a zero of \( g - a \) with multiplicity \( n \) \( (> k) \) where \( m \) is not necessarily equal to \( n \).

We write \( f \), \( g \) share \( (a, k) \) to mean that \( f \), \( g \) share the value \( a \) with weight \( k \). Clearly if \( f \), \( g \) share \( (a, k) \) then \( f \), \( g \) share \( (a, p) \) for all integers \( p, 0 \leq p < k \). Also we note that \( f \), \( g \) share a value \( a \) \( \text{IM} \) or \( \text{CM} \) if and only if \( f \), \( g \) share \( (a, 0) \) or \( (a, \infty) \), respectively.

We now state the result of Alzahary and Yi [1].

Theorem E. [1] Let \( f \) and \( g \) be two non-constant meromorphic functions sharing \((a_1, 1)\), \((a_2, \infty)\) and \((a_3, \infty)\), where \( \{a_1, a_2, a_3\} = \{0, 1, \infty\} \). Let \( a, b \) be two complex numbers such
that \(a, b \notin \{0, 1, \infty\}\). If \(f - a\) and \(g - b\) share \((0, 0)\) then \(f\) is a bilinear transformation of \(g\).

Moreover, \(f\) and \(g\) satisfy one of the following relations: (i) \(f \equiv g\), (ii) \(fg \equiv 1\), (iii) \(bf \equiv ag\), (iv) \(f + g \equiv 1\), (v) \(f \equiv ag\), (vi) \(f \equiv (1 - a)g + a\), (vii) \((1 - b)f \equiv (1 - a)g + (a - b)\), (viii) \(f(a - 1 + g) \equiv ag\), (ix) \(f((b - a)g + (a - 1)b) \equiv a(b - 1)g\), (x) \(f(g - 1) \equiv g\).

The cases (ii) and (v) may occur if \(ab = 1\), cases (iv) and (viii) may occur if \(a + b = 1\), cases (vi) and (x) may occur if \(ab = a + b\).

In the paper we further improve Theorems C and E and prove the following result.

**Theorem 1.1.** Let \(f\) and \(g\) be two distinct non-constant meromorphic functions sharing \((a_1, 1), (a_2, m), (a_3, k)\), where \((m - 1)(mk - 1) > (1 + m)^2\) and \(\{a_1, a_2, a_3\} = \{0, 1, \infty\}\). If for two values \(a, b \notin \{0, 1, \infty\}\) the functions \(f - a\) and \(g - b\) share \((0, 0)\) then \(f, g\) share \((0, \infty), (1, \infty), (\infty, \infty)\) and \(f - a, g - b\) share \((0, \infty)\). Also there exists a non-constant entire function \(\lambda\) such that \(f\) and \(g\) are one of the following forms:

(i) \(f = ae^\lambda\) and \(g = be^{-\lambda}\), where \(ab = 1\).

(ii) \(f = 1 + ae^\lambda\) and \(g = 1 + (1 - \frac{1}{b})e^{-\lambda}\), where \(ab = a + b\).

(iii) \(f = \frac{a}{e^{\lambda} - \frac{1}{b}e^{\lambda}}\) and \(g = \frac{1}{e^{\lambda} - \frac{1}{b}e^{\lambda}}\), where \(ab = 1\).

(iv) \(f = \frac{b}{(b - 1)(1 - e^{\lambda})}\) and \(g = \frac{1}{(a - 1)(1 - e^{-\lambda})}\), where \(a \neq b\).

(v) \(f = \frac{a}{b}\) and \(g = \frac{1}{e^{\lambda} - \frac{1}{b}e^{\lambda}}\), where \(ab = a + b\).

(vi) \(f = \frac{a}{b}\) and \(g = \frac{1}{e^{\lambda} - \frac{1}{b}e^{\lambda}}\), where \(ab = a + b\).

(vii) \(f = \frac{b}{(b - 1)(1 - e^{\lambda})}\) and \(g = \frac{1}{(a - 1)(1 - e^{-\lambda})}\), where \(a \neq b\).

(viii) \(f = a + e^\lambda\) and \(g = b(1 + \frac{1}{e^{\lambda}})\), where \(a + b = 1\).

(ix) \(f = \frac{a}{e^{\lambda} - \frac{1}{b}e^{\lambda}}\) and \(g = \frac{1}{(a - 1)(1 - e^{\lambda})}\), where \(a \neq b\).

The following corollary immediately follows from Theorem 1.1.

**Corollary 1.1.** Let \(f\) and \(g\) be non-constant meromorphic functions sharing \((a_1, 1), (a_2, m), (a_3, k)\), where \((m - 1)(mk - 1) > (1 + m)^2\) and \(\{a_1, a_2, a_3\} = \{0, 1, \infty\}\). If \(a \notin \{0, 1, \infty, -1, \frac{1}{2}, 2\}\) is a complex number such that \(f, g\) share \((a, 0)\) then \(f \equiv g\).

**Remark 1.1.** Theorem 1.1 and Corollary 1.1 are valid for the following pairs of minimum values of \(m\) and \(k\): \(m = 2, k = 6; m = 3, k = 4; m = 4, k = 3\) and \(m = 6, k = 2\). However the question of further reducing the weights of sharing the values still remains open.

We do not explain the standard definitions and notations of the value distribution theory as those are available in [6]. However we explain in the following definitions some notations used in the paper.

**Definition 1.2.** Let \(f\) be a meromorphic function and \(a \in \mathbb{C} \cup \{\infty\}\). For a positive integer \(p\) we denote by \(N(r, a; f | \leq p)\) \((\tilde{N}(r, a; f | \leq p))\) the counting function (reduced counting function) of those \(a\)-points of \(f\) whose multiplicities are less than or equal to \(p\).

Similarly by \(N(r, a; f | \geq p)\) \((\tilde{N}(r, a; f | \geq p))\) we denote the counting function (reduced counting function) of those \(a\)-points of \(f\) whose multiplicities are greater than or equal to \(p\).
Definition 1.3. Let \( f, g \) be two meromorphic functions and \( a, b \in \mathbb{C} \cup \{\infty\} \). We denote by \( N(r, a; f \mid = g) \) (\( \overline{N}(r, a; f \mid = g) \)) the counting function (reduced counting function) of those \( a \)-points of \( f \) which are \( b \)-points of \( g \) also.

Definition 1.4. Let \( f, g \) be two meromorphic functions and \( a, b \in \mathbb{C} \cup \{\infty\} \). For a positive integer \( p \) we denote by \( N(r, a; f \mid g = b, \geq p) \) (\( \overline{N}(r, a; f \mid g = b, \geq p) \)) the counting function (reduced counting function) of those \( a \)-points of \( f \) which are \( b \)-points of \( g \) with multiplicities not less than \( p \).

