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3-Lie algebras with an ideal N ☆

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ABSTRACT

We define the hypo-nilpotent ideal in n-Lie algebras and obtain all solvable 3-Lie algebras with an m-dimensional simplest filiform 3-Lie algebra as a maximal hypo-nilpotent ideal. We prove that the dimension of such solvable 3-Lie algebras is at most $m + 2$, and there is no solvable 3-Lie algebra with the simplest filiform 3-Lie algebra as the nilradical.

1. Introduction

Recently, the study on n-Lie algebras has attracted much attention largely due to their close connection with the Nambu mechanics and geometries [1,2], Poisson and Jacobi manifolds, and Hamiltonian mechanics [3–6].

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In 1985, Filippov [7] introduced the concept of n-Lie algebras. He classified n-Lie algebras of dimension $n+1$ over an algebraically closed field of characteristic zero. Kasymov [8] developed the structural notions such as simplicity, nilpotency, representations and Cardan subalgebras. Bai and Meng described the Frattini subalgebras, formations, and the centroid of n-Lie algebras, and constructed some finite dimensional n-Lie algebras over fields of characteristic $p \geqslant 0$ [9–11]. Ling [12] proved that for $n \geqslant 3$ all finite dimensional simple n-Lie algebras over an algebraically closed field F of characteristic 0 are isomorphic to the vector product on F^{n+1}. Up to now, infinite dimensional simple n-Lie algebras over fields of characteristic $p \geqslant 0$ are only Jacobian algebras and their quotient algebras [1–15]. There are other results on representations and structures of n-Lie algebras [16–19].

The organization for the rest of this paper is as follows. Section 2 introduces some basic notions, defines hypo-nilpotent ideal of n-Lie algebras and proves some lemmas. Section 3 studies 3-Lie algebras with the simplest filiform nilradicals. Section 4 describes the structures of solvable 3-Lie algebras with a maximal hypo-nilpotent ideal N, where N is the simplest filiform 3-Lie algebra.

Throughout this paper we consider n-Lie algebras over a field F of characteristic zero, and with $n \geqslant 3$.

2. Fundamental notions

First we introduce some notions of n-Lie algebras (see [7,8]). A vector space A over a field F is an n-Lie algebra if there is an n-ary multilinear operation $[, \ldots,]$ satisfying the following identities

$$[x_1, \ldots, x_n] = (-1)^{\tau(\sigma)}[x_{\sigma(1)}, \ldots, x_{\sigma(n)}], \tag{2.1}$$

and

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n], \tag{2.2}$$

where σ runs over the symmetric group S_n and the number $\tau(\sigma)$ is equal to 0 or 1 depending on the parity of the permutation σ.

A derivation of an n-Lie algebra A is a linear map $D : A \to A$, such that for any elements $x_1, \ldots x_n$ of A

$$D([x_1, \ldots, x_n]) = \sum_{i=1}^{n} [x_1, \ldots, D(x_i), \ldots, x_n].$$

The set of all derivations of A is a subalgebra of Lie algebra $\mathrm{gl}(A)$. This subalgebra is called the derivation algebra of A, and is denoted by $\mathrm{Der}A$. The map $\mathrm{ad}(x_1, \ldots, x_{n-1}) : A \to A$ defined by $\mathrm{ad}(x_1, \ldots, x_{n-1})(x_n) = [x_1, \ldots, x_n]$ for $x_1, \ldots, x_n \in A$ is called a left multiplication. It follows from (2.2) that $\mathrm{ad}(x_1, \ldots, x_{n-1})$ is a derivation. The set of all finite linear combinations of left multiplications is an ideal of $\mathrm{Der}A$ and is denoted by $\mathrm{ad}(A)$. Every element in $\mathrm{ad}(A)$ is by definition an inner derivation, and for $\forall x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$ of A,

$$\begin{aligned} &[\mathrm{ad}(x_1, \ldots, x_{n-1}), \mathrm{ad}(y_1, \ldots, y_{n-1})] \\ &\quad = \mathrm{ad}(x_1, \ldots, x_{n-1})\mathrm{ad}(y_1, \ldots, y_{n-1}) - \mathrm{ad}(y_1, \ldots, y_{n-1})\mathrm{ad}(x_1, \ldots, x_{n-1}) \\ &\quad = \textstyle\sum_{i=1}^{n-1} \mathrm{ad}(y_1, \ldots, [x_1, \ldots, x_{n-1}, y_i], \ldots, y_{n-1}). \end{aligned} \tag{2.2*}$$

Let $A_1, A_2, \ldots, A_n$ be subalgebras of n-Lie algebra A and $[A_1, A_2, \ldots, A_n]$ stands for the subspace of A generated by all vectors $[x_1, x_2, \ldots, x_n]$, where $x_i \in A_i$ for $i = 1, 2, \ldots, n$. The subalgebra $[A, A, \ldots, A]$ is called the derived algebra of A, and is denoted by A^1. If $A^1 = 0$, then A is called an abelian n-Lie algebra.

An ideal of an n-Lie algebra A is a subspace I such that $[I, A, \ldots, A] \subseteq I$. If $A^1 \neq 0$ and A has no ideals except for 0 and itself, then A is by definition a simple n-Lie algebra.

An ideal I of an n-Lie algebra A is called a solvable ideal, if $I^{(r)} = 0$ for some $r \geqslant 0$, where $I^{(0)} = I$ and $I^{(s)}$ is defined by induction,

$$I^{(s+1)} = [I^{(s)}, I^{(s)}, A, \ldots, A]$$

for $s \geqslant 0$. When $A = I, A$ is a solvable n-Lie algebra.

An ideal I of an n-Lie algebra A is called a nilpotent ideal, if I satisfies $I^r = 0$ for some $r \geqslant 0$, where $I^0 = I$ and I^r is defined by induction, $I^{r+1} = [I^r, I, A, \ldots, A]$ for $r \geqslant 0$. If $I = A, A$ is called a nilpotent n-Lie algebra.

The sum of two nilpotent ideals of A is nilpotent, and the largest nilpotent ideal of A is called the nilradical of A, and is denoted by $NR(A)$.

We refer to the ideals $A^{(s+1)} = [A^{(s)}, A^{(s)}, A, \ldots, A], s = 0, 1, \ldots$, and $A^{s+1} = [A^s, A, A, \ldots, A], s = 0, 1, \ldots$ respectively, as the derived series and the lower central series of A. We shall use the notations DS and CS for sets of integers denoting the dimensions of subalgebras in derived series and lower central series of A, respectively.

Denote by A^* an associative algebra generated by all operators $\mathrm{ad}(x)$, where $x = (x_1, \ldots, x_{n-1}) \in A^{(n-1)}$. If I is an ideal of A, denote by $I^*, K(I)$ and $\mathrm{ad}(I, A)$ respectively the subalgebra of A^*, the ideal of A^* and the subalgebra of $\mathrm{ad}(A)$ generated by the operators of the form $\mathrm{ad}(c, x_1, \ldots, x_{n-2}), c \in I, x_i \in A, i = 1, \ldots, n-2$. It follows at once from (2.2*) that $K(I) = I^* \cdot A^* = A^* \cdot I^*$, and $\mathrm{ad}(I, A)$ is an ideal of $\mathrm{ad}(A)$.

Lemma 2.1 [8]. *An ideal I of an n-Lie algebra A is a nilpotent ideal if and only if $K(I)$ is a nilpotent ideal of the associative algebra A^*.*

An ideal I of an n-Lie algebra A may not be a nilpotent ideal although it is a nilpotent subalgebra. This property is different from that of Lie algebras. In the following, we concern such types of ideals of n-Lie algebras.

Definition 2.1. Let A be an n-Lie algebra and I be an ideal of A. If I is a nilpotent subalgebra but is not a nilpotent ideal, then I is called a hypo-nilpotent ideal of A. If I is not properly contained in any hypo-nilpotent ideals, then I is called a maximal hypo-nilpotent ideal of A.

Remark 2.1

(1) From Definition 2.1, a hypo-nilpotent ideal of A is a proper ideal.
(2) The nilradical $NR(A)$ is properly contained in maximal hypo-nilpotent ideals. In fact, if N is a maximal hypo-nilpotent ideal, then $J = N + NR(A)$ is an ideal of A. And J is not a nilpotent ideal since the hypo-nilpotency of N. Hence J is also a hypo-nilpotent ideal. Since N is maximal, we get $J = N$. Therefore, $NR(A)$ is properly contained in N.
(3) The sum of two hypo-nilpotent ideals of A may not be hypo-nilpotent. See the following example.

Example 2.1. Let A be a 6-dimensional 4-Lie algebra over F, and its multiplication table in the basis $e_1, \ldots, e_6$ be as follows:

$$\begin{cases} [e_1, e_4, e_5, e_6] = e_1 + \alpha e_2, \\ [e_2, e_4, e_5, e_6] = e_2 + \alpha e_3, \\ [e_3, e_4, e_5, e_6] = e_3, \end{cases}$$

where $\alpha \in F, \alpha \neq 0$. Set $I_1 = Fe_1 + Fe_2 + Fe_3 + Fe_4, I_2 = Fe_1 + Fe_2 + Fe_3 + Fe_5, I_3 = Fe_1 + Fe_2 + Fe_3 + Fe_4 + Fe_6$ and $I_4 = Fe_1 + Fe_2 + Fe_3 + Fe_5 + Fe_6$. For $k = 1, 2, 3, 4, I_k$ are hypo-nilpotent ideals of A since

$$I_k^{s+1} = [I_k^s, I_k, A, A] = Fe_1 + Fe_2 + Fe_3 \neq 0,$$
$$[I_k, I_k, I_k, I_k] = 0, \quad k = 1, 2, 3, 4; s \geqslant 0.$$

A simple computation shows that $I_1 + I_2$ is a hypo-nilpotent ideal, but $I_3 + I_4$ is not nilpotent.

Definition 2.2. Let A be a nilpotent m-dimensional n-Lie algebra over $F(m > n)$. If the lower central series $A^i (i \geqslant 1)$ satisfy the following condition:

$$\dim A^i = m - n - i + 1, \quad i \geqslant 1,$$

A is called a filiform n-Lie algebra.

Definition 2.3. Let A be an m-dimensional filiform n-Lie algebra ($m \geqslant n + 2$). If there is a basis $e_1, \ldots, e_m$ of A such that

$$[e_1, \ldots, e_{n-1}, e_j] = e_{j-1}, n + 1 \leqslant j \leqslant m, \tag{2.3}$$

and other brackets of the basis vectors equal zero, then A is called the simplest filiform n-Lie algebra.

For later convenience, we give the inner derivation algebra of m-dimensional the simplest filiform 3-Lie algebra.

