We introduce $\Delta$-groups and show how they fit in the context of lattice field theory. To a topological space $M$ we associate a $\Delta$-group $\Gamma(M)$. We define the symmetric cohomology $H^{S3}(G,A)$ of a group $G$ with coefficients in a $G$-module $A$. The $\Delta$-group $\Gamma(M)$ is determined by the action of $\pi_1(M)$ on $\pi_2(M)$ and an element of $H^{S3}(\pi_1(M),\pi_2(M))$.

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Introduction

A topological lattice field theory is a prescription of initial data which allows one to construct an invariant for manifolds. The machinery works like this: one starts with a triangulation of the manifold, associates a quantity to each simplex, takes an “appropriate” sum and shows that the result is an invariant. Of course, in practice, things are a little bit more complicated. The first problem is how you decide, from combinatorial data, if two triangulations give the same manifold. This can be settled by using an Alexander type theorem [13] or Pachner’s Theorem [2,11]. The second problem is to find an algebraic input which reflects the combinatorial equivalence between two triangulations of a manifold. In dimension two it is known that topological lattice field theories are in bijection with semisimple associative algebras [8]. For three-dimensional manifolds, invariants can be obtained from various algebraic structures: Hopf algebras in [9] and [3], 6-j-symbols in [13] and [6] or finite groups and 3-cocycles in [4]. A review of these results and an example in dimension four can be found in [2].

3-algebras were introduced by Lawrence in [10] as another possible approach to the problem. A 3-algebra is a vector space $A$ together with three maps $m : A \otimes A \otimes A \to A$, $\overline{m} : A \otimes A \to A \otimes A$ and $P : A \to A$ which satisfy certain compatibilities. Geometrically $m$ represents the projection of three faces of a tetrahedron to the fourth face, $\overline{m}$ is the projection from two faces of a tetrahedron...
to the other two and \( P \) is the rotation of a face by an angle of \( \frac{2\pi}{3} \). It was shown in [10] that the \( 6j \)-symbol invariant [13], fits naturally in the context of 3-algebras. The first result about 3-algebras is a coherence type theorem which says that in a 3-algebra a product does not depend on the way we make the evaluation. This is the analog of the fact that in an associative algebra the product does not depend on the way we insert the parentheses.

In this paper we introduce \( \Delta \)-groups, a generalization of the notion of group. The most natural way to give examples of groups is to consider the fundamental group \( \pi_1(M) \) of a space \( M \). Here, we start with a space \( M \) and give a ternary operation that satisfies an "ternary-associativity" condition. It might not look like the simplest associativity we can think of but it is the natural one because of the analogy between the way is defined multiplication in the fundamental group and the way we define our ternary multiplication. Moreover this associativity fits in the language of 3-algebras.

We start the paper by introducing strong 3-algebras. These are some particular types of 3-algebras, with \( \mathfrak{m}(a \otimes b) = m \otimes id(a \otimes b) \otimes u(1) \) where \( u : k \to A \otimes A \) is a linear map. The advantage of working with strong 3-algebras is that the relations among \( HS \) elements in \( \pi_1 \) give examples of groups is to consider the fundamental group \( \pi_1(M) \) of a space \( M \). This is similar to the way one defines the cyclic cohomology for algebras (of course in that case one might not look like the simplest associativity we can think of but it is the natural one because of the analogy between the way is defined multiplication in the fundamental group and the way we define our ternary multiplication. Moreover this associativity fits in the language of 3-algebras.

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More important, once we have the definition of strong 3-algebras it is easier to introduce the set theoretical equivalent of a 3-algebra, the notion of \( \Delta \)-group. We give a construction which associates to a topological space \( M \) a \( \Delta \)-group \( \Gamma(M) \). This generalization is in the spirit of the definition of the fundamental group \( \pi_1(M) \). The idea is to replace paths between based points with \( 2 \)-paths between “based curves” (in other words, equivalence classes of maps from a 2-simplex to \( M \) which restricted to boundary are certain fixed curves). We show that every finite \( \Delta \)-group gives rise to a strong 3-algebra.

We study a certain class of \( \Delta \)-groups associated to a \( G \)-module \( A \) and show that they are classified by the symmetric cohomology \( HS^3(G, A) \). In particular we associate to every topological space an element in \( HS^3(\pi_1(M), \pi_2(M)) \). The image of this element in \( H^3(\pi_1(M), \pi_2(M)) \) is the classical \( k \)-invariant.

In the last section we define the symmetric cohomology \( HS^n(G, A) \). For this we give an action of the symmetric group \( \Sigma_{n+1} \) on \( C^n(G, A) \) and show that it is compatible with the usual differential. This is similar to the way one defines the cyclic cohomology for algebras (of course in that case one uses the action of the cyclic group \( C_{n+1} \)).

1. Preliminaries

In this section we recall a few definitions and results about 3-algebras. For more details we refer to [10]. In what follows \( k \) is a field, and \( \otimes \) means \( \otimes_k \). If \( V \) is a vector space \( \tau_{i,j} : V^{\otimes n} \to V^{\otimes n} \) is the transposition which interchanges the \( i \)th and \( j \)th positions.

**Definition 1.1.** A 3-algebra over \( k \) is a vector space \( A \) endowed with \( k \)-linear maps,

\[
P : A \to A \quad \text{(of order 3, } P^3 = id),
\]

\[
m : A \otimes A \otimes A \to A,
\]

\[
\overline{m} : A \otimes A \to A \otimes A,
\]

which satisfy the following conditions:

\begin{enumerate}
  \item \( m(m \otimes 1 \otimes 1) = m(1 \otimes 1 \otimes m) \tau_{34}(1 \otimes \overline{m} \otimes 1 \otimes 1) \tau_{34} \);
  \item \( (1 \otimes m) \tau_{23}(1 \otimes \overline{m}) = (\overline{m} (1 \otimes m) \tau_{12} (P^{-1} \otimes 1 \otimes 1 \otimes 1) (m \otimes 1 \otimes 1) (P \otimes P \otimes 1 \otimes 1)) \tau_{23} \);
  \item \( m(m \otimes 1) = (1 \otimes m) \tau_{12} (P^2 \otimes \overline{m} \otimes 1) (1 \otimes 1 \otimes \overline{m}) \tau_{23} \);
  \item \( (1 \otimes m) \tau_{12} (1 \otimes \overline{m}) = (m \otimes 1) (1 \otimes \overline{m}) (P \otimes P \otimes 1) (m \otimes 1) (1 \otimes P^{-1} \otimes 1) \);
  \item \( (1 \otimes m) \tau_{23} (\overline{m} \otimes P^2 \otimes 1) = (m \otimes 1) (1 \otimes 1 \otimes \overline{m}) \);
  \item \( P m = m(P \otimes P \otimes P) \tau_{23} \tau_{12} \);
  \item \( \overline{m} \) commutes with \( (P^2 \otimes P) \tau_{12} \).
\end{enumerate}
Definition 1.2. A 3-algebra is said to be orthogonal if:

(viii) \((1 \otimes P^2)\tau_{12} m(P \otimes P)m = Q : A \otimes A \to A \otimes A\) is a projection and \(m\) vanishes on \((\ker Q) \otimes A\).

The geometric pictures for \(m\) and \(m\) are Figs. 1 and 2, respectively.

A product in a 3-algebra corresponds to a labeled triangulation \(\Pi\) of a triangle \(T\). More exactly, each triangle of the triangulation has labels 1, 2, 3 placed on sides and is labeled with an element of \(A\). An ordered evaluation \(T\) of a product \(\Pi\) is sequence of triangulations, starting with \(\Pi\) and ending with the trivial triangulation of \(T\) such that at each step we change the triangulation by a move depicted in Fig. 1 or Fig. 2.

