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Analytical result for dimensionally regularized massless on-shell planar triple box

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Abstract

The dimensionally regularized massless on-shell planar triple box Feynman diagram with powers of propagators equal to one is analytically evaluated for general values of the Mandelstam variables s and t in a Laurent expansion in the parameter $\epsilon = (4-d)/2$ of dimensional regularization up to a finite part. An explicit result is expressed in terms of harmonic polylogarithms, with parameters 0 and 1, up to the sixth order. The evaluation is based on the method of Feynman parameters and multiple Mellin–Barnes representation. The same technique can be quite similarly applied to planar triple boxes with any numerators and integer powers of the propagators.

In the last four years, the problem of analytical evaluation was completely solved for most important classes of two-loop Feynman diagrams with four external lines within dimensional regularization [1]. In the pure massless case with all end-points on-shell, i.e., $p_i^2 = 0$, $i = 1, 2, 3, 4$, this was done in [2–7]. The corresponding analytical algorithms were successfully applied to the evaluation of various two-loop virtual corrections [8]. In the case of massless two-loop four-point diagrams with one leg off-shell the problem of evaluation was solved in [9,10], with subsequent applications [11] to the process $e^+e^- \to$ 3jets. A first result for the massive on-shell case was presented in [12]. (See [13,14] for brief reviews of results on the analytical evaluation of various double-box Feynman integrals and the corresponding methods of evaluation.)

In [14,15], first analytical results on three-loop on-shell massless four-point diagrams within dimensional regularization were obtained. The leading power asymptotic behaviour of the dimensionally regularized massless on-shell planar triple box diagram shown in Fig. 1 in the Regge limit $t/s \to 0$ was analytically evaluated in [15] with the help of the strategy of expansion by regions [16]. Then, in [14], explicit analytical results for the unexpanded master planar triple box were presented for $1/\epsilon^j$ terms of Laurent expansion in ϵ with $j = 6, 5, 4, 3$ and 2.

The purpose of this Letter is to complete this task, i.e., analytically evaluate the missing $1/\epsilon$ part and the finite part. An explicit result will be expressed in terms of harmonic polylogarithms (HPL) [17], with parameters 0

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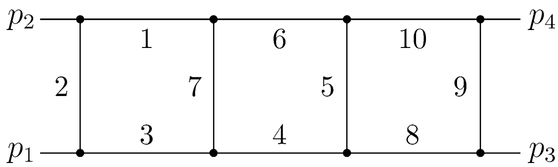

Fig. 1. Planar triple box diagram.

and 1, up to the sixth order. The evaluation is based on the technique of alpha parameters and Mellin–Barnes (MB) representation which was successfully used in [2,4,9,12] and reduces, due to taking residues and shifting contours, to a decomposition of a given MB integral into pieces where a Laurent expansion of the integrand in ϵ becomes possible.

The general planar triple box Feynman integral without numerator takes the form

$$T(a_1,\ldots,a_{10};s,t;\epsilon)=\iiint\frac{\mathrm{d}^dk\,\mathrm{d}^dl\,\mathrm{d}^dr}{(k^2)^{a_1}[(k+p_2)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}}$$
$$\times\frac{1}{[(l+p_1+p_2)^2]^{a_4}[(r-l)^2]^{a_5}(l^2)^{a_6}[(k-l)^2]^{a_7}}$$
$$\times\frac{1}{[(r+p_1+p_2)^2]^{a_8}[(r+p_1+p_2+p_3)^2]^{a_9}(r^2)^{a_{10}}},\tag{1}$$

where $s=(p_1+p_2)^2$ and $t=(p_2+p_3)^2$ are Mandelstam variables and k,l,r are the loop momenta. Usual prescriptions $k^2=k^2+i0$, $s=s+i0$, etc., are implied.

To resolve the singularity structure of Feynman integrals in ϵ it is very useful to apply the MB representation

$$\frac{1}{(X+Y)^\nu}=\frac{1}{\Gamma(\nu)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\mathrm{d}z\,\frac{Y^z}{X^{\nu+z}}\Gamma(\nu+z)\Gamma(-z),\tag{2}$$

that makes it possible to replace sums of terms raised to some power by their products in some powers, at the cost of introducing extra integrations. By a straightforward generalization of two-loop manipulations [6,12] one can introduce, in a suitable way, MB integrations, first, after the integration over one of the loop momenta, r, then after the integration over l, and complete this procedure after integration over the loop momentum k. As a result, one arrives [15] at the following sevenfold MB representation of (1):