Lemma 2.2. *Let N be m-dimensional the simplest filiform* 3*-Lie algebra with a basis $e_1, \ldots, e_m$ satisfying* (2.3). *Then the inner derivation algebra* $\mathrm{ad}(N)$ *has a basis* $\mathrm{ad}(e_1, e_2), \mathrm{ad}(e_1, e_j), \mathrm{ad}(e_2, e_j), j = 4, 5, \ldots, m$. *And with respect to the basis $e_1, \ldots, e_m$, $\mathrm{ad}(e_k, e_l)$ is represented by the following matrix form*

$$\mathrm{ad}(e_1, e_2) = \sum_{j=4}^{m} E_{jj-1}, \quad \mathrm{ad}(e_1, e_i) = E_{2i-1}, \quad \mathrm{ad}(e_2, e_i) = E_{1i-1}, \quad 4 \leqslant i \leqslant m, \tag{2.4}$$

where E_{ij} is the $(m \times m)$ matrix unit.

Proof. From (2.3) we have

$$\mathrm{ad}(e_1, e_2)(e_i) = e_{i-1}, \quad \mathrm{ad}(e_1, e_i)(e_2) = -e_{i-1}, \quad \mathrm{ad}(e_2, e_i)(e_1) = e_{i-1}, \quad 4 \leqslant i \leqslant m.$$

Then the matrix form of $\mathrm{ad}(e_i, e_j)$ with respect to the basis $e_1, \ldots, e_m$ is

$$\mathrm{ad}(e_1, e_2) = \sum_{j=4}^{m} E_{jj-1}, \quad \mathrm{ad}(e_1, e_i) = E_{2i-1}, \quad \mathrm{ad}(e_2, e_i) = E_{1i-1}, \quad 4 \leqslant i \leqslant m.$$

Therefore,

$$\mathrm{ad}(A) = F\mathrm{ad}(e_1, e_2) + \sum_{i=4}^{m} F\mathrm{ad}(e_1, e_i) + \sum_{i=4}^{m} F\mathrm{ad}(e_2, e_i). \qquad \square$$

Unless stated otherwise, in the following N is m-dimensional the simplest filiform 3-Lie algebra with the multiplication table (2.3) in the basis $e_1, \ldots, e_m$, and $m \geqslant 5$.

Now let A be an $(m + 1)$-dimensional 3-Lie algebra with ideal N. If x is a vector of A which is not contained in N. Then $x, e_1, \ldots, e_m$ is a basis of A, and the computation table in the basis $x, e_1, \ldots, e_m$ is given by

$$\begin{cases} [e_1, e_2, e_j] = e_{j-1}, & 4 \leqslant j \leqslant m, \\ [x, e_i, e_j] = \sum_{k=1}^{m} a_{ij}^k e_k, & 1 \leqslant i, j \leqslant m, \end{cases} \tag{2.5}$$

where $a_{ij}^k \in F, a_{ij}^k = -a_{ji}^k, 1 \leqslant i, j \leqslant m$. Therefore, the following $\left(\frac{m(m-1)}{2} \times m\right)$ matrix M determines the structure of A

$$\begin{pmatrix}
a_{12}^{1} & a_{12}^{2} & a_{12}^{3} & a_{12}^{4} & a_{12}^{5} & \cdot & a_{12}^{m-3} & a_{12}^{m-2} & a_{12}^{m-1} & a_{12}^{m} \\
a_{13}^{1} & a_{13}^{2} & a_{13}^{3} & a_{13}^{4} & a_{13}^{5} & \cdot & a_{13}^{m-3} & a_{13}^{m-2} & a_{13}^{m-1} & a_{13}^{m} \\
a_{14}^{1} & a_{14}^{2} & a_{14}^{3} & a_{14}^{4} & a_{14}^{5} & \cdot & a_{14}^{m-3} & a_{14}^{m-2} & a_{14}^{m-1} & a_{14}^{m} \\
a_{15}^{1} & a_{15}^{2} & a_{15}^{3} & a_{15}^{4} & a_{15}^{5} & \cdot & a_{15}^{m-3} & a_{15}^{m-2} & a_{15}^{m-1} & a_{15}^{m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & . & \vdots & \vdots & \vdots & \vdots \\
a_{1m-2}^{1} & a_{1m-2}^{2} & a_{1m-2}^{3} & a_{1m-2}^{4} & a_{1m-2}^{5} & \cdot & a_{1m-2}^{m-3} & a_{1m-2}^{m-2} & a_{1m-2}^{m-1} & a_{1m-2}^{m} \\
a_{1m-1}^{1} & a_{1m-1}^{2} & a_{1m-1}^{3} & a_{1m-1}^{4} & a_{1m-1}^{5} & \cdot & a_{1m-1}^{m-3} & a_{1m-1}^{m-2} & a_{1m-1}^{m-1} & a_{1m-1}^{m} \\
a_{1m}^{1} & a_{1m}^{2} & a_{1m}^{3} & a_{1m}^{4} & a_{1m}^{5} & \cdot & a_{1m}^{m-3} & a_{1m}^{m-2} & a_{1m}^{m-1} & a_{1m}^{m} \\
a_{23}^{1} & a_{23}^{2} & a_{23}^{3} & a_{23}^{4} & a_{23}^{5} & \cdot & a_{23}^{m-3} & a_{23}^{m-2} & a_{23}^{m-1} & a_{23}^{m} \\
a_{24}^{1} & a_{24}^{2} & a_{24}^{3} & a_{24}^{4} & a_{24}^{5} & \cdot & a_{24}^{m-3} & a_{24}^{m-2} & a_{24}^{m-1} & a_{24}^{m} \\
a_{25}^{1} & a_{25}^{2} & a_{25}^{3} & a_{25}^{4} & a_{25}^{5} & \cdot & a_{25}^{m-3} & a_{25}^{m-2} & a_{25}^{m-1} & a_{25}^{m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & . & \vdots & \vdots & \vdots & \vdots \\
a_{2m-2}^{1} & a_{2m-2}^{2} & a_{2m-2}^{3} & a_{2m-2}^{4} & a_{2m-2}^{5} & \cdot & a_{2m-2}^{m-3} & a_{2m-2}^{m-2} & a_{2m-2}^{m-1} & a_{2m-2}^{m} \\
a_{2m-1}^{1} & a_{2m-1}^{2} & a_{2m-1}^{3} & a_{2m-1}^{4} & a_{2m-1}^{5} & \cdot & a_{2m-1}^{m-3} & a_{2m-1}^{m-2} & a_{2m-1}^{m-1} & a_{2m-1}^{m} \\
a_{2m}^{1} & a_{2m}^{2} & a_{2m}^{3} & a_{2m}^{4} & a_{2m}^{5} & \cdot & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & a_{2m}^{m} \\
a_{34}^{1} & a_{34}^{2} & a_{34}^{3} & a_{34}^{4} & a_{34}^{5} & \cdot & a_{34}^{m-3} & a_{34}^{m-2} & a_{34}^{m-1} & a_{34}^{m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & . & \vdots & \vdots & \vdots & \vdots \\
a_{m-1m}^{1} & a_{m-1m}^{2} & a_{m-1m}^{3} & a_{m-1m}^{4} & a_{m-1m}^{5} & \cdot & a_{m-1m}^{m-3} & a_{m-1m}^{m-2} & a_{m-1m}^{m-1} & a_{m-1m}^{m}
\end{pmatrix}. \tag{2.5*}$$

The matrix M is called the structural matrix of A with respect to the basis $x, e_1, \ldots, e_m$. By the above notations we have following result.

Lemma 2.3. *Let A be an $(m+1)$-dimensional 3-Lie algebra with the ideal N. Then the structural matrix M is of the following form*

$$M = \begin{pmatrix}
a_{12}^{1} & a_{12}^{2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{12}^{m} \\
0 & 0 & a_{13}^{3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{14}^{4} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{1m}^{m-2} & 0 & a_{15}^{5} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & a_{1m}^{5} & a_{1m}^{6} & a_{1m}^{7} & \cdots & 0 & a_{1m-2}^{m-2} & 0 & 0 \\
0 & 0 & a_{1m}^{4} & a_{1m}^{5} & a_{1m}^{6} & \cdots & a_{1m}^{m-2} & 0 & a_{1m-1}^{m-1} & 0 \\
0 & 0 & a_{1m}^{3} & a_{1m}^{4} & a_{1m}^{5} & \cdots & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & a_{1m}^{m} \\
0 & 0 & a_{23}^{3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{24}^{4} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{2m}^{m-2} & 0 & a_{25}^{5} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & a_{2m}^{5} & a_{2m}^{6} & a_{2m}^{7} & \cdots & 0 & a_{2m-2}^{m-2} & 0 & 0 \\
0 & 0 & a_{2m}^{4} & a_{2m}^{5} & a_{2m}^{6} & \cdots & a_{2m}^{m-2} & 0 & a_{2m-1}^{m-1} & 0 \\
0 & 0 & a_{2m}^{3} & a_{2m}^{4} & a_{2m}^{5} & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & a_{2m}^{m} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{2.6}$$

where a_{ij}^k *for* $1 \leqslant i, j, k \leqslant m$ *satisfy the identities*

$$a_{1i}^i = a_{1m}^m + (m-i)a_{12}^2, a_{2i}^i = a_{2m}^m - (m-i)a_{12}^1, \quad i = 3, 4, \ldots, m; \tag{2.7}$$

$$\begin{pmatrix} a_{1m}^3 & a_{2m}^3 \\ a_{1m}^4 & a_{2m}^4 \\ \cdots & \cdots \\ a_{1m}^{m-2} & a_{2m}^{m-2} \\ a_{1m}^m & a_{2m}^m \end{pmatrix} \begin{pmatrix} a_{12}^1 \\ a_{12}^2 \end{pmatrix} = 0. \tag{2.8}$$

Proof. Firstly, since N is an ideal of A and $\dim A = m+1$, the derived algebra $A^1 = [A, A, A] = [A, A, N] \subseteq N$. Then the structural matrix M is (2.5*) with respect to the basis $x, e_1, \ldots, e_m$.

