A Hardy–Littlewood theorem for multiple series

Mikhail Dyachenko a, Sergey Tikhonov b,*

a Moscow State University, Vorob’evy Gory 117234, Russia
b Scuola Normale Superiore di Pisa, Piazza dei Cavalieri 7, 56126, Italy

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Abstract
In this paper we obtain a multi-dimensional analogue of the Hardy–Littlewood theorem on Fourier coefficients.

Keywords: Hardy–Littlewood theorem; Multiple trigonometric series; General monotone coefficients

1. Introduction

In this paper we consider double trigonometric series

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \cos mx \cos ny, \]  
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \sin mx \sin ny, \]  
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} e^{imx} e^{iny}. \]

We will suppose that

\[ a_{m,n} \rightarrow 0 \quad \text{as} \quad m + n \rightarrow \infty. \] (1.4)

As usual \( f \in L^p[0, 2\pi]^2 \), where \( 1 < p < \infty \), if

\[ \| f(x, y) \|_p = \| f(x, y) \|_{L^p[0, 2\pi]^2} = \left( \int_0^{2\pi} \int_0^{2\pi} |f(x, y)|^p \, dx \, dy \right)^{\frac{1}{p}} < \infty. \]
We denote $\Delta_{11} a_{m,n} := \Delta_{10}(\Delta_{01} a_{m,n})$, where $\Delta_{10} a_{m,n} = a_{m,n} - a_{m+1,n}$ and $\Delta_{01} a_{m,n} = a_{m,n} - a_{m,n+1}$.

Now we recall (see [3,8]) that if the sequence $\{a_{m,n}\}$ is monotone in the sense of Hardy, i.e., if $\Delta_{11} a_{m,n} \geq 0$, or more general, if

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\Delta_{11} a_{m,n}| < \infty,
$$

then series (1.1)–(1.3) converge everywhere on $(0, 2\pi)^2$, in Pringsheim’s sense, to $f_1, f_2, f_3$, respectively, i.e.,

$$
S_{MN}^i(x, y) \rightarrow f_i(x, y) \quad \text{as} \quad \min(M, N) \rightarrow \infty,
$$

where $S_{MN}^i(x, y)$ are the rectangular partial sums of order $M$ and $N$ in $x$ and $y$ respectively of series (1.1)–(1.3).

We are going to discuss the problem of the extension of one-dimensional Hardy–Littlewood theorem (see [6,13]) to higher dimensions. First, we recall the result for multiple series with monotone coefficients in the sense of Hardy.

**Theorem 1.** (See [8].) Let $1 < p < \infty$, and let $a = \{a_{m,n}\}_{m,n \in \mathbb{N}}$ satisfy $\Delta_{11} a_{m,n} \geq 0$, then ($i = 1, 2, 3$)

$$
f_i(x, y) \in \mathbb{L}_p \iff I := \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n}^p mn (mn)^{p-2} \right)^{\frac{1}{p}} < \infty.
$$

We note that from the one-dimensional prototype of (1.6) we immediately have

$$
C_1(p)n^{\frac{p-1}{p}} \leq \|D_n(x)\|_p \leq C_2(p)n^{\frac{p-1}{p}}, \quad 1 < p < \infty,
$$

where $D_n(x)$ is the Dirichlet kernel, i.e., $D_n(x) = \sum_{|k| \leq n} e^{ikx}$. In the multi-dimensional situation the estimation of the $L_p$-norm of $D_B(x)$ depends on the geometry of the set $B$ (see [1,2,4,7]).

