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Stable cohomology over local rings

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Abstract

For a commutative noetherian ring R with residue field k stable cohomology modules $\widehat{\operatorname{Ext}}{}^n_R(k,k)$ have been defined for each $n \in \mathbb{Z}$, but their meaning has remained elusive. It is proved that the k-rank of any $\widehat{\operatorname{Ext}}{}^n_R(k,k)$ characterizes important properties of R, such as being regular, complete intersection, or Gorenstein. These numerical characterizations are based on results concerning the structure of $\mathbb{Z}$-graded k-algebra carried by stable cohomology. It is shown that in many cases it is determined by absolute cohomology through a canonical homomorphism of algebras $\operatorname{Ext}_R(k,k) \to \widehat{\operatorname{Ext}}_R(k,k)$. Some techniques developed in the paper are applicable to the study of stable cohomology functors over general associative rings.

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Contents

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0. Introduction

A stable cohomology theory over an associative ring R associates to every pair (M, N) of R-modules groups $\widehat{\mathrm{Ext}}{}^n_R(M, N)$, which are defined for each $n \in \mathbb{Z}$ and vanish for all n when M or N has finite projective dimension. Different constructions, have been proposed by Benson and Carlson, Mislin, and Vogel, and all yield canonically isomorphic theories. However, there have been few applications outside of group theory and Galois theory, for which the prototype—Tate cohomology for finite groups—was created in the 1950s.

In the first four sections we develop general techniques for computing stable cohomology. We approach it through a canonical transformation $\iota : \mathrm{Ext}_R \to \widehat{\mathrm{Ext}}_R$ of absolute cohomology into stable cohomology, which we study by systematically using the compatible multiplicative structures carried by the two theories. A new feature are extensive applications of a third cohomological functor, the bounded cohomology $\overline{\mathrm{Ext}}_R$, which appears in a long exact sequence measuring the kernel and the cokernel of ι. By extending a construction of Eisenbud we show how to track changes in stable cohomology under factorizations of non-zero-divisors.

The core of the paper is its second part, devoted to stable cohomology over commutative noetherian local rings. One goal is to investigate if and how this theory reflects or detects properties of a ring or a module. A second goal is to study the structure of the local cohomology functors themselves. Historical precedent in commutative algebra points to the residue field k of a local ring R as the ultimate test case, so the focus is kept on it for much of the second part of the paper.

When applying the general machinery to a local ring R with residue field k we heavily use the fact that the absolute cohomology algebra $\mathcal{E} = \mathrm{Ext}_R(k, k)$ is the universal enveloping algebra of a graded Lie algebra. The existence of such a structure underlies a well documented successful interaction between local algebra and rational homotopy theory. Félix, Halperin, and Thomas have transplanted from algebra and systematically used in topology a notion of depth of cohomology modules. We take the concept back into algebra and use it in a different manner. Background material is developed in Appendix A and Section 5.

In Section 6 we give necessary and sufficient conditions for a local ring R to be regular (respectively, complete intersection, Gorenstein) in terms of the vanishing (respectively, size, finiteness) of $\operatorname{rank}_k \widehat{\mathrm{Ext}}{}^n_R(k, k)$ for a *single* value $n \in \mathbb{Z}$. The last result is surprising: unlike regularity or complete intersection, Gorensteinness is not recognized even by the entire sequence $(\operatorname{rank}_k \mathrm{Ext}^n_R(k, k))_{n \geqslant 0}$.

In Section 7 we start the study of the graded k-algebra $\mathcal{S} = \widehat{\mathrm{Ext}}{}^n_R(k, k)$. A result of Martsinkovsky, for which we give a short proof, shows that when R is singular the map of graded algebras $\iota : \mathcal{E} \to \mathcal{S}$ is injective. We determine $\mathrm{Coker}(\iota)$ as a left $\mathcal{E}$-module and prove that $\operatorname{depth} \mathcal{E} \geq 2$ implies $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}$ where $\mathcal{T}$ is the $\mathcal{E}$-torsion submodule of $\mathcal{S}$, and is the *unique* direct complement of $\iota(\mathcal{E})$ as a left $\mathcal{E}$-module. In Section 8 this information is used to produce a nearly complete, explicit computation of the algebra $\mathcal{S}$ for complete intersection rings.

It is natural to ask whether the results for complete intersections extend, in some form, to all singular Gorenstein rings R with $\operatorname{codim} R \geq 2$. In Section 9 we prove that $\operatorname{depth} \mathcal{E} \geq 2$ implies $\mathcal{T}$ is a two-sided ideal of $\mathcal{S}$ with $\mathcal{T}^2 = 0$, is isomorphic to a shift of $\mathrm{Hom}_k(\mathcal{E}, k)$ as left $\mathcal{E}$-module, and $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}$. A similar relationship between the Tate cohomology algebra $\widehat{\mathrm{H}}^*(G, k)$ of a finite group G and its cohomology algebra $\mathrm{H}^*(G, k)$ was discovered by Benson and Carlson. The parallel is remarkable, as $\mathrm{H}^*(G, k)$ is graded-commutative and finitely generated, while $\mathcal{E}$ may be non-finitely generated and almost always is very far from commutative.

One has $\operatorname{depth} \mathcal{E} \geq 1$ for all singular rings, so the condition $\operatorname{depth} \mathcal{E} \geq 2$ is not too special. We prove that it holds for several classes of Gorenstein rings, including those of codimension 2 or 3, those of minimal multiplicity, and the localizations of Koszul algebras. We are not aware whether splitting occurs always: Gorenstein rings with $\operatorname{depth} \mathcal{E} = 1$ are hard to come by, and for the known ones $\mathcal{E}$ splits off $\mathcal{S}$.

It was noted above that when R is not Gorenstein $\operatorname{rank}_k \widehat{\mathrm{Ext}}{}^n_R(k, k)$ is infinite for each n, so over such rings a different structure of $\mathcal{S}$ may be expected. As a test case in Section 10 we turn to Golod rings, whose homological properties are in many respects opposite to those of Gorenstein rings. We show that $\operatorname{depth} \mathcal{E} = 1$ holds for all Golod rings, and for a subclass of such rings we work out the structure of $\mathcal{S}$ in sufficient detail to prove that ι does not split as a map of left $\mathcal{E}$-module.

1. Cohomology theories

We start by describing notions concerning complexes and, more generally, DG (that is, differential graded) modules and algebras. The latter are used to describe composition products carried by the *absolute cohomology* functors. We then introduce a *bounded cohomology* theory that has not been systematically studied before. Finally, we present Vogel's construction of *stable cohomology*.

1.1. DG algebras and DG modules

To grade a complex C we use subscripts *or* superscripts. Thus, C can be written either as a sequence of maps $\partial^C_n : C_n \to C_{n-1}$, or as a sequence of maps $\partial^{-n}_C : C^{-n} \to C^{-n+1}$, with $\partial^{-n}_C = \partial^C_n$. Accordingly, an element $c \in C_n$ is assigned a lower (or homological) degree n, denoted *and* an upper (or cohomological) degree $-n$; we write $\lfloor c \rfloor = n$ and $\lceil c \rceil = -n$, respectively. When the nature of degree does not matter we use $|c|$ in place of either $\lfloor c \rfloor$ or $\lceil c \rceil$.

When $z \in C$ is a cycle $\mathrm{cl}(z)$ denotes its homology class.

For every $s \in \mathbb{Z}$ let $\Sigma^s C$ denote the complex with $(\Sigma^s C)_n = C_{n-s}$ and $\partial^{\Sigma^s C}_n = (-1)^s \partial^C_{n-s}$; let $\sigma^s : C \to \Sigma^s C$ be the bijective map $C_n \ni c \mapsto c \in (\Sigma^s C)_{n+s}$.

Bimodules have actions from the left and from the right, listed in that order. If A and A' are DG algebras and C is a DG A–A'-bimodule, then the formula

$$a \cdot \sigma^s(c) \cdot a' = (-1)^{|a|s} \sigma^s(a \cdot c \cdot a')$$

turns $\Sigma^s C$ into a DG A–A'-bimodule and σ^s into a chain map of DG bimodules. Furthermore, when B is a DG A–A-bimodule the map

$$B \otimes_A \Sigma^s C \longrightarrow \Sigma^s (B \otimes_A C)$$

given by $b \otimes \sigma^s(c) \mapsto (-1)^{|b|s} \sigma^s(b \otimes c)$ is an isomorphism of DG A–A'-modules.

For the rest of the section R denotes an associative ring, M and N are left R-modules, and $F \to M$ and $G \to N$ are projective resolutions.

1.2. Absolute Ext

Let $\mathrm{Hom}_R(F,G)$ denote the complex of abelian groups with

$$\mathrm{Hom}_R(F,G)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(F_i, G_{i+n}) = \mathrm{Hom}_R(F,G)^{-n}$$

as component of homological degree n (cohomological degree $-n$), and differential

$$\partial(\beta) = \partial^G \beta - (-1)^{|\beta|} \beta \partial^F.$$

The induced map $\mathrm{Hom}_R(F,G) \to \mathrm{Hom}_R(F,N)$ is a quasi-isomorphism, so one has

$$\mathrm{H}\big(\mathrm{Hom}_R(F,G)\big) \cong \mathrm{H}\big(\mathrm{Hom}_R(F,N)\big) = \mathrm{Ext}_R(M,N).$$

1.2.1. Composition of homomorphisms turns $\mathrm{Hom}_R(F,F)$ and $\mathrm{Hom}_R(G,G)$ into DG algebras, and $\mathrm{Hom}_R(F,G)$ into a DG $\mathrm{Hom}_R(G,G)$–$\mathrm{Hom}_R(F,F)$-bimodule. The *composition products* induced in homology can be computed from any pair of projective resolutions. They turn $\mathrm{Ext}_R(M,M)$ and $\mathrm{Ext}_R(N,N)$ into graded algebras, and $\mathrm{Ext}_R(M,N)$ into a graded $\mathrm{Ext}_R(N,N)$–$\mathrm{Ext}_R(M,M)$-bimodule.

1.2.2. The DG algebra $\mathrm{Hom}_R(G,G)$ acts on the complex G by evaluation of homomorphisms. For every complex C of right R-modules the map

$$\begin{aligned}\mathrm{Hom}_R(G,G) \otimes_{\mathbb{Z}} (C \otimes_R G) &\longrightarrow C \otimes_R G,\\ \alpha \otimes (c \otimes g) &\longmapsto (-1)^{|\alpha||c|} c \otimes \alpha(g)\end{aligned}$$

endows $C \otimes_R G$ with a structure of left DG module over $\mathrm{Hom}_R(G,G)$. Clearly, this structure is natural with respect to morphisms of complexes $C \to C'$.

1.2.3. Let L be a right R-module. Setting $C = L$ in 1.2.2 one obtains a morphism

$$\mathrm{Hom}_R(G,G) \otimes_{\mathbb{Z}} (L \otimes_R G) \longrightarrow L \otimes_R G.$$

In homology it induces for all $l, n \in \mathbb{Z}$ homomorphisms of abelian groups

$$\mathrm{Ext}_R^n(N,N) \otimes_{\mathbb{Z}} \mathrm{Tor}_l^R(L,N) \longrightarrow \mathrm{Tor}_{l-n}^R(L,N)$$

that turn $\mathrm{Tor}^R(L,N)$ into a graded left module over $\mathrm{Ext}_R(N,N)$.

1.3. Bounded Ext

A homomorphism $\beta \in \operatorname{Hom}_R(F, G)$ is *bounded* if $\beta_i = 0$ for all $i \gg 0$. The subset $\overline{\operatorname{Hom}}_R(F, G)$ of $\operatorname{Hom}_R(F, G)$, consisting of all bounded homomorphisms, is a subcomplex, with components

$$\overline{\operatorname{Hom}}_R(F, G)_n = \coprod_{i \in \mathbb{Z}} \operatorname{Hom}_R(F_i, G_{i+n}) = \overline{\operatorname{Hom}}_R(F, G)^{-n}.$$

The graded abelian group $\overline{\operatorname{Ext}}_R(M, N) = \mathrm{H}(\overline{\operatorname{Hom}}_R(F, G))$ with components

$$\overline{\operatorname{Ext}}^n_R(M, N) = \mathrm{H}^n\big(\overline{\operatorname{Hom}}_R(F, G)\big),$$

is called the *bounded cohomology* of M and N over R.

1.3.1. It is easy to see that $\overline{\operatorname{Hom}}_R(F, G)$ is a DG subbimodule of $\operatorname{Hom}_R(F, G)$ for the actions of $\operatorname{Hom}_R(F, F)$ and $\operatorname{Hom}_R(G, G)$ described in 1.2.1, so $\overline{\operatorname{Ext}}_R(M, N)$ becomes a graded $\operatorname{Ext}_R(N, N)$–$\operatorname{Ext}_R(M, M)$-bimodule.

The elementary observation below plays a pivotal role in the paper. It should be noted that the right-hand analog of this statement fails, see Example 10.9.

Lemma 1.3.2. *For every $\tau \in \overline{\operatorname{Ext}}_R(M, N)$ there exists an integer $j \geq 0$, such that*

$$\operatorname{Ext}^{\geqslant j}_R(N, N) \cdot \tau = 0.$$

Proof. By hypothesis, $\tau = \mathrm{cl}(\beta)$ for some chain map $\beta \in \operatorname{Hom}_R(F, G)$ satisfying $\beta(F) \subseteq G_{<j}$ for some $j \geq 0$. For each $\gamma \in \operatorname{Hom}_R(G, G)_n$ one then has

$$(\gamma\beta)(F) = \gamma\big(\beta(F)\big) \subseteq \gamma(G_{<j}) \subseteq G_{<j+n}.$$

Since $G_{<j+n} = 0$ for $n \leq -j$, this implies $\operatorname{Ext}^{\geqslant j}_R(N, N) \cdot \mathrm{cl}(\beta) = 0$. $\square$

Some of the DG module structures discussed so far are linked as follows:

Lemma 1.3.3. *There is a morphism of DG* $\operatorname{Hom}_R(G, G)$–$\operatorname{Hom}_R(F, F)$*-bimodules*

$$\omega : \operatorname{Hom}_R(F, R) \otimes_R G \longrightarrow \overline{\operatorname{Hom}}_R(F, G)$$

with actions on the source given by 1.2.1, 1.2.2, *and on the target by* 1.3.1.

If the R-module F_i is finite for each i, then ω is bijective.

Proof. It is easy to verify that $\omega'(\phi \otimes g)(f) = (-1)^{|g||f|}\phi(f)g$ defines a morphism

$$\omega' : \operatorname{Hom}_R(F, R) \otimes_R G \longrightarrow \operatorname{Hom}_R(F, G)$$

of DG bimodules. The image of ω' lies in $\overline{\operatorname{Hom}}_R(F, G)$, so it yields a morphism ω with the desired source and target. For each $n \in \mathbb{Z}$ the following equalities

$$\big(\mathrm{Hom}_R(F,R)\otimes_R G\big)_n = \coprod_{(-i)+j=n} \mathrm{Hom}_R(F_i,R)\otimes_R G_j,$$

$$\overline{\mathrm{Hom}}_R(F,G)_n = \coprod_{j-i=n} \mathrm{Hom}_R(F_i,G_j)$$

hold by definition. When each R-module F_i is finite, ω restricts to an isomorphism $\mathrm{Hom}_R(F_i,R)\otimes_R G_j \to \mathrm{Hom}_R(F_i,G_j)$ for each pair (i,j). □

1.4. Stable Ext

Using the subcomplex $\overline{\mathrm{Hom}}_R(F,G)$ described in 1.3, set

$$\widehat{\mathrm{Hom}}_R(F,G) = \mathrm{Hom}_R(F,G)/\overline{\mathrm{Hom}}_R(F,G).$$

Following Pierre Vogel, we define the *stable cohomology* of M and N over R to be the graded abelian group $\widehat{\mathrm{Ext}}_R(M,N) = \mathrm{H}(\widehat{\mathrm{Hom}}_R(F,G))$ with components

$$\widehat{\mathrm{Ext}}{}^n_R(M,N) = \mathrm{H}^n\big(\widehat{\mathrm{Hom}}_R(F,G)\big).$$

The assignment $(M,N) \mapsto \widehat{\mathrm{Ext}}_R(M,N)$ yields a cohomological functor, contravariant in M and covariant in N, from R-modules to graded $\mathbb{Z}$-modules.