Substituting the relations (2.5) into the Jacobi identities

$$[[x, e_1, e_2], e_1, e_j] = [[x, e_1, e_j], e_1, e_2] + [x, e_1, [e_2, e_1, e_j]], \quad 3 \leqslant j \leqslant m,$$

we obtain

$$a_{1j}^1 = a_{1j}^2 = 0, a_{1j}^{j+1} = a_{1j}^{j+2} = \cdots = a_{1j}^m = 0, \quad 3 \leqslant j \leqslant m-1;$$

and

$$a_{1j-1}^{j-1} = a_{1j}^j + a_{12}^2, \quad 4 \leqslant j \leqslant m; \qquad a_{1j}^k = a_{1j-1}^{k-1}, \quad 4 \leqslant k < j \leqslant m.$$

Secondly, imposing the Jacobi identities for $\{[x, e_1, e_2], e_2, e_j\}, 3 \leqslant j \leqslant m$, we get

$$a_{2j}^1 = a_{2j}^2 = 0, a_{2j}^{j+1} = a_{2j}^{j+2} = \cdots = a_{2j}^m = 0, \quad 3 \leqslant j \leqslant m-1;$$

and

$$a_{2j-1}^{j-1} = a_{2j}^j - a_{12}^1, \quad 4 \leqslant j \leqslant m; \qquad a_{2j}^k = a_{2j-1}^{k-1}, \quad 4 \leqslant k < j \leqslant m.$$

Now comparing the coefficients in each of the following identities,

$$\begin{aligned}
&[[x, e_1, e_2], e_4, e_m] = [[x, e_4, e_m], e_1, e_2], \\
&[[x, e_1, e_2], e_5, e_{m-1}] = [[x, e_5, e_{m-1}], e_1, e_2], \\
&[[x, e_1, e_4], e_1, e_m] = [[x, e_1, e_m], e_1, e_4], \\
&[[x, e_1, e_5], e_2, e_m] = [[x, e_2, e_m], e_1, e_5] + [x, e_{m-1}, e_5], \\
&[[x, e_1, e_m], e_2, e_5] = [[x, e_2, e_5], e_1, e_m] + [x, e_4, e_m], \\
&[[x, e_2, e_4], e_2, e_m] = [[x, e_2, e_m], e_2, e_4], \\
&[[x, e_1, e_i], e_2, e_j] = [[x, e_2, e_j], e_1, e_i] + [x, [e_1, e_2, e_j], e_i] = 0, \quad 3 \leqslant i, j \leqslant m,
\end{aligned}$$

we get $a_{2m}^2 = a_{2m}^1 = 0, a_{1m}^1 = a_{1m}^2 = 0$, and $a_{ij}^k = 0, 3 \leqslant i, j \leqslant m, 1 \leqslant k \leqslant m$.

Therefore, the matrix M has the form

$$\begin{pmatrix}
a_{12}^1 & a_{12}^2 & a_{12}^3 & a_{12}^4 & a_{12}^5 & \cdots & a_{12}^{m-3} & a_{12}^{m-2} & a_{12}^{m-1} & a_{12}^m \\
0 & 0 & a_{13}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{1m}^{m-1} & a_{14}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{1m}^{m-2} & a_{1m}^{m-1} & a_{15}^5 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{1m}^{m-3} & a_{1m}^{m-2} & a_{1m}^{m-1} & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & a_{1m}^5 & a_{1m}^6 & a_{1m}^7 & \cdots & a_{1m}^{m-1} & a_{1m-2}^{m-2} & 0 & 0 \\
0 & 0 & a_{1m}^4 & a_{1m}^5 & a_{1m}^6 & \cdots & a_{1m}^{m-2} & a_{1m}^{m-1} & a_{1m-1}^{m-1} & 0 \\
0 & 0 & a_{1m}^3 & a_{1m}^4 & a_{1m}^5 & \cdots & a_{1m}^{m-3} & a_{1m}^{m-2} & a_{1m}^{m-1} & a_{1m}^m \\
0 & 0 & a_{23}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{2m}^{m-1} & a_{24}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{2m}^{m-2} & a_{2m}^{m-1} & a_{25}^5 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & a_{2m}^{m-1} & a_{2m-2}^{m-2} & 0 & 0 \\
0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & a_{2m}^{m-1} & a_{2m-1}^{m-1} & 0 \\
0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & a_{2m}^m \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{2.9}$$

where

$$a_{1i}^i = a_{1m}^m + (m-i)a_{12}^2, a_{2i}^i = a_{2m}^m - (m-i)a_{12}^1, \quad i = 3, 4, \ldots, m.$$

Replacing x by $x - a_{2m}^{m-1}e_1 + a_{1m}^{m-1}e_2 - a_{12}^3 e_4 - a_{12}^4 e_5 - \cdots - a_{12}^{m-1} e_m$, we see that the matrix M is reduced to (2.6) and a_{ij}^k for $1 \leqslant i, j, k \leqslant m$ satisfy (2.7).

Lastly, using the Jacobi identities for vectors $\{[x, e_1, e_2], x, e_i\}, 3 \leqslant i \leqslant m$, we get

$$a_{12}^1 a_{1m}^i + a_{12}^2 a_{2m}^i = 0, a_{12}^1 a_{1m}^m + a_{12}^2 a_{2m}^m = 0, \quad i = 3, 4, \ldots, m-2.$$

Therefore, we get the matrix equation (2.8). □

From Lemma 2.3, if the left multiplications $\mathrm{ad}(x, e_i), 1 \leqslant i \leqslant m$ are restricted to the ideal N, then their matrix forms with respect to $e_1, \ldots, e_m$ are given by

$$\mathrm{ad}(x, e_1)|_N = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
a_{12}^1 & a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{12}^m \\
0 & 0 & a_{13}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{14}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{1m}^{m-2} & 0 & a_{15}^5 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & a_{1m}^5 & a_{1m}^6 & a_{1m}^7 & \cdots & 0 & a_{1m-2}^{m-2} & 0 & 0 \\
0 & 0 & a_{1m}^4 & a_{1m}^5 & a_{1m}^6 & \cdots & a_{1m}^{m-2} & 0 & a_{1m-1}^{m-1} & 0 \\
0 & 0 & a_{1m}^3 & a_{1m}^4 & a_{1m}^5 & \cdots & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & a_{1m}^m
\end{pmatrix}, \tag{2.10}$$

$$\mathrm{ad}(x,e_2)|_N = \begin{pmatrix} -a_{12}^1 & -a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -a_{12}^m \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{24}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & a_{25}^5 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & a_{2m-2}^{m-2} & 0 & 0 \\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & 0 & a_{2m-1}^{m-1} & 0 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & a_{2m}^m \end{pmatrix}, \quad (2.11)$$

and $\mathrm{ad}(x,e_i)|_N$, $3 \leqslant i \leqslant m$ are nilpotent, where a_{ij}^k for $i = 1,2;\ 2 \leqslant j \leqslant m;\ 1 \leqslant k \leqslant m$ satisfy (2.7) and (2.8).

3. 3-Lie algebras with nilradical N

In this section we study 3-Lie algebras with nilradical N.

Theorem 3.1. *There is no solvable non-nilpotent 3-Lie algebra with nilradical N.*

Proof. Firstly, let A be an $(m+k)$-dimensional 3-Lie algebra with the nilpotent ideal N, $k = 1,2$. We will prove A is nilpotent.

When $k = 1$, suppose $x, e_1, \ldots, e_m$ is a basis of A. Then the associative algebra A^* is generated by left multiplications $\mathrm{ad}(x,e_i)$ and $\mathrm{ad}(e_i,e_j)$, where $1 \leqslant i,j \leqslant m$. Therefore, we have $A^* = K(N,A)$. It follows from Lemma 2.1 that A is nilpotent.

When $k = 2$, let $x_1, x_2, e_1, \ldots, e_m$ be a basis of A. Set $B = Fx_1 + Fe_1 + \cdots + Fe_m$ and $C = Fx_2 + Fe_1 + \cdots + Fe_m$. Then B and C are $(m+1)$-dimensional subalgebras of A with the nilpotent ideal N. It follows from the result of the case $k = 1$, (2.10) and (2.11), that the matrices of $\mathrm{ad}(x_i,e_j)|_N (i = 1,2, 1 \leqslant j \leqslant m)$ with respect to $e_1, \ldots, e_m$ are of the form

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_4 & c_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & c_{m-4} & c_{m-5} & c_{m-6} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{m-3} & c_{m-4} & c_{m-5} & \cdots & c_3 & 0 & 0 & 0 \\ 0 & 0 & c_{m-2} & c_{m-3} & c_{m-4} & \cdots & c_4 & c_3 & 0 & 0 \end{pmatrix},$$

where $a, b, c_l \in F, 3 \leqslant l \leqslant m-2$. Therefore $\mathrm{ad}(x_i,e_j)$ are nilpotent maps of A for $i = 1,2;\ j = 1,\ldots,m$. Now suppose

$$[x_1,x_2,e_i] = \sum_{j=1}^m r_{ij}e_j, \quad 1 \leqslant i \leqslant m.$$

With the help of the Jacobi identities for $\{[x_1,x_2,e_i],e_1,e_2\}$, $\{[x_1,x_2,e_i],e_1,e_4\}$, $i = 1,2$; $\{[x_1,x_2,e_2],e_2,e_i\}$, $i \geqslant 4$; $\{[x_1,x_2,e_m],e_1,e_2\}$, $\{[x_1,x_2,e_m],e_1,e_4\}$, $\{[x_1,x_2,e_m],e_2,e_4\}$, we get that $\mathrm{ad}(x_1,x_2)|_N$ has the form

$$\begin{pmatrix} 0 & 0 & r_{13} & r_{14} & r_{15} & \cdots & r_{1m-3} & r_{1m-2} & r_{1m-1} & 0 \\ 0 & 0 & r_{23} & r_{24} & r_{25} & \cdots & r_{2m-3} & r_{2m-2} & r_{2m-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{63} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{73} & r_{63} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & r_{m-23} & r_{m-33} & r_{m-43} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{m-13} & r_{m-23} & r_{m-33} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{m3} & r_{m-13} & r_{m-23} & \cdots & r_{63} & 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathrm{ad}(x_1, x_2)|_N$ is nilpotent, and $\mathrm{ad}(x_1, x_2)$ is also nilpotent to A. This proves that A is nilpotent when $k = 2$.

Lastly we suppose that there is a solvable non-nilpotent $(m + k)$-dimensional 3-Lie algebra with nilradical N for $k \geqslant 3$. Let $x_1, \ldots, x_k, e_1, \ldots, e_m$ be a basis of A. Then there exist x_i, x_j such that $\mathrm{ad}(x_i, x_j)$ is not a nilpotent map of A. Set $T = Fx_i + Fx_j + Fe_1 + \cdots + Fe_m$, then N is a nilpotent ideal of $(m + 2)$-dimensional subalgebra T. From the above discussions, T is a nilpotent subalgebra. Hence there exists an integer r such that $ad^r(x_i, x_j)(T) = 0$. Since A is solvable and N is the nilradical of A, we have $[A, \ldots, A] \subseteq N$. Therefore,

$$\mathrm{ad}^{r+1}(x_i, x_j)(A) \subseteq \mathrm{ad}^r(x_i, x_j)(N) \subseteq \mathrm{ad}^r(x_i, x_j)(T) = 0.$$

This is a contradiction. □

Remark 3.1. The solvable condition in Theorem 3.1 is necessary. See the following example.