Since the condition $\Delta_{11} a_{m,n} \geq 0$ is rather restrictive and Theorem 1 has many applications, it is important to extend this result for wider classes of coefficients. It is interesting that (1.6) does not hold anymore if we substitute the condition $\Delta_{11} a_{m,n} \geq 0$ for a more general condition

$$
a_{m_1,n_1} \leq a_{m,n} \quad \text{for} \quad m_1 \geq m \quad \text{and} \quad n_1 \geq n.
$$

**Theorem 2.** (See [3].) Let $1 < p < \infty$, and let $a = \{a_{m,n}\}_{m,n \in \mathbb{N}}$ satisfy (1.8).

(a) If $\frac{4}{3} < p$, then $f_i(x, y) \in \mathbb{L}_p \iff I < \infty$.

(b) If $p \leq \frac{4}{3}$, then the previous criterion does not hold.

For $d$-dimensional case ($d \geq 2$) the critical value for $p$ is $\frac{2d}{d+1}$.

In this paper we study conditions on $\{a_{m,n}\}$ under which (1.6) holds for all $p \in (1, \infty)$. In Section 2 we prove the upper estimate $\|f\|_p \leq CI$. It turns out that the Littlewood–Paley technique allows us to avoid direct calculations and makes the proof much easier. Furthermore, we obtain the lower estimate $\|f\|_p \geq CI$. We mention here the paper [9] where the lower estimate was also investigated. The concept of general monotone sequence ($GM$-sequence) that we introduce in this paper allows us to write both the upper and lower estimates of $\|f\|_p$.

2. The upper estimate

Let $\beta = \{\beta_{m,n}\}_{m,n \in \mathbb{N}}$ be a non-negative sequence. We say that a sequence $a = \{a_{m,n}\}_{m,n \in \mathbb{N}}$ satisfies the $GM(\beta)$-condition if condition (1.4) holds and
\[
\sum_{m=1}^{\infty} \sum_{n=d}^{\infty} |\Delta^{11} a_{m,n}| \leq C^* l_d.
\]

(2.1)

Naturally, \(a \in GM(\beta)\) implies (1.5) and therefore series (1.1)–(1.3) converge in the Pringsheim sense to \(f_i, i = 1, 2, 3\).

**Theorem 3.** Let \(1 < p < \infty\), and let \(a = \{a_{m,n}\}_{m,n \in \mathbb{N}} \in GM(\beta)\), where the non-negative sequence \(\beta = \{\beta_{m,n}\}_{m,n \in \mathbb{N}}\) satisfies

\[
I(\beta) := \left( \sum_{l=1}^{\infty} \sum_{d=1}^{\infty} \beta_{l,d}^p (\log l)^{p-2} \right)^{\frac{1}{p}} < \infty.
\]

(2.2)

Then \((i = 1, 2, 3)\)

\[
\|f_i(x, y)\|_p \leq C(C^*, p) I(\beta).
\]

(2.3)

**Remark.** We note that under the conditions of Theorem 3, series (1.1)–(1.3) are the Fourier series of the corresponding functions.

**Proof.** By the Paley theorem, we have for \(2 \leq p < \infty\)

\[
\|f_1(x, y)\|_p \leq C(p) I(|a|) \leq C(C^*, p) I(\beta).
\]

Therefore, it is sufficient to prove (2.3) only for \(1 < p < 2\). Using the Littlewood–Paley theorem (see [13]) and the Jensen inequality, we write, for series (1.1) for example,

\[
\|f_1(x, y)\|_p^p \leq C(p) \left( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \|\Delta_{j,k}\|_p^p \right) \leq C(p) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \|\Delta_{j,k}\|_p^p,
\]

(2.4)

where

\[
\Delta_{j,k} = \Delta_{j,k}(x, y) = \sum_{m=2^j}^{2^{j+1}-2} \sum_{n=2^k}^{2^{k+1}-2} a_{m,n} \cos mx \cos ny.
\]