Theorem 1.3. (See [10].) Suppose that \(A\) is an orthogonal 3-algebra and \(\Pi\) is a labeled triangulation of a triangle. Then the composition of \(m\), \(m\) and \(P\) specified by an evaluation \(T\) of \(\Pi\), has a image in \(A\) which is independent of the choice of \(T\).

Let \(I\) be a set, \(f : \mathbb{I}^6 \to \mathbb{k}\) a map which will be denoted

\[(a, b, c, i, j, k) \rightarrow \begin{vmatrix} a & b & c \\ i & j & k \end{vmatrix}.

Assume that \(f\) satisfies the following identities:

\[
\begin{vmatrix} a & b & c \\ i & j & k \end{vmatrix} = \begin{vmatrix} a & k & j \\ i & c & b \end{vmatrix} = \begin{vmatrix} k & b & i \\ c & j & a \end{vmatrix} = \begin{vmatrix} j & i & c \\ b & a & k \end{vmatrix}.
\]

Also let \(w : I \to \mathbb{k}\).
Example 1.4. (See [10].) The \( k \)-linear space \( A \), generated by \( \{ e_{ijk} \mid i, j, k \in I \} \) together with maps:

\[
P(e_{ijk}) = e_{jki},
\]

\[
m(e_{akj} \otimes e_{bi} \otimes e_{j'c}) = \delta_{ii'} \delta_{jj'} \delta_{kk'} a \ b \ c \mid e_{abc},
\]

\[
\tilde{m}(e_{j_2bc} \otimes e_{b'aj_1}) = \delta_{bb'} \sum_j w^2_{j_2} a \ j \ j_1 c \ b \mid e_{j_2aj} \otimes e_{cj_1j},
\]

defines a 3-algebra if and only if for all \( a, b, c, e, f, j_1, j_2, j_3, j_{23} \in I \) we have:

\[
\sum_j w^2_{j_2} e \ j_3 \ j \ j_1 c \ j_3 c \ f \mid j_3 c \ f \ j_1 \ e \ j = j_3 c \ f \ j_2 \ j_3 \ j_2 \ j_1 e \ j. \]

2. Strong 3-algebras

In this section we study a particular type of 3-algebra for which there is a stronger relation between \( m \) and \( \tilde{m} \). The geometric interpretation of this dependence is depicted in Fig. 4.

Definition 2.1. A strong 3-algebra over a field \( k \) is a vector space \( A \) with \( k \)-linear maps:

\[
P : A \rightarrow A,
\]

\[
u : k \rightarrow A \otimes A,
\]

\[
m : A \otimes A \otimes A \rightarrow A
\]
such that \((A, P, m, \tilde{m})\) is a 3-algebra, where \( \tilde{m} : A \otimes A \rightarrow A \otimes A \) is defined by

\[
\tilde{m} = (m \otimes \text{id})(\text{id} \otimes u).
\]

It is natural to ask whether there are useful necessary and sufficient conditions which make \((A, m, u, P)\) a strong 3-algebra on \( A \). An answer will be given in Proposition 2.2. First notice that \( u \) should have a certain symmetry (see Fig. 3). This is not part of definition of a strong 3-algebra and is not clear if it is necessary but it will convenient to have the following identity:

\[
\tau_{12}u = (P \otimes P^2)u. \quad (2.1)
\]
Proposition 2.2. Let $A$ be a vector space over $k$ and let $P : A \to A$, $u : k \to A \otimes A$ and $m : A \otimes A \otimes A \to A$ be $k$-linear maps. Suppose that $u$ satisfies $(2.1)$, then $(A, P, u, m)$ is a strong 3-algebra if and only if (see Figs. 5 and 6):

\begin{align*}
P^3 &= \text{id}, \quad (2.2) \\
P(m(a \otimes b \otimes c) \otimes d \otimes e) &= m(P(b) \otimes P(c) \otimes P(a)), \quad (2.3) \\
\sum m(P(b) \otimes a \otimes u_1) \otimes u_2 &= \sum u_1 \otimes m(b \otimes u_2 \otimes a). \quad (2.4) \\
m(m(a \otimes b \otimes c) \otimes d \otimes e) &= \sum m(a \otimes m(b \otimes d \otimes u_1) \otimes m(c \otimes u_2 \otimes e)). \quad (2.5)
\end{align*}

Proof. Since our formulas involve several copies of $u(1)$ at the same time, we shall use the following notations $u(1) = \sum u_1 \otimes u_2 = \sum U_1 \otimes U_2 = \sum \tilde{u}_1 \otimes \tilde{u}_2$.

Obviously (vi) and (2.3) are the same. We have:
For (iv) we compute:

\[ m(1 \otimes 1 \otimes m) \tau_{34}(1 \otimes \tilde{m} \otimes 1 \otimes 1) \tau_{34}(a \otimes b \otimes c \otimes d \otimes e) \]

\[ = \sum m(1 \otimes 1 \otimes m) \tau_{34}(a \otimes m(b \otimes d \otimes u_1) \otimes u_2 \otimes c \otimes e) \]

\[ = \sum m(a \otimes m(b \otimes d \otimes u_1) \otimes m(c \otimes u_2 \otimes e)). \]

And so (i) follows from (2.5). We compute:

\[ \tilde{m}(P^2 \otimes P) \tau_{12}(a \otimes b) = \tilde{m}(P^2(b) \otimes P(a)) \]

\[ = \sum m(P^2(b) \otimes P(a) \otimes u_1) \otimes u_2, \]

\[ (P^2 \otimes P) \tau_{12}(a \otimes b) = \sum (P^2 \otimes P) \tau_{12}(m(a \otimes b \otimes u_1) \otimes u_2) \]

\[ = \sum P^2(u_2) \otimes P(m(a \otimes b \otimes u_1)) \]

\[ \overset{(2.3)}{=} \sum P^2(u_2) \otimes m(P(b) \otimes (a \otimes u_1) \otimes P(a)) \]

\[ \overset{(2.1)}{=} \sum u_1 \otimes m(P(b) \otimes u_2 \otimes P(a)) \]

and now (vii) follows from (2.4). To prove (v) we use (2.4) and the following two equations

\[ (1 \otimes m) \tau_{23}(\tilde{m} \otimes P^2 \otimes 1)(a \otimes b \otimes c \otimes d) = \sum (1 \otimes m) \tau_{23}(m(a \otimes b \otimes u_1) \otimes u_2 \otimes P^2(c) \otimes d) \]

\[ = \sum m(a \otimes b \otimes u_1) \otimes m(P^2(c) \otimes u_2 \otimes d), \]

\[ (m \otimes 1)(1 \otimes 1 \otimes \tilde{m})(a \otimes b \otimes c \otimes d) = \sum (m \otimes 1)(a \otimes b \otimes m(c \otimes d \otimes u_1) \otimes u_2) \]

\[ = \sum m(a \otimes b \otimes m(c \otimes d \otimes u_1)) \otimes u_2. \]