In the paper we denote by \( f \) and \( g \) two non-constant meromorphic functions defined in the open complex plane \( \mathbb{C} \) unless otherwise stated.

2. Lemmas

In this section we present some lemmas which are required in the sequel.

Lemma 2.1. [3] If \( f, g \) share \( (0, 0), (1, 0), (\infty, 0) \) then \( T(r, f) \leq 3T(r, g) + S(r, f) \) and \( T(r, g) \leq 3T(r, f) + S(r, g) \).

This shows that \( S(r, f) = S(r, g) \) and we denote them by \( S(r) \).

Lemma 2.2. [9] Let \( f, g \) share \( (0, 1), (1, m), (\infty, k) \) and \( f \not\equiv g \), where \( (m - 1)(mk - 1) > (1 + m)^2 \). Then for \( a = 0, 1, \infty \),

\[
\overline{N}(r, a; f \mid \geq 2) + \overline{N}(r, a; g \mid \geq 2) = S(r).
\]

Lemma 2.3. [10, 12] Let \( f, g \) share \( (0, 1), (1, m), (\infty, k) \) and \( f \not\equiv g \), where \( (m - 1)(mk - 1) > (1 + m)^2 \). If \( f \) is not a bilinear transformation of \( g \) then each of the following holds:

(i) \( T(r, f) + T(r, g) = N(r, 0; f \mid \leq 1) + N(r, 1; f \mid \leq 1) + N(r, \infty; f \mid \leq 1) + N_0(r) + S(r) \),
(ii) \( T(r, f) + T(r, g) = N(r, 0; g \mid \leq 1) + N(r, 1; g \mid \leq 1) + N(r, \infty; g \mid \leq 1) + N_0(r) + S(r) \),
(iii) \( T(r, f) = N(r, 0; g' \mid \leq 1) + N_0(r) + S(r) \),
(iv) \( T(r, g) = N(r, 0; f' \mid \leq 1) + N_0(r) + S(r) \),
(v) \( N_1(r) = S(r) \),
(vi) \( N_0(r, 0; g' \mid \geq 2) = S(r) \),
(vii) \( N_0(r, 0; f' \mid \geq 2) = S(r) \),
(viii) \( \overline{N}(r, 0; g' \mid \geq 2) = S(r) \),
(ix) \( \overline{N}(r, 0; f' \mid \geq 2) = S(r) \),
(x) \( N(r, 0; f - g \mid \geq 2) = S(r) \),
(xi) \( N(r, 0; f - g \mid g = \infty) = S(r) \),
(xii) \( N(r, 0; f - g \mid f = \infty) = S(r) \),

where \( N_0(r) \) (\( N_1(r) \)) denotes the counting function of those simple (multiple) zeros of \( f - g \) which are not the zeros of \( f(f' - 1) \) and \( \frac{1}{f'} \); also \( N_0(r, 0; g' \mid \geq 2) \) (\( N_0(r, 0; f' \mid \geq 2) \)) is the counting function of those multiple zeros of \( g'(f') \) which are not the zeros of \( f(f - 1) \).
Lemma 2.4. [11] Let \( f, g \) share \((0, 1), (1, m), (\infty, k)\) and \( f \neq g \), where \((m - 1)(mk - 1) > (1 + m)^2\). If \( \alpha = \frac{r - 1}{s - 1} \) and \( \beta = \frac{r}{s} \) then \( \overline{N}(r, a; \alpha) + \overline{N}(r, a; \beta) = S(r) \) for \( a = 0, \infty \).

Lemma 2.5. Let \( f, g \) share \((0, 1), (1, m), (\infty, k)\), where \((m - 1)(mk - 1) > (1 + m)^2\). Furthermore let \( f - a \) and \( g - b \) share \((0, 0)\), where \( a \) and \( b \) are distinct complex numbers such that \( a, b \notin [0, 1, \infty] \). If \( f \) is not a bilinear transformation of \( g \) and \( \frac{\alpha' \beta'}{\alpha \beta'} \) is non-constant then \( N(r, a; f | \geq 2) + N(r, b; g | \geq 2) = S(r) \).

Proof. Let \( \alpha \) and \( \beta \) be defined as in Lemma 2.4. Since \( f \) is not a bilinear transformation of \( g \), it follows that \( \alpha, \beta \) and \( \alpha \beta \) are non-constant. Also \( f = \frac{1 - \alpha}{1 - \alpha \beta} \) and \( g = \frac{(1 - \alpha) \beta}{1 - \alpha \beta} \). Let \( \psi = \frac{\alpha' \beta'}{\alpha \beta'} \). Then

\[
T(r, \psi) = T \left( r, \frac{1}{\psi} \right) + O(1) = T \left( r, \frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} \right) + O(1)
\]

\[
\leq T \left( r, \frac{\alpha}{\alpha'} \right) + T \left( r, \frac{\beta}{\beta'} \right) + O(1) = T \left( r, \frac{\alpha'}{\alpha} \right) + T \left( r, \frac{\beta'}{\beta} \right) + O(1) \tag{2.1}
\]

From the definitions of \( \alpha \) and \( \beta \) and by Lemma 2.1 we see that \( S(r, \alpha) \) and \( S(r, \beta) \) are replaceable by \( S(r) \). So by Lemma 2.4 we get

\[
T \left( r, \frac{\alpha'}{\alpha} \right) = S(r, \alpha') + m \left( r, \frac{\alpha'}{\alpha} \right) = \overline{N}(r, 0; \alpha) + \overline{N}(r, \infty; \alpha) + S(r, \alpha) = S(r).
\]

Similarly we get \( T(r, \frac{\beta'}{\beta}) = S(r) \). So from (2.1) we obtain \( T(r, \psi) = S(r) \). Also \( f - g = \frac{(1 - \alpha)(1 - \beta)}{1 - \alpha \beta}, \ g - 1 = \frac{\beta - 1}{1 - \alpha \beta} \) and

\[
\frac{g'}{g} = \frac{\beta'(1 - \alpha \beta) + (\beta - 1)(\alpha' \beta + \alpha \beta')}{\beta(1 - \alpha)(1 - \alpha \beta)}.
\]

Therefore

\[
\frac{g'(g - f)}{g(g - 1)} = \frac{(1 - \alpha)(\alpha \beta' + \alpha' \beta) - \alpha' \beta(1 - \alpha \beta)}{\alpha \beta(1 - \alpha \beta)}.
\]