$$T(a_1,\ldots,a_8;s,t;\epsilon)$$
$$=\frac{(i\pi^{d/2})^3(-1)^a}{\prod_{j=2,5,7,8,9,10}\Gamma(a_j)\Gamma(4-a_{589(10)}-2\epsilon)(-s)^{a-6+3\epsilon}}$$
$$\times\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)}$$
$$\times\frac{\Gamma(2-a_{12}-\epsilon+z_2)\Gamma(2-a_{23}-\epsilon+z_3)\Gamma(a_7+w-z_4)\Gamma(-z_5)\Gamma(-z_6)}{\Gamma(4-a_{123}-2\epsilon+w-z_4)\Gamma(a_6-z_5)\Gamma(a_4-z_6)}$$
$$\times\Gamma(+a_{123}-2+\epsilon+z_4)\Gamma(w+z_2+z_3+z_4-z_7)\Gamma(2-a_{59(10)}-\epsilon-z_5-z_7)$$
$$\times\Gamma(2-a_{589}-\epsilon-z_6-z_7)\Gamma(a_{467}-2+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(a_9+z_7)$$
$$\times\Gamma(a_5+z_5+z_6+z_7)\Gamma(4-a_{467}-2\epsilon+z_5+z_6+z_7)\Gamma(a_{589(10)}-2+\epsilon+z_5+z_6+z_7)$$
$$\times\Gamma(2-a_{67}-\epsilon-w-z_2+z_5+z_7)\Gamma(2-a_{47}-\epsilon-w-z_3+z_6+z_7),\tag{3}$$

where $a=\sum_{i=1}^{10} a_i$, $a_{589(10)}=a_5+a_8+a_9+a_{10}$, $a_{123}=a_1+a_2+a_3$, etc., and integration contours are chosen in the standard way.

In the case of the master triple box, we set $a_i=1$ for $i=1,2,\dots,10$ to obtain

$$
\begin{aligned}
T^{(0)}(s,t;\epsilon) &\equiv T(1,\dots,1;s,t;\epsilon) \\
&= \frac{\left(i\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{4+3\epsilon}} \frac{1}{(2\pi i)^7} \int\limits_{-i\infty}^{+i\infty} \mathrm{d}w \prod_{j=2}^{7} \mathrm{d}z_j \left(\frac{t}{s}\right)^w \frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)} \\
&\quad \times \frac{\Gamma(-\epsilon+z_2)\Gamma(-\epsilon+z_3)\Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4)\Gamma(1+\epsilon+z_4)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)} \\
&\quad \times \frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(w+z_2+z_3+z_4-z_7)}{\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_5+z_6+z_7)} \\
&\quad \times \Gamma(-1-\epsilon-z_5-z_7)\Gamma(-1-\epsilon-z_6-z_7)\Gamma(1+z_7) \\
&\quad \times \Gamma(1+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(-\epsilon-w-z_2+z_5+z_7) \\
&\quad \times \Gamma(-\epsilon-w-z_3+z_6+z_7)\Gamma(1+z_5+z_6+z_7)\Gamma(2+\epsilon+z_5+z_6+z_7). \qquad (4)
\end{aligned}
$$

Observe that, because of the presence of the factor $\Gamma(-2\epsilon)$ in the denominator, we are forced to take some residue in order to arrive at a non-zero result at $\epsilon=0$, so that the integral is effectively sixfold.

Then the standard procedure of taking residues and shifting contours is applied, with the goal to obtain a sum of integrals where one may expand integrands in Laurent series in ϵ. One- and two-loop examples of such procedures can be found, e.g., in [13]. The poles in ϵ are not visible at once, at a first integration over one of the MB variables. However, the rule for finding a mechanism of the generation of poles is based on the simple observation that a product of two gamma functions $\Gamma(a+z)\Gamma(b-z)$, where z is a MB integration variable and a and b depend on the rest of the variables, generates a pole of the type $\Gamma(a+b)$. This means that any contour in the next integrations should be chosen according to this dependence. So, the first step is an analysis of various pairs of gamma functions and various orders of integration in (4). The analysis of the integrand shows that the following four gamma functions play a crucial role for the generation of poles in ϵ: $\Gamma(-\epsilon+z_{2,3})$ and $\Gamma(-1-\epsilon-z_{6,5}-z_7)$. The first decomposition of the integral (4) arises when one either takes a residue at the first pole of one of these gamma functions or shifts the corresponding contour, i.e., changes the nature of this pole. As a result, Eq. (4) is decomposed as $2T_{0001}+2T_{0010}+2T_{0011}+T_{0101}+2T_{0110}+2T_{0111}+T_{1010}+2T_{1011}+T_{1111}$, where a symmetry of the integrand is taken into account. Here the value 1 of an index means that a residue is taken and 0 means a shifting of a contour. The first two indices correspond to the gamma functions $\Gamma(-\epsilon+z_2)$ and $\Gamma(-1-\epsilon-z_5-z_7)$ and the second two indices to $\Gamma(-\epsilon+z_3)$ and $\Gamma(-1-\epsilon-z_6-z_7)$, respectively. The term T_{0000} is absent because it is zero at $\epsilon=0$ due to $\Gamma(-2\epsilon)$ in the denominator.