For (iv) we compute:

\[ (1 \otimes \tilde{m}) \tau_{12}(1 \otimes \tilde{m})(a \otimes b \otimes c) = \sum (1 \otimes \tilde{m})(m(b \otimes c \otimes u_1) \otimes a \otimes u_2) \]

\[ = \sum m(b \otimes c \otimes u_1) \otimes m(a \otimes u_2 \otimes U_1) \otimes U_2, \]

\[ (\tilde{m} \otimes 1)(1 \otimes \tilde{m})(P \otimes P \otimes 1)(\tilde{m} \otimes 1)(a \otimes P^{-1}(b) \otimes c) \]

\[ = \sum (\tilde{m} \otimes 1)(1 \otimes \tilde{m})(P \otimes P \otimes 1)(m(a \otimes P^2(b) \otimes u_1) \otimes u_2 \otimes c) \]

\[ \overset{(2.3)}{=} \sum (\tilde{m} \otimes 1)(1 \otimes \tilde{m})(m(b \otimes P(u_1) \otimes P(a)) \otimes P(u_2) \otimes c) \]

\[ = \sum (\tilde{m} \otimes 1)(m(b \otimes P(u_1) \otimes P(a)) \otimes m(P(u_2) \otimes c \otimes U_1) \otimes U_2) \]

\[ = \sum m(b \otimes P(u_1) \otimes P(a)) \otimes m(P(u_2) \otimes c \otimes U_1) \otimes \tilde{u}_1) \otimes \tilde{u}_2 \otimes U_2 \]

\[ \overset{(2.3)(2.5)}{=} \sum m(b \otimes c \otimes m(P(a) \otimes U_1 \otimes \tilde{u}_1)) \otimes \tilde{u}_2 \otimes U_2 \]

\[ \overset{(2.4)}{=} \sum m(b \otimes c \otimes u_1) \otimes m(a \otimes u_2 \otimes U_1) \otimes U_2 \]
so we get (iv). For (iii) we check the following two equalities:

\[
\tilde{m}(m \otimes 1)(a \otimes b \otimes c \otimes d) = \tilde{m}(m(a \otimes b \otimes c) \otimes d) = \sum m(m(a \otimes b \otimes c) \otimes d \otimes u_1) \otimes u_2
\]

\[
(2.5) \sum m(a \otimes m(b \otimes d \otimes U_1) \otimes m(c \otimes U_2 \otimes u_1)) \otimes u_2
\]

\[
(2.4) \sum m(a \otimes m(b \otimes d \otimes U_1) \otimes u_1) \otimes m(P^2(c) \otimes u_2 \otimes U_2).
\]

\[
(1 \otimes m)\tau_{12}((P^2 \otimes \tilde{m} \otimes 1)(1 \otimes 1 \otimes \tilde{m})\tau_{12}\tau_{23}(a \otimes b \otimes c \otimes d)
\]

\[
= \sum (1 \otimes m)\tau_{12}((P^2 \otimes \tilde{m} \otimes 1)(1 \otimes 1 \otimes \tilde{m})(c \otimes a \otimes b \otimes d)
\]

\[
= \sum (1 \otimes m)\tau_{12}((P^2 \otimes \tilde{m} \otimes 1)(c \otimes a \otimes m(b \otimes d \otimes u_1) \otimes u_2)
\]

\[
= \sum (1 \otimes m)\tau_{12}((P^2(c) \otimes m(a \otimes m(b \otimes d \otimes u_1) \otimes U_1) \otimes U_2 \otimes u_2)
\]

\[
= \sum m(a \otimes m(b \otimes d \otimes u_1) \otimes U_1) \otimes m(P^2(c) \otimes u_2 \otimes U_2)
\]

Finally we have for (ii):

\[
(1 \otimes m)\tau_{23}((\tilde{m} \otimes 1 \otimes 1)(a \otimes b \otimes c \otimes d) = \sum (1 \otimes m)\tau_{23}(m(a \otimes b \otimes u_1) \otimes u_2 \otimes c \otimes d)
\]

\[
= \sum (1 \otimes m)(m(a \otimes b \otimes u_1) \otimes c \otimes u_2 \otimes d)
\]

\[
= \sum m(a \otimes b \otimes u_1) \otimes m(c \otimes u_2 \otimes d)
\]

\[
(2.3) \sum m(a \otimes b \otimes m(P^2(c) \otimes d \otimes u_1)) \otimes u_2.
\]

and

\[
\tilde{m}(1 \otimes m)\tau_{12}((P^2 \otimes 1 \otimes 1 \otimes 1)((\tilde{m} \otimes 1 \otimes 1)(P \otimes P \otimes 1 \otimes 1)\tau_{23}(a \otimes b \otimes c \otimes d)
\]

\[
= \tilde{m}(1 \otimes m)\tau_{12}((P^2 \otimes 1 \otimes 1 \otimes 1)((\tilde{m} \otimes 1 \otimes 1)(P(a) \otimes P(c) \otimes b \otimes d)
\]

\[
= \sum \tilde{m}(1 \otimes m)\tau_{12}((P^2 \otimes 1 \otimes 1 \otimes 1)(m(P(a) \otimes P(c) \otimes u_1) \otimes u_2 \otimes b \otimes d)
\]

\[
(2.3) \sum \tilde{m}(1 \otimes m)\tau_{12}(m(P^2(u_1) \otimes a \otimes c) \otimes u_2 \otimes b \otimes d)
\]

\[
= \sum \tilde{m}(1 \otimes m)(u_2 \otimes m(P^2(u_1) \otimes a \otimes c) \otimes b \otimes d)
\]

\[
= \sum \tilde{m}(u_2 \otimes m(m(P^2(u_1) \otimes a \otimes c) \otimes b \otimes d))
\]

\[
= \sum m(u_2 \otimes m(m(P^2(u_1) \otimes a \otimes c) \otimes b \otimes d) \otimes U_1) \otimes U_2
\]

\[
(2.3) \sum m(m(a \otimes u_2 \otimes P(c)) \otimes m(P^2(u_1) \otimes b \otimes d) \otimes U_1) \otimes U_2
\]

\[
(2.3,2.5) \sum m(a \otimes b \otimes m(P(c) \otimes d \otimes U_1)) \otimes U_2
\]

which establishes (ii) and completes our proof. □
Example 2.3. Consider the construction from Example 1.4. We define \( u : k \to A \otimes A \) by
\[
 u(1) = \sum_{j,u,v} w^j e_{uvj} \otimes e_{u,jv}.
\]
Then \( (A, m, u, P) \) is a strong 3-algebra.

The following example is inspired by the Dijkgraaf–Witten invariant associated to a finite group \( G \) and a 3-cocycle \([4]\).

Example 2.4. Let \( G \) be a finite group and \( \alpha : G \times G \times G \to k \) a 3-cocycle. We consider the vector space \( k[G^{(3−1)}] \) which has a basis indexed by the triples \((g, h, k)\) with the property \( khg = 1 \) (notice that \( k[G^{(3−1)}] \) has dimension \( |G|^2 \)).

Define three linear maps \( P : k[G^{(3−1)}] \to k[G^{(3−1)}], u : k \to k[G^{(3−1)}] \otimes k[G^{(3−1)}] \) and \( m : k[G^{(3−1)}] \otimes k[G^{(3−1)}] \to k[G^{(3−1)}] \) determined by:
\[
P((g, h, k)) = (h, k, g),
\]
\[
u(1) = \sum_g (g, h, (hg)^{-1}) \otimes (g^{-1}, hg, h^{-1}),
\]
\[
m((x, y, z), (p, q, r), (a, b, c)) = \delta(ax)\delta(br)\delta(py)\alpha(z, p, b)(x, q, c)
\]
(here \( \delta(x) = 1 \) if \( x = 1 \) and \( \delta(x) = 0 \) if \( x \neq 1 \), hence \( \delta(ax) = 1 \) if \( a = x^{-1} \)). Then \( k[G^{(3−1)}] \) is a strong 3-algebra if and only if we have: \( \alpha(g, h, k) = \alpha(gh, k, (hk)^{-1}) = \alpha((hk)^{-1}, g^{-1}, gh) = \alpha(hk, k^{-1}, (gh)^{-1}) \).