1.4.1. As $\overline{\mathrm{Hom}}_R(F,G)$ is a DG subbimodule of $\mathrm{Hom}_R(F,G)$ for the left action of $\mathrm{Hom}_R(G,G)$ and the right action of $\mathrm{Hom}_R(F,F)$, see 1.3.1, one sees that $\widehat{\mathrm{Ext}}_R(M,M)$ and $\widehat{\mathrm{Ext}}_R(N,N)$ are graded algebras, and $\widehat{\mathrm{Ext}}_R(M,N)$ is a graded $\widehat{\mathrm{Ext}}_R(N,N)$–$\widehat{\mathrm{Ext}}_R(M,M)$-bimodule.

Stable cohomology over general associative rings took a long time to emerge, and then it appeared in several avatars. We give a short, incomplete list of sources.

1.4.2. Historically the first example of stable cohomology is Tate's cohomology theory $\widehat{\mathrm{H}}^n(G,-)$ for modules over a finite group G: One has $\widehat{\mathrm{H}}^n(G,-) = \widehat{\mathrm{Ext}}{}^n_{\mathbb{Z}G}(\mathbb{Z},-)$, where $\mathbb{Z}G$ is the group ring of G; see [13, Chapter XII]. Tate's construction is based on complete resolutions of $\mathbb{Z}$. Buchweitz [12] extended the technique to define a two-variable theory over two-sided noetherian Gorenstein rings.

The functors $\widehat{\mathrm{Ext}}{}^n_R(-,-)$ were introduced by Vogel in the mid-1980s. The first published account appears only in [19], where it is called 'Tate–Vogel cohomology.' Different approaches were independently proposed by Benson and Carlson [8] and by Mislin [26]; background and details can be found in Kropholler's survey [22, §4].[2] We have settled on the name 'stable cohomology' to emphasize the fact that $\widehat{\mathrm{Ext}}{}^0_R(M,N)$ is a group of homomorphisms of objects in a *stabilization* of the category of R-modules, see Beligiannis [7, §§3, 5] for details.

2. Comparisons

In this section R is an associative ring, M and N are left R-modules, $F \to M$ and $G \to N$ denote projective resolutions. The objective is to describe important links between the cohomology theories introduced in Section 1.

[2] Where Definition 4.2.2 contains a typo: $\Omega^{i+n}N$ should be changed to $\Omega^{i-n}N$.

2.1. By construction, there is an exact sequence of DG bimodules

$$0 \longrightarrow \overline{\mathrm{Hom}}_R(F,G) \longrightarrow \mathrm{Hom}_R(F,G) \longrightarrow \widehat{\mathrm{Hom}}_R(F,G) \longrightarrow 0 \tag{2.1.1}$$

that is unique up to homotopy. It defines an exact sequence

$$\begin{aligned} \overline{\mathrm{Ext}}_R(M,N) \xrightarrow{\eta} \mathrm{Ext}_R(M,N) \xrightarrow{\iota} \widehat{\mathrm{Ext}}_R(M,N) \\ \xrightarrow{\eth} \Sigma\,\overline{\mathrm{Ext}}_R(M,N) \xrightarrow{\Sigma\eta} \Sigma\,\mathrm{Ext}_R(M,N) \end{aligned} \tag{2.1.2}$$

of graded $\mathrm{Ext}_R(N,N)$–$\mathrm{Ext}_R(M,M)$-bimodules. Thus, there is an exact sequence

$$\begin{aligned} \cdots \longrightarrow \overline{\mathrm{Ext}}{}^n_R(M,N) \xrightarrow{\eta^n} \mathrm{Ext}^n_R(M,N) \xrightarrow{\iota^n} \widehat{\mathrm{Ext}}{}^n_R(M,N) \\ \xrightarrow{\eth^n} \overline{\mathrm{Ext}}{}^{n+1}_R(M,N) \xrightarrow{\eta^{n+1}} \mathrm{Ext}^{n+1}_R(M,N) \longrightarrow \cdots \end{aligned} \tag{2.1.3}$$

of abelian groups, and the latter is natural in both module arguments.

Furthermore, $\mathrm{Ext}_R(M,M) \to \widehat{\mathrm{Ext}}_R(M,M)$ is a homomorphism of graded algebras, and $\iota : \mathrm{Ext}_R(M,N) \to \widehat{\mathrm{Ext}}_R(M,N)$ is an equivariant homomorphism of graded $\widehat{\mathrm{Ext}}_R(N,N)$–$\widehat{\mathrm{Ext}}_R(M,M)$-bimodules.

It is easy to determine if η is an isomorphism; see also [22, (4.2.4)], [33, (4.5.1)].

Proposition 2.2. *The following conditions are equivalent.*

(i) *M has finite projective dimension.*
(ii) $\widehat{\mathrm{Ext}}{}^n_R(M,-) = 0$ *for every $n \in \mathbb{Z}$.*
(iii) $\widehat{\mathrm{Ext}}{}^n_R(-,M) = 0$ *for every $n \in \mathbb{Z}$.*
(iv) $\widehat{\mathrm{Ext}}{}^0_R(M,M) = 0$.
(v) $\eta^n : \overline{\mathrm{Ext}}{}^n_R(M,-) \to \mathrm{Ext}^n_R(M,-)$ *is an isomorphism for every $n \in \mathbb{Z}$.*
(vi) $\eta^n : \overline{\mathrm{Ext}}{}^n_R(-,M) \to \mathrm{Ext}^n_R(-,M)$ *is an isomorphism for every $n \in \mathbb{Z}$.*

Proof. Choosing a finite resolution $F \to M$ one gets $\overline{\mathrm{Hom}}_R(F,G) = \mathrm{Hom}_R(F,G)$ for every resolution G. Thus, (i) implies (v) and (vi). The exact sequence (2.1.3) shows that (v) implies (ii), and (vi) implies (iii). It is clear that (ii) or (iii) implies (iv). If (iv) holds, then for some $\gamma \in \mathrm{Hom}_R(F,F)^1$ and some $p \geq 0$ the morphism $\beta = \mathrm{id}^F - \partial\gamma + \gamma\partial : F \to F$ satisfies $\beta_i = 0$ for all $i \geq p$. Thus, one gets

$$0 = \mathrm{H}^p\big(\mathrm{Hom}_R(\beta,-)\big) = \mathrm{H}^p\big(\mathrm{Hom}_R\big(\mathrm{id}^F,-\big)\big) = \mathrm{id}^{\mathrm{H}^p(\mathrm{Hom}_R(\beta,-))}$$

hence $\mathrm{Ext}^p_R(M,-) = \mathrm{H}^p(\mathrm{Hom}_R(F,-)) = 0$; that is, $\mathrm{pd}_R M < p$, so (i) holds. □

Next we give a criterion for ι to be an isomorphism in high degrees.

Theorem 2.2. *For an integer m the following conditions are equivalent.*

(i) $\iota^n : \mathrm{Ext}^n_R(M,-) \to \widehat{\mathrm{Ext}}{}^n_R(M,-)$ *is an isomorphism for all $n > m$ and ι^m is an epimorphism.*
(ii) $\mathrm{Ext}^n_R(M,P) = 0$ *for all $n > m$ and every projective R-module P.*
(iii) $\overline{\mathrm{Ext}}{}^n_R(M,-) = 0$ *for all $n > m$.*

When M has a resolution by finite projective modules they are also equivalent to

(ii′) $\operatorname{Ext}^n_R(M,R)=0$ *for all* $n>m$.

Proof. The exact sequence (2.1.3) shows that (i) and (iii) are equivalent. If (iii) holds, then so does (ii), because $\operatorname{Ext}^n_R(M,P)\cong\overline{\operatorname{Ext}}{}^n_R(M,P)$ by Proposition 2.2. It is clear that (ii) implies (ii′). The converse holds because the hypothesis on M in (ii′) implies that the functor $\operatorname{Ext}^n_R(M,-)$ commutes with all direct sums.

To prove that (ii) implies (iii) fix an integer $n>m$ and choose a chain map $\alpha\in\overline{\operatorname{Hom}}_R(F,G)^n$. Thus, for some fixed $s\geq n$ and all $j\geq s$ one has $\alpha_j=0$, while

$$\partial^G_{j+1-n}\alpha_{j+1}-(-1)^n\alpha_j\partial^F_{j+1}=0\quad\text{holds for all } j\in\mathbb{Z}. \tag{2.2.1$_j$}$$

We need to find a homomorphism $\beta\in\overline{\operatorname{Hom}}_R(F,G)^{n-1}$ that satisfies

$$\partial^G_{j+2-n}\beta_{j+1}-(-1)^{n-1}\beta_j\partial^F_{j+1}=\alpha_{j+1}\quad\text{for all } j\in\mathbb{Z}. \tag{2.2.2$_j$}$$

Set $\beta_j=0$ for $j\geq s$ and assume by descending induction on j that we already have maps β_j satisfying $(2.2.2)_j$ for some integer $i\in[n,s]$ and all $j\geq i$. Set $\delta^h=(-1)^{n+1}\operatorname{Hom}_R(\partial^F_{h+1},G_{h-n})$ for each h. Using $(2.2.1)_i$ and $(2.2.2)_i$ we get

$$\begin{aligned}\delta^i\big(\alpha_i-\partial^G_{i+1-n}\beta_i\big)&=(-1)^{n+1}\alpha_i\partial^F_{i+1}+(-1)^n\partial^G_{i+1-n}\beta_i\partial^F_{i+1}\\&=-\partial^G_{i+1-n}\alpha_{i+1}+(-1)^n\partial^G_{i+1-n}\beta_i\partial^F_{i+1}\\&=-\partial^G_{i+1-n}\big(\alpha_{i+1}+(-1)^{n-1}\beta_i\partial^F_{i+1}\big)\\&=-\partial^G_{i+1-n}\partial^G_{i+2-n}\beta_{i+1}\\&=0.\end{aligned}$$

On the other hand, since one has $\operatorname{Ext}^i_R(M,G_{i-n})=0$ the sequence

$$\operatorname{Hom}_R(F_{i-1},G_{i-n})\xrightarrow{\delta^{i-1}}\operatorname{Hom}_R(F_i,G_{i-n})\xrightarrow{\delta^i}\operatorname{Hom}_R(F_{i+1},G_{i-n})$$

is exact, so there exists a homomorphism $\beta_{i-1}:F_{i-1}\to G_{i-n}$, such that

$$\alpha_i-\partial^G_{i+1-n}\beta_i=\delta^{i-1}(\beta_{i-1})=-(-1)^{n-1}\beta_{i-1}\partial^F_i.$$

Thus, β_{i-1} satisfies $(2.2.2)_{i-1}$, so the induction step is complete. As a result, for each $j\geq n-1$ we now have a homomorphism $\beta_j:F_j\to G_{j-(n-1)}$ satisfying the equality $(2.2.2)_j$. As $G_{j-(n-1)}=0$ for $j<n-1$, setting $\beta_j=0$ we extend the equality to all $j\in\mathbb{Z}$. We have proved $\overline{\operatorname{Ext}}{}^n_R(M,N)=0$, as desired. □

2.3. A *complete resolution* of M is a morphism of complexes $\nu:T\to F$ such that ν_i is bijective for all $i\gg 0$, each T_i is projective, and for all $n\in\mathbb{Z}$ and every projective R-module P one has $\mathrm{H}_n(T)=0=\mathrm{H}_n(\operatorname{Hom}_R(T,P))$; see [14, (1.1)]. (In some contexts it is assumed, in addition, that the R-modules T_n are also finite; no such hypothesis is needed or made here.) When such a complete resolution exists, $\mathrm{H}_{-n}(\mathrm{Hom}_R(M,N))$ is called the nth *Tate cohomology* of M with coefficients in N.

Cornick and Kropholler [14, (1.2)] prove that when Tate cohomology is defined it is naturally isomorphic to stable cohomology. We deduce this from Theorem 2.2:

Corollary 2.4. *If $\nu : T \to F$ is a complete resolution of M, then one has*

$$\mathrm{H}^n\big(\mathrm{Hom}_R(T,N)\big) \cong \widehat{\mathrm{Ext}}{}^n_R(M,N) \quad \text{for each } n \in \mathbb{Z}.$$

Proof. Fix $n \in \mathbb{Z}$ and set $K = \mathrm{Coker}(\partial^T_n)$. For each $i \geq 1$ one then has

$$\mathrm{H}^{n-1+i}\big(\mathrm{Hom}_R(T,N)\big) = \mathrm{Ext}^i_R(K,N), \tag{2.4.1$_i$}$$

because $\Sigma^{-(n-1)}(T_{\geqslant n-1})$ is a projective resolution of K. From the condition on T one gets $\mathrm{Ext}^i_R(K,P) = 0$ for all $i \geq 1$ and every projective R-module P, so

$$\mathrm{Ext}^1_R(K,N) \cong \widehat{\mathrm{Ext}}{}^1_R(K,N) \tag{2.4.2}$$

holds by the theorem. Choose $p \geq n$ with ν_i bijective for $i \geq p$. The R-module $L = \mathrm{Ker}(\partial^F_{p-1})$ is then isomorphic to $\mathrm{Ker}(\partial^T_{p-1})$, so there exist exact sequences

$$0 \longrightarrow L \longrightarrow T_{p-1} \longrightarrow \cdots \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow K \longrightarrow 0,$$
$$0 \longrightarrow L \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

In view of 2.2, the iterated connecting maps defined by these sequences yield

$$\widehat{\mathrm{Ext}}{}^1_R(K,N) \cong \widehat{\mathrm{Ext}}{}^{n-p}_R(L,N) \cong \widehat{\mathrm{Ext}}{}^n_R(M,N). \tag{2.4.3}$$

To finish the proof, concatenate the isomorphisms $(2.4.1)_1$, (2.4.2), and (2.4.3). □

3. Additional structures

In this section we discuss the existence of finer natural structures on stable cohomology groups, such as rings of operators or internal gradings.

Proposition 3.1. *Let R be an algebra over a commutative ring K.*

(1) *The exact sequence* (2.1.2) *is one of graded K-modules, and the various pairings of cohomology groups are K-bilinear.*

If, in addition, K is noetherian, R is finite as a K-module, M and N are finite R-modules, and n is an integer, then the following assertions also hold.

(2) *The K-modules $\widehat{\mathrm{Ext}}{}^n_R(M,N)$ and $\overline{\mathrm{Ext}}{}^{n+1}_R(M,N)$ are finite simultaneously.*
(3) *When $\mathrm{Ext}^{\gg 0}_R(M,R) = 0$ the K-module $\widehat{\mathrm{Ext}}{}^n_R(M,N)$ is finite for every $n \in \mathbb{Z}$.*

Proof. (1) This is due to the fact that the relevant maps in cohomology are induced by morphisms of complexes or K-modules.

(2) By (1), the maps in the exact sequence (2.1.3) are K-linear, and under our hypotheses the K-modules $\operatorname{Ext}_R^n(M,N)$ and $\operatorname{Ext}_R^{n+1}(M,N)$ are noetherian.