Also

\[
(f - \psi) \left( \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} \right) = \frac{(1 - \alpha)(\alpha \beta' + \alpha' \beta) - \alpha' \beta(1 - \alpha \beta)}{\alpha \beta(1 - \alpha \beta)}.
\]

Hence

\[
(f - \psi) \left( \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} \right) = \frac{g'(g - f)}{g(g - 1)} \tag{2.2}
\]

Let \( z_0 \) be a double zero of \( g - b \) which is neither a zero nor a pole of \( \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} \). Then \( z_0 \) is a simple zero of \( \frac{g'(g - f)}{g(g - 1)} \) and so by (2.2) \( z_0 \) is a simple zero of \( f - \psi \). Since \( f - a, g - b \) share \((0, 0)\) and \( \alpha - \psi = (f - \psi) - (f - a) \), it follows that \( z_0 \) is a zero of \( a - \psi \). If \( N(r, b; g | = 2) \) denotes the counting function of double zeros of \( g - b \) (counted according to multiplicity) then from above we get

\[
N(r, b; g | = 2) \leq 2N(r, a; \psi) + S(r) = S(r).
\]
Now by Lemma 2.3(vi) we get
\[ N(r, b; g \mid \geq 2) = N(r, b; g \mid = 2) + N(r, b; g \mid \geq 3) \leq 2N_0(r, 0; g' \mid \geq 2) + S(r) = S(r). \]
Similarly we can prove that \( N(r, a; f \mid \geq 2) = S(r). \) This proves the lemma.

**Lemma 2.6.** Let \( f \) and \( g \) be two distinct meromorphic functions sharing \((0, 1), (1, m), (\infty, k)\) and \((c, 0)\), where \((m - 1)(mk - 1) > (1 + m)^2\) and \(c \neq 0, 1, \infty\) is a complex number. Then \( f, g \) share \((0, \infty), (1, \infty), (\infty, \infty)\) and \((c, \infty)\).

**Proof.** Let \( F = \frac{f'(f - c)}{f(f - 1)} - \frac{g'(g - c)}{g(g - 1)} \). If \( F \equiv 0 \) then clearly \( f, g \) share \((c, \infty)\). We suppose that \( F \not\equiv 0 \). Since
\[ F = \left( \frac{cf'}{f} - \frac{(c - 1)f'}{f - 1} \right) - \left( \frac{cg'}{g} - \frac{(c - 1)g'}{g - 1} \right), \]
we see that \( m(r, F) = S(r) \). Since all the possible poles of \( F \) are simple and \( f, g \) share \((0, 1), (1, m), (\infty, k)\), we get by Lemma 2.2
\[ N(r, F) = \overline{N}(r, F) \leq \overline{N}(r, 1; f \mid \geq 2) + \overline{N}(r, 0; f \mid \geq 2) + \overline{N}(r, \infty; f \mid \geq 2) = S(r). \]
Hence \( T(r, F) = S(r) \). Also we see that \( \overline{N}(r, c; f) = \overline{N}(r, c; g) \leq N(r, 0; F) \leq T(r, F) + O(1) = S(r) \). Let
\[ G_1 = \frac{f'(f - 1)}{f(f - c)} - \frac{g'(g - 1)}{g(g - c)} \quad \text{and} \quad G_2 = \frac{f'f}{(f - 1)(f - c)} - \frac{g'g}{(g - 1)(g - c)}. \]
If \( G_1 \equiv 0 \) or \( G_2 \equiv 0 \) then clearly \( f, g \) share \((c, \infty)\). We suppose that \( G_1 \not\equiv 0 \) and \( G_2 \not\equiv 0 \). Also we can easily see that \( m(r, G_1) = S(r) \) and \( m(r, G_2) = S(r) \). Since \( G_1 \) and \( G_2 \) can have only simple poles we get by Lemma 2.2
\[ N(r, G_1) = \overline{N}(r, G_1) \leq \overline{N}(r, 0; f \mid \geq 2) + \overline{N}(r, \infty; f \mid \geq 2) + \overline{N}(r, c; f) = S(r) \]
and
\[ N(r, G_2) = \overline{N}(r, G_2) \leq \overline{N}(r, 1; f \mid \geq 2) + \overline{N}(r, \infty; f \mid \geq 2) + \overline{N}(r, c; f) = S(r). \]
Therefore \( T(r, G_1) = S(r) \) and \( T(r, G_2) = S(r) \) and so \( \overline{N}(r, 0; f) = \overline{N}(r, 0; g) \leq N(r, 0; G_2) = S(r) \) and \( \overline{N}(r, 1; f) = \overline{N}(r, 1; g) \leq N(r, 0; G_1) = S(r) \). This shows that \( \overline{N}(r, c; f) + \overline{N}(r, 0; f) = \overline{N}(r, 1; f) = S(r) \), which is impossible. Therefore \( f, g \) share \((c, \infty)\).

Let \( f_1 = \frac{f - c}{f - c} \) and \( g_1 = \frac{g - c}{g - c} \). Then \( f_1 \) and \( g_1 \) share \((0, \infty), (1, m), (\infty, k)\) and \((c, 1), 1 \). So as above \( f_1 \) and \( g_1 \) share \((c - 1, \infty)\) and so \( f, g \) share \((0, \infty)\). Therefore \( f \) and \( g \) share \((0, \infty), (1, m), (\infty, k)\) and \((c, \infty)\).

Let \( f_2 = \frac{f}{f - c} \) and \( g_2 = \frac{g}{g - c} \). Then \( f_2 \) and \( g_2 \) share \((0, \infty), (1, k), (\infty, \infty)\) and \((\frac{1}{1 - c}, m)\). Hence as above we see that \( f_2 \) and \( g_2 \) share \((\frac{1}{1 - c}, \infty)\) and so \( f, g \) share \((1, \infty)\). Therefore \( f \) and \( g \) share \((0, \infty), (1, \infty), (\infty, k)\) and \((c, \infty)\).

Finally let \( f_3 = \frac{f - c}{f - c} \) and \( g_3 = \frac{g - c}{g - c} \). Then \( f_3 \) and \( g_3 \) share \((0, \infty), (1, k), (\infty, \infty)\) and \((\frac{c - 1}{c}, k)\). Hence as above we see that \( f_3, g_3 \) share \((\frac{c - 1}{c}, \infty)\) and so \( f, g \) share \((\infty, \infty)\). Therefore \( f \) and \( g \) share \((0, \infty), (1, \infty), (\infty, \infty)\) and \((c, \infty)\). This proves the lemma.