Example 3.1. Let A be an $(m + 4)$-dimensional 3-Lie algebra with basis $x_1, x_2, x_3, x_4, e_1, \ldots, e_m$, and its multiplication table be

$$\begin{cases} [x_1, x_2, x_4] = x_3, \\ [x_1, x_3, x_4] = x_2, \\ [x_2, x_3, x_4] = x_1, \\ [x_4, e_1, e_2] = e_3, \\ [e_1, e_2, e_j] = e_{j-1}, \quad 4 \leqslant j \leqslant m, \end{cases}$$

and other brackets of the basis vectors equal 0. By a direct computation we get that N is the nilradical of A, and

$$A^{(1)} = Fx_1 + Fx_2 + Fx_3 + Fe_3 + \cdots + Fe_{m-1}, A^{(s)} = Fx_1 + Fx_2 + Fx_3 \neq 0, \quad s > 1.$$

It follows that A is an unsolvable 3-Lie algebra.

4. 3-Lie algebras with a maximal hypo-nilpotent ideal *N*

In this section we study 3-Lie algebras with a maximal hypo-nilpotent ideal N. We construct all such solvable 3-Lie algebras and give them a simple classification.

Theorem 4.1. *Let A be an $(m + 1)$-dimensional 3-Lie algebra with a maximal hypo-nilpotent ideal N. Then A is solvable, and up to isomorphisms, one and only one of the following possibilities for the structural matrix M of A holds*

$$M_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & b_3 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & b_4 & b_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & b_{m-4} & b_{m-5} & b_{m-6} & \cdots & 0 & 1 & 0 & 0 \\
0 & 0 & b_{m-3} & b_{m-4} & b_{m-5} & \cdots & b_3 & 0 & 1 & 0 \\
0 & 0 & b_{m-2} & b_{m-3} & b_{m-4} & \cdots & b_4 & b_3 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & c_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & c_4 & c_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & c_{m-4} & c_{m-5} & c_{m-6} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & c_{m-3} & c_{m-4} & c_{m-5} & \cdots & c_3 & 0 & 0 & 0 \\
0 & 0 & c_{m-2} & c_{m-3} & c_{m-4} & \cdots & c_4 & c_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$M_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & m-3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m-4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m-5 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$M_3 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & m-3+\alpha & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & m-4+\alpha & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & m-5+\alpha & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2+\alpha & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1+\alpha & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$M_4 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\\
0 & 0 & m-2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & m-3 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & m-4 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix},$$

where $b_i, c_i, \alpha \in F, 3 \leqslant i \leqslant m-2$; $\alpha \neq 0$.

Proof. From Lemma 2.3, the structural matrix M with respect to a basis $x, e_1, \ldots, e_m$ of A is of the form (2.6) and satisfies conditions (2.7) and (2.8). Since A is not nilpotent, we have that $a_{12}^1, a_{12}^2, a_{1m}^m$ and a_{2m}^m can not be equal to zero simultaneously.

Now we determine the structural matrix M according to the solutions of (2.8).

(I). In the case of $a_{12}^1 = a_{12}^2 = 0$, then $a_{1m}^m \neq 0$ or $a_{2m}^m \neq 0$. Therefore, the structural matrix M is given by

$$M=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{12}^m\\
0 & 0 & a_{1m}^m & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a_{1m}^m & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & a_{1m}^{m-2} & 0 & a_{1m}^m & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & a_{1m}^5 & a_{1m}^6 & a_{1m}^7 & \cdots & 0 & a_{1m}^m & 0 & 0\\
0 & 0 & a_{1m}^4 & a_{1m}^5 & a_{1m}^6 & \cdots & a_{1m}^{m-2} & 0 & a_{1m}^m & 0\\
0 & 0 & a_{1m}^3 & a_{1m}^4 & a_{1m}^5 & \cdots & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & a_{1m}^m\\
0 & 0 & a_{2m}^m & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a_{2m}^m & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & a_{2m}^{m-2} & 0 & a_{2m}^m & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & a_{2m}^m & 0 & 0\\
0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & 0 & a_{2m}^m & 0\\
0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & a_{2m}^m\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Without lose of generality we may assume that $a_{1m}^m \neq 0$ (similar for $a_{2m}^m \neq 0$). Taking a linear transformation of the basis $x, e_1, \ldots, e_m$ by replacing x and e_2 by $\frac{1}{a_{1m}^m}x + \frac{a_{12}^m}{a_{1m}^m}\left(\sum_{j=4}^{m-1} \frac{a_{1m}^{j-1}}{a_{1m}^m} e_j\right)$ and $e_2 - \frac{a_{2m}^m}{a_{1m}^m}e_1 - \frac{a_{12}^m}{a_{1m}^m}e_m$ respectively, we get the structural matrix

$$M_1=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & b_3 & 0 & 1 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & b_4 & b_3 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & b_{m-4} & b_{m-5} & b_{m-6} & \cdots & 0 & 1 & 0 & 0\\
0 & 0 & b_{m-3} & b_{m-4} & b_{m-5} & \cdots & b_3 & 0 & 1 & 0\\
0 & 0 & b_{m-2} & b_{m-3} & b_{m-4} & \cdots & b_4 & b_3 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & c_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & c_4 & c_3 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & c_{m-4} & c_{m-5} & c_{m-6} & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & c_{m-3} & c_{m-4} & c_{m-5} & \cdots & c_3 & 0 & 0 & 0\\
0 & 0 & c_{m-2} & c_{m-3} & c_{m-4} & \cdots & c_4 & c_3 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix},$$

where $b_i, c_i \in F, i = 3, \ldots, m-2$.

(II). If $a_{12}^1 \neq 0$ or $a_{12}^2 \neq 0$, then the rank r of the matrix $\begin{pmatrix} a_{1m}^3 & a_{2m}^3\\ a_{1m}^4 & a_{2m}^4\\ \cdots & \cdots\\ a_{1m}^{m-2} & a_{2m}^{m-2}\\ a_{1m}^m & a_{2m}^m \end{pmatrix}$ is equal to 0 or 1.

When $r = 0$, we might as well suppose $a^2_{12} \neq 0$ (similar for $a^1_{12} \neq 0$), then we have $a^i_{1m} = a^m_{1m} = a^i_{2m} = a^m_{2m} = 0, i = 3, 4, \ldots, m-2$ and the matrix M is of the form

$$\begin{pmatrix}
a^1_{12} & a^2_{12} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a^m_{12} \\
0 & 0 & (m-3)a^2_{12} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (m-4)a^2_{12} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (m-5)a^2_{12} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2a^2_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a^2_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -(m-3)a^1_{12} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -(m-4)a^1_{12} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -(m-5)a^1_{12} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -2a^1_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -a^1_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Now substituting $\frac{1}{a^2_{12}}x$ and $\frac{a^1_{12}}{a^2_{12}}e_1 + e_2 + \frac{a^m_{12}}{a^2_{12}}e_m$ for x and e_2 respectively, the structural matrix M is reduced to the form

$$M_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & m-3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m-4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m-5 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}.$$

When $r = 1$, without lose of generality we may assume that there is an $a^i_{1m} \neq 0$ for some i, $3 \leqslant i \leqslant m$. Then we have $a^i_{2m} = ka^i_{1m}, i = 3, 4, \ldots, m-2, m$; $k \in F$. Replacing e_2 by $e_2 - ke_1$, the structural matrix M is reduced to

$$\begin{pmatrix}
0 & a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{12}^m \\
0 & 0 & a_{13}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{14}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{1m}^{m-2} & 0 & a_{15}^5 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & a_{1m}^5 & a_{1m}^6 & a_{1m}^7 & \cdots & 0 & a_{1m-2}^{m-2} & 0 & 0 \\
0 & 0 & a_{1m}^4 & a_{1m}^5 & a_{1m}^6 & \cdots & a_{1m}^{m-2} & 0 & a_{1m-1}^{m-1} & 0 \\
0 & 0 & a_{1m}^3 & a_{1m}^4 & a_{1m}^5 & \cdots & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & a_{1m}^m \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix},$$

where $a_{1i}^i = a_{1m}^m + (m-i)a_{12}^2, i = 3, 4, \ldots, m$, and $a_{12}^2 \neq 0$.

If $a_{1m}^m \neq 0$, then for any l satisfying $3 \leqslant l \leqslant m-2$, we take a series of linear transformations defined by

$$\tilde{e}_k = e_k, \quad 1 \leqslant k \leqslant l+1; \qquad \tilde{e}_k = e_k - \frac{a_{1m}^{m-l+1}}{(l-1)a_{12}^2} e_{k-l+1}, \quad l+2 \leqslant k \leqslant m.$$

Then the basis vectors $\tilde{e}_1, \ldots, \tilde{e}_m$ satisfy (2.3). And with respect to the basis $x, \tilde{e}_1, \ldots, \tilde{e}_m$, the structural matrix of A is

$$\begin{pmatrix}
0 & a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{12}^m \\
0 & 0 & a_{13}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{14}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{15}^5 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{1m-2}^{m-2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{1m-1}^{m-1} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{1m}^m \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix},$$

where $a_{1i}^i = a_{1m}^m + (m-i)a_{12}^2, i = 3, 4, \ldots, m$.

In the case of $a_{1m}^m \neq a_{12}^2$, replacing x and $\tilde{e}_2$ by $\frac{1}{a_{12}^2}x$ and $\tilde{e}_2 - \frac{a_{12}^m}{a_{1m}^m - a_{12}^2}\tilde{e}_m$ respectively, we get the structural matrix

$$M_3 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & m-3+\alpha & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m-4+\alpha & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m-5+\alpha & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2+\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1+\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix},$$

where $\alpha \in F, \alpha \neq 0, 1$.

In the case of $a_{1m}^m = a_{12}^2, a_{12}^m \neq 0$, substituting $\frac{1}{a_{12}^2}x$ and $\frac{a_{12}^m}{a_{12}^2}\tilde{e}_i$ for x and $\tilde{e}_i, i = 3, 4, \ldots, m$, we get the structural matrix

$$M_4 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
0 & 0 & m-2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m-3 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m-4 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}.$$

In the case of $a_{1m}^m = a_{12}^2, a_{12}^m = 0$, substituting $\frac{1}{a_{12}^2}x$ for x, we get that the structural matrix of A is M_3 with $\alpha = 1$.