(3) Let $F \to M$ and $G \to N$ be resolutions by finite projective R-modules. Choose $m \geq 1$ so that $\mathrm{H}_{-j}(\operatorname{Hom}_R(F,R)) = \operatorname{Ext}_R^j(M,R) = 0$ for all $j > m$; then $\operatorname{Hom}_R(F,R)$ is quasi-isomorphic to the complex C of right R-modules defined by

$$C_i = \begin{cases} \operatorname{Hom}_R(F_{-i},R) & \text{for } 0 \geq i \geq -m; \\ \operatorname{Im}\operatorname{Hom}_R(\partial^F_{m+1},R) & \text{for } i = -m-1; \\ 0 & \text{for } i < -m-1 \text{ or } i > 0. \end{cases}$$

As a consequence, $\operatorname{Hom}_R(F,R) \otimes_R G$ is quasi-isomorphic to $C \otimes_R G$. For each $n \in \mathbb{Z}$ this gives the second isomorphism below; Lemma 1.3.3 provides the first one:

$$\begin{aligned} \overline{\operatorname{Ext}}{}_R^n(M,N) &= \mathrm{H}^n\big(\overline{\operatorname{Hom}}_R(F,G)\big) \\ &\cong \mathrm{H}^n\big(\operatorname{Hom}_R(F,R) \otimes_R G\big) \\ &\cong \mathrm{H}^n(C \otimes_R G). \end{aligned}$$

As $(C \otimes_R G)^n = \coprod_{j=0}^m C^j \otimes_R G^{n-j}$ is a finite K-module for each n, we see that $\overline{\operatorname{Ext}}{}_R^n(M,N)$ is a finite K-module; by (2), so is $\widehat{\operatorname{Ext}}{}_R^n(M,N)$. □

Stable cohomology behaves predictably under flat base change.

Proposition 3.2. *Let R be a commutative noetherian ring, M an R-module that admits a resolution by finite projective modules, and $R \to R'$ a homomorphism of rings such that the right R-module R' is flat.*

For each R-module N there is then a commutative diagram

$$\begin{array}{ccc} R' \otimes_R \operatorname{Ext}_R(M,N) & \xrightarrow{R' \otimes_R \iota} & R' \otimes_R \widehat{\operatorname{Ext}}_R(M,N) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \operatorname{Ext}_{R'}(R' \otimes_R M, R' \otimes_R N) & \xrightarrow{\iota} & \widehat{\operatorname{Ext}}_{R'}(R' \otimes_R M, R' \otimes_R N). \end{array}$$

When $N = M$ all the maps in the diagram are homomorphisms of graded algebras.

Proof. Set $(-)' = (R' \otimes_R -)$. Let $F \to M$ be a resolution by finite projective R-modules and $G \to N$ be a projective resolution. In the commutative square

$$\begin{array}{ccc} R' \otimes_R \operatorname{Hom}_R(F,G) & \xrightarrow{\simeq} & R' \otimes_R \operatorname{Hom}_R(F,N) \\ \big\downarrow & & \big\downarrow\cong \\ \operatorname{Hom}_{R'}(F',G') & \xrightarrow{\simeq} & \operatorname{Hom}_{R'}(F',N') \end{array}$$

the isomorphism is due to the choice of F. Thus, the left vertical map is a quasi-isomorphism. It appears in the following commutative diagram where the vertical arrows are induced by the map

$\alpha \mapsto \alpha'$ and the rows are exact, see (2.1.1):

$$\begin{tikzcd}
0 \arrow[r] & R' \otimes_R \overline{\mathrm{Hom}}_R(F,G) \arrow[r] \arrow[d, "\cong"'] & R' \otimes_R \mathrm{Hom}_R(F,G) \arrow[r] \arrow[d, "\simeq"'] & R' \otimes_R \widehat{\mathrm{Hom}}_R(F,G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\mathrm{Hom}}_{R'}(F',G') \arrow[r] & \mathrm{Hom}_{R'}(F',G') \arrow[r] & \widehat{\mathrm{Hom}}_{R'}(F',G') \arrow[r] & 0.
\end{tikzcd}$$

The flatness of R' implies that $F' \to M'$ and $G' \to N'$ are R'-projective resolutions, and that the homology of the right-hand square above is the desired diagram. □

Next we turn to cohomology of graded objects.

3.3. We say that the ring R is *internally graded* if $R = \bigoplus_{i\in\mathbb{Z}}^{\infty} R_i$ as abelian groups, and $R_i R_j \subseteq R_{i+j}$ holds for all i, j. Internal gradings for M, N are defined similarly. By convention, we allow M_i to be written also as M^{-i}. As usual, we let $M(s)$ denote the graded R-module with $M(s)_i = M_{s+i}$ for all $i \in \mathbb{Z}$.

Assume R, M, and N are internally graded. A homomorphism $\beta : M \to N$ is *homogeneous* of internal degree $-j$ if $\beta(M_i) \subseteq N_{i-j}$ holds for each $i \in \mathbb{Z}$. All such maps form an abelian subgroup $\mathrm{Homgr}_R(M,N)^j$ of $\mathrm{Hom}_R(M,N)$. Clearly, the sum of these subgroups is direct, so $\mathrm{Hom}_R(M,N)$ contains as a subgroup the group

$$\mathrm{Homgr}_R(M,N) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Homgr}_R(M,N)^j.$$

When M is finitely presented, one has $\mathrm{Homgr}_R(M,N) = \mathrm{Hom}_R(M,N)$.

Every graded R-module M has a *graded free resolution* $F \to M$ that is, a resolution in which each F_i is a graded free R-module and differentials are homogeneous of internal degree 0. It produces a subcomplex $\mathrm{Homgr}_R(F,N)$ of $\mathrm{Hom}_R(F,N)$, consisting of graded abelian groups and homomorphisms of internal degree 0.

Assume that each R-module F_i is finite. One then has $\mathrm{Homgr}_R(F,N) = \mathrm{Hom}_R(F,N)$, and hence the absolute Ext groups inherit an internal grading:

$$\mathrm{Ext}_R^n(M,N) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}_R^n(M,N)^j.$$

For each $n \in \mathbb{Z}$ one also has equalities

$$\mathrm{Hom}_R(F,G)_n = \prod_{i\in\mathbb{Z}} \bigoplus_{j\in\mathbb{Z}} \mathrm{Homgr}_R(F_i, G_{i+n})^j.$$

However, there is no induced internal grading on the right-hand side, so extra steps are needed to introduce such a grading on stable cohomology groups.

Proposition 3.4. *Assume R is an internally graded ring, M, N are internally graded R-modules, and M has a graded free resolution $F \to M$ where each F_i is finite* (*as is the case, for example, when R is left noetherian and M is finite*).

For each $n \in \mathbb{Z}$ the abelian groups $\widehat{\operatorname{Ext}}^n_R(M,N)$ and $\overline{\operatorname{Ext}}^n_R(M,N)$ then have natural internal gradings, which are preserved by the homomorphisms in the exact sequence (2.1.3) *and are additive under the various products.*

If M has a complete resolution T by finite projective graded R-modules with differentials ∂^T of degree 0, *then the internal gradings of $\mathrm{H}^n(\operatorname{Homgr}_R(T,N))$ and $\widehat{\operatorname{Ext}}^n_R(M,N)$ are preserved by the isomorphisms of Corollary* 2.4.

Proof. Let $G \to N$ be graded free resolution. For all $j, n \in \mathbb{Z}$ the subgroups

$$\operatorname{Homgr}_R(F,G)^j_n = \prod_{i\in\mathbb{Z}} \operatorname{Homgr}_R(F_i, G_{i+n})^j$$

of $\operatorname{Hom}_R(F,G)_n$ form a subcomplex $\operatorname{Homgr}_R(F,G)^j$ of $\operatorname{Hom}_R(F,G)$. The Comparison Theorem for graded resolutions shows that the canonical morphism $\operatorname{Homgr}_R(F,G)^j \to \operatorname{Homgr}_R(F,N)^j$ is a quasi-isomorphism. It follows that the complex $\operatorname{Homgr}_R(F,G) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Homgr}_R(F,G)^j$ appears in a commutative diagram

$$\begin{tikzcd}
\operatorname{Homgr}_R(F,G) \arrow[r, "\simeq"] \arrow[d] & \operatorname{Homgr}_R(F,N) \arrow[d, equal] \\
\operatorname{Hom}_R(F,G) \arrow[r, "\simeq"] & \operatorname{Hom}_R(F,N)
\end{tikzcd}$$

where the horizontal maps are quasi-isomorphisms, and the equality is due to the finiteness of the modules F_i. Thus, the left vertical map is a quasi-isomorphism.

Setting $\overline{\operatorname{Homgr}}_R(F,G)_n = \bigoplus_{j\in\mathbb{Z}} \coprod_{i\in\mathbb{Z}} \operatorname{Homgr}_R(F_i, G_{i+n})^j$ for each $n \in \mathbb{Z}$ one gets an internally graded subcomplex $\overline{\operatorname{Homgr}}_R(F,G)$ of the internally graded complex $\operatorname{Homgr}_R(F,G)$. Thus, $\widehat{\operatorname{Homgr}}_R(F,G) = \operatorname{Homgr}_R(F,G)/\overline{\operatorname{Homgr}}_R(F,G)$ is an internally graded complex. On the other hand, one has equalities

$$\begin{aligned}
\overline{\operatorname{Homgr}}_R(F,G)_n &= \bigoplus_{j\in\mathbb{Z}} \coprod_{i\in\mathbb{Z}} \operatorname{Homgr}_R(F_i, G_{i+n})^j \\
&= \coprod_{i\in\mathbb{Z}} \bigoplus_{j\in\mathbb{Z}} \operatorname{Homgr}_R(F_i, G_{i+n})^j \\
&= \coprod_{i\in\mathbb{Z}} \operatorname{Hom}_R(F_i, G_{i+n}) = \overline{\operatorname{Hom}}_R(F,G)_n.
\end{aligned}$$

Putting the preceding observations together, one gets a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{\operatorname{Homgr}}_R(F,G) \arrow[r] \arrow[d, equal] & \operatorname{Homgr}_R(F,G) \arrow[r] \arrow[d, "\simeq"'] & \widehat{\operatorname{Homgr}}_R(F,G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\operatorname{Hom}}_R(F,G) \arrow[r] & \operatorname{Hom}_R(F,G) \arrow[r] & \widehat{\operatorname{Hom}}_R(F,G) \arrow[r] & 0.
\end{tikzcd}$$

It implies that the vertical arrow on the right-hand side is a quasi-isomorphism.

The first assertion of the proposition follows. The remaining ones are easy consequences of the quasi-isomorphisms above and the definition of products, respectively, the definition of the isomorphisms $\mathrm{H}^n(\mathrm{Hom}_R(T,N)) \cong \widehat{\mathrm{Ext}}{}^n_R(M,N)$. $\square$

4. Non-zero-divisors

A staple in basic homological algebra are 'change-of-rings theorems' that track the behavior of cohomology groups under passage to quotients modulo non-zero-divisors. An essential ingredient in such results is the functoriality of absolute Ext groups in the ring argument. Stable cohomology does not enjoy a similar property, so we approach change of rings through the natural homomorphisms

$$\iota : \mathrm{Ext}_R(M,N) \longrightarrow \widehat{\mathrm{Ext}}_R(M,N)$$

from (2.1). To simplify notation, we let $\hat{\alpha}$ denote the image of α under ι.

For absolute cohomology the following result is due to Gulliksen [20].

Theorem 4.1. *Let Q be an associative ring, $f \in Q$ a central non-zero-divisor, and set $R = Q/(f)$. For all R-modules M, N there exist elements*

$$\vartheta^M \in \mathrm{Ext}^2_R(M,M) \quad \textit{and} \quad \vartheta^N \in \mathrm{Ext}^2_R(N,N)$$

with the following properties:

(1) *Every ξ in $\overline{\mathrm{Ext}}_R(M,N)$ (respectively, $\mathrm{Ext}_R(M,N)$, $\widehat{\mathrm{Ext}}_R(M,N)$) satisfies*

$$\vartheta^N \cdot \xi = \xi \cdot \vartheta^M.$$

(2) *ϑ^M is in the center of $\mathrm{Ext}_R(M,M)$ and ϑ^N in that of $\mathrm{Ext}_R(N,N)$.*

(3) *$\hat{\vartheta}^M$ is in the center of $\widehat{\mathrm{Ext}}_R(M,M)$ and $\hat{\vartheta}^N$ in that of $\widehat{\mathrm{Ext}}_R(N,N)$.*

When there is no ambiguity, we let ϑ denote either one of ϑ^N or ϑ^M, and $(\ )_\vartheta$ the functor of graded localization at the multiplicatively closed set $\{\vartheta^i \mid i \geq 0\}$.

The last assertion of the corollary is due to Buchweitz, see [12, (10.2.3)].

Corollary 4.2. *There are induced structures of graded algebras on $\mathrm{Ext}_R(N,N)_\vartheta$ and $\mathrm{Ext}_R(M,M)_\vartheta$ (respectively, $\widehat{\mathrm{Ext}}_R(N,N)_\vartheta$ and $\widehat{\mathrm{Ext}}_R(M,M)_\vartheta$), and an induced left–right-bimodule structure on $\mathrm{Ext}_R(M,N)_\vartheta$ (respectively, $\widehat{\mathrm{Ext}}_R(M,N)_\vartheta$).*

The map ι from 2.1 *induces isomorphisms of graded algebras*

$$\mathrm{Ext}_R(N,N)_\vartheta \longrightarrow \widehat{\mathrm{Ext}}_R(N,N)_\vartheta,$$
$$\mathrm{Ext}_R(M,M)_\vartheta \longrightarrow \widehat{\mathrm{Ext}}_R(M,M)_\vartheta$$

and an equivariant isomorphism of graded bimodules over them,

$$\mathrm{Ext}_R(M,N)_\vartheta \longrightarrow \widehat{\mathrm{Ext}}_R(M,N)_\vartheta.$$

If M or N has finite projective dimension over Q, then

$$\widehat{\operatorname{Ext}}_R(M,N)_\vartheta = \widehat{\operatorname{Ext}}_R(M,N).$$

The notation of the theorem is in force for the rest of this section. In the proofs, presented later in this section, we use a construction of Eisenbud [15, (1.1)]:

4.3. Let F be a *liftable* complex of projective R-modules, meaning that there is a graded projective Q-module $\widetilde{F}$ with $R\otimes_Q \widetilde{F} = F$; for example, every complex of free modules is liftable. Since each $\widetilde{F}_i$ is projective, one can choose a map

$$\tilde{\partial}^F \in \operatorname{Hom}_Q(\widetilde{F},\widetilde{F})_{-1} \quad \text{with } R\otimes_Q \tilde{\partial}^F = \partial^F.$$

As $R\otimes_Q (\tilde{\partial}^F)^2 = (R\otimes_Q \tilde{\partial}^F)^2 = 0$, for each $x\in \widetilde{F}_n$ there exists $y\in \widetilde{F}_{n-2}$ satisfying $(\tilde{\partial}^F)^2(x) = fy$. As f is a non-zero-divisor on $\widetilde{F}_{n-2}$ (it is one on Q and $\widetilde{F}_{n-2}$ is projective), y is defined uniquely and hence depends Q-linearly on x. Setting $\tilde{\theta}^F(x) = y$ one gets a homomorphism $\tilde{\theta}^F \in \operatorname{Hom}_Q(\widetilde{F},\widetilde{F})_{-2}$. As f is central, we get

$$f\big(\tilde{\theta}^F\tilde{\partial}^F\big) = \big(f\tilde{\theta}^F\big)\tilde{\partial}^F = \big(\tilde{\partial}^F\big)^3 = \tilde{\partial}^F\big(f\tilde{\theta}^F\big) = f\big(\tilde{\partial}^F\tilde{\theta}^F\big).$$

As f is a non-zero-divisor, this implies $\tilde{\theta}^F\tilde{\partial}^F = \tilde{\partial}^F\tilde{\theta}^F$. Thus, one gets a chain map

$$\theta^F = R\otimes_Q \tilde{\theta}^F \in \operatorname{Hom}_R(F,F)_{-2}.$$

The first assertion of the next result is [15, (1.3)]. An adaptation of the original argument allows us to handle the other two cases as well.