Applying the Abel transformation, we have

\[
\Delta_{j,k} = \sum_{m=2^j}^{2^{j+1}-2} \sum_{n=2^k}^{2^{k+1}-2} \Delta^{11} a_{m,n} \delta_{m,n} + \sum_{m=2^j}^{2^{j+1}-2} \Delta^{10} a_{m,2^{k+1}-1} \delta_{m,2^{k+1}-1} + \sum_{n=2^k}^{2^{k+1}-2} \Delta^{01} a_{2^{j+1}-1,n} \delta_{2^{j+1}-1,n}
\]

where \(\delta_{m,n} := \sum_{s=2^j}^{2^{j+1}-2} \sum_{t=2^k}^{2^{k+1}-2} \cos sz \cos ty\). Therefore, writing the Dirichlet kernel as \(D_n(z) = 1 + 2 \sum_{l=1}^{\infty} \cos lz\), we have (see (1.7))

\[
\|\Delta_{j,k}\|_p^p \leq C(p) \max_{(m,n) \in [2^{-1},2^{-1}] \times [2^{k-1},2^{k+1}]} \left( \|D_m(x) D_n(y)\|_p^p \left( \sum_{m=2^j}^{2^{j+1}-2} \sum_{n=2^k}^{2^{k+1}-2} |\Delta^{11} a_{m,n}| \right) \right) \leq C(p) 2^{(j+k)(p-1)} \left( \sum_{m=2^j}^{2^{j+1}-2} \sum_{n=2^k}^{2^{k+1}-2} |\Delta^{11} a_{m,n}| \right) \left( \sum_{m=2^j}^{2^{j+1}-2} \sum_{n=2^k}^{2^{k+1}-2} |\Delta^{11} a_{m,n}| \right)^{\frac{1}{p}}.
\]

Finally, by (2.2) and (2.4), we obtain

\[
\|f_1(x, y)\|_p \leq C(C^*, p) \left( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (jk)^{p-2} \left( \sum_{m=2^j}^{2^{j+1}-2} \sum_{n=2^k}^{2^{k+1}-2} |\Delta^{11} a_{m,n}| \right) \right)^{\frac{1}{p}} \leq C(C^*, p) I(\beta).
\]

In a similar way we get the same inequalities for \(f_2 \) and \(f_3\). This completes the proof of Theorem 1. □

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1 By \(C(s, t, \ldots)\) we denote the positive constants that are dependent only on \(s, t, \ldots\) and may be different in different formulas.
Example. We will say that the sequence \( a = \{a_{m,n}\}_{m,n \in \mathbb{N}} \) satisfies the \( GM^* \)-condition if \( a \in GM(\beta) \) with \( \beta_{l,d} := |a_{l,d}| \), i.e.,

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\Delta^{11} a_{m,n}| \leq C^* |a_{l,d}|.
\]

Clearly, \( \{a: \Delta^{11} a_{m,n} \geq 0\} \subseteq GM^* \).

Furthermore, one can write the following generalization of \( GM^* \).

Definition. We say that the sequence \( a = \{a_{m,n}\}_{m,n \in \mathbb{N}} \) satisfies the \( GM^2 \)-condition \(^2\) if condition (1.4) holds and condition (2.1) holds with

\[
\beta = \left\{ \beta_{l,d} := |a_{l,d}| + \sum_{m=l+1}^{\infty} \frac{|a_{m,d}|}{m} + \sum_{n=d+1}^{\infty} \frac{|a_{l,n}|}{n} + \sum_{m=l+1}^{\infty} \sum_{n=d+1}^{\infty} \frac{|a_{m,n}|}{mn} \right\}.
\]

This definition is a multivariate version of the one-dimensional general monotone sequences defined in [10].

We have \( GM^* \subseteq GM^2 \). We also remark that for the class \( GM^2 \) the upper estimate in the Hardy–Littlewood theorem still holds.