**Note 2.1.** Lemma 2.6 improves 3CM + 1IM = 4CM theorem.
Lemma 2.7. [13] Let \( f_1 \) and \( f_2 \) be two non-constant meromorphic functions satisfying for \( i = 1, 2, \)

\[
\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S(r; f_1, f_2).
\]

If \( f_1 f_2' - 1 \) is not identically zero for all integers \( s \) and \( t \) \(|s| + |t| > 0\) then for any positive \( \varepsilon \) we have

\[
\overline{N}_0(r, 1; f_1, f_2) \leq \varepsilon T(r; f_1, f_2) + S(r; f_1, f_2).
\]

where \( \overline{N}_0(r, 1; f_1, f_2) \) denotes the reduced counting function of \( f_1 \) and \( f_2 \) related to the common 1-points and \( T(r; f_1, f_2) = T(r, f_1) + T(r, f_2), S(r; f_1, f_2) = o[T(r; f_1, f_2)] \) as \( r \to \infty \) possibly outside a set of finite linear measure.

The following lemma is a variant of Lemma 2.7 [1] and Theorem 2.5 [6].

Lemma 2.8. Let \( f \) and \( g \) be two non-constant meromorphic functions. If \( a_1, a_2, a_3 \) are distinct meromorphic functions such that \( T(r, a_j) = S(r; f, g) \) for \( j = 1, 2, 3 \) then

\[
T(r, f) \leq \overline{N}(r, 0; f - a_1) + \overline{N}(r, 0; f - a_2) + \overline{N}(r, 0; f - a_3) + S(r; f, g).
\]

Lemma 2.9. [15, p. 301] Let \( f \) be a non-constant meromorphic function satisfying the Riccati differential equation

\[
f' = a + bf + cf^2,
\]

where \( a, b, c \neq 0 \) are meromorphic functions such that \( T(r, a) + T(r, b) + T(r, c) = S(r, f) \).

Further let \( \rho \) be a meromorphic function with \( T(r, \rho) = S(r, f) \).

(i) If \( \rho \) satisfies the differential equation (2.3) then \( \overline{N}(r, 0; f - \rho) = S(r, f) \).

(ii) If \( \rho \) does not satisfy the differential equation (2.3) then \( \overline{N}(r, 0; f - \rho) = T(r, f) + S(r, f) \).

Lemma 2.10. [11] Let \( f \) and \( g \) be distinct meromorphic functions sharing \( 0, 0 \), \( 1, 0 \) and \( \infty, 0 \). If \( f \) is a bilinear transformation of \( g \) then \( f \) and \( g \) satisfy one of the following: (i) \( fg \equiv 1 \), (ii) \((f - 1)(g - 1) \equiv 1 \), (iii) \( f + g \equiv 1 \), (iv) \( f \equiv cg \), (v) \( f - 1 \equiv c(g - 1) \), (vi) \((c - 1)f + 1 \equiv (c - 1)g - c \) with \( c \neq 0 \) \((0, 1, \infty) \) is a constant.

Lemma 2.11. Let \( f \) and \( g \) be distinct non-constant meromorphic functions sharing \( 0, 0 \), \( 1, 0 \) and \( \infty, 0 \). If \( f \) is a bilinear transformation of \( g \) and \( f - a \), \( g - b \) share \( 0, 0 \), where \( a, b \neq 0, 1, \infty \), then there exists a non-constant entire function \( \lambda \) such that \( f \) and \( g \) are one of the following forms:

(i) \( f = ae^\lambda \) and \( g = be^{-\lambda} \), where \( ab = 1 \).

(ii) \( f = 1 + ae^\lambda \) and \( g = 1 + (1 - \frac{1}{b})e^{-\lambda} \), where \( ab = a + b \).

(iii) \( f = \frac{a}{a+e^\lambda} \) and \( g = \frac{e^\lambda}{1-b+e^\lambda} \), where \( a + b = 1 \).

(iv) \( f = \frac{e^\lambda - a}{e^\lambda - 1} \) and \( g = \frac{be^{\lambda} - 1}{e^{\lambda} - b} \), where \( ab = 1 \).

(v) \( f = \frac{be^{\lambda} - a}{be^{\lambda} - b} \) and \( g = \frac{be^\lambda - a}{a e^\lambda - a} \), where \( a \neq b \).
exists a non-constant entire function $\frac{be^\lambda}{e^\lambda-1}$, where $ab = a + b$. (vi)

$$f = \frac{a}{1-e^\lambda}$$ and $g = \frac{be^\lambda}{e^\lambda-1},$ where $a \neq b$. (vii)

$$f = \frac{b-a}{(b-1)(1-e^\lambda)}$$ and $g = \frac{(b-a)e^\lambda}{(a-1)(1-e^\lambda)},$ where $a \neq b$. (viii)

Further, $f$ and $g$ share $(0, \infty)$, $(1, \infty)$, $(\infty, \infty)$ and $f - a$, $g - b$ share $(0, \infty)$.

Proof. Clearly $f$ and $g$ satisfy one of the relations given in Lemma 2.10.

Suppose that $fg \equiv 1$. Then $f$ and $g$ do not assume the values 0 and $\infty$. Hence there exists a non-constant entire function $\lambda$ such that $f = ae^\lambda$ and $g = \frac{1}{a} e^{-\lambda}$. Since $f - a$ and $g - b$ share $(0, 0)$ we see that $ab = 1$. Therefore $f = ae^\lambda$ and $g = be^{-\lambda}$, where $ab = 1$. This is the possibility (i).

Suppose that $(f - 1)(g - 1) \equiv 1$. Then $f$ and $g$ do not assume the values 1 and $\infty$. Hence there exists a non-constant entire function $\lambda$ such that $f = 1 + ae^\lambda$ and $g = 1 + \frac{1}{a} e^{-\lambda}$. Since $f - a$ and $g - b$ share $(0, 0)$, we see that $ab = a + b$. Therefore $f = 1 + ae^\lambda$ and $g = 1 + (1 - \frac{1}{b}) e^{-\lambda}$, where $ab = a + b$. This is the possibility (ii).

Suppose that $f + g \equiv 1$. Then $f$ and $g$ do not assume the values 0 and 1. So there exists a non-constant entire function $\lambda$ such that $f = \frac{a}{a+e^\lambda}$ and $g = \frac{e^\lambda - a}{ae^\lambda - a}$. Since $f - a$ and $g - b$ share $(0, 0)$, we get $a + b = 1$. Therefore $f = \frac{a}{a+e^\lambda}$ and $g = \frac{e^\lambda - a}{1 - be^{-\lambda}}$, where $a + b = 1$. This is the possibility (iii).