Lemma 4.4. *Let F and G be liftable complexes of projective R-modules and let γ be a homomorphism in* $\operatorname{Hom}_R(F,G)$.

If γ is a chain map, then $\theta^G\gamma$ and $\gamma\theta^F$ are homotopic in $\operatorname{Hom}_R(F,G)$.

If γ is a bounded chain map, then $\theta^G\gamma$ and $\gamma\theta^F$ are homotopic in $\overline{\operatorname{Hom}}_R(F,G)$.

If $\hat{\gamma}$ is a chain map, then $\theta^G\hat{\gamma}$ and $\hat{\gamma}\theta^F$ are homotopic in $\widehat{\operatorname{Hom}}_R(F,G)$.

Proof. Set $n = \lfloor\gamma\rfloor$ and assume first $\hat{\gamma}$ is a chain map. Thus, the map

$$\delta = \partial^G\gamma - (-1)^n\gamma\partial^F \in \operatorname{Hom}_R(F,G)_{n-1} \tag{4.4.1}$$

satisfies $\delta_i = 0$ for all $i\geq j$ and a fixed $j\in\mathbb{Z}$. Choose $\tilde{\gamma}\in \operatorname{Hom}_Q(\widetilde{F},\widetilde{G})_n$ with $R\otimes_Q\tilde{\gamma} = \gamma$ and $\tilde{\delta}\in\overline{\operatorname{Hom}}_Q(\widetilde{F},\widetilde{G})_{n-1}$ with $R\otimes_Q\tilde{\delta} = \delta$ and $\tilde{\delta}_i = 0$ for all $i\geq j$. There exits then a unique $\tilde{\tau}\in\operatorname{Hom}_Q(\widetilde{F},\widetilde{G})_{n-1}$ satisfying

$$\tilde{\partial}^G\tilde{\gamma} = (-1)^n\tilde{\gamma}\tilde{\partial}^F + \tilde{\delta} + f\tilde{\tau}.$$

Using the relation above and the equalities $(\tilde{\partial}^F)^2 = f\tilde{\theta}^F$ and $(\tilde{\partial}^G)^2 = f\tilde{\theta}^G$, we get

$$\begin{aligned}f\big(\tilde\theta^G\tilde\gamma\big) &= \big(\tilde\partial^G\big)^2\tilde\gamma\\ &= (-1)^n\tilde\partial^G\tilde\gamma\tilde\partial^F + \tilde\partial^G\tilde\delta + \tilde\partial^G f\tilde\tau\\ &= \tilde\gamma\big(\tilde\partial^F\big)^2 + (-1)^n\tilde\delta\tilde\partial^F + (-1)^n f\tilde\tau\tilde\partial^F + \tilde\partial^G\tilde\delta + f\tilde\partial^G\tilde\tau\\ &= f\big(\tilde\gamma\tilde\theta^F + \tilde\partial^G\tilde\tau + (-1)^n\tilde\tau\tilde\partial^F\big) + \big(\tilde\partial^G\tilde\delta + (-1)^n\tilde\delta\tilde\partial^F\big).\end{aligned}$$

Since f is a non-zero-divisor on $\widetilde{G}$, the preceding computation yields

$$\tilde\theta^G\tilde\gamma - \tilde\gamma\tilde\theta^F - \big(\tilde\partial^G\tilde\tau - (-1)^{n+1}\tilde\tau\tilde\partial^F\big) \in \overline{\mathrm{Hom}}_Q(\widetilde{F}, \widetilde{G}).$$

The map $\tau = R\otimes_Q \tilde\tau : F \to G$ then satisfies

$$\theta^G\gamma - \gamma\theta^F - \big(\partial^G\tau - (-1)^{n+1}\tau\partial^F\big) \in \overline{\mathrm{Hom}}_R(F, G).$$

In other words, $\hat\tau \in \widehat{\mathrm{Hom}}_R(F, G)$ is a homotopy between $\theta^G\hat\gamma$ and $\hat\gamma\theta^F$.

If γ is a chain map, then (4.4.1) holds with $\delta = 0$, so in the computation above one can choose $\tilde\delta = 0$. The resulting τ is a homotopy between $\theta^G\gamma$ and $\gamma\theta^F$.

When γ is a bounded chain map the map $\tilde\gamma$ can be chosen to be bounded as well, and then $\tilde\tau$ is necessarily bounded, so the homotopy τ is in $\overline{\mathrm{Hom}}_R(F, G)$. □

Proof of Theorem 4.1. Let $F \to M$ be a liftable projective resolution (for example, choose F to be free a resolution), and set $\vartheta^M = \mathrm{cl}(\theta^F) \in \mathrm{Ext}^2_R(M, M)$.

First we show that ϑ^M does not depend on the choice of liftable resolution. If $F' \to M$ is one, then pick a morphisms of complexes $\gamma : F \to F'$ lifting id^M. By Lemma 4.4, the maps $\theta^{F'}\gamma$ and $\gamma\theta^F$ are homotopic. Thus, the isomorphisms

$$\mathrm{H}\big(\mathrm{Hom}_R(F', F')\big) \xrightarrow{\cong} \mathrm{H}\big(\mathrm{Hom}_R(F, F')\big) \xleftarrow{\cong} \mathrm{H}\big(\mathrm{Hom}_R(F, F)\big).$$

map $\mathrm{cl}(\theta^{F'})$ and $\mathrm{cl}(\theta^F)$ to the same element, which was to be shown.

The other assertions of the theorem follow directly from the lemma. □

One can produce liftable resolutions using a construction of Shamash [28, Section 3]; we describe it next, following the simplified exposition in [4].

4.5. Let E be a projective resolution of M over Q.

By induction, one gets for each $i \geq 0$ a map $\sigma^{(i)} \in \mathrm{Hom}_Q(E, E)_{2i-1}$, such that

$$\sigma^{(0)} = \partial^E \quad\text{and}\quad \sum_{h=0}^{i}\sigma^{(h)}\sigma^{(i-h)} = \begin{cases} f\,\mathrm{id}^E & \text{for } i = 1;\\ 0 & \text{for } i \geq 2.\end{cases}$$

Let D be a graded $\mathbb{Z}$-module, such that for each $i \geq 0$ the $\mathbb{Z}$-module D_{2i} is free with a single basis element $y^{(i)}$. Let $\widetilde{F}$ be the graded projective Q-module with

$$\widetilde{F}_n = \bigoplus_{i\geqslant 0} E_{n-2i}\otimes_{\mathbb{Z}} D_{2i}.$$

For every $i \geq 0$ and each $e \in E_{n-2i}$ the formula

$$\tilde{\partial}_n\big(e \otimes y^{(i)}\big) = \sum_{h\geqslant 0} \sigma^{(h)}(e) \otimes y^{(i-h)}$$

defines a Q-linear map $\tilde{\partial} : \widetilde{F} \to \widetilde{F}$ of degree -1. A direct computation yields

$$\tilde{\partial}^2\big(e \otimes y^{(i)}\big) = fe \otimes y^{(i-1)}.$$

As a consequence, one obtains a complex of projective R-modules

$$(F, \partial) = (R \otimes_Q \widetilde{F}, R \otimes_Q \tilde{\partial}).$$

It is proved in [4, (3.1.3)] that this a resolution of M over R. This projective resolution is clearly liftable, and one can define a map θ^F as in (4.3) by setting

$$\theta^F\big(r \otimes e \otimes y^{(i)}\big) = r \otimes e \otimes y^{(i-1)}.$$

Proof of Corollary 4.2. The multiplicativity properties follow from the definitions of the various products in cohomology and the centrality of the element ϑ.

The exact sequence (2.1.2) is the homology sequence of an exact sequence of DG modules over $\mathrm{Hom}_R(N,N)$. Thus, its maps commute with left multiplication by the element $\vartheta = \vartheta^N \in \mathrm{Ext}^2_R(N,N)$. Localizing at $\{\vartheta^i\}_{i\geqslant 0}$ the exact sequence 2.1 we obtain an exact sequence of $\mathrm{Ext}_R(N,N)_\vartheta$–$\mathrm{Ext}_R(M,M)_\vartheta$ bimodules. Lemma 1.3.2 implies that in this sequence $\overline{\mathrm{Ext}}_R(M,N)_\vartheta$ vanishes, so ι_ϑ is bijective.

For the last assertion, we show that if $\mathrm{pd}_Q M$ is finite, then $\hat{\vartheta} \in \widehat{\mathrm{Ext}}_R(M,M)$ is invertible. Let E be a finite projective resolution of M over Q and set

$$\zeta\big(r \otimes e \otimes y^{(i)}\big) = r \otimes e \otimes y^{(i+1)}$$

in the notation of 4.5. This formula defines a map $\zeta \in \mathrm{Hom}_Q(F,F)_2$, such that

$$(\partial\zeta - \zeta\partial)(F) \subseteq \big(R \otimes_Q E \otimes_{\mathbb{Z}} \mathbb{Z}y^{(0)}\big) \supseteq \big(\zeta\theta^F - \mathrm{id}^F\big)(F) \quad \text{and} \quad \theta^F\zeta = \mathrm{id}^F.$$

Thus, $\partial(\hat{\zeta}) = 0$ and $\hat{\zeta}\hat{\theta}^F = \hat{\theta}^F\hat{\zeta} = \widehat{\mathrm{id}^F}$ in $\widehat{\mathrm{Hom}}_R(F,F)$, as desired. □

As another application of Shamash's construction, we derive an exact sequence used several times in the paper, for which many other proofs are known.

Proposition 4.6. *There is an exact sequence of graded bimodules*

$$\begin{aligned}\Sigma^{-2}\,\mathrm{Ext}_R(M,N) \xrightarrow{\lambda} \mathrm{Ext}_R(M,N) \longrightarrow \mathrm{Ext}_Q(M,N)&\\ \longrightarrow \Sigma^{-1}\,\mathrm{Ext}_R(M,N) \xrightarrow{\lambda} \Sigma\,\mathrm{Ext}_R(M,N)&\end{aligned}$$

over $\mathrm{Ext}_R(N,N)$–$\mathrm{Ext}_R(M,M)$*, where* λ *is given by multiplication with* ϑ*.*

Proof. Using the notation of 4.5, we form a sequence

$$0 \longrightarrow R \otimes_Q E \xrightarrow{\alpha} F \xrightarrow{\theta^F} \Sigma^2 F \longrightarrow 0$$

of morphisms complexes of R-modules, where $\alpha(r \otimes e) = r \otimes e \otimes y^{(0)}$. It is clear that the underlying sequence of graded R-modules is split exact. Let $G \to N$ be a projective resolution. The induced sequence of complexes of abelian groups

$$0 \longrightarrow \Sigma^{-2} \operatorname{Hom}_R(F, G) \longrightarrow \operatorname{Hom}_R(F, G) \longrightarrow \operatorname{Hom}_R(R \otimes_Q E, G) \longrightarrow 0$$

is then exact, and its cohomology exact sequence is the desired one. □

5. Depth of cohomology modules

Let $(R, \mathfrak{m}, k)$ be a local ring,[3] M a finite R-module, and set

$$\mathcal{E} = \operatorname{Ext}_R(k, k) \quad \text{and} \quad \mathcal{M} = \operatorname{Ext}_R(M, k).$$

The *depth* of $\mathcal{M}$ over $\mathcal{E}$ is defined by means of the formula

$$\operatorname{depth}_{\mathcal{E}} \mathcal{M} = \inf\bigl\{n \in \mathbb{N} \bigm| \operatorname{Ext}^n_{\mathcal{E}}(k, \mathcal{M}) \neq 0\bigr\}.$$

We systematically write $\operatorname{depth} \mathcal{E}$ in place of $\operatorname{depth}_{\mathcal{E}} \mathcal{E}$.

The use of depth to study the structure of $\mathcal{E}$ was pioneered by Félix, Jacobsson, Halperin, Löfwall, and Thomas in the important paper [16]. Their main result is the finiteness of $\operatorname{depth} \mathcal{E}$; to prove it they develop methods for obtaining upper bounds on depth. To study stable cohomology we mostly need lower bounds.

General properties of depth of graded modules used in the paper are collected in Appendix A. In this section we focus on additional properties stemming from the cohomological nature of $\mathcal{E}$ and $\mathcal{M}$.

5.1. Universal enveloping algebras

Let π be a graded Lie algebra over k, such that $\operatorname{rank}_k \pi^i$ is finite for all $i \in \mathbb{Z}$ and $\pi^i = 0$ for $i \leq 0$, and let $\mathcal{D}$ be the universal enveloping algebra of π; for definitions of these notions see [4, (10.1.2)].[4]

5.1.1. The k-algebra $\mathcal{D}$ has an increasing multiplicative filtration, whose pth stage is the k-linear span of products involving at most p elements of π. By the Poincaré–Birkhoff–Witt Theorem, see [30, Thm. 2, Cor.], the associated graded k-algebra is isomorphic to $\Lambda_k(\pi^{\text{odd}}) \otimes_k \mathsf{S}_k(\pi^{\text{even}})$, where Λ_k and S_k denote, respectively, exterior algebra and symmetric algebra functors over k.

Directly from the Poincaré–Birkhoff–Witt isomorphism one gets:

[3] Recall that this means that R is a commutative noetherian ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

[4] Where condition (3) contains a typo: both $-$ signs should be changed to $+$ signs.

5.1.2. There is an equality of formal power series

$$\sum_{n=0}^{\infty}\bigl(\operatorname{rank}_k \mathcal{D}^n\bigr)t^n = \frac{\prod_{i\geqslant 0}(1+t^{2i+1})^{\operatorname{rank}_k \pi^{2i+1}}}{\prod_{i\geqslant 0}(1-t^{2i+2})^{\operatorname{rank}_k \pi^{2i+2}}}.$$

5.1.3. If $\mathcal{D}'$ is the universal enveloping algebra of a graded Lie subalgebra π' of π, then $\mathcal{D}$ is free as a left $\mathcal{D}'$-module and as a right $\mathcal{D}'$-module. In particular, every $\zeta \in \pi^{\mathrm{even}} \setminus \{0\}$ is a left non-zero-divisor and a right non-zero-divisor.

A theorem of Milnor and Moore, André, and Sjödin, see [30] or [4, (10.2.1.5)] for proofs, introduces graded Lie algebras into the study of local rings.

5.1.4. The graded k-algebra $\mathcal{E} = \operatorname{Ext}_R(k,k)$ is the universal enveloping algebra of a graded Lie algebra, denoted π_R and called the *homotopy Lie algebra* of R.

In low degrees the components of π_R are easy to describe.

5.1.5. A *minimal Cohen presentation* of the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is an isomorphism $\widehat{R} \cong Q/\mathfrak{a}$, where $(Q,\mathfrak{n},k)$ is a complete regular local ring and $\mathfrak{a}$ is an ideal contained in $\mathfrak{n}^2$. Cohen's Structure Theorem shows that one always exists.