Each of these terms is further appropriately decomposed and, eventually, one is left with integrals where integrands can be expanded in ϵ. These resulting terms involve up to five integrations. Taking some of these integrations with the help of the first and the second Barnes lemmas, one reduces all the integrals to no more than twofold MB integrals of gamma functions and their derivatives. In some of them, one more integration can be also performed in gamma functions. Then the last integration, over w is performed by taking residues and summing up resulting series, in terms of HPL. Keeping in mind the Regge limit, $t/s\to 0$, let us, for definiteness, decide to close the contour of the final integration, over w, to the right and obtain power series in t/s. The coefficients of these series are (up to $(-1)^n$) linear combinations of $1/n^6, S_1(n)/n^5,\dots,S_1(n)S_3(n)/n^2,\dots$, where $S_k(n)=\sum_{j=1}^n j^{-k}$. Summing up these series gives results in terms of HPL of the variable $-t/s$ which can be analytically continued to any domain from the region $|t/s|<1$.

In the twofold MB integrals where one more integration (over a variable different from w) can be hardly performed in gamma functions, one performs it with w in a vicinity of an integer point $w=n=0,1,2,\dots$, in expansion in $z=w-n$, with a sufficient accuracy. Then one obtains powers series where, in addition to $1/n^6$, $S_1(n)/n^5, \ldots$, quantities like $S_{ik}(n) = \sum_{j=1}^n j^{-i} S_k(j)$, $S_{ikl}(n) = \sum_{j=1}^n j^{-i} S_{kl}(j)$ appear. These series are also summed up in terms of HPL. Here one could use procedures connected with nested sums, realized in FORM [18] and described in [19] (see also [20]). (I preferred, however, to derive and check the necessary summation formulae myself because do not use FORM.)

Eventually we arrive at the following result:

$$T^{(0)}(s,t;\epsilon) = -\frac{(i\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon})^3}{s^3(-t)^{1+3\epsilon}} \sum_{i=0}^{6} \frac{c_j(x,L)}{\epsilon^j}, \tag{5}$$