If $a_{1m}^m = 0$, similar to the above discussions, we get that the structural matrix of A is M_2.
Lastly by a direct computation, we get DS and CS corresponding to all cases $M_i, i = 1,2,3,4$:

$$M_1 : DS = [m+1, m-2, 0], CS = [m+1, m-2, m-2, \cdots];$$
$$M_2 : DS = [m+1, m-2, m-4, 0], CS = [m+1, m-2, m-2, \cdots];$$
$$M_3 : DS = [m+1, m-1, m-3, 0], CS = [m+1, m-1, m-1, \cdots];$$
$$M_4 : DS = [m+1, m-1, m-3, 0], CS = [m+1, m-1, m-1, \cdots].$$

It follows from $M_i^{(4)} = 0$ and $M_i^r = M_i^2 \neq 0$ when $r \geqslant 3$, that A with structural matrix M_i is solvable but not nilpotent for $i = 1,2,3,4$. Since A in the case of M_3 is spilt, it is not isomorphic to the case M_4. Therefore, the 3-Lie algebra A with the structural matrix M_i is not isomorphic to the case M_j when $i \neq j, 1 \leqslant i,j \leqslant 4$. □

Theorem 4.2. *Let A be an $(m+2)$-dimensional 3-Lie algebra with a maximal hypo-nilpotent ideal N. Then A is solvable, and there exists a basis $x_1, x_2, e_1, \ldots, e_m$ such that the multiplication table of A is given by*:

$$\begin{cases} [e_1, e_2, e_j] = e_{j-1}, 4 \leqslant j \leqslant m, \\ [x_1, e_1, e_2] = e_2, \\ [x_1, e_1, e_i] = (m-i)e_i, i = 3, 4, \ldots, m, \\ [x_2, e_1, e_i] = e_i, i = 3, 4, \ldots, m, \end{cases} \tag{4.1}$$

and other brackets of the basis vectors are equal to zero.

Corollary 4.1. *There is an unique $(m+2)$-dimensional 3-Lie algebra with a maximal hypo-nilpotent ideal N up to isomorphisms.*

The proof of Theorem 4.2. Since $\dim A = m+2$, we have $A^1 = [A,A,A] = [A,A,N] \subseteq N$. Let $x_1, x_2, e_1, \ldots, e_m$ be a basis of A. Then $A_1 = Fx_1 + N, A_2 = Fx_2 + N$ are ideals and N is a maximal hypo-nilpotent ideal of A_1 and A_2 respectively.

Firstly, we prove that there exist a basis $e_1, \ldots, e_m$ of N satisfying (2.3), and vectors x_i of $A_i (i = 1,2)$ which are not contained in N such that $\mathrm{ad}(x_1, e_1)|_N$ and $\mathrm{ad}(x_2, e_1)|_N$ are linear nil-independent. We might as well suppose that the structural matrix of A_1 in the basis $x_1, e_1, \ldots, e_m$ is $M_i (i = 1,2,3,4)$, and the structural matrix of A_2 in the basis $x_2, e_1, \ldots, e_m$ is (2.6) (If the structural matrix of A_2 in the basis $x_2, e_1, \ldots, e_m$ is (2.9), then we get the structural matrix (2.6) in the basis $x_2', e_1, \ldots, e_m$, where $x_2' = x_2 - a_{2m}^{m-1} e_1 + a_{1m}^{m-1} e_2 - a_{12}^3 e_4 - a_{12}^4 e_5 - \cdots - a_{12}^{m-1} e_m$.).

(1). Suppose that the structural matrix of A_1 in the basis $x_1, e_1, \ldots, e_m$ is M_1. Then

$$ad(x_1, e_1)|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_4 & b_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & b_{m-4} & b_{m-5} & b_{m-6} & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & b_{m-3} & b_{m-4} & b_{m-5} & \cdots & b_3 & 0 & 1 & 0 \\ 0 & 0 & b_{m-2} & b_{m-3} & b_{m-4} & \cdots & b_4 & b_3 & 0 & 1 \end{pmatrix},$$

$$\mathrm{ad}(x_2,e_1)|_N=\begin{pmatrix} 0&0&0&0&0&\cdots&0&0&0&0\\ a_{12}^1&a_{12}^2&0&0&0&\cdots&0&0&0&a_{12}^m\\ 0&0&a_{13}^3&0&0&\cdots&0&0&0&0\\ 0&0&0&a_{14}^4&0&\cdots&0&0&0&0\\ 0&0&a_{1m}^{m-2}&0&a_{15}^5&\cdots&0&0&0&0\\ 0&0&a_{1m}^{m-3}&a_{1m}^{m-2}&0&\cdots&0&0&0&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&0&a_{1m}^5&a_{1m}^6&a_{1m}^7&\cdots&0&a_{1m-2}^{m-2}&0&0\\ 0&0&a_{1m}^4&a_{1m}^5&a_{1m}^6&\cdots&a_{1m}^{m-2}&0&a_{1m-1}^{m-1}&0\\ 0&0&a_{1m}^3&a_{1m}^4&a_{1m}^5&\cdots&a_{1m}^{m-3}&a_{1m}^{m-2}&0&a_{1m}^m \end{pmatrix},$$

$$ad(x_1,e_2)|_N=\begin{pmatrix} 0&0&0&0&0&\cdots&0&0&0&0\\ 0&0&0&0&0&\cdots&0&0&0&0\\ 0&0&0&0&0&\cdots&0&0&0&0\\ 0&0&0&0&0&\cdots&0&0&0&0\\ 0&0&c_3&0&0&\cdots&0&0&0&0\\ 0&0&c_4&c_3&0&\cdots&0&0&0&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&0&c_{m-4}&c_{m-5}&c_{m-6}&\cdots&0&0&0&0\\ 0&0&c_{m-3}&c_{m-4}&c_{m-5}&\cdots&c_3&0&0&0\\ 0&0&c_{m-2}&c_{m-3}&c_{m-4}&\cdots&c_4&c_3&0&0 \end{pmatrix},$$

$$\mathrm{ad}(x_2,e_2)|_N=\begin{pmatrix} -a_{12}^1&-a_{12}^2&0&0&0&\cdots&0&0&0&-a_{12}^m\\ 0&0&0&0&0&\cdots&0&0&0&0\\ 0&0&a_{23}^3&0&0&\cdots&0&0&0&0\\ 0&0&0&a_{24}^4&0&\cdots&0&0&0&0\\ 0&0&a_{2m}^{m-2}&0&a_{25}^5&\cdots&0&0&0&0\\ 0&0&a_{2m}^{m-3}&a_{2m}^{m-2}&0&\cdots&0&0&0&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&0&a_{2m}^5&a_{2m}^6&a_{2m}^7&\cdots&0&a_{2m-2}^{m-2}&0&0\\ 0&0&a_{2m}^4&a_{2m}^5&a_{2m}^6&\cdots&a_{2m}^{m-2}&0&a_{2m-1}^{m-1}&0\\ 0&0&a_{2m}^3&a_{2m}^4&a_{2m}^5&\cdots&a_{2m}^{m-3}&a_{2m}^{m-2}&0&a_{2m}^m \end{pmatrix},$$

and a_{ij}^k satisfying (2.7) and (2.8).

If $|\mathrm{ad}(x_1,e_1)|_N$ and $\mathrm{ad}(x_2,e_1)|_N$ are linear nil-dependent, then we have $\begin{cases} k_2a_{12}^2=0,\\ k_1+k_2a_{1m}^m=0,\end{cases}$ where $k_1,k_2\in F$ are not equal to zero simultaneously. Therefore, $a_{12}^2=0, k_2\neq 0$. From (2.8) we obtain $a_{12}^1a_{1m}^j=0$ for $j=3,\ldots,m-2,m$. If $a_{12}^1=0$, then

$$\mathrm{ad}(x_2,e_1)|_N=\begin{pmatrix} 0&0&0&0&0&\cdots&0&0&0&0\\ 0&0&0&0&0&\cdots&0&0&0&a_{12}^m\\ 0&0&a_{1m}^m&0&0&\cdots&0&0&0&0\\ 0&0&0&a_{1m}^m&0&\cdots&0&0&0&0\\ 0&0&a_{1m}^{m-2}&0&a_{1m}^m&\cdots&0&0&0&0\\ 0&0&a_{1m}^{m-3}&a_{1m}^{m-2}&0&\cdots&0&0&0&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&0&a_{1m}^5&a_{1m}^6&a_{1m}^7&\cdots&0&a_{1m}^m&0&0\\ 0&0&a_{1m}^4&a_{1m}^5&a_{1m}^6&\cdots&a_{1m}^{m-2}&0&a_{1m}^m&0\\ 0&0&a_{1m}^3&a_{1m}^4&a_{1m}^5&\cdots&a_{1m}^{m-3}&a_{1m}^{m-2}&0&a_{1m}^m \end{pmatrix},$$

$$\operatorname{ad}(x_2,e_2)|_N=\begin{pmatrix}0&0&0&0&0&\cdots&0&0&0&-a_{12}^m\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&a_{2m}^m&0&0&\cdots&0&0&0&0\\0&0&0&a_{2m}^m&0&\cdots&0&0&0&0\\0&0&a_{2m}^{m-2}&0&a_{2m}^m&\cdots&0&0&0&0\\0&0&a_{2m}^{m-3}&a_{2m}^{m-2}&0&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&a_{2m}^5&a_{2m}^6&a_{2m}^7&\cdots&0&a_{2m}^m&0&0\\0&0&a_{2m}^4&a_{2m}^5&a_{2m}^6&\cdots&a_{2m}^{m-2}&0&a_{2m}^m&0\\0&0&a_{2m}^3&a_{2m}^4&a_{2m}^5&\cdots&a_{2m}^{m-3}&a_{2m}^{m-2}&0&a_{2m}^m\end{pmatrix}.$$

From $[[x_1,x_2,e_1],e_2,e_3]=0$, we get $a_{2m}^m=0$,

$$\operatorname{ad}(x_2,e_2)|_N=\begin{pmatrix}0&0&0&0&0&\cdots&0&0&0&-a_{12}^m\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&a_{2m}^{m-2}&0&0&\cdots&0&0&0&0\\0&0&a_{2m}^{m-3}&a_{2m}^{m-2}&0&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&a_{2m}^5&a_{2m}^6&a_{2m}^7&\cdots&0&0&0&0\\0&0&a_{2m}^4&a_{2m}^5&a_{2m}^6&\cdots&a_{2m}^{m-2}&0&0&0\\0&0&a_{2m}^3&a_{2m}^4&a_{2m}^5&\cdots&a_{2m}^{m-3}&a_{2m}^{m-2}&0&0\end{pmatrix}.$$