3. \( \Delta \)-Groups

It is well known that to every finite group \( G \) one can associate the group algebra \( kG \). It would be nice to have a similar construction for 3-algebras. For this we need to replace groups with some other set theoretical structure, which we will call a \( \Delta \)-group.

In this section we associate to every path connected topological space \( M \) a \( \Delta \)-group \( \Gamma(M) \) which will be, in some sense, the higher-dimensional analog of the fundamental group \( \pi_1(M) \). We first describe the construction and then give the actual definition of \( \Delta \)-groups. We conclude by showing that from every finite \( \Delta \)-group one can construct a strong 3-algebra.

Let \( M \) be a path connected topological space such that no element from \( \pi_1(M) \) has order 2. Take \( m_0 \) to be a base point in \( M \). Let \( \Omega(M, m_0) \) be the set of all closed paths in \( M \) starting at \( m_0 \) (i.e. \( \gamma : [0, 1] \to M \) such that \( \gamma(0) = \gamma(1) = m_0 \)). Consider the map \( pr : \Omega(M, m_0) \to \pi_1(M) \) that sends a path to its homotopy class. We fix a section \( s : \pi_1(M) \to \Omega(M, m_0) \) satisfying these two conditions:
\[
s(\alpha^{-1}) = s(\alpha) \circ (t \to 1 - t),
\]
\[
s(1) = \text{constant map} (t \to m_0).
\]

Here we use the fact that \( \pi_1(M) \) has no elements of order 2. Set \( B(M) = s(\pi_1(M)) \).

Consider the standard 2-simplex \( \Delta_2 = \{(x_0, x_1, x_2) \mid x_0 + x_1 + x_2 = 1, x_j \geq 0 \} \). We denote by \([0, 1]\) and \([0, 2]\) the three vertexes of the simplex and by \([0, 1], [1, 2]\) and \([0, 2]\) the corresponding edges; for example \( [0, 1] \) is the path \( \gamma(t) = (t, 1 - t, 0) \in \Delta_2 \).

For \( \alpha, \beta \in B(M) \) we define \( \Gamma(\alpha, \beta) \) to be the set of homotopy equivalence classes of maps \( \alpha : \Delta_2 \to M \) such that \( \alpha_{[0, 1]} = \alpha, \alpha_{[1, 2]} = \beta \) and \( \alpha_{[0, 2]} = \beta \alpha \). Here by \( \beta \alpha \) we mean the element \( s(pr(\beta)pr(\alpha)) \in B(M) \). \( \Gamma(\alpha, \beta) \) is never empty because in \( \pi_1(M) \) we have \( pr(\beta \alpha)^{-1}pr(\beta)pr(\alpha) = 1 \).

Let \( p, q : \Delta_2 \to \Delta_2 \) be the maps defined by
Fig. 7. $m(m(a, b, c), d, e) = m(a, m(b, d, f), m(c, U(f), e))$.

$p(x_0, x_1, x_2) = (x_1, x_2, x_0)$ and $q(x_0, x_1, x_2) = (x_1, x_0, x_2)$.

Notice that $p^3 = id_\Delta$, $q^2 = id_\Delta$ and that they give a representation of the symmetric group $\Sigma_3$.

We define $P : \Gamma(\alpha, \beta) \to \Gamma(\beta, (\beta\alpha)^{-1})$ by $P(a) = ap$ and $Q : \Gamma(\alpha, \beta) \to \Gamma(\alpha^{-1}, \beta\alpha)$ by $Q(a) = aq$. Obviously we have:

\[
p^3 = id, \quad Q^2 = id \quad \text{and} \quad QP = P^2 Q.
\]

Consider now the 3-dimensional simplex $\Delta_3 = \{(x_0, x_1, x_2, x_3) \mid x_0 + x_1 + x_2 + x_3 = 1, \ x_i \geq 0\}$. Take $a \in \Gamma(\alpha, \beta^{-1})$, $b \in \Gamma(\beta, \gamma^{-1})$ and $c \in \Gamma(\beta^{-1}\alpha, \gamma^{-1}\beta)$.

We want to define a map $\omega : \Delta_3 \to M$ by gluing $a$, $b$ and $c$ on three faces and then extending the map to the rest of the simplex. First we define $\omega_{[1,2,0]} = a$. Since $a \in \Gamma(\alpha, \beta^{-1})$ it follows that $\omega_{[0,2]} = \beta^{-1}$ and because $b \in \Gamma(\beta, \gamma^{-1})$ and $\beta^{-1}(t) = \beta(1-t)$ we can extend $\omega$ such that $\omega_{[0,2,3]} = b$. Using a similar argument we can assume that $\omega_{[1,0,3]} = c$. Finally, if $x \in \Delta_3$ is a point on the segment $[(1,0,0,0), (0,\frac{1}{2}, \frac{1}{2}, \frac{1}{2})]$ we extend $\omega$ to the whole of $\Delta_3$ such that $\omega_{[1,2,x]} = a$, $\omega_{[x,2,3]} = b$, and $\omega_{[1,x,3]} = c$. From the construction we can see that the homotopy class of $\omega_{[1,2,3]}$ depends only on the homotopy class of $a$, $b$ and $c$. Thus we have defined a map

\[
m : \Gamma(\alpha, \beta^{-1}) \times \Gamma(\beta, \gamma^{-1}) \times \Gamma(\beta^{-1}\alpha, \gamma^{-1}\beta) \to \Gamma(\alpha, \gamma^{-1}),
\]

\[
m(a, b, c) = \omega_{[1,2,3]}.
\]

From definition it follows that $Q(f) = f_{[1,0,2]}$. We will give now a different construction of $Q(f)$ that we need in the next paragraph. For every $\alpha, \beta \in B(M)$ and $f \in \Gamma(\alpha, \beta)$ we construct $U(f) \in \Gamma(\alpha^{-1}, \beta\alpha)$ in the following way: first we define $\theta : \Delta_3 \to M$ such that $\theta_{[1,2]} = m_0$. For every point $x$ on the edge $[1,2]$ we put $\theta_{[0,2,x]} = \alpha$ and $\theta_{[x,3]} = \beta$. We extend $\theta$ such that $\theta_{[0,1,3]} = f$. Up to homotopy any extension of $\theta$ to the whole of $\Delta_3$ will give the same restriction on $[2,0,3]$. And so we have a map:

\[
U : \Gamma(\alpha, \beta) \to \Gamma(\alpha^{-1}, \beta\alpha),
\]

\[
U(f) = \theta_{[2,0,3]}.
\]

It is not difficult to see that $U(f) = Q(f)$.

In Fig. 7 we have two triangulations of the 3-dimensional ball $B_3$. On the left-hand side we have two 3-simplexes $[0,2,1,3]$ and $[3,2,1,4]$ which are glued along the face $[2,1,3]$. On the right-hand side we have four 3-simplexes $[0,5,6,4]$, $[5,0,1,4]$, $[6,2,0,4]$, and $[0,2,1,4]$ which are glued along the faces $[0,5,4]$, $[6,0,4]$, $[0,1,4]$ and $[2,0,4]$, respectively.

We consider $a \in \Gamma(\alpha, \beta^{-1})$, $b \in \Gamma(\beta, \gamma^{-1})$, $c \in \Gamma(\beta^{-1}\alpha, \gamma^{-1}\beta)$, $d \in \Gamma(\gamma, \delta^{-1})$, $e \in \Gamma(\gamma^{-1}\alpha, \delta^{-1}\gamma)$ and $f \in \Gamma(\gamma^{-1}\beta, \delta^{-1}\gamma)$.
We can define two maps on $B_3$ such that $[2, 1, 0]$ is mapped to $a$, $[0, 1, 3(5)]$ to $b$, $[2, 0, 3(6)]$ to $c$, $[3(5), 1, 4]$ to $d$, $[2, 3(6), 4]$ to $e$ and $[0, 5, 4]$ to $f$. Moreover we send $[5, 6]$ to $m_0$ and for every point $x \in [5, 6]$ we send $[0, x]$ to $\gamma^{-1} \beta$ and $[x, 4]$ to $\delta^{-1} \gamma$. It follows that $[6, 0, 4]$ must go to $Q(f)$.