where γ_{E} is the Euler constant, $x = -t/s$, $L = \ln(s/t)$, and

$$\begin{aligned}
c_6 &= \frac{16}{9}, \qquad c_5 = -\frac{5}{3}L, \qquad c_4 = -\frac{3}{2}\pi^2, \\
c_3 &= 3\big(H_{0,0,1}(x) + L H_{0,1}(x)\big) + \frac{3}{2}\big(L^2+\pi^2\big)H_1(x) - \frac{11}{12}\pi^2 L - \frac{131}{9}\zeta_3, \\
c_2 &= -3\big(17H_{0,0,0,1}(x) + H_{0,0,1,1}(x) + H_{0,1,0,1}(x) + H_{1,0,0,1}(x)\big) \\
&\quad - L\big(37H_{0,0,1}(x) + 3H_{0,1,1}(x) + 3H_{1,0,1}(x)\big) - \frac{3}{2}\big(L^2+\pi^2\big)H_{1,1}(x) \\
&\quad - \left(\frac{23}{2}L^2 + 8\pi^2\right)H_{0,1}(x) - \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta_3\right)H_1(x) + \frac{49}{3}\zeta_3 L - \frac{1411}{1080}\pi^4, \\
c_1 &= 3\big(81H_{0,0,0,0,1}(x) + 41H_{0,0,0,1,1}(x) + 37H_{0,0,1,0,1}(x) + H_{0,0,1,1,1}(x) + 33H_{0,1,0,0,1}(x) + H_{0,1,0,1,1}(x) \\
&\qquad + H_{0,1,1,0,1}(x) + 29H_{1,0,0,0,1}(x) + H_{1,0,0,1,1}(x) + H_{1,0,1,0,1}(x) + H_{1,1,0,0,1}(x)\big) \\
&\quad + L\big(177H_{0,0,0,1}(x) + 85H_{0,0,1,1}(x) + 73H_{0,1,0,1}(x) + 3H_{0,1,1,1}(x) \\
&\qquad\quad + 61H_{1,0,0,1}(x) + 3H_{1,0,1,1}(x) + 3H_{1,1,0,1}(x)\big) \\
&\quad + \left(\frac{119}{2}L^2 + \frac{139}{12}\pi^2\right)H_{0,0,1}(x) + \left(\frac{47}{2}L^2 + 20\pi^2\right)H_{0,1,1}(x) + \left(\frac{35}{2}L^2 + 14\pi^2\right)H_{1,0,1}(x) \\
&\quad + \frac{3}{2}\big(L^2+\pi^2\big)H_{1,1,1}(x) + \left(\frac{23}{2}L^3 + \frac{83}{12}\pi^2 L - 96\zeta_3\right)H_{0,1}(x) + \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta_3\right)H_{1,1}(x) \\
&\quad + \left(\frac{9}{8}L^4 + \frac{25}{8}\pi^2 L^2 - 58\zeta_3 L + \frac{13}{8}\pi^4\right)H_1(x) - \frac{503}{1440}\pi^4 L + \frac{73}{4}\pi^2\zeta_3 - \frac{301}{15}\zeta_5, \\
c_0 &= -\big(951H_{0,0,0,0,0,1}(x) + 819H_{0,0,0,0,1,1}(x) + 699H_{0,0,0,1,0,1}(x) + 195H_{0,0,0,1,1,1}(x) \\
&\qquad + 547H_{0,0,1,0,0,1}(x) + 231H_{0,0,1,0,1,1}(x) + 159H_{0,0,1,1,0,1}(x) + 3H_{0,0,1,1,1,1}(x) \\
&\qquad + 363H_{0,1,0,0,0,1}(x) + 267H_{0,1,0,0,1,1}(x) + 195H_{0,1,0,1,0,1}(x) + 3H_{0,1,0,1,1,1}(x) \\
&\qquad + 123H_{0,1,1,0,0,1}(x) + 3H_{0,1,1,0,1,1}(x) + 3H_{0,1,1,1,0,1}(x) + 147H_{1,0,0,0,0,1}(x) \\
&\qquad + 303H_{1,0,0,0,1,1}(x) + 231H_{1,0,0,1,0,1}(x) + 3H_{1,0,0,1,1,1}(x) + 159H_{1,0,1,0,0,1}(x) \\
&\qquad + 3H_{1,0,1,0,1,1}(x) + 3H_{1,0,1,1,0,1}(x) + 87H_{1,1,0,0,0,1}(x) + 3H_{1,1,0,0,1,1}(x) \\
&\qquad + 3H_{1,1,0,1,0,1}(x) + 3H_{1,1,1,0,0,1}(x)\big) \\
&\quad - L\big(729H_{0,0,0,0,1}(x) + 537H_{0,0,0,1,1}(x) + 445H_{0,0,1,0,1}(x) + 133H_{0,0,1,1,1}(x) \\
&\qquad\quad + 321H_{0,1,0,0,1}(x) + 169H_{0,1,0,1,1}(x) + 97H_{0,1,1,0,1}(x) + 3H_{0,1,1,1,1}(x) \\
&\qquad\quad + 165H_{1,0,0,0,1}(x) + 205H_{1,0,0,1,1}(x) + 133H_{1,0,1,0,1}(x) + 3H_{1,0,1,1,1}(x) \\
&\qquad\quad + 61H_{1,1,0,0,1}(x) + 3H_{1,1,0,1,1}(x) + 3H_{1,1,1,0,1}(x)\big)
\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{531}{2}L^2+\frac{89}{4}\pi^2\right)H_{0,0,0,1}(x)-\left(\frac{311}{2}L^2+\frac{619}{12}\pi^2\right)H_{0,0,1,1}(x)-\left(\frac{247}{2}L^2+\frac{307}{12}\pi^2\right)H_{0,1,0,1}(x)\\
&-\left(\frac{71}{2}L^2+32\pi^2\right)H_{0,1,1,1}(x)-\left(\frac{151}{2}L^2-\frac{197}{12}\pi^2\right)H_{1,0,0,1}(x)-\left(\frac{107}{2}L^2+50\pi^2\right)H_{1,0,1,1}(x)\\
&-\left(\frac{35}{2}L^2+14\pi^2\right)H_{1,1,0,1}(x)-\frac{3}{2}\left(L^2+\pi^2\right)H_{1,1,1,1}(x)\\
&-\left(\frac{119}{2}L^3+\frac{317}{12}\pi^2L-455\zeta_3\right)H_{0,0,1}(x)-\left(\frac{47}{2}L^3+\frac{179}{12}\pi^2L-120\zeta_3\right)H_{0,1,1}(x)\\
&-\left(\frac{35}{2}L^3+\frac{35}{12}\pi^2L-156\zeta_3\right)H_{1,0,1}(x)-\left(\frac{3}{2}L^3+\pi^2L-3\zeta_3\right)H_{1,1,1}(x)\\
&-\left(\frac{69}{8}L^4+\frac{101}{8}\pi^2L^2-291\zeta_3L+\frac{559}{90}\pi^4\right)H_{0,1}(x)-\left(\frac{9}{8}L^4+\frac{25}{8}\pi^2L^2-58\zeta_3L+\frac{13}{8}\pi^4\right)H_{1,1}(x)\\
&-\left(\frac{27}{40}L^5+\frac{25}{8}\pi^2L^3-\frac{183}{2}\zeta_3L^2+\frac{131}{60}\pi^4L-\frac{37}{12}\pi^2\zeta_3+57\zeta_5\right)H_1(x)\\
&+\left(\frac{223}{12}\pi^2\zeta_3+149\zeta_5\right)L+\frac{167}{9}\zeta_3^2-\frac{624607}{544320}\pi^6. \qquad (6)
\end{aligned}
$$