Suppose that $f \equiv cg$. Then $f$ does not assume the values 1 and $c$. Hence there exists a non-constant entire function $\lambda$ such that $f = \frac{e^\lambda - c}{e^\lambda - 1}$ and $g = \frac{e^\lambda - c}{ae^\lambda - a}$. Since $f - a$ and $g - b$ share $(0, 0)$, we get $ab = 1$. Therefore $f = \frac{e^\lambda - a}{e^\lambda - 1}$ and $g = \frac{be^\lambda - 1}{e^\lambda - 1}$, where $ab = 1$. This is the possibility (iv).

Suppose that $f$ assumes the value $a$. Since $f - a$ and $g - b$ share $(0, 0)$, we see that $bc = a$ and so $a \neq b$. Therefore from above we get $f = \frac{be^\lambda - a}{be^\lambda - b}$ and $g = \frac{be^\lambda - a}{ae^\lambda - a}$, where $a \neq b$. This is the possibility (v).

Suppose that $f - 1 \equiv c(g - 1)$. Then $f$ does not assume the values 0 and $1 - c$. So there exists a non-constant entire function $\lambda$ such that $f = \frac{1-c}{1-e^\lambda}$ and $g = \frac{(1-c)e^\lambda}{c(1-e^\lambda)}$. (vi)

Suppose that $f$ does not assume the value $a$. Then $c = 1 - a$ and so $f = \frac{a}{1-e^\lambda}$ and $g = \frac{ae^\lambda}{(1-a)(1-e^\lambda)}$. Since $f - a$ and $g - b$ share $(0, 0)$, $g$ does not assume the value $b$ and so we get $ab = a + b$. Therefore $f = \frac{a}{1-e^\lambda}$ and $g = \frac{be^\lambda}{1-e^\lambda}$, where $ab = a + b$. This is the possibility (vi).

Suppose that $f$ assumes the value $a$. Since $f - a$ and $g - b$ share $(0, 0)$, we get $c(b - 1) = a - 1$ and so $a \neq b$. Therefore $f = \frac{b-a}{(b-1)(1-e^\lambda)}$ and $g = \frac{(b-a)e^\lambda}{(a-1)(1-e^\lambda)}$, where $a \neq b$. This is the possibility (vii).

Suppose that $(c - 1)f + 1) \equiv (c - 1)g - c \equiv 0$. Then $f$ does not assume the values $\infty$ and $\frac{1}{1-c}$. So there exists a non-constant entire function $\lambda$ such that $f = \frac{1}{1-c} + e^\lambda$ and $g = \frac{c}{c-1}(1 + \frac{1}{1-(1-c)e^\lambda})$.

Suppose that $f$ does not assume the value $a$. Then $a(1 - c) = 1$ and so $f = a + e^\lambda$ and $g = (1 - a)(1 + \frac{a}{e^\lambda})$. Since $f - a$ and $g - b$ share $(0, 0)$, $g$ does not assume the value $b$ and so $a + b = 1$. Therefore $f = a + e^\lambda$ and $g = b(1 + \frac{1-b}{e^\lambda})$, where $a + b = 1$. This is the possibility (viii).
Suppose that \( f \) assumes the value \( a \). Since \( f - a \) and \( g - b \) share \((0, 0)\), we get \( ca(b - 1) = b(a - 1) \) and so \( a \neq b \). Therefore \( f = e^\rho - \frac{a(b - 1)}{a-b} \) and \( g = \frac{b(a-1)}{a-b} \{1 - \frac{a(b-1)}{(a-b)e^\rho}\} \), where \( a \neq b \). This is the possibility (ix).

Since \( f \) and \( g \) are one of (i)–(ix), we can easily verify that \( f \) and \( g \) share \((0, \infty), (1, \infty), (\infty, \infty)\) and \( f - a, g - b \) share \((0, \infty)\). This proves the lemma.

**Lemma 2.12.** Let \( f \) be a non-constant meromorphic function satisfying the Riccati differential equation

\[
f' = a + bf + cf^2,
\]

where \( a, b, c (\neq 0) \) are meromorphic functions such that \( T(r, a) + T(r, b) + T(r, c) = S(r, f) \). Further let \( \rho \) be a meromorphic function with \( T(r, \rho) = S(r, f) \). If \( \rho \) does not satisfy the above Riccati differential equation then \( N(r, 0; f - \rho | \geq 2) = S(r, f) \).

**Proof.** Putting \( f = h + \rho \) in the Riccati differential equation we get

\[
h' = \mu + (b + 2c\rho)h + ch^2,
\]

where \( \mu = -\rho' + a + bp + cp^2 \) and \( T(r, \mu) = S(r, f) \).

Since \( \rho \) does not satisfy the differential equation, we get \( \mu \neq 0 \). Let \( z_* \) be a zero of \( h \) with multiplicity \( p \) \((\geq 2)\) which is not a pole of \( b + 2c\rho \) and \( c \). Then from above we see that \( z_* \) is a zero of \( \mu \) with multiplicity \( p - 1 \). Therefore

\[
N(r, 0; f - \rho | \geq 2) = N(r, 0; h | \geq 2) \leq 2T(r, \mu) + 2T(r, b + 2c\rho) + 2T(r, c) = S(r, f).
\]

This proves the lemma.

3. **Proof of Theorem 1.1**

We may suppose that \( \frac{a^\beta}{\alpha\beta + a^\beta} \) is non-constant. For, otherwise \( f \) and \( g \) share \( 0, 1, \infty \) CM and the result follows from Theorem C and Lemma 2.11.

We show that \( f \) is a bilinear transformation of \( g \) so that by Lemma 2.11 the theorem follows.

Let \( a_1 = 0, a_2 = 1 \) and \( a_3 = \infty \).

If \( a = b \) then by Lemma 2.6 and Theorem A \( f \) becomes a bilinear transformation of \( g \).

Suppose that \( a \neq b \). Also we suppose that \( f \) is not a bilinear transformation of \( g \). Suppose that \( \alpha \) and \( \beta \) are defined as in Lemma 2.4. Then \( \alpha, \beta \) and \( \alpha \beta \) are non-constant. Further we see that

\[
f = \frac{1-g}{1-\alpha g} \quad \text{and} \quad g = \frac{1-\alpha\beta}{1-g}. \quad (1-\alpha) \frac{1-\beta}{1-\alpha \beta}
\]

Since \( f - g = \frac{(1-\alpha)(1-\beta)}{1-\alpha \beta} \) and \( g - 1 = \frac{1-\alpha\beta}{1-\alpha} \), we see that \( 1 - \alpha \beta = \frac{g-f}{f(g-1)} \). Hence \( F = (f - a) \frac{g-f}{f(g-1)} \).