It is obvious that up to homotopy the image of $[2, 1, 4]$ from the two maps is the same element in $\Gamma(\alpha, \delta^{-1})$. More exactly we have:

$$m(m(a, b, c), d, e) = m(a, m(b, d, f), m(c, Q(f), e)).$$

It is easy to see that:

$$P(m(a, b, c)) = m(P(b), P(c), P(a)),
\quad Q(m(b, a, f)) = m(Q(b), Q(f), Q(a)).$$

We want to prove an analog for formula (2.4). The staring point is again Example 2.4 where we take $\alpha$ to be the trivial 3-cocycle. If we write the condition (2.4) for $a = (u, v, t)$ and $b = (x, y, z) \in k[G^{(3-1)}]$ we get that for every $k_1 = (g_1, h_1, (h_1g_1)^{-1})$ there is an element $k_2 = (g_2, h_2, (h_2g_2)^{-1})$ such that the following equalities hold:

$$m((y, z, x), (u, v, t), (g_1, h_1, (h_1g_1)^{-1})) = (g_2, h_2, (h_2g_2)^{-1}),
\quad (g_1^{-1}h_1g_1, h_1^{-1}) = m((x, y, z), (g_2^{-1}, h_2g_2, h_2^{-1}), (u, v, t)).$$

This can be written as $m(P(b), a, k_1) = k_2$ and $Q(k_1) = m(b, Q(k_2), a)$. We combine the two equalities to get

$$Q(k_1) = m(b, Q(m(P(b), a, k_1)), a)$$

or after some changes of variables

$$f = m(m(f, a, b), P^2 Q(a), P Q(b)). \quad (3.6)$$

One can see that Eq. (3.6) is also true for every $f \in \Gamma(\alpha, \beta^{-1})$, $a \in \Gamma(\beta, \gamma^{-1})$ and $b \in \Gamma(\beta^{-1} \alpha, \gamma^{-1} \beta)$.

We are ready to give the formal definition of a $\Delta$-group.

**Definition 3.1.** Let $G$ be a group. A $\Delta$-group based at $G$ is a collection of sets $T = \{T(g, h)\}_{g, h \in G}$ together with operations:

$$m : T(g, h^{-1}) \times T(h, k^{-1}) \times T(h^{-1}g, k^{-1}h) \to T(g, k^{-1}),
\quad P : T(g, h) \to T(h, g^{-1}h^{-1}),
\quad Q : T(g, h) \to T(g^{-1}, hg),$$

such that the following identities hold:

$$P^2 = id, \quad Q^2 = id, \quad P^2 Q = Q P, \quad (3.7)
\quad P(m(a, b, c)) = m(P(b), P(c), P(a)), \quad (3.8)
\quad Q(m(a, b, c)) = m(Q(a), Q(c), Q(b)). \quad (3.9)$$
\[ m(m(a, b, c), d, e) = m(a, m(b, d, f), m(c, Q(f), e)), \quad (3.10) \]
\[ m(m(f, a, b), p^2Q(a), PQ(b)) = f. \quad (3.11) \]

**Remark 3.2.** Condition (3.7) ensures that \( P \) and \( Q \) define an action of the symmetric group \( \Sigma_3 \).

**Example 3.3.** If \((A, +)\) is a commutative group, we define
\[ m(a, b, c) = a + b + c, \quad P(a) = a, \quad Q(a) = -a \]
then \( T(1, A) = \{T(1, 1) = A\} \) becomes a \( \Delta \)-group based at the trivial group \( 1 \).

**Example 3.4.** If \( G \) is a group set \( T(g, h) = \{(g, h, (hg)^{-1})\} \) for all \( g, h \in G \) and \( T(G, 0) = \{T(g, h)\}_{g,h \in G} \).
Then \( T(G, 0) \) is a \( \Delta \)-group based at \( G \) with
\[ m\left(\left((g, h^{-1}, g^{-1}h), (h, k^{-1}, h^{-1}k), (h^{-1}g, k^{-1}h, g^{-1}k)\right)\right) = (g, k^{-1}, g^{-1}k), \]
\[ P\left(\left((g, h, g^{-1}h^{-1})\right)\right) = (h, g^{-1}h^{-1}, g), \]
\[ Q\left(\left((g, h, g^{-1}h^{-1})\right)\right) = (g^{-1}, hg, h^{-1}). \]
We call \( T(G, 0) \) the trivial \( \Delta \)-group based at \( G \).

**Example 3.5.** Let \( G \) be a group and \((A, +)\) a \( G \)-module. Set \( T(g, h) = \{(a, (g, h)) \mid a \in A\} \) for all \( g, h \in G \) and \( T(G, A) = \{T(g, h)\}_{g,h \in G} \).
Then \( T(G, A) \) is a \( \Delta \)-group based at \( G \) with
\[ m\left(\left(\langle a, (g, h^{-1})\rangle, \langle b, (h, k^{-1})\rangle, \langle c, (h^{-1}g, k^{-1}h)\rangle\right)\right) = \langle a + (g^{-1}h)b + c, (g, k^{-1})\rangle, \]
\[ P\left(\left(\langle a, (g, h^{-1})\rangle\right)\right) = \langle ga, (h^{-1}, g^{-1}h)\rangle, \]
\[ Q\left(\left(\langle a, (g, h^{-1})\rangle\right)\right) = \langle -(ga), (g^{-1}, h^{-1}g)\rangle. \]
In the next section we shall give a more general example \( T(G, A, \alpha) \) corresponding to special 3-cocycles \( \alpha \in C^3(G, A) \), so we postpone the verification until then. Also, at that point, it will become clear how these examples were conceived.

We can now formulate the main result of this section:

**Theorem 3.6.** Let \( M \) be a path connected topological space with the property that \( \pi_1(M) \) has no element of order two. Then \( \Gamma(M) = \{\Gamma(\alpha, \beta)\}_{\alpha, \beta} \) is a \( \Delta \)-group based at \( \pi_1(M) \). Moreover if \( f : M \to N \) is a map between two such spaces then \( f^*: \Gamma(M) \to \Gamma(N) \) is a morphism of \( \Delta \)-groups.

**Proof.** It follows from the above construction. \( \square \)

**Remark 3.7.** It is easy to see that \( \Gamma(M) \) does not depend on the set of paths \( B(M) \). If \( B'(M) \) is another set of based curves, we can take a set of homotopies between pairs of elements from \( B(M) \) and \( B'(M) \) and then using these homotopies we can construct an isomorphism between \( \Gamma(M) \) and \( \Gamma'(M) \).

**Remark 3.8.** We have the following exact sequence of \( \Delta \)-groups
\[ 0 \to T(1, \pi_2(M)) \xrightarrow{i} \Gamma(M) \xrightarrow{p} T(\pi_1(M), 0) \to 1. \]
Here by a short exact sequence we mean that $i$ and $p$ are morphisms, $i$ is one to one, $p$ is onto and $\text{Im}(i) = \text{Ker}(p)$ where $\text{Ker}(p) = \{x \in \Gamma(M) \mid p(x) = (0, (1, 1))\}$. In a further paper we intend to elaborate on exact sequence in the category of $\Delta$-groups.