There are isomorphisms of k-vector spaces, see e.g. [4, (10.2.1.2), (7.1.5)]:

$$\pi_R^1 \cong \operatorname{Hom}_k\bigl(\mathfrak{m}/\mathfrak{m}^2,k\bigr) \cong \operatorname{Hom}_k\bigl(\mathfrak{n}/\mathfrak{n}^2,k\bigr), \tag{5.1.5.1}$$

$$\pi_R^2 \cong \operatorname{Hom}_k(\mathfrak{a}/\mathfrak{n}\mathfrak{a},k). \tag{5.1.5.2}$$

Recall that the number $\operatorname{codim} R = \operatorname{edim} R - \dim R$ is called the *codimension* of R. Krull's Principal Ideal Theorem and the catenarity of the regular ring Q give

$$\operatorname{rank}_k \pi_R^2 \geq \operatorname{height}(\mathfrak{a}) = \dim Q - \dim \widehat{R} = \operatorname{edim} R - \dim R = \operatorname{codim} R. \tag{5.1.5.3}$$

A specific subalgebra of $\mathcal{E}$ will prove useful in computations.

Lemma 5.1.6. *The graded subspace $\pi_R^{\geqslant 2}$ of π_R is a Lie ideal, and the universal enveloping algebra $\mathcal{D}$ of $\pi_R^{\geqslant 2}$ satisfies* $\operatorname{depth}\mathcal{D} = \operatorname{depth}\mathcal{E}$.

Proof. For degree reasons, $\pi_R^{\geqslant 2}$ is a graded Lie ideal of π_R. Thus, one has $\mathcal{D}^{\geqslant 1}\mathcal{E} = \mathcal{E}\mathcal{D}^{\geqslant 1}$. The Poincaré–Birkhoff–Witt Theorem 5.1.1 implies that the right $\mathcal{D}$-module $\mathcal{E}$ is free and $\mathcal{E}/(\mathcal{E}\cdot\mathcal{D}^{\geqslant 1}) \cong \Lambda_k(\pi^1)$ holds. Now apply Corollary A.8. □

The following known simple consequence of 5.1.4 is used throughout the paper.

Lemma 5.1.7. *The ring R is singular if and only if some element $\vartheta \in \mathcal{E}^2$ is a left and right non-zero-divisor on $\mathcal{E}$, if and only if* $\operatorname{depth}\mathcal{E} \geq 1$.

Proof. When R is singular so is $\widehat{R}$, hence $\pi_R^2 \neq 0$ by (5.1.5.2), so 5.1.3 yields a left and right non-zero-divisor $\vartheta \in \mathcal{E}^2$; in view of A.5 the existence of ϑ implies $\operatorname{depth}\mathcal{E} \geq 1$. On the other hand, $\operatorname{depth}\mathcal{E} \geq 1$ implies R is singular: assuming the contrary one gets $\mathcal{E}^i = 0$ for $i \gg 0$, hence $\Gamma\mathcal{E} = \mathcal{E} \neq 0$, contradicting A.5. □

The next result is due to Martsinkovsky [25, Theorem 6]. We give a short proof.

Theorem 5.1.8. *When R is singular for each $n \in \mathbb{Z}$ there is an exact sequence*

$$0 \longrightarrow \operatorname{Ext}_R^n(k,k) \xrightarrow{\iota} \widehat{\operatorname{Ext}}{}_R^n(k,k) \xrightarrow{\eth} \overline{\operatorname{Ext}}{}_R^{n-1}(k,k) \longrightarrow 0.$$

Proof. We prove $\eta = 0$ in the exact sequence of graded left $\operatorname{Ext}_R(k,k)$-modules

$$\overline{\operatorname{Ext}}{}_R(k,k) \xrightarrow{\eta} \operatorname{Ext}_R(k,k) \xrightarrow{\iota} \widehat{\operatorname{Ext}}{}_R(k,k) \xrightarrow{\eth} \Sigma\,\overline{\operatorname{Ext}}{}_R(k,k) \xrightarrow{\Sigma\eta} \Sigma\,\widehat{\operatorname{Ext}}{}_R(k,k)$$

given by 2.1. In view of A.5, Lemmas 1.3.2 and 5.1.7 imply the equalities

$$\overline{\operatorname{Ext}}{}_R(k,k) = \Gamma\,\overline{\operatorname{Ext}}{}_R(k,k) \quad \text{and} \quad \Gamma \operatorname{Ext}_R(k,k) = 0,$$

where Γ denotes the section functor A.4. The $\operatorname{Ext}_R(k,k)$-linearity of η now yields

$$\eta\big(\overline{\operatorname{Ext}}{}_R(k,k)\big) = \eta\big(\Gamma\,\overline{\operatorname{Ext}}{}_R(k,k)\big) \subseteq \Gamma \operatorname{Ext}_R(k,k) = 0. \qquad \square$$

5.2. Regular elements

Fix an element $g \in \mathfrak{m}$, let M be an R-module annihilated by g. We view M also as a module over $R' = R/(g)$, and set:

$$\mathcal{E}' = \operatorname{Ext}_{R'}(k,k) \quad \text{and} \quad \mathcal{M}' = \operatorname{Ext}_{R'}(M,k).$$

The canonical homomorphism of rings $R \to R'$ induces a homomorphism

$$\rho^k : \mathcal{E}' \longrightarrow \mathcal{E}$$

of graded k-algebras and a ρ^k-equivariant homomorphism of graded modules

$$\rho^M : \mathcal{M}' \longrightarrow \mathcal{M}.$$

5.2.1. If g is R-regular Theorem 4.1 and Proposition 4.6 yield an exact sequence

$$\Sigma^{-2}\mathcal{M}' \xrightarrow{\lambda^M} \mathcal{M}' \xrightarrow{\rho^M} \mathcal{M} \longrightarrow \Sigma^{-1}\mathcal{M}' \xrightarrow{\Sigma\lambda^M} \Sigma\mathcal{M}'$$

of graded left $\mathcal{E}'$-modules, with λ^M induced by multiplication on $\mathcal{M}'$ with a central element $\vartheta' \in \mathcal{E}'^2$. Thus, one then has $\operatorname{Ker}(\rho^k) = \vartheta'\mathcal{E}' = \mathcal{E}'\vartheta'$ and $\operatorname{Ker}(\rho^M) = \vartheta'\mathcal{M}'$.

It is useful to know how depth changes when passing from the $\mathcal{E}$-module $\mathcal{M}$ to the $\mathcal{E}'$-module $\mathcal{M}'$. In two cases we provide complete—and contrasting—answers.

Proposition 5.2.2. *If $g \notin \mathfrak{m}^2$ is R-regular, then ρ^k and ρ^M are injective,*

$$\operatorname{depth}\mathcal{E}' = \operatorname{depth}\mathcal{E} \quad \textit{and} \quad \operatorname{depth}_{\mathcal{E}'}\mathcal{M}' = \operatorname{depth}_{\mathcal{E}'}\mathcal{M}.$$

Proof. It is proved in [3, (2.8)] that ρ^k and ρ^M are injective. Set $\mathcal{D} = \rho^k(\mathcal{E}')$.

The exact sequence in 5.2.1 and A.2 yield $\operatorname{depth}_{\mathcal{D}} \mathcal{M}' = \operatorname{depth}_{\mathcal{D}} \mathcal{M}$.

As ρ^k is injective, applied to $M = k$ the same sequence yields

$$\sum_{n=0}^{\infty} (\operatorname{rank}_k \mathcal{E}^n) t^n = (1+t) \sum_{n=0}^{\infty} (\operatorname{rank}_k \mathcal{D}^n) t^n.$$

As $\mathcal{D}$ is the universal enveloping algebra of the Lie subalgebra $\pi' = \rho^k(\pi_{R'}) \subseteq \pi_R$, from 5.1.2 we deduce $\operatorname{rank}_k \pi_R^1 = \operatorname{rank}_k \pi'^1 + 1$ and $\operatorname{rank}_k \pi_R^n = \operatorname{rank}_k \pi'^n$ for all $n \geq 2$. Thus, $\pi_R = \pi' \oplus k\varepsilon$ for some $\varepsilon \in \pi_R^1$, so π' is an ideal for degree reasons, and hence $\mathcal{D}$ is normal in $\mathcal{E}$. The sequence in 5.2.1 also produces an isomorphism $\mathcal{E} \cong \mathcal{D} \oplus \Sigma^{-1}\mathcal{D}$ of left $\mathcal{D}$-modules, so Corollary A.8 gives $\operatorname{depth} \mathcal{D} = \operatorname{depth} \mathcal{E}$. $\square$

Proposition 5.2.3. *If $g \in \mathfrak{m}^2$ is R-regular, then the central element $\vartheta' \in \mathcal{E}'^2$ from* 5.2.1 *is central also in $\mathcal{S}$, is regular on $\mathcal{E}'$, and one has*

$$\mathcal{E}'/(\vartheta') \cong \mathcal{E} \quad \textit{and} \quad \operatorname{depth} \mathcal{E}' = \operatorname{depth} \mathcal{E} + 1.$$

When $g \in \mathfrak{m} \operatorname{Ann}_R M$ the element ϑ' is a regular also on $\mathcal{M}'$ and one has

$$\mathcal{M}'/\vartheta'\mathcal{M}' \cong \mathcal{M} \quad \textit{and} \quad \operatorname{depth}_{\mathcal{E}'} \mathcal{M}' = \operatorname{depth}_{\mathcal{E}} \mathcal{M} + 1.$$

Proof. By Theorem 4.1 the element ϑ' is central in $\mathcal{S}$, and by [3, (2.8)] it is regular on $\mathcal{E}'$ and $\mathcal{M}'$. The isomorphisms come from 5.2.1, the equalities from A.3(2). $\square$

A last variation on the preceding theme is proved by elementary arguments:

Lemma 5.2.4. *If $M = L/gL$ for some R-module L and an L-regular element $g \in \mathfrak{m}$, and $\mathcal{L}$ denotes the $\mathcal{E}$-module* $\operatorname{Ext}_R(L,k)$*, then one has* $\operatorname{depth}_{\mathcal{E}} \mathcal{M} = \operatorname{depth}_{\mathcal{E}} \mathcal{L}$.

When g is also R-regular, one has $\operatorname{depth}_{\mathcal{E}'} \mathcal{M}' = \operatorname{depth}_{\mathcal{E}'} \mathcal{L}$ *as well.*

Proof. For the first equality apply A.2 to the exact sequence of graded $\mathcal{E}$-modules

$$0 \longrightarrow \Sigma^{-1}\mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L} \longrightarrow 0.$$

It is induced by exact sequence $0 \to L \xrightarrow{g} L \to M \to 0$ of R-modules because one has $\operatorname{Ext}_R(g\operatorname{id}^M, k) = 0$. For the second equality, note the isomorphism $\mathcal{M}' \cong \mathcal{L}$. $\square$

5.3. Residue field extensions

For depth computations one can sometimes adjust the ring R while preserving essential homological properties.

Lemma 5.3.1. *When $R \to (R', \mathfrak{m}', k')$ is a flat homomorphism of local rings with $\mathfrak{m}R' = \mathfrak{m}'$ there is a commutative diagram of homomorphisms of graded k'-algebras*

$$\begin{tikzcd} k' \otimes_k \mathcal{E} \arrow[r, "k' \otimes_k \iota"] \arrow[d, "\cong"'] & k' \otimes_k \mathcal{S} \arrow[d, "\cong"] \\ \mathcal{E}' \arrow[r, "\iota"] & \mathcal{S}' \end{tikzcd}$$

where $\mathcal{S} = \widehat{\operatorname{Ext}}_R(k,k)$, $\mathcal{E}' = \operatorname{Ext}_{R'}(k',k')$ and $\mathcal{S}' = \widehat{\operatorname{Ext}}_{R'}(k',k')$; in particular,

$$\operatorname{depth} \mathcal{E}' = \operatorname{depth} \mathcal{E} \quad \textit{and} \quad \operatorname{depth}_{\mathcal{E}'} \mathcal{S}' = \operatorname{depth}_{\mathcal{E}} \mathcal{S}.$$

Proof. Proposition 3.2 provides the commutative diagram. It implies the equalities of depths, in view of standard change of rings formulas. □

Recall that $\operatorname{edim} R$ denotes the *embedding dimension* of R, that is, the minimal number of generators of $\mathfrak{m}$, and $\operatorname{mult} R$ denotes the multiplicity of R.

Lemma 5.3.2. *There exists a complete local ring R' with algebraically closed residue field,* $\operatorname{mult} R = \operatorname{mult} R'$, $\operatorname{edim} R' = \operatorname{edim} R - \operatorname{depth} R$, $\operatorname{depth} R' = 0$, *and*

$$\operatorname{depth}\big(\operatorname{Ext}_{R'}(k',k')\big) = \operatorname{depth} \mathcal{E}.$$

Proof. Let k' be an algebraic closure of k. There always is a flat local homomorphism $R \to S$ with $S/\mathfrak{m}S = k'$, see [10, Chapitre IX, Appendice, Théorème 1, Corollaire]. One then has $\operatorname{depth} \operatorname{Ext}_{\widehat{S}}(k',k') = \operatorname{depth} \mathcal{E}$, see Lemma 5.3.1, and $\operatorname{depth} \widehat{S} = \operatorname{depth} R$.

If $\operatorname{depth} R$ is positive, then $\operatorname{mult} T = \operatorname{mult} S$ for $T = S/(g)$ and some regular element $g \notin (\mathfrak{n}\widehat{S})^2$. In that case $\operatorname{edim} T = \operatorname{edim} R - 1$ and $\operatorname{depth} T = \operatorname{depth} R - 1$ also hold, while Proposition 5.2.2 gives $\operatorname{depth} \operatorname{Ext}_T(k',k') = \operatorname{depth} \operatorname{Ext}_{\widehat{S}}(k',k')$. □

6. Finiteness of stable cohomology

In this section $(R, \mathfrak{m}, k)$ is a local ring. Classical results characterize ring-theoretical properties of R in terms of vanishing of absolute Ext modules. Here we establish analogs for stable Ext modules. Remarkably, the key turns out to be a better understanding of bounded cohomology.

Theorem 6.1. *For each R-module N there is an isomorphism*

$$\overline{\operatorname{Ext}}_R(k,N) \cong \operatorname{Ext}_R(k,R) \otimes_k \operatorname{Tor}^R(k,N) \tag{6.1.1}$$

of graded left $\operatorname{Ext}_R(N,N)$-modules. In particular, there is an isomorphism

$$\overline{\operatorname{Ext}}{}^n_R(k,N) \cong \coprod_{i-j=n}^{\infty} \operatorname{Ext}^i_R(k,R) \otimes_k \operatorname{Tor}^R_j(k,N) \tag{6.1.2}$$

of k-vector spaces for every $n \in \mathbb{Z}$.

Remark. The reader will notice that the graded algebra $\operatorname{Ext}_R(k,k)$ acts from the right on both $\overline{\operatorname{Ext}}_R(k,N)$ and $\operatorname{Ext}_R(k,R)$, and from the left on $\operatorname{Tor}^R(k,N)$. We do not know whether these structures are related.

Proof. Let $G \to N$ and $F \to k$ be free resolutions, with F_n finite for each n, and let $R \to J$ be an injective resolution. These resolutions induce quasi-isomorphisms

$$\begin{aligned}\operatorname{Hom}_R(F,R)\otimes_R G &\simeq \operatorname{Hom}_R(F,J)\otimes_R G\\ &\simeq \operatorname{Hom}_R(k,J)\otimes_R G\\ &\cong \operatorname{Hom}_R(k,J)\otimes_k (k\otimes_R G)\end{aligned}$$

that commute with the action of $\operatorname{Hom}_R(G,G)$. From Lemma 1.3.3 and the Künneth formula one now obtains isomorphisms of graded left $\operatorname{Ext}_R(N,N)$-modules

$$\overline{\operatorname{Ext}}_R(k,N)\cong \mathrm{H}\big(\operatorname{Hom}_R(F,R)\otimes_R G\big)\cong \operatorname{Ext}_R(k,R)\otimes_k \operatorname{Tor}^R(k,N). \qquad \square$$

The *Bass numbers* of R, defined by $\mu^n = \operatorname{rank}_k \operatorname{Ext}^n_R(k,R)$, appear in some of the most useful characterizations of the Gorenstein property, recalled below.