Here $\zeta_3=\zeta(3)$, $\zeta_5=\zeta(5)$ and $\zeta(z)$ is the Riemann zeta function. The functions $H_{a_1,a_2,\ldots,a_n}(x)\equiv H(a_1,a_2,\ldots,a_n;x)$, with $a_i=1,0,-1$, are HPL [17] which are recursively defined by

$$H(a_1,a_2,\ldots,a_n;x)=\int_0^x f(a_1;t)H(a_2,\ldots,a_n;t),$$

where

$$f(\pm1;x)=\frac{1}{1\mp x},\qquad f(0;x)=\frac{1}{x},\qquad H(\pm1;x)=\mp\ln(1\mp x),\qquad H(0;x)=\ln x.$$

In (6), only HPL with parameters 0 and 1 are involved. If a given HPL involves only parameters $a_i=0$ and 1 and the number of these parameters is less or equal to four, it can be expressed [17] in terms of usual polylogarithms $\mathrm{Li}(x)$ [21] and generalized polylogarithms [22]

$$S_{a,b}(x)=\frac{(-1)^{a+b-1}}{(a-1)!b!}\int_0^1\frac{\ln^{a-1}(t)\ln^b(1-xt)}{t}\,\mathrm{d}t.$$

(See [14] where the coefficients c_j, with $j\geqslant 2$ are expressed in terms of (generalized) polylogarithms.)

The above result was confirmed [23] with the help of numerical integration in the space of alpha parameters [24]. Another natural check of the result is its agreement with the leading power Regge asymptotic behaviour [15] which was evaluated by an independent method based on the strategy of expansion by regions [16].

The procedure described above can be applied, in a similar way, to the calculation of any massless planar on-shell triple box. At a first step, one has to take care of the following four gamma functions in (3):

$$\Gamma(2-a_{12}-\epsilon+z_2),\quad \Gamma(2-a_{23}-\epsilon+z_3),\quad \Gamma(2-a_{59(10)}-\epsilon-z_5-z_7),\quad \Gamma(2-a_{589}-\epsilon-z_6-z_7).$$

This procedure gives a decomposition similar to $2T_{0001}+2T_{0010}+\cdots$. Next steps will be also generalizations of the corresponding steps in the evaluation of (4). Hopefully, such a procedure can be made automatic by means of computer algebra.

The result presented above shows that analytical calculations of four-point on-shell massless Feynman diagrams at the three-loop level are quite possible so that one may think of evaluating three-loop virtual corrections to various scattering processes.