**Theorem 4.** Let \( 1 < p < \infty \), and let a non-negative sequence \( a = \{a_{m,n}\}_{m,n \in \mathbb{N}} \in GM^2 \). Then \( (i = 1, 2, 3) \)

\[
\| f_i(x, y) \|_p \leq C(C^*, p) \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}|^p (mn)^{p-2} \right)^{\frac{1}{p}} =: C(C^*, p) I.
\]

Indeed, we only have to check the accuracy of \( I(\beta) \leq C(p) I(|a|) \equiv C(p) I \). It follows immediately from the following Hardy type inequality (see for example [13, Chapter I])

\[
\sum_{n=1}^{\infty} n^\tau \left( \sum_{k=n}^{\infty} \frac{|a_k|}{k} \right)^p \leq C(p, \tau) \sum_{k=1}^{\infty} k^\tau |a_k|^p \quad \text{for all } \tau > -1, \; p > 1.
\]

Actually, in this case\(^3\), \( I(\beta) \approx I(|a|) \) because we obviously have \( |a_{l,d}| \leq \beta_{l,d} \).

In the next section we will show that the \( GM^2 \)-condition guarantees that the inverse part in the Hardy–Littlewood criterion is also true.

3. The lower estimate

**Theorem 5.** Let \( 1 < p < \infty \), and let a non-negative sequence \( a = \{a_{m,n}\}_{m,n \in \mathbb{N}} \in GM^2 \). Then \( (i = 1, 2, 3) \)

\[
\| f_i(x, y) \|_p \geq C(C^*, p) \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}|^p (mn)^{p-2} \right)^{\frac{1}{p}} =: C(C^*, p) I.
\]

**Proof.** First, we note that for all \( s \leq l \) and \( t \leq d \) we have

\[
a_{l,d} = \left| \sum_{m=s+1}^{\infty} \sum_{n=t+1}^{\infty} \Delta^{11} a_{m,n} \right| \leq \sum_{m=s}^{\infty} \sum_{n=t}^{\infty} |\Delta^{11} a_{m,n}|
\]

\[
\leq C^* \left( a_{s,t} + \sum_{m=s+1}^{\infty} \frac{a_{m,t}}{m} + \sum_{n=t+1}^{\infty} \frac{a_{s,n}}{n} + \sum_{m=s+1}^{\infty} \sum_{n=t+1}^{\infty} \frac{a_{m,n}}{mn} \right)
\]

\(^2\) Here 2 indicates the dimension.

\(^3\) As usual, \( F \asymp G \) means that \( F \leq CG \) and \( G \leq CF \).
Therefore,
\[
\frac{1}{ld} \sum_{s=1}^{l} \sum_{t=1}^{d} a_{s,t} \geq \frac{1}{C^*} a_{l,d} - \frac{1}{ld} \sum_{s=1}^{l} \sum_{t=1}^{d} (J_1 + J_2 + J_3).
\]  
(3.1)
We divide the rest of the proof into three steps.

**Step 1.** Using the Hardy inequality [13, Chapter I]
\[
\sum_{n=1}^{\infty} n^\tau \left( \frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^p \leq C(p, \tau) \sum_{k=1}^{\infty} k^\tau |a_k|^p \quad \text{for all } \tau < p - 1, \ p > 1,
\]  
(3.2)
we get
\[
\sum_{l,d=1}^{\infty} (ld)^{p-2} \left( \frac{1}{ld} \sum_{s=1}^{l} \sum_{t=1}^{d} J_1 \right)^p \leq C(p) \sum_{s,d=1}^{\infty} (sd)^{p-2} \left( \frac{1}{d} \sum_{t=1}^{d} J_1 \right)^p,
\]  
(3.3)
\[
\sum_{l,d=1}^{\infty} (ld)^{p-2} \left( \frac{1}{ld} \sum_{s=1}^{l} \sum_{t=1}^{d} J_2 \right)^p \leq C(p) \sum_{l,t=1}^{\infty} (lt)^{p-2} \left( \frac{1}{l} \sum_{s=1}^{l} J_2 \right)^p,
\]  
(3.4)
\[
\sum_{l,d=1}^{\infty} (ld)^{p-2} \left( \frac{1}{ld} \sum_{s=1}^{l} \sum_{t=1}^{d} J_3 \right)^p \leq C(p) \sum_{s,t=1}^{\infty} (st)^{p-2} J_3^p.
\]  
(3.5)