6.2. The ring R is *Gorenstein* if it satisfies the following equivalent conditions:

(i) $\mu^n = 1$ for $n = \operatorname{depth} R$ and $\mu^n = 0$ for all $n \neq \operatorname{depth} R$.
(ii) $\mu^n = 0$ for some $n > \operatorname{depth} R$.
(iii) $\operatorname{id}_R R < \infty$.

Corollary 6.3. *If the k-vector space $\widehat{\operatorname{Ext}}^n_R(k,N)$ is finite for some $n \in \mathbb{Z}$, then N has finite projective dimension or R is Gorenstein.*

Remark. It is tempting to ask whether finiteness conditions on $\widehat{\operatorname{Ext}}^n_R(M,k)$ imply homological restrictions on the module M or the ring R. A negative answer is given in Example 6.9.

Proof. When $\operatorname{rank}_k \widehat{\operatorname{Ext}}^n_R(k,N)$ is finite so is $\operatorname{rank}_k \overline{\operatorname{Ext}}^{n+1}_R(k,N)$; see Proposition 3.1(2). In view of Eq. (6.1.2) this implies $\operatorname{Tor}^R_j(k,N)=0$ for all $j \gg 0$ or $\mu^i = 0$ for all $i \gg 0$; that is, N has finite projective dimension or R is Gorenstein. $\square$

Even the entire sequence $(\operatorname{rank}_k \operatorname{Ext}^n_R(k,k))_{n\geqslant 0}$ does not recognize the Gorenstein property of R, see Example 6.10, so the next result is rather surprising. The expression for $\operatorname{rank}_k \widehat{\operatorname{Ext}}^n_R(k,k)$ in the next result is known, see [6, (9.2)].

Theorem 6.4. *The ring R is Gorenstein if* $\operatorname{rank}_k \widehat{\operatorname{Ext}}^n_R(k,k) < \infty$ *for some $n \in \mathbb{Z}$, only if the module $\widehat{\operatorname{Ext}}^n_R(M,N)$ is finite for every $n \in \mathbb{Z}$ and all finite modules M, N.*

If R is Gorenstein and singular, then for $d =$ depth R *and each $n \in \mathbb{Z}$ one has*

$$\operatorname{rank}_k \widehat{\operatorname{Ext}}^n_R(k,k) = \operatorname{rank}_k \operatorname{Ext}^n_R(k,k) + \operatorname{rank}_k \operatorname{Ext}^{d-1-n}_R(k,k).$$

Proof. If $\operatorname{rank}_k \widehat{\operatorname{Ext}}{}^n_R(k,k)$ is finite for some $n \in \mathbb{Z}$, then R is Gorenstein by Corollary 6.3. If R is Gorenstein, then it has finite injective dimension as R-module, see 6.2, so by Theorem 3.1(3) the module $\widehat{\operatorname{Ext}}{}^n_R(M,N)$ is finite when M and N are.

Theorem 5.1.8 provides the first equality in the following chain:

$$\begin{aligned}\operatorname{rank}_k \widehat{\operatorname{Ext}}{}^n_R(k,k) &= \operatorname{rank}_k \operatorname{Ext}^n_R(k,k) + \operatorname{rank}_k \overline{\operatorname{Ext}}{}^{n-1}_R(k,k)\\ &= \operatorname{rank}_k \operatorname{Ext}^n_R(k,k) + \operatorname{rank}_k \operatorname{Tor}^R_{d-1-n}(k,k)\\ &= \operatorname{rank}_k \operatorname{Ext}^n_R(k,k) + \operatorname{rank}_k \operatorname{Ext}^{d-1-n}_R(k,k).\end{aligned}$$

For the others use formulas (6.1.2) and 6.2(i), and vector space duality. □

Theorem 6.5. *The ring R is regular if $\widehat{\operatorname{Ext}}{}^n_R(k,k) = 0$ for some $n \in \mathbb{Z}$, only if $\widehat{\operatorname{Ext}}{}^n_R(M,N) = 0$ for every $n \in \mathbb{Z}$ and all R-modules M, N.*

Proof. When R is regular it has finite global dimension, so $\widehat{\operatorname{Ext}}{}^n_R(M,N) = 0$ for all M, N, and n by 2.2. When R is singular $\operatorname{Ext}^i_R(k,k) \neq 0$ for all $i \geq 0$. Assuming $\widehat{\operatorname{Ext}}{}^n_R(k,k) = 0$ for some n, part (3) of the theorem shows that R is Gorenstein. Part (2) yields $\operatorname{Ext}^n_R(k,k) = 0 = \operatorname{Ext}^{d-1-n}_R(k,k)$, implying $d < 0$, which is absurd. □

It is well known that polynomial growth of the sequence $(\operatorname{rank}_k \operatorname{Ext}^n_R(k,k))_{n\geqslant 0}$ recognizes a smaller class of rings, whose definition we proceed to recall.

6.6. The ring R is *complete intersection* if in some minimal Cohen presentation of $\widehat{R} \cong Q/\mathfrak{a}$, see 5.1.5, the ideal $\mathfrak{a}$ is generated by a Q-regular set.

Next we strengthen the recognition criterion for complete intersections.

Theorem 6.7. *Let R be a local ring, set $d = \dim R$ and $c = \operatorname{codim} R$.*

For each $n \in \mathbb{Z}$ there is an inequality

$$\operatorname{rank}_k \operatorname{Ext}^n_R(k,k) \geq \sum_{i=0}^{d} \binom{d}{i}\binom{c+n-i-1}{c-1}. \tag{6.7.1$_n$}$$

When R is complete intersection equalities hold for all $n \geq 0$.

If equality holds for a single $n \geq 2$, then R is complete intersection.

Remark 6.7.2. For every ring R easy computations yield $\operatorname{rank}_k \operatorname{Ext}^0_R(k,k) = 1$ and $\operatorname{rank}_k \operatorname{Ext}^1_R(k,k) = \operatorname{edim} R = c + d$. This shows that in $(6.7.1)_n$ equality always holds for $n = 0, 1$, so the condition on n cannot be dropped from the last assertion.

Proof of Theorem 6.7. Let $\widehat{R} \cong Q/\mathfrak{a}$ be a minimal Cohen presentation and set $r = \operatorname{rank}_k \mathfrak{a}/\mathfrak{n}\mathfrak{a}$. As $c + d = \operatorname{edim} R = \operatorname{rank}_k \pi^1_R$ and $r = \operatorname{rank}_k \pi^2_R$, see 5.1.5, from 5.1.2 one gets the first coefficient-wise inequality of formal power series below:

$$\begin{aligned}\sum_{n=0}^{\infty} \operatorname{rank}_k \operatorname{Ext}^n_R(k,k)t^n &= \frac{(1+t)^{c+d}}{(1-t^2)^r}\,\frac{\prod_{i\geqslant 1}(1+t^{2i+1})^{\operatorname{rank}_k \pi^{2i+1}}}{\prod_{i\geqslant 1}(1-t^{2i+2})^{\operatorname{rank}_k \pi^{2i+2}}}\\ &\succcurlyeq \frac{(1+t)^{c+d}}{(1-t^2)^r}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1+t)^d}{(1-t)^c}\frac{1}{(1-t^2)^{r-c}} \\
&\succcurlyeq \frac{(1+t)^d}{(1-t)^c} + (r-c)t^2\frac{(1+t)^d}{(1-t)^c}.
\end{aligned}
$$

The second inequality holds because $r-c$ is non-negative, see (5.1.5.3).

Comparing coefficients, one obtains for every $n \geq 0$ a numerical inequality

$$
\operatorname{rank}_k \operatorname{Ext}^n_R(k,k) \geq \sum_{i=0}^{d}\binom{d}{i}\binom{c+n-i-1}{c-1} + (r-c)\sum_{i=0}^{d}\binom{d}{i}\binom{c+n-i-3}{c-1},
$$

which shows that the inequality in $(6.7.1)_n$ holds for every $n \geq 0$.

If equality holds for some $n \geq 2$, then the last formula yields $r = c$, so by the Cohen–Macaulay Theorem the ideal $\mathfrak{a}$ is generated by a regular sequence.

When R is complete intersection one has $r = c$, and Tate [32, Theorem 6] proves

$$
\sum_{n=0}^{\infty} \operatorname{rank}_k \operatorname{Ext}^n_R(k,k)\, t^n = \frac{(1+t)^d}{(1-t)^c}.
$$

This equality of power series means that $(6.7.1)_n$ holds for each $n \geq 0$. $\square$

Regular local rings are precisely the complete intersection rings of codimension 0, and every complete intersection ring is Gorenstein. This hierarchy may also be observed by comparing the next result with Theorems 6.5 and 6.4.

Theorem 6.8. *For each $n \in \mathbb{Z}$ there is an inequality*

$$
\operatorname{rank}_k \widehat{\operatorname{Ext}}^n_R(k,k) \geq \sum_{i=0}^{d}\binom{d}{i}\left(\binom{c+n-i-1}{c-1} + \binom{c+d-n-i-2}{c-1}\right). \qquad (6.8.1)_n
$$

When R is complete intersection equalities hold for each $n \in \mathbb{Z}$.

If equality holds for a single $n \leq d-3$ or $n \geq 2$, then R is complete intersection.

Remark 6.8.2. For $d \geq 4$ there are no restrictions on n in the last assertion, so the value of $\operatorname{rank}_k \widehat{\operatorname{Ext}}^n_R(k,k)$ for *any* $n \in \mathbb{Z}$ determines whether R is complete intersection. On the other hand, when d satisfies $0 \leq d \leq 3$ Remark 6.7.2 and Theorem 6.4 show that for each n in the non-empty interval $[d-2, 1]$ the value of $\operatorname{rank}_k \widehat{\operatorname{Ext}}^n(k,k)$ is the same for all Gorenstein rings with $\dim R = d$.

Proof of Theorem 6.8. Theorems 6.4 and 6.7 show that inequalities always hold in $(6.8.1)_n$, and they become equalities when R is complete intersection.

If equality holds in $(6.8.1)_n$ for some $n \geq 2$ (respectively, $n \leq d-3$), then Theorem 6.4 implies that equality holds in $(6.7.1)_n$ with $n \geq 2$ (respectively, in $(6.7.1)_{d-1-n}$ with $d-1-n \geq 2$), so R is complete intersection by Theorem 6.7. $\square$

To put in context some results in this section we provide two examples. The first one and Corollary 6.3 shows that finiteness of stable cohomology is not symmetric.

Example 6.9. Let $(S, \mathfrak{n}, k)$ be an arbitrary local ring and set $R = S[x]/(x^2)$. The R-module $M = R/(x)$ then has $\widehat{\operatorname{Ext}}^n_R(M, k) \cong k$ for every $n \in \mathbb{Z}$.

Indeed, the following sequence clearly is a complete resolution of M over R:

$$T = \cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow \cdots$$

so Corollary 2.4 yields $\widehat{\operatorname{Ext}}^n_R(M, k) \cong \mathrm{H}^n(\operatorname{Hom}_R(T, k)) \cong k$ for each $n \in \mathbb{Z}$.

The next example should be compared to Theorems 6.4 and 6.7.

Example 6.10. Let k be a field and e a non-negative integer. The ring

$$R = \frac{k[t_1, \ldots, t_e]}{(\{t_i^2 - t_{i+1}^2\}_{1 \le i \le e-1} \cup \{t_i t_j\}_{1 \le i < j \le e})}$$

is artinian and Gorenstein for every $e \ge 2$, the ring

$$S = \frac{k[t_1, \ldots, t_e]}{(\{t_1^2\} \cup \{t_1 t_j\}_{3 \le j \le e} \cup \{t_i t_j\}_{2 \le i \le j \le e})}$$

is artinian, not Gorenstein for each $e \ge 3$, and the following equalities

$$\sum_{n=0}^{\infty} \operatorname{rank}_k \operatorname{Ext}^n_R(k, k) t^n = \frac{1}{1 - et + t^2} = \sum_{n=0}^{\infty} \operatorname{rank}_k \operatorname{Ext}^n_S(k, k) t^n,$$

hold for all $e \ge 2$, see [24, Theorem 2] and [18, Corollary, p. 38], respectively.

7. Structure of stable cohomology algebras

This is the first of four sections devoted to the structure of the stable cohomology algebra of a local ring $(R, \mathfrak{m}, k)$. We fix the following notation.

7.1. As in 2.1, let ι be the canonical homomorphism of graded k-algebras

$$\iota : \mathcal{E} \to \mathcal{S} \quad \text{where } \mathcal{E} = \operatorname{Ext}_R(k, k) \text{ and } \mathcal{S} = \widehat{\operatorname{Ext}}_R(k, k).$$

To describe the position of $\iota(\mathcal{E})$ in $\mathcal{S}$ we use the *left torsion submodule* of $\mathcal{S}$:

$$\mathcal{T} = \{\sigma \in \mathcal{S} \mid \mathcal{E}^{\geqslant i} \cdot \sigma = 0 \text{ for some } i \ge 0\}.$$

Note that one has $\mathcal{T} = \Gamma \mathcal{S}$, see A.4, and that $\mathcal{T}$ is an $\mathcal{E}$-subbimodule of $\mathcal{S}$.

Set $\mathcal{I} = \operatorname{Hom}_k(\mathcal{E}, k)$. This is a graded $\mathcal{E}$-bimodule with the canonical actions:

$$\begin{aligned} (\varepsilon \cdot e)(\varepsilon') &= (-1)^{|\varepsilon|(|e| + |\varepsilon'|)} e(\varepsilon' \cdot \varepsilon) \\ (e \cdot \varepsilon)(\varepsilon') &= e(\varepsilon \cdot \varepsilon') \end{aligned} \qquad \text{for all } \varepsilon, \varepsilon' \in \mathcal{E} \text{ and } e \in \mathcal{I}.$$

The left action of $\mathcal{E}$ on $\mathcal{I}$ is of prime importance in later developments.

Regular rings are excluded from the theorem because for them $\mathcal{S}=0$, see Theorem 6.5.

Theorem 7.2. *Let R be a singular local ring and set* depth $R=d$.

(1) *There is an exact sequence of left $\mathcal{E}$-modules*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\iota} \mathcal{S} \xrightarrow{\eth} \coprod_{i=d-1}^{\infty} \left(\Sigma^{-i}\mathcal{I}\right)^{\mu^{i+1}} \longrightarrow 0$$

where $\mu^i = \operatorname{rank}_k \operatorname{Ext}^i_R(k,R)$, and there are equalities

$$\mathcal{S} = \iota(\mathcal{E}) + \mathcal{E}\cdot\mathcal{S}^{<0} \quad \text{and} \quad \iota(\mathcal{E}) \cap \mathcal{T} = 0.$$

(2) *If $\mathcal{S}=\iota(\mathcal{E})\oplus\mathcal{T}'$ for some graded left $\mathcal{E}$-submodule $\mathcal{T}'\subseteq\mathcal{S}$, then $\mathcal{T}'=\mathcal{T}$ and*

$$\mathcal{T} \cong \coprod_{i=d-1}^{\infty} \left(\Sigma^{-i}\mathcal{I}\right)^{\mu^{i+1}} \quad \text{as graded left } \mathcal{E}\text{-modules.}$$

(3) *If* depth $\mathcal{E}\geq 2$, *then $\mathcal{S}=\iota(\mathcal{E})\oplus\mathcal{T}$ as graded $\mathcal{E}$-bimodules.*

(4) *If $\widehat{R}=Q/(f)$ for some singular local ring $(Q,\mathfrak{n},k)$ and a non-zero-divisor $f\in\mathfrak{n}^2$, then* depth $\mathcal{E}\geq 2$ *and $\mathcal{T}$ is a two-sided ideal of $\mathcal{S}$, such that*

$$\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T} \quad \text{and} \quad \mathcal{T}\cdot\mathcal{T} = 0.$$

For the proof we need a lemma. It is well known and easy to show that $\operatorname{Ext}_R(k,k)$ and $\operatorname{Tor}^R(k,k)$ are dual graded vector spaces. A more precise statement is contained in the next result, which represents a version of Tate duality. It can be derived, with a little work, from [23, (2.1)]. We provide a direct argument.