Since by Lemma 2.4 \( \bar{N}(r, \infty; F) = S(r) \) and \( w \) has only simple poles, we get

\[
T(r, w) = m(r, w) + N(r, w) = \bar{N}(r, 0; F) + S(r).
\]

By Lemma 2.3(x), (xii), Lemmas 2.2 and 2.5 we get

\[
\bar{N}(r, 0; F | \geq 2) \leq N(r, a; f | \geq 2) + N(r, 0; f - g | \geq 2) + \bar{N}(r, \infty; f | \geq 2) + N(r, 0; f - g | f = \infty) = S(r).
\]
Hence from (3.1) and (3.2) we get in view of Lemma 2.3(x)
\[
T(r, w) = N(r, 0; F \mid \leq 1) + S(r) \\
= N(r, a; f \mid \leq 1) + N_0(r) + N_2(r) + S(r),
\]
(3.3)
where \( N_2(r) \) is the counting function of those simple poles of \( f \) which are nonzero regular points of \( f - g \).

By the second fundamental theorem, Lemma 2.2, Lemma 2.3(i), (iv), (vii) and Lemma 2.5 we obtain
\[
2T(r, f) \leq N(r, a; f \mid \leq 1) + N(r, 0; f \mid \leq 1) + N(r, 1; f \mid \leq 1) + N(r, \infty; f \mid \leq 1) \\
- N_0(r, 0; f' \mid \leq 1) + S(r, f) \\
= N(r, a; f \mid \leq 1) + T(r, f) + T(r, g) - N_0(r, 0; f' \mid \leq 1) + S(r) \\
= N(r, a; f \mid \leq 1) + T(r, f) + N(r, 0; f' \mid \leq 1) - N_0(r, 0; f' \mid \leq 1) + S(r),
\]
where \( N_0(r, 0; f' \mid \leq 1) \) denotes the counting function of those simple zeros of \( f' \) which are not the zeros of \( f(f - 1) \).

Now by Lemma 2.2 we get
\[
N(r, 0; f' \mid \leq 1) \leq N_0(r, 0; f' \mid \leq 1) + \overline{N}(r, 0; f \mid \geq 2) + \overline{N}(r, 1; f \mid \geq 2) \\
= N_0(r, 0; f' \mid \leq 1) + S(r).
\]
Hence from above we get
\[
T(r, f) = N(r, a; f \mid \leq 1) + S(r).
\]
(3.4)
Similarly
\[
T(r, g) = N(r, b; g \mid \leq 1) + S(r).
\]
(3.5)

From (3.3) and (3.4) we obtain
\[
T(r, w) = T(r, f) + N_0(r) + N_2(r) + S(r).
\]
(3.6)

Also since \( f - a \) and \( g - b \) share \((0, 0)\), we see that \( \overline{N}(r, a; f) = \overline{N}(r, b; g) \). Hence we get from (3.4) and (3.5) in view of Lemma 2.5
\[
T(r, f) = T(r, g) + S(r).
\]
(3.7)

Let
\[
\tau_1 = \frac{a - 1}{b - 1}(\gamma - b\delta),
\]
\[
\tau_2 = \frac{a - 1}{2(b - 1)}\{\gamma' + \gamma^2 - b(\delta' + \delta^2)\}
\]
and
\[
\tau_3 = \frac{a - 1}{6(b - 1)}\{\gamma'' + 3\gamma\gamma' + \gamma^3 - b(\delta'' + 3\delta\delta' + \delta^3)\},
\]
where \( \gamma = \frac{\alpha'}{\alpha} \) and \( \delta = \frac{(a\beta')}{a\beta} = \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} \). Using Lemma 2.4 we can verify that \( T(r, \gamma) = S(r) \) and \( T(r, \delta) = S(r) \).
Let $\tau_1 \equiv 0$. Then $b = \frac{a' \beta}{\alpha \beta + a' \alpha}$. Since $f - g = \frac{(1 - \alpha)(1 - \beta)}{1 - \alpha \beta}$, $f - 1 = \frac{\alpha(\beta - 1)}{1 - \alpha \beta}$ and

$$
\frac{f'}{f} = -\frac{\alpha' (1 - \alpha \beta) + (1 - \alpha)(\alpha' \beta + \alpha \beta')}{(1 - \alpha \beta)(1 - \alpha)}.
$$

we get

$$
\frac{f'(f - g)}{f(f - 1)} = \frac{\alpha'(1 - \alpha \beta) + (\alpha - 1)(\alpha' \beta + \alpha \beta')}{\alpha(1 - \alpha \beta)}.
$$

Also $g - b = \beta \frac{(1 - \alpha)(\alpha \beta + a' \alpha) - \alpha'(1 - \beta)(\alpha \beta + a' \alpha)}{(1 - \alpha \beta)(\alpha \beta + a' \alpha)}$ and so $(g - b) \delta = \frac{f'(g - f)}{f'(f - 1)}$. This shows that a zero of $g - b$ which is neither a zero nor a pole of $\delta$ is a zero of $f'$. Since $f - a$ and $g - b$ share $(0, 0)$, it follows that $N(r, a; f | \leq 1) = S(r)$, which is impossible by (3.4). Therefore $\tau_1 \neq 0$.

Let $z_0$ be a simple zero of $f - a$ and $\tau_1(z_0) \neq 0$. Then $g(z_0) = b$ and so $a(z_0) = \frac{a - 1}{b - 1}$ and $\beta(z_0) = \frac{b}{a}$. Expanding $F$ around $z_0$ in Taylor's series we get

$$
-F(z) = \tau_1(z_0)(z - z_0) + \tau_2(z_0)(z - z_0)^2 + \tau_3(z_0)(z - z_0)^3 + O((z - z_0)^4).
$$

(3.8)

Hence in some neighbourhood of $z_0$ we get

$$
w(z) = \frac{1}{z - z_0} + \frac{B(z_0)}{2} + C(z_0)(z - z_0) + O((z - z_0)^2),
$$

(3.9)

where $B = \frac{2\tau_2}{\tau_1}$ and $C = \frac{2\tau_3}{\tau_1} - (\frac{\tau_2}{\tau_1})^2$.

We define

$$
H = w' + w^2 - Bw - A,
$$

(3.10)

where $A = 3C - \frac{B^2}{4} - B'$. Clearly $T(r, A) + T(r, B) + T(r, C) = S(r)$ and since $w = \frac{F'}{T}$ and $F = (f - a) \frac{g - f}{f(g - 1)}$, by Lemma 2.1 and (3.6) we get $S(r, w) = S(r)$.