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References

[1] G. 't Hooft, M. Veltman, Nucl. Phys. B 44 (1972) 189;
C.G. Bollini, J.J. Giambiagi, Nuovo Cimento B 12 (1972) 20.
[2] V.A. Smirnov, Phys. Lett. B 460 (1999) 397.
[3] V.A. Smirnov, O.L. Veretin, Nucl. Phys. B 566 (2000) 469.
[4] J.B. Tausk, Phys. Lett. B 469 (1999) 225.
[5] C. Anastasiou, E.W.N. Glover, C. Oleari, Nucl. Phys. B 575 (2000) 416;
C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi, J.B. Tausk, Nucl. Phys. B 580 (2000) 577;
T. Gehrmann, E. Remiddi, Nucl. Phys. B (Proc. Suppl.) 89 (2000) 251.
[6] C. Anastasiou, J.B. Tausk, M.E. Tejeda-Yeomans, Nucl. Phys. B (Proc. Suppl.) 89 (2000) 262.
[7] T. Gehrmann, E. Remiddi, Nucl. Phys. B 640 (2002) 379.
[8] Z. Bern, L. Dixon, A. Ghinculov, Phys. Rev. D 63 (2001) 053007;
C. Anastasiou, E.W.N. Glover, C. Oleari, M.E. Tejeda-Yeomans, Nucl. Phys. B 601 (2001) 318;
C. Anastasiou, E.W.N. Glover, C. Oleari, M.E. Tejeda-Yeomans, Nucl. Phys. B 601 (2001) 341;
C. Anastasiou, E.W.N. Glover, C. Oleari, M.E. Tejeda-Yeomans, Nucl. Phys. B 605 (2001) 486;
E.W.N. Glover, C. Oleari, M.E. Tejeda-Yeomans, Nucl. Phys. B 605 (2001) 467;
Z. Bern, A. De Freitas, L.J. Dixon, JHEP 0109 (2001) 037;
Z. Bern, A. De Freitas, L.J. Dixon, A. Ghinculov, H.L. Wong, JHEP 0111 (2001) 031;
Z. Bern, A. De Freitas, L.J. Dixon, JHEP 0203 (2002) 018;
C. Anastasiou, E.W.N. Glover, M.E. Tejeda-Yeomans, Nucl. Phys. B 629 (2002) 255;
E.W.N. Glover, hep-ph/0211412;
E.W.N. Glover, M.E. Tejeda-Yeomans, hep-ph/0304169;
Z. Bern, A. De Freitas, L.J. Dixon, hep-ph/0304168.
[9] V.A. Smirnov, Phys. Lett. B 491 (2000) 130;
V.A. Smirnov, Phys. Lett. B 500 (2001) 330.
[10] T. Gehrmann, E. Remiddi, Nucl. Phys. B 601 (2001) 248;
T. Gehrmann, E. Remiddi, Nucl. Phys. B 601 (2001) 287.
[11] L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis, E. Remiddi, Nucl. Phys. B 627 (2002) 107;
L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis, E. Remiddi, Nucl. Phys. B 642 (2002) 227.
[12] V.A. Smirnov, Phys. Lett. B 524 (2002) 129.
[13] V.A. Smirnov, hep-ph/0209177.
[14] V.A. Smirnov, hep-ph/0209295.
[15] V.A. Smirnov, Phys. Lett. B 547 (2002) 239.
[16] M. Beneke, V.A. Smirnov, Nucl. Phys. B 522 (1998) 321;
V.A. Smirnov, E.R. Rakhmetov, Teor. Mat. Fiz. 120 (1999) 64;
V.A. Smirnov, Phys. Lett. B 465 (1999) 226;
V.A. Smirnov, Applied Asymptotic Expansions in Momenta, Masses, Springer, Berlin, 2002.
[17] E. Remiddi, J.A.M. Vermaseren, Int. J. Mod. Phys. A 15 (2000) 725.
[18] J.A.M. Vermaseren, Symbolic Manipulation with FORM, Version 2, Computer Algebra Nederland, Amsterdam, 1991.
[19] J.A.M. Vermaseren, Int. J. Mod. Phys. A 14 (1999) 2037.
[20] S. Moch, P. Uwer, S. Weinzierl, J. Math. Phys. 43 (2002) 3363;
S. Moch, P. Uwer, S. Weinzierl, Phys. Rev. D 66 (2002) 114001.

[21] L. Lewin, Polylogarithms and Associated Functions, North-Holland, Amsterdam, 1981.
[22] K.S. Kolbig, J.A. Mignaco, E. Remiddi, B.I.T. 10 (1970) 38;
K.S. Kolbig, Math. Comput. 39 (1982) 647;
A. Devoto, D.W. Duke, Riv. Nuovo Cimento 7 (1984) 1.
[23] T. Binoth, G. Heinrich, hep-ph/0305234.
[24] T. Binoth, G. Heinrich, Nucl. Phys. B 585 (2000) 741.