Then $-a_{1m}^m\operatorname{ad}(x_1,e_1)|_N+\operatorname{ad}(x_2,e_1)|_N$ and $-a_{1m}^m\operatorname{ad}(x_1,e_2)|_N+\operatorname{ad}(x_2,e_2)|_N$ are nilpotent, that is $I=F(-a_{1m}^m x_1+x_2)+N$ is an $(m+1)$-dimensional hypo-nilpotent ideal of A. This is a contradiction. Therefore, $a_{12}^1\neq 0, a_{1m}^3=a_{1m}^4=\cdots=a_{1m}^{m-2}=a_{1m}^m=0$ and

$$\operatorname{ad}(x_2,e_1)|_N=\begin{pmatrix}0&0&0&0&0&\cdots&0&0&0&0\\a_{12}^1&0&0&0&0&\cdots&0&0&0&a_{12}^m\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\end{pmatrix},$$

$$\operatorname{ad}(x_2,e_2)|_N=\begin{pmatrix}-a_{12}^1&0&0&0&0&\cdots&0&0&0&-a_{12}^m\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&a_{23}^3&0&0&\cdots&0&0&0&0\\0&0&0&a_{24}^4&0&\cdots&0&0&0&0\\0&0&a_{2m}^{m-2}&0&a_{25}^5&\cdots&0&0&0&0\\0&0&a_{2m}^{m-3}&a_{2m}^{m-2}&0&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&a_{2m}^5&a_{2m}^6&a_{2m}^7&\cdots&0&a_{2m-2}^{m-2}&0&0\\0&0&a_{2m}^4&a_{2m}^5&a_{2m}^6&\cdots&a_{2m}^{m-2}&0&a_{2m-1}^{m-1}&0\\0&0&a_{2m}^3&a_{2m}^4&a_{2m}^5&\cdots&a_{2m}^{m-3}&a_{2m}^{m-2}&0&a_{2m}^m\end{pmatrix}.$$

Let $\tilde{e_1} = e_1 + e_2, \tilde{e_i} = e_i, i = 2, 3, \ldots, m$. Then $\tilde{e_1}, \ldots, \tilde{e_m}$ is a basis of N satisfying (2.3) and

$$ad(x_1, \tilde{e_1})|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & d_4 & d_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & d_{m-4} & d_{m-5} & d_{m-6} & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & d_{m-3} & d_{m-4} & d_{m-5} & \cdots & d_3 & 0 & 1 & 0 \\ 0 & 0 & d_{m-2} & d_{m-3} & d_{m-4} & \cdots & d_4 & d_3 & 0 & 1 \end{pmatrix},$$

where $d_i = b_i + c_i, i = 3, 4, \ldots, m-2$,

$$\mathrm{ad}(x_2, \tilde{e_1})|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{12}^1 & -a_{12}^1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{12}^m \\ 0 & 0 & a_{23}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{24}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & a_{25}^5 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & a_{2m-2}^{m-2} & 0 & 0 \\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & 0 & a_{2m-1}^{m-1} & 0 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & a_{2m}^m \end{pmatrix},$$

$$ad(x_1, \tilde{e_2})|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_4 & c_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & c_{m-4} & c_{m-5} & c_{m-6} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{m-3} & c_{m-4} & c_{m-5} & \cdots & c_3 & 0 & 0 & 0 \\ 0 & 0 & c_{m-2} & c_{m-3} & c_{m-4} & \cdots & c_4 & c_3 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}(x_2, \tilde{e_2})|_N = \begin{pmatrix} -a_{12}^1 & a_{12}^1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -a_{12}^m \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{24}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & a_{25}^5 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & a_{2m-2}^{m-2} & 0 & 0 \\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & 0 & a_{2m-1}^{m-1} & 0 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & a_{2m}^m \end{pmatrix},$$

where $a_{2j}^j = a_{2m}^m - (m-j)a_{12}^1, 3 \leqslant j \leqslant m$. Since $a_{12}^1 \neq 0, ad(x_1, \tilde{e_1})|_N$ and $ad(x_2, \tilde{e_1})|_N$ are linear nil-independent.

(2). When the structural matrix of A_1 is M_2. Then

$$ad(x_1,e_1)|_N=\begin{pmatrix}0&0&0&0&0&\cdots&0&0&0&0\\0&1&0&0&0&\cdots&0&0&0&0\\0&0&m-3&0&0&\cdots&0&0&0&0\\0&0&0&m-4&0&\cdots&0&0&0&0\\0&0&0&0&m-5&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&0&\cdots&3&0&0&0\\0&0&0&0&0&\cdots&0&2&0&0\\0&0&0&0&0&\cdots&0&0&1&0\\0&0&0&0&0&\cdots&0&0&0&0\end{pmatrix},$$

$$ad(x_1,e_2)|_N=\begin{pmatrix}0&-1&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\end{pmatrix}.$$

Since $[ad(x_1,e_1)|_N, ad(x_2,e_1)|_N]=$

$$\begin{pmatrix}0&0&0&0&0&\cdots&0&0&0&0\\a_{12}^1&0&0&0&0&\cdots&0&0&0&a_{12}^m\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&-2a_{1m}^{m-2}&0&0&\cdots&0&0&0&0\\0&0&-3a_{1m}^{m-3}&-2a_{1m}^{m-2}&0&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&(5-m)a_{1m}^5&(6-m)a_{1m}^6&(7-m)a_{1m}^7&\cdots&0&0&0&0\\0&0&(4-m)a_{1m}^4&(5-m)a_{1m}^5&(6-m)a_{1m}^6&\cdots&-2a_{1m}^{m-2}&0&0&0\\0&0&(3-m)a_{1m}^3&(4-m)a_{1m}^4&(5-m)a_{1m}^5&\cdots&-3a_{1m}^{m-3}&-2a_{1m}^{m-2}&0&0\end{pmatrix}$$

and $[\mathrm{ad}(x_1,e_1)|_N, \mathrm{ad}(x_2,e_1)|_N]$ is a finitely linear combination of $\{\mathrm{ad}(e_1,e_2)|_N, \mathrm{ad}(e_1,e_i)|_N, \mathrm{ad}(e_2,e_j)|_N, i,j=4,5,\ldots,m\}$, $[\mathrm{ad}(x_1,e_1)|_N, \mathrm{ad}(x_2,e_1)|_N]$ is of the form

$$\begin{pmatrix}0&0&\alpha_1&\alpha_2&\alpha_3&\cdots&\alpha_{m-5}&\alpha_{m-4}&\alpha_{m-3}&0\\0&0&\beta_1&\beta_2&\beta_3&\cdots&\beta_{m-5}&\beta_{m-4}&\beta_{m-3}&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&\gamma&0&0&\cdots&0&0&0&0\\0&0&0&\gamma&0&\cdots&0&0&0&0\\0&0&0&0&\gamma&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&0&\cdots&\gamma&0&0&0\\0&0&0&0&0&\cdots&0&\gamma&0&0\\0&0&0&0&0&\cdots&0&0&\gamma&0\end{pmatrix}.$$

Comparing the elements of two matrices above, we obtain $a_{1m}^3=\cdots=a_{1m}^{m-2}=0, a_{12}^1=a_{12}^m=0$. Then

$$\mathrm{ad}(x_2,e_1)|_N=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & a_{13}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & a_{14}^4 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a_{15}^5 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{1m-2}^{m-2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{1m-1}^{m-1} & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{1m}^m \end{pmatrix}.$$

If $\mathrm{ad}(x_1,e_1)|_N$ and $\mathrm{ad}(x_2,e_1)|_N$ are linear nil-dependent, then there exist $k_1,k_2\in F$ and $k_1\neq 0$ or $k_2\neq 0$ such that $\begin{cases} k_1+k_2a_{12}^2=0,\\ k_2a_{1m}^m=0.\end{cases}$. Therefore, $a_{1m}^m=0, k_2\neq 0$. From (2.8) we have $a_{12}^2=0$ or $a_{2m}^m=0$.

If $a_{12}^2=0$, then $\mathrm{ad}(x_2,e_1)|_N=0$,

$$\mathrm{ad}(x_2,e_2)|_N=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & a_{2m}^m & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & a_{2m}^m & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & a_{2m}^{m-2} & 0 & a_{2m}^m & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & a_{2m}^m & 0 & 0\\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & 0 & a_{2m}^m & 0\\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & a_{2m}^m \end{pmatrix}.$$

From $[[x_1,x_2,e_1],e_2,e_3]=0$, we get $a_{2m}^m=0$, and $\mathrm{ad}(x_2,e_2)|_N$ is nilpotent. Then A_2 is an $(m+1)$-dimensional hypo-nilpotent ideal of A. This is a contradiction. Therefore, we have $a_{12}^2\neq 0$ and $a_{2m}^m=0$, and

$$\mathrm{ad}(x_2,e_1)|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & (m-3)a_{12}^2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & (m-4)a_{12}^2 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & (m-5)a_{12}^2 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 3a_{12}^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2a_{12}^2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{12}^2 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}(x_2,e_2)|_N=\begin{pmatrix} 0 & -a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & a_{2m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & a_{2m}^{5} & a_{2m}^{6} & a_{2m}^{7} & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & a_{2m}^{4} & a_{2m}^{5} & a_{2m}^{6} & \cdots & a_{2m}^{m-2} & 0 & 0 & 0\\ 0 & 0 & a_{2m}^{3} & a_{2m}^{4} & a_{2m}^{5} & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & 0 \end{pmatrix}.$$

It follows that $-a_{12}^2\mathrm{ad}(x_1,e_1)|_N+\mathrm{ad}(x_2,e_1)|_N$ and $-a_{12}^2\mathrm{ad}(x_1,e_2)|_N+\mathrm{ad}(x_2,e_2)|_N$ are nilpotent, and $I=F(-a_{12}^2x_1+x_2)+N$ is an $(m+1)$-dimensional hypo-nilpotent ideal of A. This is a contradiction. Therefore, $\mathrm{ad}(x_1,e_1)|_N$ and $\mathrm{ad}(x_2,e_1)|_N$ are linear nil-independent.

(3). If the structural matrix of A_1 in the basis $x_1,e_1,\ldots,e_m$ is M_3 or M_4, then $\mathrm{ad}(x_1,e_1)|_N$ and $\mathrm{ad}(x_2,e_1)|_N$ are linear nil-independent by similar discussions to the case (2).