It is now easy to verify that $z_0$ is a zero of $H$. Let $H \neq 0$. Then

$$
N(r, a; f | \leq 1) \leq N(r, 0; H) \leq T(r, H) + O(1) = N(r, H) + S(r).
$$

(3.11)

From (3.4) and (3.11) we obtain

$$
T(r, f) \leq N(r, H) + S(r).
$$

(3.12)

Let $z_1$ be a pole of $F$. Then $z_1$ is a simple pole of $w$. So if $z_1$ is not a pole of $A$ and $B$ then $z_1$ is a pole of $H$ with multiplicity at most two. Hence by Lemma 2.4 we get

$$
N(r, \infty; H | F = \infty) \leq 2 \bar{N}(r, \infty; F) + S(r) = S(r).
$$

(3.13)

Let $z_2$ be a multiple zero of $F$. Then $z_2$ is a simple pole of $w$. So if $z_2$ is not a pole of $A$ and $B$ then $z_2$ is a pole of $H$ of multiplicity at most two. Hence by (3.2) we get

$$
N(r, \infty; H | F = 0, r \geq 2) \leq 2 \bar{N}(r, 0; F | \geq 2) + S(r) = S(r).
$$

(3.14)

Let $z_3$ be a simple zero of $F$ which is not a pole of $A$ and $B$. Then in some neighbourhood of $z_3$ we get $F(z) = (z - z_3)\phi(z)$, where $\phi$ is analytic at $z_3$ and $\phi(z_3) \neq 0$. Therefore in some neighbourhood of $z_3$ we obtain

$$
H(z) = \left( 2\frac{\phi'}{\phi} - B \right) \frac{1}{z - z_3} + \psi,
$$

where $\psi$ is analytic at $z_3$. Hence

$$
N(r, \infty; H | \psi = \infty) \leq 2 \bar{N}(r, \infty; \psi) + S(r) = S(r).
$$

(3.15)
where \( \psi = (\frac{\phi'}{\phi})' + (\frac{\phi'}{\phi})^2 - \frac{B\phi'}{\phi} - A \). This shows that \( z_3 \) is a pole of \( H \) with multiplicity at most one.

Since a simple zero of \( f - a \) is a zero of \( H, F = (f - a) \frac{g - f}{f(g - 1)}, \) and \( N(r, 0; F | f = t) \leq N(r, 0; f - g | \geq 2) \) for \( t = 0, 1 \), we get from (3.13) and (3.14) in view of Lemmas 2.3(x) and 2.5

\[
N(r, H) = N(r, \infty; H | F = \infty) + N(r, \infty; H | F = 0) + S(r).
\]

(3.15)

From (3.12) and (3.15) we obtain

\[
T(r, f) \leq N_0(r) + N_2(r) + S(r).
\]

(3.16)

From the definitions of \( \alpha \) and \( \beta \) we see by Lemma 2.1 that

\[
T(r, \alpha) \leq 4T(r, f) + S(r)
\]

(3.17)

and

\[
T(r, \beta) \leq 4T(r, f) + S(r).
\]

(3.18)

Also since \( f = \frac{1-\alpha}{1-\alpha\beta} \) and \( g = \frac{(1-\alpha)\beta}{1-\alpha\beta} \), we get

\[
T(r, f) \leq 2T(r, \alpha) + 2T(r, \beta) + O(1)
\]

(3.19)

and

\[
T(r, g) \leq 2T(r, \alpha) + 2T(r, \beta) + O(1).
\]

(3.20)

From (3.17)–(3.20) we see that \( S(r, f) = S(r, g) = S(r) = S(r; \alpha, \beta) \). Since \( N_0(r) + N_2(r) \leq \overline{N}_0(r, 1; \alpha, \beta) \), we obtain from (3.16)–(3.18)

\[
T(r; \alpha, \beta) \leq 8\overline{N}_0(1; \alpha, \beta) + S(r; \alpha, \beta).
\]

So by Lemma 2.7 there exist two integers \( s \) and \( t \) (\(|s| + |t| > 0\)) such that \( \alpha^s \beta^t \equiv 1 \) and so

\[
(f - 1)^s f^t \equiv (g - 1)^s g^t.
\]

(3.21)

Since \( f \) is not a bilinear transformation of \( g \), it follows that \( s \neq 0, t \neq 0 \) and \( s \pm t \neq 0 \). Hence from (3.21) we see that \( f, g \) share \((0, \infty), (1, \infty)\) and \((\infty, \infty)\). This contradicts Theorem C.

Now we suppose that \( H \equiv 0 \). Then \( w \) satisfies the Riccati differential equation

\[
w' = A + Bw - w^2.
\]

(3.22)

From the definitions of \( F \) and \( w \) we can easily deduce the following

\[
F(w - \delta) = (\delta - \gamma)(\alpha - \phi_1),
\]

(3.23)

\[
F(w - \gamma) = a(\delta - \gamma)(\alpha\beta - \phi_2)
\]

(3.24)

and

\[
Fw = a\delta\alpha(\beta - \phi_3),
\]

(3.25)

where \( \phi_1 = \frac{(1-a)\delta}{\delta - \gamma}, \phi_2 = \frac{(a-1)\gamma}{a(\delta - \gamma)} \) and \( \phi_3 = \frac{\gamma}{a\delta} \). Since \( \alpha, \beta \) and \( \alpha\beta \) are non-constant, we see that \( \phi_i \neq 0, \infty \) for \( i = 1, 2, 3 \).
Since $T(r, \phi_1) = S(r) = S(r; \alpha, \beta)$, we get by Lemmas 2.4 and 2.8

$$T(r, \alpha) \leq \bar{N}(r, 0; \alpha) + \bar{N}(r, \infty; \alpha) + \bar{N}(r, 0; \alpha - \phi_1) + S(r; \alpha, \beta)$$

$$= \bar{N}(r, 0; \alpha - \phi_1) + S(r)$$

$$\leq N(r, 0; \alpha - \phi_1) + S(r)$$

and so

$$T(r, \alpha) = \bar{N}(r, 0; \alpha - \phi_1) + S(r) = N(r, 0; \alpha - \phi_1) + S(r). \quad (3.26)$$