**Step 2.** We use
\[
\sum_{k=n+1}^{\infty} \frac{a_k}{k} = \sum_{k=n+1}^{\infty} \frac{1}{k} \left( \frac{1}{k} \sum_{l=1}^{k} a_l \right) - \frac{1}{n} \sum_{l=1}^{n} a_l.
\]  
(3.6)
Then for \( b_{k}(a_{s,t}) := \frac{1}{k} \sum_{l=1}^{k} a_{l,t} \) we have \( J_1 = \sum_{k=s}^{\infty} \frac{b_k(a_{s,t})}{k+1} - b_k(a_{s,t}) \). Hence,
\[
\sum_{s,d=1}^{\infty} (sd)^{p-2} \left( \frac{1}{sd} \sum_{k=1}^{s} a_{k,t} \right)^p \leq C(p) \sum_{s,d=1}^{\infty} (sd)^{p-2} \left( \frac{1}{sd} \sum_{k=1}^{s} \sum_{t=1}^{d} a_{k,t} \right)^p + C(p) \sum_{s,d=1}^{\infty} (sd)^{p-2} \left( \frac{1}{sd} \sum_{k=1}^{s} \sum_{t=1}^{d} a_{k,t} \right)^p \equiv C(p) J,
\]  
(3.7)
where we use inequality (2.5). In a similar way we obtain the following inequalities
\[
\sum_{l,t=1}^{\infty} (lt)^{p-2} \left( \frac{1}{l} \sum_{s=1}^{l} J_2 \right)^p + \sum_{s,t=1}^{\infty} (st)^{p-2} J_3^p \leq C(p) J.
\]  
(3.8)

**Step 3.** Thus, combining (3.3)–(3.5) with (3.7)–(3.8), we obtain
\[
\sum_{l,d=1}^{\infty} (ld)^{p-2} \left( \frac{1}{ld} \sum_{s=1}^{l} \sum_{t=1}^{d} (J_1 + J_2 + J_3) \right)^p \leq C(p) J.
\]
Now, we apply the multi-dimensional analogue of Hardy’s theorem on Fourier coefficients proved by Nursultanov in [9, Theorem 5(b)]:
\[
J \leq C(p) \| f \|_p^p, \quad p > 2.
\]
Finally, this estimate and inequality (3.1) imply \( I \leq C(C^*, p) \| f \|_p^p \) for \( p > 2 \). The same inequality for \( 1 < p \leq 2 \) follows from the Paley theorem. This completes the proof of Theorem 5. □
4. Remarks

1. Using the Littlewood–Paley technique and the method of the proof of Theorem 5, one can obtain the Hardy–Littlewood type theorem for multiple series with respect to the regular systems.

Definition. An orthonormal system \( \{ \varphi_k(x) \}_{k \in \mathbb{N}} \) is called regular, if there exists a constant \( C \) such that
\[
\left| \int_V \varphi_k(x) \, dx \right| \leq C \min(\{|V|, 1/k\}) \quad \text{for any closed interval } V \text{ from } [0, 1]
\]
and
\[
\left( \sum_{k \in W} \varphi_k(\cdot)^* \right)^* (t) \leq C \min(\{|W|, 1/t\}) \quad \text{for any closed interval } W \subset \mathbb{N} \text{ and } t \in (0, 1).
\]
Here \( f^* \) denotes the decreasing rearrangement of \( f \).

