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Note

Matrices Permutable to $*$ Matrices

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A (0, 1) matrix (a_{ij}) is said to be a $*$ matrix iff $a_{ij} = 1$ implies $a_{i'j'} = 1$ for all (i', j') satisfying $1 \leqslant i' \leqslant i$, $1 \leqslant j' \leqslant j$. (0, 1) matrices permutable to $*$ matrices are characterized and counted. Commutivity of matrices which permute to $*$ matrices is also discussed.

In [1] we defined $*$ matrices to be matrices (a_{ij}) with $a_{ij} = 0$ or 1 and $a_{i'j'} = 1$ if $a_{ij} = 1$ and $1 \leqslant i' \leqslant i$, $1 \leqslant j' \leqslant j$. For example the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a $*$ matrix. We note that $*$ matrices are maximal in the sense of Ryser [3, p. 62]. In this note we wish to consider the number $p(m, n)$ of $m \times n$ (0,1) matrices from which a $*$ matrix can be obtained by permuting rows and columns. We say such a matrix permutes to a $*$ matrix. For example the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

permutes to the $*$ matrix A. We observe that B contains no submatrix equal to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The following theorem shows that this is a characteristic property of matrices permutable to $*$ matrices.

THEOREM 1. *A (0, 1) matrix permutes to a $*$ matrix if and only if every 2×2 submatrix is unequal to* $\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$.

Proof. Clearly if a matrix has $\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ as a submatrix then so does every permutation and hence no permutation can be a $*$ matrix. Conversely suppose a matrix has neither as a submatrix. Permute the columns so that each column has at least as many 1's as the column to the right of it. Now permute the rows so that the first column has no 0 above a 1. If none of the first $k - 1$ columns has a 0 above a 1 then rows may be permuted so that none of the first k columns has a 0 above a 1. This procedure leads to the desired $*$ matrix.

We note that if in a matrix permutable to a $*$ matrix the 0's are replaced by 1's and the 1's by 0's then by the above characterization the resulting matrix also permutes to a $*$ matrix. Hence $p(m, n)$ is even. A generalization of this result is a consequence of the following theorem.

$S(n, k)$ is a Stirling number of the second kind for $1 \leqslant k \leqslant n$ and is defined to be 0 for $k > n$.

THEOREM 2.

$$p(m, n) = \sum_{k \geqslant 0} (k!)^2 S(m + 1, k + 1) S(n + 1, k + 1)$$
$$= \sum_{k \geqslant 1} (-1)^{n+k} k!(k + 1)^m S(n, k).$$

Proof. For convenience in classifying the matrices counted by $p(m, n)$ according to the number of distinct row sums and column sums we consider $(m + 1) \times (n + 1)$ matrices obtained by bordering the given matrices with a row and column of zeroes; clearly this has no effect on the count.

Let p_k be the number of such (0, 1) matrices which permute to a $*$ matrix having k "corners," i.e., exactly k entries $a_{ij} = 1$ with $a_{i+1,j} = 0 = a_{i,j+1}$. Now since each matrix counted by p_k has exactly $k + 1$ distinct row sums (a zero row sum is guaranteed), an ordered partition of the set $M = \{1,..., m + 1\}$ into $k + 1$ parts with $m + 1$ in the $(k + 1)$-th part is defined by letting the i-th part of the partition consist of the indices of those rows with the i-th largest row sum. Similarly an ordered partition, of which there are $k! S(n + 1, k + 1)$, of the set $N = \{1,..., n + 1\}$ into $k + 1$ parts with $n + 1$ in the $(k + 1)$-th part is determined by the columns since there are $k + 1$ distinct column sums also.

As an example the rows of the previously defined matrix B determine the ordered partition $R = (\{1, 4\}, \{2\}, \{3\}, \{5\})$ of $\{1,..., 5\}$ while the columns of B determine the ordered partition $C = (\{4\}, \{2\}, \{1, 5\}, \{3, 6\})$ of $\{1,..., 6\}$.