Secondly, without lose of generality, suppose that $x_1,x_2,e_1,\ldots,e_m$ is a basis of A, and the structural matrices of $A_1=Fx_1+N$ and $A_2=Fx_2+N$ are of the form (2.6), and $\mathrm{ad}(x_1,e_1)|_N$ and $\mathrm{ad}(x_2,e_1)|_N$ are linear nil-independent. Then we have

$$\mathrm{ad}(x_1,e_1)|_N=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ a_{12}^1 & a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{12}^m\\ 0 & 0 & a_{13}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & a_{14}^4 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & a_{1m}^{m-2} & 0 & a_{15}^5 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & a_{1m}^{5} & a_{1m}^{6} & a_{1m}^{7} & \cdots & 0 & a_{1m-2}^{m-2} & 0 & 0\\ 0 & 0 & a_{1m}^{4} & a_{1m}^{5} & a_{1m}^{6} & \cdots & a_{1m}^{m-2} & 0 & a_{1m-1}^{m-1} & 0\\ 0 & 0 & a_{1m}^{3} & a_{1m}^{4} & a_{1m}^{5} & \cdots & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & a_{1m}^{m} \end{pmatrix},$$

$$\mathrm{ad}(x_2,e_1)|_N=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ b_{12}^1 & b_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_{12}^m\\ 0 & 0 & b_{13}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & b_{14}^4 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & b_{1m}^{m-2} & 0 & b_{15}^5 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & b_{1m}^{5} & b_{1m}^{6} & b_{1m}^{7} & \cdots & 0 & b_{1m-2}^{m-2} & 0 & 0\\ 0 & 0 & b_{1m}^{4} & b_{1m}^{5} & b_{1m}^{6} & \cdots & b_{1m}^{m-2} & 0 & b_{1m-1}^{m-1} & 0\\ 0 & 0 & b_{1m}^{3} & b_{1m}^{4} & b_{1m}^{5} & \cdots & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & b_{1m}^{m} \end{pmatrix},$$

where $a_{1i}^i=a_{1m}^m+(m-i)a_{12}^2, b_{1i}^i=b_{1m}^m+(m-i)b_{12}^2$ for $i=3,4,\ldots,m$, and a_{1m}^m,b_{1m}^m are not equal to zero simultaneously.

We might as well suppose $a_{1m}^m=0,b_{1m}^m\neq 0$ (If $a_{1m}^m\neq 0$ and $b_{1m}^m\neq 0$, then by substituting $x_1'=x_1-\frac{a_{1m}^m}{b_{1m}^m}x_2$ for x_1 we obtain $a_{1m}^m=0,b_{1m}^m\neq 0$, and $\{\mathrm{ad}(x_1',e_1),\mathrm{ad}(x_2,e_1)\}$ are also linear nil-independent.), then $a_{12}^2\neq 0$. Without lose of generality we may further assume that $a_{1m}^m=0,a_{12}^2=1,b_{1m}^m=1,b_{12}^2=0$. From Theorem 4.1, $\mathrm{ad}(x_1,e_i)|_N$ for $i=1,2$ can be written as follows

$$\mathrm{ad}(x_1,e_1)|_N=\mathrm{diag}(0,1,m-3,m-4,\ldots,2,1,0),$$

$$\mathrm{ad}(x_1,e_2)|_N=\begin{pmatrix}0&-1&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\end{pmatrix}.$$

Then we have

$$[\mathrm{ad}(x_1,e_1)|_N,\mathrm{ad}(x_2,e_1)|_N]=\mathrm{ad}(x_1,e_1)|_N\mathrm{ad}(x_2,e_1)|_N-\mathrm{ad}(x_2,e_1)|_N\mathrm{ad}(x_1,e_1)|_N=$$
$$\begin{pmatrix}0&0&0&0&0&\cdots&0&0&0&0\\b_{12}^1&0&0&0&0&\cdots&0&0&0&b_{12}^m\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&-2b_{1m}^{m-2}&0&0&\cdots&0&0&0&0\\0&0&-3b_{1m}^{m-3}&-2b_{1m}^{m-2}&0&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&(5-m)b_{1m}^5&(6-m)b_{1m}^6&(7-m)b_{1m}^7&\cdots&0&0&0&0\\0&0&(4-m)b_{1m}^4&(5-m)b_{1m}^5&(6-m)b_{1m}^6&\cdots&-2b_{1m}^{m-2}&0&0&0\\0&0&(3-m)b_{1m}^3&(4-m)b_{1m}^4&(5-m)b_{1m}^5&\cdots&-3b_{1m}^{m-3}&-2b_{1m}^{m-2}&0&0\end{pmatrix}.$$

On the other hand the product $[\mathrm{ad}(x_1,e_1)|_N,\mathrm{ad}(x_2,e_1)|_N]$ is a finitely linear combination of $\{\mathrm{ad}(e_1,e_2)|_N,\mathrm{ad}(e_1,e_i)|_N,\mathrm{ad}(e_2,e_j)|_N,i,j=4,5,\ldots,m\}$. Then $[\mathrm{ad}(x_1,e_1)|_N,\mathrm{ad}(x_2,e_1)|_N]$ has the form

$$\begin{pmatrix}0&0&\alpha_1&\alpha_2&\alpha_3&\cdots&\alpha_{m-5}&\alpha_{m-4}&\alpha_{m-3}&0\\0&0&\beta_1&\beta_2&\beta_3&\cdots&\beta_{m-5}&\beta_{m-4}&\beta_{m-3}&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&\gamma&0&0&\cdots&0&0&0&0\\0&0&0&\gamma&0&\cdots&0&0&0&0\\0&0&0&0&\gamma&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&0&\cdots&\gamma&0&0&0\\0&0&0&0&0&\cdots&0&\gamma&0&0\\0&0&0&0&0&\cdots&0&0&\gamma&0\end{pmatrix}.$$

Comparing the elements of two matrices above, we obtain $b_{1m}^3=\cdots=b_{1m}^{m-2}=0, b_{12}^1=b_{12}^m=0$. Then we have

$$\mathrm{ad}(x_2,e_1)|_N=\mathrm{diag}(0,0,1,1,\ldots,1,1,1),$$
$$\mathrm{ad}(x_2,e_2)|_N=\begin{pmatrix}0&0&0&0&0&\cdots&0&0&0&0\\0&0&0&0&0&\cdots&0&0&0&0\\0&0&b_{2m}^m&0&0&\cdots&0&0&0&0\\0&0&0&b_{2m}^m&0&\cdots&0&0&0&0\\0&0&b_{2m}^{m-2}&0&b_{2m}^m&\cdots&0&0&0&0\\0&0&b_{2m}^{m-3}&b_{2m}^{m-2}&0&\cdots&0&0&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&b_{2m}^5&b_{2m}^6&b_{2m}^7&\cdots&0&b_{2m}^m&0&0\\0&0&b_{2m}^4&b_{2m}^5&b_{2m}^6&\cdots&b_{2m}^{m-2}&0&b_{2m}^m&0\\0&0&b_{2m}^3&b_{2m}^4&b_{2m}^5&\cdots&b_{2m}^{m-3}&b_{2m}^{m-2}&0&b_{2m}^m\end{pmatrix}.$$

Summarizing, we get

$$[x_1,e_1,e_2]=e_2, [x_1,e_1,e_i]=(m-i)e_i,\quad i=3,4,\ldots,m;$$
$$[x_2,e_1,e_i]=e_i,\ i=3,4,\ldots,m;\quad [x_2,e_2,e_j]=0,\ j=1,2;\quad [x_2,e_2,e_3]=b_{2m}^m e_3,$$
$$[x_2,e_2,e_4]=b_{2m}^m e_4, [x_2,e_2,e_k]=\sum_{j=3}^{k-2} b_{2m}^{m-k+j}e_j+b_{2m}^m e_k,\quad k=5,6,\cdots,m.$$

Now suppose

$$[x_1,x_2,e_i]=\sum_{j=1}^m r_{ij}e_j,\quad i=1,\ldots,m.$$

From $\left[\sum_{j=1}^m r_{1j}e_j,e_2,e_m\right]=[[x_1,e_2,e_m],x_2,e_1]+[x_1,[x_2,e_2,e_m],e_1]+[x_1,x_2,[e_1,e_2,e_m]]$, we get $r_{11}=r_{m-1m-1},r_{m-11}=r_{m-12}=r_{m-1m}=0,r_{m-1i}=b_{2m}^i(m-i),i=3,4,\ldots,m-2$.

From $\left[\sum_{j=1}^m r_{2j}e_j,e_1,e_m\right]=[[x_1,e_1,e_m],x_2,e_2]+[x_1,[x_2,e_1,e_m],e_2]+[x_1,x_2,[e_2,e_1,e_m]]$, we get $r_{22}=r_{m-1m-1},r_{m-1i}=r_{m-1m}=0,i=1,2,\ldots,m-2$.

Therefore, $b_{2m}^i=0$, for $i=3,4,\ldots,m-2$. Again by $[[x_1,x_2,e_1],e_2,e_3]=0$, we get $b_{2m}^m=0$.

By imposing Jacobi identities on $\{[x_1,x_2,e_2],e_1,e_2\},\{[x_1,x_2,e_1],e_2,e_4\},\{[x_1,x_2,e_2],e_1,e_4\}$, and $\{[x_1,x_2,e_1],x_1,e_2\}$, we get $r_{2i}=0$ for $4\leqslant i\leqslant m$; $r_{3j}=0,1\leqslant j\leqslant m$ and $j\neq 3$; $r_{11}=r_{22}=r_{33}$, and $r_{11}=r_{23}=0$ respectively.

Again using the Jacobi identities on $\{[x_1,x_2,e_1],e_1,e_i\},\{[x_1,x_2,e_1],e_2,e_i\},\{[x_1,x_2,e_2],e_1,e_i\}$, $\{[x_1,x_2,e_2],e_2,e_i\},\{[x_1,x_2,e_m],e_1,e_i\},\{[x_1,x_2,e_m],e_2,e_i\}$ for $1\leqslant i\leqslant m, i\neq 3$; and $\{[x_1,x_2,e_1],x_1,e_m\}$, we get $r_{ij}=0$ when $i\neq 1$ and $j\neq 3$, that is

$$[x_1,x_2,e_1]=r_{13}e_3, [x_1,x_2,e_i]=0,\quad i=2,\ldots,m.$$

After replacing x_1 by $x_1-r_{13}e_3$, we get $[x_1,x_2,e_1]=0$. Therefore, A has the multiplication table (4.1) in a basis $x_1,x_2,e_1,\ldots,e_m$ and A is solvable. □

Theorem 4.3. *Let A be a solvable $(m+k)$-dimensional 3-Lie algebra with a maximal hypo-nilpotent ideal N. Then we have $k=1$ or 2.*

Proof. We suppose $k\geqslant 3$. Let $x_1,\ldots,x_k,e_1,\cdots e_m$ be a basis of A. Thanks to the solvability of A and Remark 2.1, we have $[A,A,A]\subseteq N$. By Theorem 4.2, we might as well suppose

$$\mathrm{ad}(x_1,e_1)|_N=\mathrm{diag}(0,1,m-3,m-4,\ldots,2,1,0),$$
$$\mathrm{ad}(x_2,e_1)|_N=\mathrm{diag}(0,0,1,1,\ldots,1,1,1), \mathrm{ad}(x_2,e_2)|_N=0,$$
$$\mathrm{ad}(x_1,e_2)|_N=\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

in the basis $e_1,\ldots,e_m$ of N.