Remark 3.9. Let $M$ be a topological space. If $\pi_1(M) = 1$ then $\Gamma(M) = T(1, \pi_2(M))$ as in Example 3.3. If $\pi_2(M) = 0$ then $\Gamma(M) = T(\pi_1(M), 0)$ as in Example 3.4.

We end this section with an example of a $3$-algebra associated to a finite $\Delta$-group. This is similar to the construction of the group algebra $kG$ associated to a group $G$.

Example 3.10. Suppose that $T$ is a finite $\Delta$-group based at $G$ and $k$ is a field of characteristic $0$. Define $A = \coprod_{g,h \in G} \coprod_{x \in T(g,h)} kx$. We extend $m$ and $P$ linearly to the whole vector space $A$. Define $u : k \to A \otimes A$,

$$u(1) = \sum_{g,h \in G} \frac{1}{\# T(g,h)} \sum_{x \in T(g,h)} x \otimes Q(x).$$

Straightforward computations show that $(A, m, u, P)$ is a strong $3$-algebra.

4. $\Gamma(M) \simeq T(\pi_1(M), \pi_2(M), \alpha)$

In this section we study the structure of $\Gamma(M)$. We will prove that it is determined by the action of $\pi_1(M)$ on $\pi_2(M)$ and a certain $3$-cocycle.

In what follows $\pi_1(M)$ has a multiplicative operation, $\pi_2(M)$ has an additive operation, $g, h, \ldots$ are elements from $\pi_1$ and $a, b, \ldots, f$ are elements of $\pi_2$. These conventions allows us to use $ga$ for the action of $\pi_1$ on $\pi_2$ without confusion.

Let $M$ be a path connected topological space such that there is no element of order $2$ or $3$ in $\pi_1(M)$. Fix an element $x(g, h^{-1})$ in each $\Gamma(g, h^{-1})$. We may assume without lose of generality that $P(x(g, h^{-1})) = x(h^{-1}, g^{-1}h)$ and $Q(x(g, h^{-1})) = x(g^{-1}, h^{-1}g)$. If $g \neq h$, $g \neq 1$ and $h \neq 1$ this follows since $\pi_1(M)$ has no elements of order $2$ or $3$. For $g = h$ we use the fact that if $\gamma \in B(M)$ represents $g$ then $\gamma(1 - t) \in B(M)$ represents $g^{-1}$. We define $a(g, g^{-1}) : \Delta_2 \to M$ such that for every $x \in [0, 2]$ we have $a_x(x, 1) = \gamma$. This map will have the symmetry required. The case $g = 1$ or $h = 1$ is similar.

Notice that any other element $y \in \Gamma(g, h^{-1})$ differs from $x(g, h^{-1})$ by an element of $\pi_2(M)$; the only problem is where we glue this bubble. By convention we assume that the element of $\pi_2$ is always at the $[0]$ corner of our two simplex (see Fig. 8). This means that we can identify the set $\Gamma(M)(g, h^{-1})$ with $\pi_2(M)$. For convenience we denote such an element by $(a, (g, h^{-1}))$. If one wants to move the bubble from the $[0]$ corner to the $[1]$ corner then one has to take into consideration the action of $\pi_1(M)$ on $\pi_2(M)$. Having this in mind, it is easy to see that:

$$P((a, (g, h^{-1}))) = (ga, (h^{-1}, g^{-1}h)), \tag{4.12}$$

$$Q((a, (g, h^{-1}))) = (-ga, (g^{-1}, h^{-1}g)). \tag{4.13}$$
To find the multiplication, we look to Fig. 9. There are two distinct problems. The first one is how to multiply the $x(g, h^{-1})$'s among themselves and the second is how to add the elements of $\pi_2(M)$. Fortunately the two problems are independent.

First take $x(g, h^{-1}), x(h, k^{-1}), x(h^{-1}g, k^{-1}h) \in \Gamma(M)$. The product of these three elements belongs to $\Gamma(M)(g, k^{-1})$ so it is just $x(g, k^{-1})$ plus an element $y(g, h, k) \in \pi_2(M)$ which is glued in the $[0]$ corner. We define $\alpha : G \times G \times G \to A$ by $\alpha(g^{-1}h, h^{-1}k, k^{-1}) = y(g, h, k)$. To be more precise we have:

$$
\alpha(g, h, k) = y((ghk)^{-1}, (hk)^{-1}, k^{-1}).
$$

(4.14)

We will prove later that this is a 3-cocycle.

Now let $(a, (g, h^{-1})), (b, (h, k^{-1})), (c, (h^{-1}g, k^{-1}h)) \in \Gamma(M)$. The $\pi_2(M)$ component of the first and third element are already in the $[0]$ corner. However for the second element we have to move $b$ along $g^{-1}h$. This means that the $\pi_2(M)$ contribution to the product is: $a + (g^{-1}h)b + c$. To conclude we have that in $\Gamma(M)$ the multiplication is defined by

$$
m((a, (g, h^{-1})), (b, (h, k^{-1})), (c, (h^{-1}g, k^{-1}h)))
= (a + (g^{-1}h)b + c + \alpha(g^{-1}h, h^{-1}k, k^{-1})). (g, k^{-1})).
$$

(4.15)

It is clear now how we constructed Example 3.5 and why Remark 3.9 is true.

More generally, suppose that $G$ is a group, $A$ a $G$-module and $\alpha : G \times G \times G \to A$. We want to see under what conditions the above maps define a $\Delta$-group $T(G, A, \alpha)$.

**Proposition 4.1.** Let $T(G, A, \alpha)(g, h) = (a, (g, h)) \mid a \in A$. If $P, Q$ and $m$ are defined by (4.12), (4.13) and (4.15), then $T(G, A, \alpha)$ is a $\Delta$-group if and only if $\alpha \in Z^3(G, A)$ and we have:

$$
\alpha(x, y, z) = xy\alpha(y^{-1}, yz, (xyz)^{-1}) = -\alpha(x, yz, z^{-1}) = -\alpha(xy, y^{-1}, yz).
$$

(4.16)

**Proof.** First we check that $P^3 = id$.

$$
P^3((a, (g, h^{-1}))) = P^2((ga, (h^{-1}, g^{-1}h)))
= P((h^{-1}ga, (g^{-1}h, g)))
= (g^{-1}hh^{-1}ga, (g, h^{-1}))
= (a, (g, h^{-1})).
$$
Similarly one has that $Q^2 = \text{id}$ and $Q^2 P = PQ$. We check (3.8).

$$P(m((a, (g, h^{-1})), (b, (h, k^{-1})), (c, (h^{-1} g, k^{-1} h))))$$
$$= P((a + (g^{-1} h) b + c + \alpha(g^{-1} h, h^{-1} k, k^{-1}), (g, k^{-1})))$$
$$= (g(a + (g^{-1} h) b + c + \alpha(g^{-1} h, h^{-1} k, k^{-1})), (k^{-1}, g^{-1} k))$$
$$= (ga + hb + gc + \alpha(g^{-1} h, h^{-1} k, k^{-1}), (k^{-1}, g^{-1} k)),$$
$$m(P(b, (h, k^{-1})), P(c, (h^{-1} g, k^{-1} h)), P(a, (g, h^{-1})))$$
$$= m((hb, (k^{-1}, h^{-1} k)), (h^{-1} gc, (k^{-1} h, g^{-1} k)), (ga, (h^{-1}, g^{-1} h)))$$
$$= (hb + k(h^{-1} k)^{-1} h^{-1} gc + ga + \alpha(h, h^{-1} g, g^{-1} k), (k^{-1}, g^{-1} k))$$
$$= (hb + gc + ga + \alpha(h, h^{-1} g, g^{-1} k), (k^{-1}, g^{-1} k)).$$