Conversely suppose the ordered partitions

$$R = (\{r_1, ..., r_{i_1}\}, ..., \{r_{i_k+1}, ..., r_{i_{k+1}} = m + 1\})$$

and

$$C = (\{c_1, ..., c_{j_1}\}, ..., \{c_{j_k+1}, ..., c_{j_{k+1}} = n + 1\})$$

of M and N, respectively, into $k + 1$ parts are given. Let R' and C' be the ordered partitions

$$(\{1, ..., i_1\}, ..., \{i_k + 1, ..., i_{k+1} = m + 1\})$$

and

$$(\{1, ..., j_1\}, ..., \{j_k + 1, ..., j_{k+1} = n + 1\})$$

of M and N, respectively. It suffices now to show there is a unique $*$ matrix A determining R' and C' for then the matrix B obtained from A by applying the row and column permutations $s \to r_s$ and $t \to c_t$ is the unique matrix determining R and C. But the corners of any $*$ matrix determining R' and C' must lie in rows $i_1, ..., i_k$ and columns $j_1, ..., j_k$. Hence A must be that $(m + 1) \times (n + 1)$ $*$ matrix with corners at $(i_1, j_k), ..., (i_k, j_1)$.

In the example we have $R' = (\{1, 2\}, \{3\}, \{4\}, \{5\})$ and

$$C' = (\{1\}, \{2\}, \{3, 4\}, \{5, 6\}).$$

The matrix determining R' and C' has corners at (2, 4), (3, 2), and (4, 1) and therefore is the $*$ matrix A previously defined.

The second equality of the theorem can be derived from the identity

$$k!\, S(n + 1, k + 1) = \sum_{j \geqslant k} (-1)^{n+j} \binom{j}{k} j!\, S(n, j),$$

which is equivalent to one found in [2, p. 209]. The derivation involves multiplying both sides by $k!S(m + 1, k + 1)$, summing on k, and using the identity

$$\sum_{k \geqslant 0} \binom{j}{k} k!\, S(m + 1, k + 1) = (j + 1)^m,$$

which is easily derived by using the recurrence relation and usual equation of definition of the Stirling numbers.

The following result is an immediate consequence of the theorem.

COROLLARY 3.

$$p(m + p - 1, n) \equiv p(m, n) (\text{mod}\, p) \textit{ for all primes } p.$$

If A and B are permutable to $*$ matrices we may have $AB = BA$ without having $A = B$ or either of them equal to the zero matrix. For example, if C is any $n \times n$ $*$ matrix and D is the $n \times n$ zero matrix then trivially the matrices $\left[\begin{smallmatrix} C & D \\ D & D \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} D & C \\ D & D \end{smallmatrix}\right]$ commute since their product is the $2n \times 2n$ zero matrix. If, however, $A = PA'P^T$, $B = PB'P^T$ for $n \times n$ $*$ matrices A' and B' and some permutation matrix P (P^T is its transpose) then we must have $A = B$ or $A = 0$ or $B = 0$. This follows from the fact that $PAP^TPBP^T = PBP^TPAP^T$ if and only if $AB = BA$ and from the following theorem:

THEOREM 4. *Two $*$ matrices A and B commute if and only if $A = B$ or one of them is the zero matrix.*

Proof. Let k be the least integer for which the k-th row of A is not equal to the k-th row of B. Let r_k be the k-th row sum of A. We may assume B has k-th row sum $r_k + r$ with $r > 0$. Then the $(r_k + 1)$-th column sum of A is at most $k - 1$ while that of B is at least k. Commutativity then requires $\min(r_k, k) = \min(r_k + r, k - 1)$ which implies $r_k \leqslant k - 1$ and this in turn gives $r_k = k - 1$. If $k = 1$ then A is the zero matrix. For $k > 1$ both A and B have their first column sums c and c' equal to or greater than k. Hence

$$k - 1 = \min(r_k, c') = \min(r_k + r, c) \geqslant k,$$

which is impossible.

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