From (3.23) we get in view of (3.2)

$$\bar{N}(r, 0; w - \delta) \leq \bar{N}(r, 0; \alpha - \phi_1) + \bar{N}(r, 0; \delta - \gamma) + S(r)$$

$$= \bar{N}(r, 0; \alpha - \phi_1) + S(r)$$

$$= \bar{N}(r, 0; F(w - \delta)) + S(r)$$

$$\leq \bar{N}(r, 0; w - \delta) + \bar{N}(r, 0; F | \geq 2) + S(r)$$

$$= \bar{N}(r, 0; w - \delta) + S(r)$$

and so from (3.26) we get

$$T(r, \alpha) = \bar{N}(r, 0; w - \delta) + S(r). \quad (3.27)$$

Also by Lemma 2.4 and the second fundamental theorem we get

$$T(r, \alpha) = \bar{N}(r, 1; \alpha) + S(r). \quad (3.28)$$

Since $\alpha - 1 = \frac{f - g}{g - 1}$ and by Lemma 3(x)

$$\bar{N}(r, 0; \frac{f - g}{g - 1} | g = 1) \leq N(r, 0; f - g | \geq 2) = S(r),$$

we get by Lemmas 2.2 and 2.3(v), (xi)

$$\bar{N}(r, 1; \alpha) = N_0(r) + N(r, 0; f | \leq 1) + N_2(r) + S(r). \quad (3.29)$$

Therefore we obtain from (3.27)–(3.29)

$$\bar{N}(r, 0; w - \delta) = N(r, 0; f | \leq 1) + N_0(r) + N_2(r) + S(r). \quad (3.30)$$

In a similar manner using (3.24) and (3.25) we get

$$\bar{N}(r, 0; w - \gamma) = \bar{N}(r, 0; f | \leq 1) + N_0(r) + N_2(r) + S(r), \quad (3.31)$$

$$\bar{N}(r, 0; w) = N(r, 1; f | \leq 1) + N_0(r) + N_2(r) + S(r), \quad (3.32)$$

$$\bar{N}(r, 1; \alpha \beta) = N(r, \infty; f | \leq 1) + N_0(r) + N_2(r) + S(r), \quad (3.33)$$

$$\bar{N}(r, 1; \beta) = N(r, 1; f | \leq 1) + N_0(r) + N_2(r) + S(r), \quad (3.34)$$

$$T(r, \alpha \beta) = \bar{N}(r, 1; \alpha \beta) + S(r). \quad (3.35)$$

and

$$T(r, \beta) = \bar{N}(r, 1; \beta) + S(r). \quad (3.36)$$
Let \( w = 0 \) be a solution of (3.22). Then by Lemma 2.9, (3.32), (3.34) and (3.36) we get 
\[
T(r, \beta) = S(r).
\]
Since \( f - a \) and \( g - b \) share \((0, 0)\) we obtain 
\[
N(r, a; f | \leq 1) \leq N\left(r, \frac{b}{a}; \beta\right) \leq T(r, \beta) = S(r),
\]
which by (3.4) implies a contradiction.

Let \( w = \gamma \) be a solution of (3.22). Then by Lemma 2.9, (3.31), (3.33) and (3.35) we get 
\[
T(r, \alpha \beta) = S(r).
\]
Since \( f - a \) and \( g - b \) share \((0, 0)\) we obtain 
\[
N(r, a; f | \leq 1) \leq N\left(r, \frac{a - 1}{b - 1}; \alpha \beta\right) \leq T(r, \alpha \beta) = S(r),
\]
which by (3.4) implies a contradiction.

Let \( w = \delta \) be a solution of (3.22). Then by Lemma 2.9, (3.28)–(3.30) we get 
\[
T(r, \alpha) = S(r).
\]
Since \( f - a \) and \( g - b \) share \((0, 0)\), we obtain 
\[
N(r, a; f | \leq 1) \leq N\left(r, \frac{a - 1}{b - 1}; \alpha\right) \leq T(r, \alpha) = S(r),
\]
which by (3.4) implies a contradiction.

Therefore \( w = 0, w = \gamma \) and \( w = \delta \) are not solutions of (3.22). Now by Lemma 2.12 we obtain 
\[
\begin{align*}
N(r, 0; w - \gamma) &= \overline{N}(r, 0; w - \gamma) + S(r), \quad (3.37) \\
N(r, 0; w - \gamma) &= \overline{N}(r, 0; w - \gamma) + S(r) \quad (3.38)
\end{align*}
\]
and 
\[
N(r, 0; w) = \overline{N}(r, 0; w) + S(r). \quad (3.39)
\]

Since \( 0, \gamma \) and \( \delta \) do not satisfy (3.22), by Lemma 2.9, (3.6), (3.30)–(3.32) and (3.37)–(3.39) we obtain 
\[
\begin{align*}
T(r, f) &= N(r, 0; f | \leq 1) + S(r), \quad (3.40) \\
T(r, f) &= N(r, \infty; f | \leq 1) + S(r) \quad (3.41)
\end{align*}
\]
and 
\[
T(r, f) = N(r, 1; f | \leq 1) + S(r). \quad (3.42)
\]

Now by Lemma 2.3(i), (3.7) and (3.40)–(3.42) we get 
\[
3T(r, f) = N(r, 0; f | \leq 1) + N(r, \infty; f | \leq 1) + N(r, 1; f | \leq 1) + S(r)
\]
\[
= T(r, f) + T(r, g) - N_0(r) + S(r)
\]
\[
= 2T(r, f) - N_0(r) + S(r)
\]
and so \( T(r, f) + N_0(r) = S(r) \), which is a contradiction. Therefore \( f \) is a bilinear transformation of \( g \).

Let \( a_1 = 1, a_2 = 0 \) and \( a_3 = \infty \). We define \( f_1 = 1 - f \) and \( g_1 = 1 - g \). Then \( f_1, g_1 \) share \((0, 1), (1, m), (\infty, k)\) and \( f_1 - 1 + a, g - 1 + b \) share \((0, 0)\). Hence \( f_1 \) is a bilinear transformation of \( g_1 \) and so \( f \) is a bilinear transformation of \( g \).
Let $a_1 = \infty$, $a_2 = 1$ and $a_3 = 0$. We define $f_2 = \frac{1}{f}$ and $g_2 = \frac{1}{g}$. Then $f_2$, $g_2$ share $(0, 1)$, $(1, m)$, $(\infty, k)$ and $f_2 - \frac{1}{a}$, $g_2 - \frac{1}{b}$ share $(0, 0)$. Hence $f_2$ is a bilinear transformation of $g_2$ and so $f$ is a bilinear transformation of $g$.

Since $m$ and $k$ are interchangeable, we need not consider the other permutations of $a_1, a_2, a_3$. This proves the theorem.

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**References**