Lemma 7.3. *The graded left $\mathcal{E}$-modules* $\operatorname{Tor}^R(k,k)$ *and $\mathcal{I}$ described in* 1.2.3 *and* 7.1, *respectively, have the following properties.*

(1) *There is a natural isomorphism $\delta:\operatorname{Tor}^R(k,k)\cong\mathcal{I}$.*

(2) *If ϑ is a right non-zero-divisor on $\mathcal{E}$, then $\mathcal{I}=\vartheta^i\cdot\mathcal{I}$ for each $i\geq 0$.*

Proof. (1) Let $\varkappa: G\to k$ be a minimal free resolution.

Setting $C=k$ in 1.2.2 one obtains the first map below:

$$\operatorname{Hom}_R(G,G)\otimes_R(k\otimes_R G) \longrightarrow k\otimes_R G \xrightarrow{k\otimes_R\varkappa} k\otimes_R k = k.$$

In homology the composition induces a morphism of graded k-vector spaces

$$\mathcal{E}\otimes_k \operatorname{Tor}^R(k,k) \longrightarrow k$$

and hence a morphism of graded k-vector spaces

$$\delta : \operatorname{Tor}^R(k,k) \longrightarrow \mathcal{I} \quad \text{given by } \delta(1\otimes g)\big(\operatorname{cl}(\alpha)\big) = (-1)^{|g||\alpha|}\varkappa\alpha(g).$$

A routine computation shows that δ is $\mathcal{E}$-linear.

Any non-zero cycle in $k\otimes_R G_n$ has the form $1\otimes g$ for some $g \in G_n \setminus \mathfrak{m}G_n$. Choose an R-linear map $\beta : G_n \to k$ with $\beta(g) = 1$. The Comparison Theorem for resolutions yields a chain map $\alpha : G \to G$ of degree $-n$ with $\varkappa\alpha_n = \beta$; in particular,

$$\delta(1\otimes g)\big(\operatorname{cl}(\alpha)\big) = \pm\beta(g) = \pm 1 \neq 0.$$

Thus, the k-linear map $\delta_n : k\otimes_R G_n \to \mathcal{I}^n$ is injective. Since the k-vector spaces $\operatorname{Tor}^R_n(k,k)$ and $\operatorname{Ext}^n_R(k,k)$ have the same finite rank, δ_n is even bijective.

(2) When ϑ is a right non-zero-divisor on $\mathcal{E}$ the map $\varepsilon \mapsto \varepsilon\cdot\vartheta^i$ is a k-linear injection $\mathcal{E}\to\mathcal{E}$. Its dual is a surjection $\mathcal{I}\to\mathcal{I}$ given by $\upsilon \mapsto \pm\vartheta^i\cdot\upsilon$, so $\mathcal{I} = \vartheta^i\cdot\mathcal{I}$. □

Proof of Theorem 7.2. (1) Theorem 5.1.8 and 2.1 yield an exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{\iota} \mathcal{S} \xrightarrow{\eth} \Sigma\mathcal{B} \longrightarrow 0 \tag{7.3.1}$$

of graded $\mathcal{E}$-bimodules with $\mathcal{B} = \overline{\operatorname{Ext}}_R(k,k)$; the proof of the theorem also shows

$$\mathcal{B} = \Gamma\mathcal{B} \quad \text{and} \quad \Gamma\mathcal{E} = 0. \tag{7.3.2}$$

Theorem 6.1 and Lemma 7.3 yield isomorphisms of left $\mathcal{E}$-modules

$$\begin{aligned}\Sigma^{-1}\operatorname{Coker}(\iota) &\cong \mathcal{B} \cong \operatorname{Ext}_R(k,R)\otimes_k \operatorname{Tor}^R(k,k) \cong \operatorname{Ext}_R(k,R)\otimes_k \mathcal{I}\\ &\cong \coprod_{i=d}^{\infty}\big(\Sigma^{-i}\mathcal{I}\big)^{\mu^i}.\end{aligned} \tag{7.3.3}$$

From the definition of Γ and (7.3.2) we obtain

$$\iota(\mathcal{E})\cap\mathcal{T} = \Gamma\big(\iota(\mathcal{E})\big) \cong \Gamma\mathcal{E} = 0. \tag{7.3.4}$$

Next we set $\mathcal{N} = \mathcal{E}\cdot\mathcal{S}^{<0}$ and prove $\mathcal{S} = \iota(\mathcal{E}) + \mathcal{N}$. Note that for each $n<0$ the map $\eth^n$ yields $\mathcal{N}^n = \mathcal{S}^n \cong \mathcal{B}^{n-1}$. Let $\vartheta \in \mathcal{E}^2$ be a right non-zero-divisor, see Lemma 5.1.7. As $\Sigma\mathcal{B}$ is a direct sum of shifts of $\mathcal{I}$, Lemma 7.3(2) implies

$$\Sigma\mathcal{B} = \vartheta^i\,\Sigma\mathcal{B} \quad \text{for each } i \geq 0. \tag{7.3.5}$$

Let σ be an arbitrary element of $\mathcal{S}$. By the remarks above, one has $\eth(\sigma) = \vartheta^i\upsilon$ for some $\upsilon \in (\Sigma\mathcal{B})^{<0}$ and $i \geq 0$, and $\eth(\nu) = \upsilon$ for some $\nu \in \mathcal{N}$. Thus,

$$\eth\big(\sigma - \vartheta^i\nu\big) = \eth(\sigma) - \vartheta^i\upsilon = 0,$$

hence $\sigma - \vartheta^i\nu$ is in $\operatorname{Ker}(\eth) = \iota(\mathcal{E})$, so one gets $\sigma \in \vartheta^i\nu + \iota(\mathcal{E}) \subseteq \mathcal{N} + \iota(\mathcal{E})$.

(2) By hypothesis, $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}'$ for some left $\mathcal{E}$-submodule $\mathcal{T}' \subseteq \mathcal{S}$, so (7.3.1) yields $\mathcal{T}' \cong \Sigma\mathcal{B}$. From (7.3.2) we get $\mathcal{T}' = \Gamma\mathcal{T}' \subseteq \Gamma\mathcal{S} = \mathcal{T}$, hence $\iota(\mathcal{E}) \cap \mathcal{T}' = 0$ by (7.3.4). These relations imply $\mathcal{T}' = \mathcal{T}$ and yield a direct sum of $\mathcal{E}$-bimodules

$$\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}. \tag{7.3.6}$$

The expression for $\mathcal{T}$ comes from the equality above and the isomorphisms in (7.3.3).

(3) If $\operatorname{depth}\mathcal{E} \geq 2$ holds, then one has $\operatorname{Ext}^1_{\mathcal{E}}(\Sigma\mathcal{B}, \mathcal{E}) = 0$, see Proposition A.6, so the sequence (7.3.1) of graded left $\mathcal{E}$-modules splits, hence $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}$, see (2).

(4) By Lemma 5.3.1 we may assume $R = Q/(f)$, so we obtain

$$\operatorname{depth}\mathcal{E} = \operatorname{depth}\operatorname{Ext}_Q(k,k) + 1 \geq 2$$

by referring to Proposition 5.2.3 and Lemma 5.1.7. Part (3) now yields a direct sum decomposition (7.3.6) of $\mathcal{E}$-bimodules. Next we prove $\mathcal{T} \cdot \mathcal{T} = 0$.

Let $\vartheta \in \mathcal{E}^2$ be the central non-zero-divisor defined by the equality $R = Q/(f)$, see Proposition 5.2.3. Let τ be an element of $\mathcal{T}$. Choosing $i \geq 0$ so that $\vartheta^i \cdot \tau = 0$ holds, we get $\tau \cdot \vartheta^i = \vartheta^i \cdot \tau = 0$. On the other hand, (7.3.1) implies $\mathcal{T} \cong \Sigma\mathcal{B}$ as left $\mathcal{E}$-modules, so from (7.3.5) we get $\mathcal{T} = \vartheta^i \cdot \mathcal{T}$. Thus, we obtain

$$\tau \cdot \mathcal{T} = \tau \cdot \big(\vartheta^i \cdot \mathcal{T}\big) = \big(\tau \cdot \vartheta^i\big) \cdot \mathcal{T} = 0 \cdot \mathcal{T} = 0.$$

As $\tau \in \mathcal{T}$ was chosen arbitrarily, we conclude $\mathcal{T} \cdot \mathcal{T} = 0$, as desired.

To finish the proof we apply the remark below. □

Remark 7.3.7. If $\mathcal{S} = \iota(\mathcal{E}) + \mathcal{T}$ and $\mathcal{T} \cdot \mathcal{T} = 0$, then $\mathcal{T}$ is a two-sided ideal of $\mathcal{S}$.

Indeed, $\mathcal{T}$ is stable under multiplication on either side with elements of $\mathcal{E}$, because it is an $\mathcal{E}$-bimodule, and by its own elements, as $\mathcal{T} \cdot \mathcal{T} = 0$; thus, $\mathcal{S} \cdot \mathcal{T} \subseteq \mathcal{T} \supseteq \mathcal{T} \cdot \mathcal{S}$.

The following example provides applications for Theorem 7.2(3).

Example 7.4. Let k be a field, let S and T be localizations of finitely generated k-algebras at k-rational maximal ideals. In this case, $S \otimes_k T$ is a local ring.

Set $\mathcal{F} = \operatorname{Ext}_S(k,k)$ and $\mathcal{G} = \operatorname{Ext}_T(k,k)$. Standard Künneth arguments give an isomorphism $\operatorname{Ext}_{S\otimes_k T}(k,k) \cong \mathcal{F} \otimes_k \mathcal{G}$ of graded k-algebras, so [16, (3.1.iii)] yields

$$\operatorname{depth}\operatorname{Ext}_{S\otimes_k T}(k,k) = \operatorname{depth}\mathcal{F} + \operatorname{depth}\mathcal{G}.$$

Thus, for singular S and T one has $\operatorname{depth}\operatorname{Ext}_{S\otimes_k T}(k,k) \geq 2$, see Lemma 5.1.7.

Applications of Theorem 7.2(4) are discussed in the next section.

8. Stable cohomology algebras of complete intersection rings

In this section we provide an explicit and nearly complete computation of the structure of the stable cohomology algebra of a complete intersection local ring. This is achieved by combining results from this paper with already available information on the absolute cohomology algebra. We fix notation for the entire section.

8.1. Let $(R, \mathfrak{m}, k)$ be a complete intersection local ring. Fix a presentation $\widehat{R} = Q/(\boldsymbol{f})$ with $(Q, \mathfrak{n}, k)$ a regular local ring and $\boldsymbol{f} = \{f_1, \ldots, f_c\}$ is a Q-regular sequence in $\mathfrak{n}^2$. One then has $\operatorname{edim} R = e$, $\operatorname{codim} R = c$, and $\dim R = e - c$.

Fix a minimal generating set $\{t_1, \ldots, t_e\}$ for $\mathfrak{n}$ and write

$$f_h \equiv \sum_{1 \leqslant i \leqslant j \leqslant e} a_{hij} t_i t_j \pmod{\mathfrak{n}^3} \quad \text{with } a_{hij} \in Q \text{ for } 1 \le h \le c.$$

As in 7.1, we set $\mathcal{E} = \operatorname{Ext}_R(k, k)$ and $\mathcal{S} = \widehat{\operatorname{Ext}}_R(k, k)$.

Sjödin [29] has completely described the algebra $\mathcal{E}$; we recall his result below.

8.2. The algebra $\mathcal{E} = \operatorname{Ext}_R(k, k)$ is generated by elements $\xi_1, \ldots, \xi_e$ of degree 1 and $\vartheta_1, \ldots, \vartheta_c$ of degree 2, subject only to the relations

$$\begin{aligned} \xi_i^2 &= \sum_{h=1}^{c} \bar{a}_{hii} \vartheta_h \quad \text{for } 1 \le i \le e; \\ \xi_i \xi_j + \xi_j \xi_i &= \sum_{h=1}^{c} \bar{a}_{hij} \vartheta_h \quad \text{for } 1 \le i < j \le e; \\ \vartheta_h \xi_i &= \xi_i \vartheta_h \quad \text{for } 1 \le h \le c \text{ and } 1 \le i \le e; \\ \vartheta_g \vartheta_h &= \vartheta_h \vartheta_g \quad \text{for } 1 \le g \le h \le c, \end{aligned}$$

where $\bar{a}_{hij}$ denotes the image of a_{hij} in k; see [29, Theorem 5] or [4, (10.2.2)].

The next lemma determines the applicability of Theorem 7.2(4).

Lemma 8.3. *For R as in* 8.1 *one has* $\operatorname{depth} \mathcal{E} = \operatorname{codim} R$.

Proof. By Lemma 5.3.1 we may assume $R = Q/(\boldsymbol{f})$ for a regular local $(Q, \mathfrak{n}, k)$ ring and a Q-regular set $\boldsymbol{f} \subseteq \mathfrak{n}^2$, with $\operatorname{card}(\boldsymbol{f}) = \operatorname{codim} R$. Proposition 5.2.3 and Lemma 5.1.7 now yield $\operatorname{depth} \mathcal{E} = \operatorname{depth} \operatorname{Ext}_Q(k, k) + \operatorname{codim} R = \operatorname{codim} R$. $\square$

For complete intersection rings of codimension one, also known as *hypersurface rings*, $\mathcal{S}$ was computed by Buchweitz [12, (10.2.3)]. We recover his result:

Proposition 8.4. *Let R be a singular hypersurface ring of dimension d.*

The k-algebra $\mathcal{S}$ has generators $\xi_1, \ldots, \xi_e$ of degree 1*, a generator $\vartheta = \vartheta_1$ of degree* 2 *and a generator ϑ' of degree -2, subject to the relations in* 8.2 *and to*

$$\vartheta \vartheta' = 1 = \vartheta' \vartheta.$$

In particular, for each $n \in \mathbb{Z}$ one has $\operatorname{rank}_k \mathcal{S}^n = 2^d$ *and* $\operatorname{Ext}^n_{\mathcal{E}}(k, \mathcal{S}) = 0$.

Proof. By Lemma 5.3.1 we may assume $R \cong Q/(f)$ with $(Q, \mathfrak{n}, k)$ regular local and $f \in \mathfrak{n}^2 \setminus \{0\}$. By Corollary 4.2 the algebra $\mathcal{S}$ is the localization of $\mathcal{E}$ at $\{\vartheta^s \mid s \ge 0\}$, so the presentation of $\mathcal{E}$ in 8.2 yields the presentation above.

The k-rank of $\mathcal{S}^n$ can be obtained from this presentation, or directly from Theorem 6.8. As ϑ is a central element in $\mathcal{E}$ and is invertible in $\mathcal{S}$, A.3(1) yields

$$\operatorname{depth}_{\mathcal{E}} \mathcal{S} = \operatorname{depth}_{\mathcal{E}} (\mathcal{S}/\mathcal{S}\vartheta) + 1 = \operatorname{depth}_{\mathcal{E}} 0 + 1 = \infty. \qquad \square$$

In higher codimension the structure of stable cohomology is completely different.

Proposition 8.5. *Let R be a complete intersection ring with* $\operatorname{codim} R = c \geq 2$.