In particular, we obtain the Hardy–Littlewood criterion for the double Walsh series
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \psi_m(x) \psi_n(y), \quad (4.1)
\]
where \( \psi_l(z) \) are the generalized Walsh functions in the Paley enumeration (see, for example, [12]): if a non-negative sequence \( a = \{a_{m,n}\}_{m,n \in \mathbb{N}} \in GM^2 \) and \( 1 < p < \infty \), then \( \|f(x, y)\|_p \asymp \sum_{m,n=1}^{\infty} |a_{m,n}|^p (mn)^{p/2} \), where \( f(x, y) \) is the sum of (4.1).

2. The \( n \)-dimensional version of the Hardy–Littlewood theorem reads as follows.
A non-negative sequence \( a = \{a_m\}, m = (m_1, \ldots, m_n) \in \mathbb{N}^n, n \geq 1 \), satisfies the \( GM^n \)-condition if
\[
a_m \to 0 \quad \text{as } |m| \equiv \sum_{j=1}^n m_j \to \infty
\]
and
\[
\sum_{m=k}^{\infty} \Delta^{(n)} a_m \leq C^* \left( \sum_{i=1}^{n} \sum_{m_i=k+1}^{\infty} \frac{a_{k_1, \ldots, k_i-1, m_i, k_i+1, \ldots, k_n}}{m_i} + \sum_{1 \leq i < j \leq n} \sum_{m_i=k+1}^{\infty} \sum_{m_j=k+1}^{\infty} \frac{a_{k_1, \ldots, m_i, \ldots, m_j, \ldots, k_n}}{m_i m_j} \right. \\
+ \left. \sum_{m=k+1}^{\infty} \frac{a_{m_1, \ldots, m_n}}{m_1 \cdots m_n} \right),
\]
where \( \sum_{m=1}^{\infty} = \sum_{m_1=i_1}^{\infty} \cdots \sum_{m_n=i_n}^{\infty} \) and the operator \( \Delta^{(n)} \) is defined as
\[
\Delta^{(n)} \equiv \prod_{j=1}^{n} \Delta^j \quad \text{and} \quad \Delta^j a_m = a_m - a_{m_1, \ldots, m_{j-1}, m_j+1, m_j+2, \ldots, m_n}.
\]

Theorem 6. Let \( 1 < p < \infty \), and let \( a \in GM^n, n \geq 1 \). Then
\[
f \in L_p[0, 2\pi]^n \quad \text{iff} \quad \sum_{m=1}^{\infty} a_m^p \left( \prod_{j=1}^{n} m_j \right)^{p-2} < \infty
\]
for the following type series \( (N = \{1, 2, \ldots, n\}, B \subseteq N) \)
\[
\sum_{m=1}^{\infty} a_m \prod_{j \in B} \cos m_j x_j \prod_{j \in N \setminus B} \sin m_j x_j. \quad (4.2)
\]
This criterion generalizes all the results that we know, even for \( n = 1 \).

3. We would also like to mention one simple corollary of the previous theorem which concerns the Hardy transform of Fourier series.

The well-known Hardy [5] result is the following: if \( f \in L_p \), \( 1 < p < \infty \),

\[
 f(x) \sim \sum_{m=1}^{\infty} a_m \cos mx,
\]

then the series

\[
 \sum_{m=1}^{\infty} A_m \cos mx, \quad A_m := \frac{1}{m} \sum_{k=1}^{m} a_k,
\]

is the Fourier series of a function \( Hf \) from \( L_p \). It is clear that the reverse part is not true. However, the norms of \( f \) and \( Hf \) are equivalent for the series with \( GM^n \) coefficients. We write this result for any dimension.