And so, in order to have (3.8), $\alpha$ must satisfy the following identity:

$$g\alpha(g^{-1} h, h^{-1} k, k^{-1}) = \alpha(h, h^{-1} g, g^{-1} k). \tag{4.17}$$

A similar computation shows that (3.9) is equivalent to

$$g\alpha(g^{-1} h, h^{-1} k, k^{-1}) = -\alpha(h, h^{-1} k, k^{-1} g). \tag{4.18}$$

Let us look at (3.10).

$$m(m((a, (g, h^{-1})), (b, (h, k^{-1})), (c, (h^{-1} g, k^{-1} h))), (d, (k, l^{-1})), (e, (k^{-1} g, l^{-1} k)))$$
$$= m((a + g^{-1} h b + c + \alpha(g^{-1} h, h^{-1} k, k^{-1}), (g, k^{-1})), (d, (k, l^{-1})), (e, (k^{-1} g, l^{-1} k)))$$
$$= (a + g^{-1} h b + c + \alpha(g^{-1} h, h^{-1} k, k^{-1}) + g^{-1} kd + e + \alpha(g^{-1} k, k^{-1} l, l^{-1}), (g, l^{-1})),$$
$$m((a, (g, h^{-1})), m((b, (h, k^{-1})), (d, (k, l^{-1})), (f, (k^{-1} h, l^{-1} k))), m((c, (h^{-1} g, k^{-1} h))),$$
$$Q((f, (k^{-1} h, l^{-1} k))), (e, (k^{-1} g, l^{-1} k))))$$
$$= m((a, (g, h^{-1})), (b + h^{-1} kd + f + \alpha(h^{-1} k, k^{-1} l, l^{-1}), (h, l^{-1})),$$
$$c - (g^{-1} h) f + e + \alpha(g^{-1} k, k^{-1} l, l^{-1} h), (h^{-1} g, l^{-1} h)))$$
$$= (a + (g^{-1} h)(b + h^{-1} kd + f + \alpha(h^{-1} k, k^{-1} l, l^{-1}))) + c - (g^{-1} h) f + e$$
$$+ \alpha(g^{-1} k, k^{-1} l, l^{-1} h) + \alpha(g^{-1} h, h^{-1} l, l^{-1}), (g, l^{-1})).$$

And so (3.10) is equivalent to

$$\alpha(g^{-1} h, h^{-1} k, k^{-1}) + \alpha(g^{-1} k, k^{-1} l, l^{-1})$$
$$= g^{-1} h \alpha(h^{-1} k, k^{-1} l, l^{-1}) + \alpha(g^{-1} k, k^{-1} l, l^{-1} h) + \alpha(g^{-1} h, h^{-1} l, l^{-1}). \tag{4.19}$$

Finally one can prove that (3.11) is equivalent to

$$\alpha(g^{-1} h, h^{-1} k, k^{-1}) = -\alpha(g^{-1} k, k^{-1} h, h^{-1}). \tag{4.20}$$
Making an appropriate change of variables (4.17), (4.18) and (4.20) can be written as:
\[
\alpha(x, y, z) = xy\alpha(y^{-1}, yz, (xyz)^{-1}) = -\alpha(x, yz, z^{-1}) = -\alpha(xy, y^{-1}, yz)
\] (4.21)
while (4.19) becomes
\[
\alpha(x, y, zt) + \alpha(xy, z, t) = x\alpha(y, z, t) + \alpha(xy, z, (yz)^{-1}) + \alpha(x, yz, t).
\] (4.22)
Using (4.21) twice we get
\[
\alpha(x, y, z) = \alpha(xy, z, (yz)^{-1})
\]
and so \(\alpha\) is a 3-cocycle.

**Remark 4.2.** For \(G, A\) and \(\alpha\) as above we have an exact sequence of \(\Delta\)-groups:
\[
0 \to T(1, A) \to T(G, A, \alpha) \to T(G, 0) \to 1.
\]
Take \(f : T(G, A, \alpha) \to T(G, A, \beta)\) which is compatible with the short exact sequence. This means that:
\[
f((a, (g, h^{-1}))) = (a + \sigma(g^{-1}h, h^{-1}), (g, h^{-1}))
\]
where \(\sigma : G \times G \to A\). Because \(f\) is a morphism of \(\Delta\)-groups, \(f\) must be compatible with \(P, Q\) and \(m\). This yields the following conditions:
\[
\sigma(g, h) = g\sigma(h, (gh)^{-1}) = -(gh)\sigma(h^{-1}, g^{-1}),
\] (4.23)
\[
\sigma(gh, k) + \alpha(g, h, k) = \sigma(h, hk) + g\sigma(h, k) + \sigma(gh, h^{-1}) + \beta(g, h, k).
\] (4.24)
From (4.23) it follows that
\[
\sigma(g, h) = -\sigma(gh, h^{-1})
\]
and so (4.24) says that \(\alpha\) and \(\beta\) are equivalent 3-cocycles. To explain the roles of (4.21) and (4.23) we have to define the first few terms of the “symmetric cohomology”, which will be defined in full in the next section.

Let \(G\) be a group, \(A\) a \(G\)-module and \(C^n(G, A) = \{\alpha : G^n \to A\}\). For \(n = 1, 2\) or 3 we have an action of the symmetric group \(\Sigma_{n+1}\) on \(C^n(G, A)\):
If \(\phi \in C^1(G, A)\),
\[
((1, 2)\phi)(g) = -g\phi(g^{-1}),
\]
if \(\sigma \in C^2(G, A)\),
\[
((1, 2)\sigma)(x, y) = -x\sigma(x^{-1}, xy),
\]
\[
((2, 3)\sigma)(x, y) = -\sigma(xy, y^{-1}),
\]
if \(\alpha \in C^3(G, A)\),
We define the symmetric cochain groups to be \( CS^n(G, A) = (C^n(G, A))^{S_2} \) \( (n = 1, 2 \text{ or } 3) \). It is then easy to check that \( CS^n(G, A) \) is a subcomplex of the usual cochain complex. We call its cohomology the symmetric cohomology and denote it by

\[
HS^n(G, A) = \frac{ZS^n(G, A)}{BS^n(G, A)},
\]

With these notations we have the following result:

**Theorem 4.3.** The map \( f : T(G, A, \alpha) \to T(G, A, \beta) \) is an isomorphism if and only if \( [\alpha] = [\beta] \) in \( HS^3(G, A) \).

**Proof.** Straightforward. \( \square \)

**Corollary 4.4.** Suppose that \( \pi_1(M) \) has no elements of order 2 or 3. Then the element \([\alpha] \in HS^3(\pi_1(M), \pi_2(M)) \) defined by (4.14) is an invariant of the space \( M \).

**Remark 4.5.** There is a natural map from \( HS^3(\pi_1(M), \pi_2(M)) \) to \( H^3(\pi_1(M), \pi_2(M)) \). One can see directly from the construction that the image of \( \alpha \) in \( H^3(\pi_1(M), \pi_2(M)) \) is the classical \( k \)-invariant introduced by Eilenberg and MacLane in [5].

It is well known that to a topological space \( M \) one can associate a 2-group by taking as 2-morphisms maps from the 2-cube \([0, 1] \times [0, 1] \) to \( M \) (see [1] for details). It can be shown that this 2-group is determined by the action of \( \pi_1(M) \) on \( \pi_2(M) \) and a cohomology class \( \alpha \in H^3(\pi_1(M), \pi_2(M)) \) (the classical \( k \)-invariant).