By (2.9), $\mathrm{ad}(x_3, e_1)|_N$ and $\mathrm{ad}(x_3, e_2)|_N$ are of the form

$$\mathrm{ad}(x_3, e_1)|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{12}^1 & a_{12}^2 & a_{12}^3 & a_{12}^4 & a_{12}^5 & \cdots & a_{12}^{m-3} & a_{12}^{m-2} & a_{12}^{m-1} & a_{12}^m \\ 0 & 0 & a_{13}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m}^{m-1} & a_{14}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m}^{m-2} & a_{1m}^{m-1} & a_{15}^5 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m}^{m-3} & a_{1m}^{m-2} & a_{1m}^{m-1} & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{1m}^5 & a_{1m}^6 & a_{1m}^7 & \cdots & a_{1m}^{m-1} & a_{1m-2}^{m-2} & 0 & 0 \\ 0 & 0 & a_{1m}^4 & a_{1m}^5 & a_{1m}^6 & \cdots & a_{1m}^{m-2} & a_{1m}^{m-1} & a_{1m-1}^{m-1} & 0 \\ 0 & 0 & a_{1m}^3 & a_{1m}^4 & a_{1m}^5 & \cdots & a_{1m}^{m-3} & a_{1m}^{m-2} & a_{1m}^{m-1} & a_{1m}^m \end{pmatrix},$$

$$\mathrm{ad}(x_3, e_2)|_N = \begin{pmatrix} -a_{12}^1 & -a_{12}^2 & -a_{12}^3 & -a_{12}^4 & -a_{12}^5 & \cdots & -a_{12}^{m-3} & -a_{12}^{m-2} & -a_{12}^{m-1} & -a_{12}^m \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-1} & a_{24}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & a_{2m}^{m-1} & a_{25}^5 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & a_{2m}^{m-1} & a_{2m-2}^{m-2} & 0 & 0 \\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & a_{2m}^{m-1} & a_{2m-1}^{m-1} & 0 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & a_{2m}^m \end{pmatrix},$$

where $a_{1i}^i = a_{1m}^m + (m-i)a_{12}^2, a_{2i}^i = a_{2m}^m - (m-i)a_{12}^1, i = 3, 4, \ldots, m$.

Let $x_3' = x_3 - a_{2m}^{m-1}e_1 + a_{1m}^{m-1}e_2 - a_{12}^3 e_4 - a_{12}^4 e_5 - \cdots - a_{12}^{m-1}e_m$. Then

$$\mathrm{ad}(x_3', e_1)|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{12}^1 & a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{12}^m \\ 0 & 0 & a_{13}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{14}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m}^{m-2} & 0 & a_{15}^5 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{1m}^5 & a_{1m}^6 & a_{1m}^7 & \cdots & 0 & a_{1m-2}^{m-2} & 0 & 0 \\ 0 & 0 & a_{1m}^4 & a_{1m}^5 & a_{1m}^6 & \cdots & a_{1m}^{m-2} & 0 & a_{1m-1}^{m-1} & 0 \\ 0 & 0 & a_{1m}^3 & a_{1m}^4 & a_{1m}^5 & \cdots & a_{1m}^{m-3} & a_{1m}^{m-2} & 0 & a_{1m}^m \end{pmatrix},$$

$$\mathrm{ad}(x_3', e_2)|_N = \begin{pmatrix} -a_{12}^1 & -a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -a_{12}^m \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{24}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & a_{25}^5 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & a_{2m-2}^{m-2} & 0 & 0 \\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & 0 & a_{2m-1}^{m-1} & 0 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & a_{2m}^m \end{pmatrix},$$

where $a_{1i}^{i} = a_{1m}^{m} + (m-i)a_{12}^{2}, a_{2i}^{i} = a_{2m}^{m} - (m-i)a_{12}^{1}, i = 3, 4, \ldots, m$. From

$$[\mathrm{ad}(x_3', e_1)|_N, \mathrm{ad}(x_1, e_1)|_N] = \mathrm{ad}(x_3', e_1)|_N \mathrm{ad}(x_1, e_1)|_N - \mathrm{ad}(x_1, e_1)|_N \mathrm{ad}(x_3', e_1)|_N =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -a_{12}^{1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -a_{12}^{m} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2a_{1m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 3a_{1m}^{m-3} & 2a_{1m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & (m-5)a_{1m}^{5} & (m-6)a_{1m}^{6} & (m-7)a_{1m}^{7} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & (m-4)a_{1m}^{4} & (m-5)a_{1m}^{5} & (m-6)a_{1m}^{6} & \cdots & 2a_{1m}^{m-2} & 0 & 0 & 0 \\ 0 & 0 & (m-3)a_{1m}^{3} & (m-4)a_{1m}^{4} & (m-5)a_{1m}^{5} & \cdots & 3a_{1m}^{m-3} & 2a_{1m}^{m-2} & 0 & 0 \end{pmatrix}, \tag{4.2}$$

and the product $[\mathrm{ad}(x_3', e_1)|_N, \mathrm{ad}(x_1, e_1)|_N]$ is a finitely linear combination of $\{\mathrm{ad}(e_1, e_2)|_N, \mathrm{ad}(e_1, e_i)|_N, \mathrm{ad}(e_2, e_j)|_N, i, j = 4, 5, \ldots, m\}$, we have the matrix form of $[\mathrm{ad}(x_3', e_1)|_N, \mathrm{ad}(x_1, e_1)|_N]$ in the basis $e_1, \ldots, e_m$ is as follows

$$\begin{pmatrix} 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{m-5} & \alpha_{m-4} & \alpha_{m-3} & 0 \\ 0 & 0 & \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{m-5} & \beta_{m-4} & \beta_{m-3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma & 0 \end{pmatrix}. \tag{4.3}$$

Comparing (4.2) and (4.3), we get

$$\mathrm{ad}(x_3', e_1)|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_{12}^{2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{13}^{3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{14}^{4} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{15}^{5} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{1m-2}^{m-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{1m-1}^{m-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{1m}^{m} \end{pmatrix},$$

$$\mathrm{ad}(x_3', e_2)|_N = \begin{pmatrix} 0 & -a_{12}^{2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{2m}^{m} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & a_{2m}^{m} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^{5} & a_{2m}^{6} & a_{2m}^{7} & \cdots & 0 & a_{2m}^{m} & 0 & 0 \\ 0 & 0 & a_{2m}^{4} & a_{2m}^{5} & a_{2m}^{6} & \cdots & a_{2m}^{m-2} & 0 & a_{2m}^{m} & 0 \\ 0 & 0 & a_{2m}^{3} & a_{2m}^{4} & a_{2m}^{5} & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & a_{2m}^{m} \end{pmatrix}.$$

By $[[x_1, x_3', e_1], e_2, e_3] = 0$, we obtain $a_{2m}^m = 0$, and

$$\mathrm{ad}(x_3', e_2)|_N = \begin{pmatrix} 0 & -a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & 0 \end{pmatrix}.$$

Therefore, $\mathrm{ad}(x_3', e_1)|_N - a_{1m}^m \mathrm{ad}(x_2, e_1)|_N - a_{12}^2 \mathrm{ad}(x_1, e_1)|_N = 0$ and $\mathrm{ad}(x_3', e_2)|_N - a_{1m}^m \mathrm{ad}(x_2, e_2)|_N - a_{12}^2\, \mathrm{ad}(x_1, e_2)|_N$ is nilpotent. It follows that $I = F(x_3' - a_{1m}^m x_2 - a_{12}^2 x_1) + N$ is an $(m+1)$-dimensional hypo-nilpotent ideal of A. This is a contradiction. Therefore, we have $k \leqslant 2$. □

Corollary 4.2. *There are no $(m+k)$-dimensional solvable 3-Lie algebras with a maximal hypo-nilpotent ideal N when $k \geqslant 3$.*

In the following we give all solvable 3-Lie algebras with a 5-dimensional the simplest filiform maximal hypo-nilpotent ideal N.

1. Let A be a 6-dimensional 3-Lie algebra with a 5-dimensional hypo-nilpotent ideal N, and $x, e_1, e_2, e_3, e_4, e_5$ be a basis of A. Then up to isomorphisms, one and only one of following possibilities holds:

$$(M_1).\ \begin{cases} [e_1, e_2, e_j] = e_{j-1}, j = 4, 5, \\ [x, e_1, e_3] = e_3, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = be_3 + e_5, \\ [x, e_2, e_5] = ce_3; \end{cases} \qquad (M_2).\ \begin{cases} [e_1, e_2, e_j] = e_{j-1}, j = 4, 5, \\ [x, e_1, e_2] = e_2, \\ [x, e_1, e_3] = 2e_3, \\ [x, e_1, e_4] = e_4; \end{cases}$$

$$(M_3).\ \begin{cases} [e_1, e_2, e_j] = e_{j-1}, j = 4, 5, \\ [x, e_1, e_2] = e_2, \\ [x, e_1, e_3] = (\alpha + 2)e_3, \\ [x, e_1, e_4] = (\alpha + 1)e_4, \\ [x, e_1, e_5] = \alpha e_5; \end{cases} \qquad (M_4).\ \begin{cases} [e_1, e_2, e_j] = e_{j-1}, j = 4, 5, \\ [x, e_1, e_2] = e_2 + e_5, \\ [x, e_1, e_3] = 3e_3, \\ [x, e_1, e_4] = 2e_4, \\ [x, e_1, e_5] = e_5; \end{cases}$$

where $b, c, \alpha \in F, \alpha \neq 0$.

2. Let A be a 7-dimensional 3-Lie algebra with a 5-dimensional maximal hypo-nilpotent ideal N. Then there is a basis $x_1, x_2, e_1, \ldots, e_5$ of A such that the multiplication table is as follows:

$$\begin{cases} [e_1, e_2, e_j] = e_{j-1}, j = 4, 5, \\ [x_1, e_1, e_2] = e_2, \\ [x_1, e_1, e_i] = (5 - i)e_i, i = 3, 4, 5, \\ [x_2, e_1, e_i] = e_i, i = 3, 4, 5. \end{cases}$$

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