**Corollary 1.** Let \( 1 < p < \infty \), and let \( a \in GM^n \), \( n \geq 1 \). Then the series (\( N = \{1, 2, \ldots, n\}, B \subseteq N \))

\[
 \sum_{m=1}^{\infty} A_m \prod_{j \in B} \cos mj x_{j} \prod_{j \in N \setminus B} \sin mj x, \quad A_m := \frac{1}{m_1 \cdots m_n} \sum_{k=1}^{m} a_k
\]

is the Fourier series of a function \( Hf \in L_p \) if and only if \( f \in L_p \), \( f \sim (4.2) \). Moreover,

\[
 \| f \|_p \approx \| Hf \|_p \approx \left( \sum_{m=1}^{\infty} a_m \left( \prod_{j=1}^{n} m_j \right)^{p-2} \right)^{\frac{1}{p}}.
\]

**Proof.** The key observation is that \( \{A_m\} \in GM^n \). For simplicity, we only consider the two-dimensional case. First, it is clear that

\[
 |\Delta_{11} A_{m,n}| \leq \frac{A_{m,n}}{(m+1)(n+1)} + \frac{1}{m(m+1)(n+1)} \sum_{l=1}^{m} a_{l,n+1} + \frac{1}{(m+1)n(n+1)} \sum_{d=1}^{n} a_{m+1,d} \\
 + \frac{a_{m+1,n+1}}{(m+1)(n+1)} =: K_1 + K_2 + K_3 + K_4.
\]

Then, using several times (3.6), we have

\[
 \sum_{m=M}^{\infty} \sum_{n=N}^{\infty} K_4 \leq A_{M,N} + \sum_{m=M+1}^{\infty} A_{m,N} + \sum_{n=N+1}^{\infty} A_{M,n} + \sum_{m=M+1}^{\infty} \sum_{n=N+1}^{\infty} \frac{A_{m,n} m n}{m n} =: K.
\]

The same arguments imply

\[
 \sum_{m=M}^{\infty} \sum_{n=N}^{\infty} K_2 \leq \sum_{m=M}^{\infty} \sum_{n=N}^{\infty} K_4 + \frac{1}{M} \sum_{l=1}^{M} \sum_{n=N}^{\infty} a_{l,n+1} \leq C K.
\]

In a similar way we obtain the same inequality for \( K_3 \). Thus, collecting these estimates, we have

\[
 \sum_{m=M}^{\infty} \sum_{n=N}^{\infty} |\Delta_{11} A_{m,n}| \leq C^* K,
\]

i.e., \( \{A\} \in GM^2 \).

Thus, taking into account Theorem 6, we obtain

\[
 \| f \|_p \approx \sum_{m=1}^{\infty} a_m \left( \prod_{j=1}^{n} m_j \right)^{p-2} \quad \text{and} \quad \| Hf \|_p \approx \sum_{m=1}^{\infty} A_m \left( \prod_{j=1}^{n} m_j \right)^{p-2}.
\]

Hence, by (3.2), the inequality \( \| Hf \|_p \leq C(C^*, p) \| f \|_p \) holds.
On the other hand, we follow the scheme of the proof of Theorem 5 to obtain $\|f\|_p \leq C(C^*, p)\|Hf\|_p$. Indeed, we use the inequality

$$a_{l,d} \leq C_* \sum_{s=1}^{l} \sum_{t=1}^{d} (a_{s,t} + J_1 + J_2 + J_3),$$

to get (see (3.2)–(3.8))

$$\|f\|_p \leq C(C^*, p) \left[ \sum_{l,d=1}^{\infty} (ld)^{p-2} \left( \left( \sum_{s=1}^{l} \sum_{t=1}^{d} (J_1 + J_2 + J_3) \right)^p + A_{l,d}^p \right) \right]^{\frac{1}{p}} \leq C(C^*, p)\|Hf\|_p,$$

which finishes the proof.

We also remark that Theorem 6 and $\{A_m\} \in GM^n$ imply

$$\|Hf\|_p \approx \left( \sum_{m=1}^{\infty} A_m^p \left( \prod_{j=1}^{n} m_j \right)^{p-2} \right)^{\frac{1}{p}}$$

for any non-negative sequence $\{a_m\}$ and $p \in (1, \infty)$.

For an analogue of the Hardy theorem for the regular systems see [11]. Then the reverse statement can be obtained in the same way as Corollary 1.

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**References**