Our main construction from this paper it is obvious based on a similar idea. The difference is that we take maps from a 2-simplex to \( M \). This allows us to construct the two maps \( P \) and \( Q \) that seem to have no equivalent in the definition of 2-groups. It is not clear if our construction can be extended to the case of a topological space \( M \) with elements of order 2 or 3 in \( \pi_1(M) \). Also it is not clear if the morphism \( HS^3(\pi_1(M), \pi_2(M)) \to H^3(\pi_1(M), \pi_2(M)) \) is always injective. A positive answer would mean that the invariant from Corollary 4.4 is exactly the classical \( k \)-invariant.

We intend to further investigate the relation between \( \Delta \)-groups and 2-groups.

5. Symmetric cohomology for groups

Let \( G \) be a group and \( A \) a \( G \)-module. In this section we define the symmetric cohomology of the group \( G \) with coefficients in \( A \). For \( n = 1, 2 \text{ or } 3 \), \( G = \pi_1(M) \), and \( A = \pi_2(M) \) we get the symmetric cohomology defined in the previous section.

We recall a few facts about the usual cohomology. We set \( C^n(G, A) = \{ \sigma : G^n \to A \} \) and define \( \partial_n : C^n(G, A) \to C^{n+1}(G, A) \) by

\[
\partial_n(\sigma)(g_1, \ldots, g_{n+1}) = g_1\sigma(g_2, \ldots, g_{n+1}) - \sigma(g_1g_2, g_3, \ldots, g_{n+1}) + \cdots + (-1)^n\sigma(g_1, \ldots, g_ng_{n+1}) + (-1)^{n+1}\sigma(g_1, \ldots, g_n).
\]

Define \( d^i : C^n(G, A) \to C^{n+1}(G, A) \) by
First we check that the square of the action of the transpositions are easy to check. This allows us to develop the whole theory of the symmetric cohomology. Homology groups are denoted by $H^n(G, A)$. Notice that $\partial_n(\sigma) = \sum_{i=0}^{n-1} (-1)^i d_i$. It is well known that in this way we obtain a complex and its differential. This allows us to develop the whole theory of the symmetric cohomology.

We give here an action of $\Sigma_{n+1}$ on $C^n(G, A)$ (for every $n$) and prove that it is compatible with the differential. This allows us to develop the whole theory of the symmetric cohomology.

It is enough to describe the action of the transpositions $(i, i+1)$ for $1 \leq i \leq n$. For $\sigma \in C^n(G, A)$ we define:

\[
(1, 2)\sigma(g_1, g_2, g_3, \ldots, g_n) = -g_1 \sigma((g_1)^{-1}, g_1 g_2, g_3, \ldots, g_n),
\]

\[
(i, i+1)\sigma(g_1, g_2, g_3, \ldots, g_n) = -\sigma(g_1, \ldots, g_{i-1} g_i, g_i^{-1} g_{i+1}, g_{i+1}^{-1}, g_{i+2}, \ldots, g_n),
\]

for $1 < i < n$,

\[
(n, n+1)\sigma(g_1, g_2, g_3, \ldots, g_n) = -\sigma(g_1, g_2, g_3, \ldots, g_{n-1} g_n, (g_n)^{-1}).
\]

**Proposition 5.1.** The above formulas define an action of $\Sigma_{n+1}$ on $C^n(G, A)$ which is compatible with the differential $\partial$.

**Proof.** First we check that the square of the action of the transposition $(i, i+1)$ is the identity:

\[
((i, i+1)((i, i+1)\sigma))(g_1, g_2, \ldots, g_n)
= -((i, i+1)\sigma)(g_1, \ldots, g_{i-1} g_i, g_i^{-1} g_{i+1}, \ldots, g_n)
= -(-\sigma(g_1, \ldots, g_{i-1} g_i g_i^{-1}, (g_i^{-1})^{-1}, g_{i+1}^{-1} g_{i+1}, \ldots, g_n))
= \sigma(g_1, \ldots, g_n).
\]

For the braid relation we have:

\[
((i, i+1)((i+1, i+2)((i, i+1)\sigma)))(g_1, g_2, \ldots, g_n)
= -((i+1, i+2)((i, i+1)\sigma))(g_1, \ldots, g_{i-1} g_i, g_i^{-1}, g_i g_{i+1}, \ldots, g_n)
= ((i, i+1)\sigma)(g_1, \ldots, g_{i-1} g_i, g_i^{-1} g_{i+1}, g_{i+1}^{-1} g_{i+2}, \ldots, g_n)
= -\sigma(g_1, \ldots, g_{i-1} g_i g_{i+1}, g_{i+1}^{-1} g_{i+2}, g_{i+2}, \ldots, g_n)
= -\sigma(g_1, \ldots, g_{i-1} g_i g_{i+1}, g_{i+1}^{-1}, g_{i+1} g_{i+2}, \ldots, g_n)
\]

and similarly

\[
((i+1, i+2)((i, i+1)((i+1, i+2)\sigma)))(g_1, g_2, \ldots, g_n)
= -\sigma(g_1, \ldots, g_{i-1} g_i g_{i+1}, g_{i+1}^{-1} g_{i+2}, g_{i+2}, \ldots, g_n).
\]

So $(i, i+1)((i+1, i+2)((i, i+1)\sigma)) = (i+1, i+2)((i, i+1)((i+1, i+2)\sigma))$. All the other relations are easy to check.
We also want to prove that this action is compatible with $\partial$. More exactly, if $\sigma \in C^n(G, A)$ is invariant under the action of $\Sigma_{n+1}$ then we wish to prove that $\partial(\sigma)$ is invariant under the action of $\Sigma_{n+2}$. We can check that

\[
(i, i + 1)(d^j(\sigma)) = d^j((i, i + 1)\sigma) \quad \text{if} \quad i < j,
\]

\[
(i, i + 1)(d^j(\sigma)) = d^j((i - 1, i)\sigma) \quad \text{if} \quad j + 2 \leq i,
\]

\[
(i, i + 1)(d^l(\sigma)) = -d^l(\sigma),
\]

\[
(i, i + 1)(d^l(\sigma)) = -d^{l-1}(\sigma).
\]

Now if we take $\sigma$ which is invariant by $\Sigma_{n+1}$ and use the fact that $\partial(\sigma) = \sum_{i=0}^{n+1}(-1)^j d^j$ we get $(i, i + 1)(\partial(\sigma)) = \partial(\sigma)$, which finishes the proof.

**Definition 5.2.** The subcomplex of the invariants will be called the symmetric cochain and will be denoted by $CS^n(G, A) = C^n(G, A)^{\Sigma_{n+1}}$. Its homology is called the symmetric cohomology of $G$ with coefficients in $A$ and is denoted $H^n(G, A)$.

**Remark 5.3.** There is a natural map from $H^n(G, A)$ to $H^n(G, A)$. When $n = 1$ or $n = 2$ it is easy to check that the map is injective. However for $n \geq 3$ it is not clear if this fact is still true.

**Remark 5.4.** In [14] Shaun Van Ault wrote a computer program which computes the symmetric cohomology $H^n(G, Z)$. He found examples of groups with the symmetric cohomology different than the usual one:

\[
H^{2k}(Z_2, Z) = Z_2 \quad \text{and} \quad H^{2k}(Z_2, Z) = 0,
\]

\[
H^2(Z_4, Z) = Z_4 \quad \text{and} \quad H^2(Z_4, Z) = Z_2.
\]

**Remark 5.5.** In [7] it was introduced the notion of crossed simplicial groups and subsequently a cohomology theory for certain symmetric objects. There are some similarities between our symmetric cohomology and the one from [7] that suggest a deeper connection.

**Remark 5.6.** In [12] we investigate the relation between symmetric cohomology the ordinary cohomology. For example we prove that if $G$ has no elements of order 2 then $H^2(S(G, A)) \cong H^2(G, A)$.

**References**