The graded k-algebra $\mathcal{S}$ has the form $\mathcal{S} \cong \mathcal{E} \oplus \mathcal{T}$, where $\mathcal{E}$ is the graded algebra from 8.2, *$\mathcal{T} = \bigcup_{i\geqslant 1}\{\tau \in \mathcal{S} \mid \mathcal{S}^{\geqslant i}\tau = 0\}$ is an ideal with $\mathcal{T} \cdot \mathcal{T} = 0$, $\mathcal{T} \cong \operatorname{Hom}_k(\mathcal{E}, k)$ as graded left $\mathcal{E}$-module, and $\tau \cdot \vartheta_h = \vartheta_h \cdot \tau$ for all $\tau \in \mathcal{T}$.*

Proof. Theorem 7.2(4) gives everything but the last assertion, which amounts to saying that each element $\iota(\vartheta_h)$ is central in $\mathcal{S}$. To see this note that $\vartheta_h \in \mathcal{E}^2$ can be defined by the canonical surjection $Q/(\boldsymbol{f} \setminus \{f_h\}) \to R$ and apply Theorem 4.1. $\square$

Still missing from Proposition 8.5 for a full description of the structure of $\mathcal{S}$ are data on the products $\tau \cdot \xi_j$. Specifically, we ask:

Question 8.6. Is $\mathcal{T}$ isomorphic to $\Sigma^{1-d} \operatorname{Hom}_k(\mathcal{E}, k)$ as graded $\mathcal{E}$-bimodules?

Here is a case where a positive answer is available from other sources.

Example 8.7. Let k be a field of characteristic $p > 0$, let $u_1, \ldots, u_e$ be positive integers, and set $q_i = p^{u_i}$. For $R = k[x_1, \ldots, x_e]/(x_1^{q_e}, \ldots, x_1^{q_1})$ the algebra $\mathcal{S}$ is graded-commutative, so one has $\mathcal{T} \cong \operatorname{Hom}_k(\mathcal{E}, k)$ as graded $\mathcal{E}$-bimodules.

Indeed, one has $R \cong kG$, where G is the abelian group $\prod_{i=1}^{e} \mathbb{Z}/(p^{u_i})$, and compatible isomorphisms of graded k-algebras $\mathcal{E} \cong \mathrm{H}^*(G, k)$ and $\mathcal{S} \cong \widehat{\mathrm{H}}^*(G, k)$, with the ordinary cohomology and the Tate cohomology of G; see [11, (V.4.6)], [11, (VI.6.2)] and [9, (6.11)]. The algebra $\widehat{\mathrm{H}}^*(G, k)$ is graded-commutative; see [13, (XII.5.3)].

9. Stable cohomology algebras of Gorenstein rings

In this section $(R, \mathfrak{m}, k)$ is a Gorenstein local ring of dimension d.

The leading theme here would come as no surprise: stable cohomology is simpler over Gorenstein rings. Numerically, this has already been made precise by Theorem 6.4. We back it up by showing that the structure of the stable cohomology algebra of a Gorenstein ring is much more rigid than might have been expected *a priori*. Although not as explicit as the in the special case of complete intersections, treated in Section 8, the results here are significantly sharper than those in Section 7.

The following notation and terminology is in force for the entire section.

9.1. Set $\mathcal{E} = \operatorname{Ext}_R(k, k)$ and $\mathcal{S} = \widehat{\operatorname{Ext}}_R(k, k)$, let $\iota : \mathcal{E} \to \mathcal{S}$ denote the canonical homomorphism of graded k-algebras, $\mathcal{T}$ the torsion subbimodule $\Gamma\mathcal{S} \subseteq \mathcal{S}$ and $\mathcal{I}$ the graded left $\mathcal{E}$-module $\operatorname{Hom}_k(\mathcal{E}, k)$ with the natural action, see 7.1

We say that the algebra $\mathcal{S}$ *splits* if $\mathcal{T}$ is a two-sided ideal of $\mathcal{S}$, such that

$$\mathcal{T} \cdot \mathcal{T} = 0, \qquad \mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}, \quad \text{and} \quad \mathcal{T} \cong \Sigma^{1-d}\mathcal{I} \quad \text{as graded left } \mathcal{E}\text{-modules}.$$

The structure of $\mathcal{S}$ for Gorenstein rings R with $\operatorname{codim} R \le 1$ is completely known, see Proposition 8.4, so this case is excluded from the next theorem.

Theorem 9.2. *Let R be a Gorenstein local ring with $\dim R = d$ and $\operatorname{codim} R \geq 2$.*

(1) *There is an exact sequence of left $\mathcal{E}$-modules*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\iota} \mathcal{S} \xrightarrow{\eth} \Sigma^{1-d}\mathcal{I} \longrightarrow 0$$

and there are equalities $\mathcal{S} = \iota(\mathcal{E}) + \mathcal{E} \cdot \mathcal{S}^{<0}$ and $\iota(\mathcal{E}) \cap \mathcal{T} = 0$.

(2) *If $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}'$ for some graded left $\mathcal{E}$-submodule $\mathcal{T}' \subseteq \mathcal{S}$, then $\mathcal{T}' = \mathcal{T}$ and the algebra $\mathcal{S}$ splits.*

(3) *If $\operatorname{depth} \mathcal{E} \geq 2$, then the algebra $\mathcal{S}$ splits.*

(4) *If $\zeta \cdot \mathcal{E} = \mathcal{E} \cdot \zeta$ for some left non-zero-divisor $\zeta \in \mathcal{E}^{\geqslant 1}$, then $\mathcal{S}$ splits and*

$$\mathcal{T} = \mathcal{E} \cdot \mathcal{S}^{<0}.$$

The proof is presented after a few remarks.

Remark 9.2.1. The theorem above should be compared to Theorem 7.2, applied to a Gorenstein ring R. Part (1) is a simple specialization. On the other hand, in Theorem 9.2 the conclusions of parts (2) through (4) are stronger, while the hypothesis of (4) is weaker, see Proposition 5.2.3.

Remark 9.2.2. Theorem 9.2 offers striking parallels to results of Benson and Carlson [8] relating the structure of the Tate cohomology algebra $\widehat{\mathrm{H}}^*(G, k)$ of a finite group G to that of the absolute cohomology algebra $\mathrm{H}^*(G, k)$.

Such similarities are unexpected, as the corresponding algebras have completely different properties. Indeed, $\mathrm{H}^*(G, k)$ and $\widehat{\mathrm{H}}^*(G, k)$ are graded commutative algebras, and the first one is also finitely generated over $\mathrm{H}_0(G, k) = k$. In stark contrast, the algebra $\operatorname{Ext}_R(k, k)$ may not be finitely generated; it is noetherian if and only if R is complete intersection; it is commutative if and only if $\widehat{R} \cong Q/\mathfrak{a}$, with $(Q, \mathfrak{n}, k)$ regular and $\mathfrak{a}$ generated by a Q-regular sequence contained in $\mathfrak{n}^3$.

Remark 9.2.3. Parts (3) and (4) of Theorem 9.2 do not cover all cases when the algebra $\mathcal{S}$ splits; see Example 9.13 and Question 9.3.

Proof of Theorem 9.2. (2) If $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}'$ as left $\mathcal{E}$-modules, then Theorem 7.2(2) yields an equality $\mathcal{T}' = \mathcal{T}$ and a decomposition $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}$ as $\mathcal{E}$-bimodules.

To prove $\mathcal{T} \cdot \mathcal{T} = 0$ we fix a right non-zero divisor $\vartheta \in \mathcal{E}^2$, see Lemma 5.1.7. Let τ be an element of $\mathcal{T}$. As $\mathcal{T}$ is an $\mathcal{E}$-subbimodule of $\mathcal{S}$, and for $j \geq d$ one has $\mathcal{T}^j \cong \mathcal{I}^{j+1-d} = 0$, we get $\tau \cdot \vartheta^i \in \mathcal{T}^{\lceil \tau \rceil + 2i} = 0$ for all $i \gg 0$. Fix an integer i with this property. Lemma 7.3(2) yields $\mathcal{I} = \vartheta^i \cdot \mathcal{I}$, so (1) implies $\mathcal{T} = \vartheta^i \cdot \mathcal{T}$, and hence

$$\tau \cdot \mathcal{T} = \tau \cdot \left(\vartheta^i \cdot \mathcal{T}\right) = \left(\tau \cdot \vartheta^i\right) \cdot \mathcal{T} = 0 \cdot \mathcal{T} = 0.$$

As $\tau \in \mathcal{T}$ was arbitrary, this implies $\mathcal{T} \cdot \mathcal{T} = 0$, so $\mathcal{T}$ is an ideal by Remark 7.3.7.

(3) This follows from (2), in view of Theorem 7.2(3).

(4) Set $\mathcal{N} = \mathcal{E} \cdot \mathcal{S}^{<0}$. As $\iota(\mathcal{E}) + \mathcal{N} = \mathcal{S}$ by Theorem 7.2(1), it suffices to prove $\iota(\mathcal{E}) \cap \mathcal{N} = 0$, see (2). Assuming the contrary means that one can write

$$0 \neq \sum_{h=1}^{u} \varepsilon_h \cdot \nu_h \in \iota(\mathcal{E})$$

with $\varepsilon_h \in \mathcal{E}$, $\nu_h \in \mathcal{S}^{<0}$, and $\varepsilon_h \cdot \nu_h \neq 0$. Set $s = \lceil \zeta \rceil$ and choose $i \geq 0$ so that

$$is + \lceil \nu_h \rceil > d \quad \text{holds for } h = 1, \dots, u.$$

As ζ is a non-zero-divisor on $\mathcal{E}$ and $\zeta \cdot \mathcal{E} = \mathcal{E} \cdot \zeta$, there exist $\varepsilon'_h \in \mathcal{E}$ such that

$$0 \neq \sum_{h=1}^{u} \zeta^i \cdot \varepsilon_h \cdot \nu_h = \sum_{h=1}^{u} \varepsilon'_h \cdot \zeta^i \cdot \nu_h.$$

Choose $l \in [1, u]$ with $\varepsilon'_l \cdot \zeta^i \cdot \nu_l \neq 0$, then set $\nu = \nu_l$ and $n = \lceil \nu \rceil$; one then has

$$0 \neq \zeta^i \cdot \nu \in \mathcal{S}^{is+n} \subseteq \mathcal{S}^{>d} = \iota(\mathcal{E})^{>d} \subseteq \iota(\mathcal{E})^{\geqslant 1}.$$

The map $\eth$ in (1) induces a bijection $\mathcal{S}^{<0} \to (\Sigma^{1-d}\mathcal{I})^{<0}$. Set $r = \operatorname{rank}_k \mathcal{N}^n$. As $\mathcal{I} = \zeta^{2r}\mathcal{I}$ by Lemma 7.3(2), we may choose $\nu' \in \mathcal{N}^{n-2rs}$ with $\zeta^{2r} \cdot \nu' = \nu$.

Let $\mathcal{D}$ be the universal enveloping algebra of the graded Lie subalgebra π_R^{even} of π_R. By 5.1.3, it has no zero-divisors (different from 0) and $\mathcal{E}$ is a free graded $\mathcal{D}$-module. Thus, $\mathcal{D}^{2j}$ and $\mathcal{D}^{2j} \cdot \zeta^i \cdot \nu$ are isomorphic k-spaces for each $j \in \mathbb{Z}$. From

$$\begin{aligned}
\mathcal{D}^{2rs} &\cong \mathcal{D}^{2rs} \cdot \zeta^i \cdot \nu_1 = \mathcal{D}^{2rs} \cdot \zeta^{i+2r} \cdot \nu' \\
&\subseteq \mathcal{E}^{2rs} \cdot \zeta^{i+2r} \cdot \nu' = \zeta^{i+2r} \cdot \mathcal{E}^{2rs} \cdot \nu' \\
&\subseteq \zeta^{i+2r} \cdot \mathcal{N}^n
\end{aligned}$$

we get $\operatorname{rank}_k \mathcal{D}^{2rs} \leq \operatorname{rank}_k \mathcal{N}^n = r$. Set $r_i = \operatorname{rank}_k \pi_R^i$. We have $r_2 \geq \operatorname{codim} R \geq 2$, see (5.1.5.3), so 5.1.2 yields coefficient-wise inequalities of formal power series

$$\begin{aligned}
\sum_{j=0}^{\infty} \left(\operatorname{rank}_k \mathcal{D}^{2j}\right) t^{2j} &= \frac{1}{(1-t^2)^2} \cdot \frac{1}{(1-t^2)^{r_2-2} \prod_{j \geqslant 2} (1-t^{2j})^{r_{2j}}} \\
&\succcurlyeq \frac{1}{(1-t^2)^2} = \sum_{j=0}^{\infty} (j+1) t^{2j}.
\end{aligned}$$

In particular, we get $\operatorname{rank}_k \mathcal{D}^{2rs} > r$, and hence the desired contradiction. □

We proceed to test the hypotheses of the preceding theorem. In the rest of this section the discussion revolves around the following question:

Question 9.3. Does the cohomology algebra $\mathcal{S} = \widehat{\mathrm{Ext}}_R(k,k)$ split for every Gorenstein local ring $(R, \mathfrak{m}, k)$ with $\operatorname{codim} R \geq 2$?

We start with a case when Theorem 9.2(4) applies.

Example 9.4. If $\widehat{R} \cong Q/\mathfrak{a}$ is a minimal Cohen presentation and the $\widehat{R}$-module $\mathfrak{a}/\mathfrak{a}^2$ has a non-zero free direct summand, then Iyengar [21, Main Theorem] provides a non-zero central element in π_R^2, hence a central non-zero-divisor in $\mathcal{E}$, cf. 5.1.3.

It might be noted that $\mathfrak{a}/\mathfrak{a}^2$ has a non-zero free direct summand whenever $\widehat{R}$ is isomorphic to $Q/(f)$ for some local ring $(Q, \mathfrak{n}, k)$ and non-zero-divisor $f \in \mathfrak{n}^2$. This case is covered already by Theorem 7.2(4), but it is not known whether all free direct summands of $\mathfrak{a}/\mathfrak{a}^2$ arise in this way.

The first application of Theorem 9.2(3) mirrors Example 7.4.

Example 9.5. Let k be a field, and let S and T be localizations of finitely generated k-algebras at k-rational maximal ideals. If S and T are singular Gorenstein rings, then so is the local ring $S \otimes_k T$, and Example 7.4 yields

$$\operatorname{depth} \mathrm{Ext}_{S\otimes_k T}(k,k) \geq 2.$$

To exhibit further classes of Gorenstein rings satisfying the hypothesis of Theorem 9.2(3) we use Koszul duality, see [31, §5] for a concise introduction.

9.6. Let k be a field, and let $A = \bigoplus_{i\geqslant 0} A_i$ be a commutative internally graded k-algebra, see (3.3), with $A_0 = k$, $A = k[A_1]$, and $\operatorname{rank}_k A_1 < \infty$.

Recall that A is said to be *Koszul* if $\mathrm{Ext}^n_A(k,k)^i = 0$ for $n \neq i$. Its *Koszul dual* is the internally graded k-algebra B with $B_i = \mathrm{Ext}^i_A(k,k)^i$ and multiplication induced by the composition product of $\mathrm{Ext}_A(k,k)$; the algebra B is Koszul as well, and its own Koszul dual is canonically isomorphic to A.

If A is a Gorenstein Koszul algebra with $\operatorname{rank}_k A < \infty$, then

$$\mathrm{Ext}^n_B(k,B) \cong \begin{cases} k & \text{for } n = \sup\{n \in \mathbb{N} \mid A_n \neq 0\}; \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, this follows by Koszul duality from [31, (5.10), (4.3.1)].

Proposition 9.7. *Let R be the localization of a commutative Gorenstein graded k-algebra $A = \bigoplus_{i\geqslant 0} A_i$ at the maximal ideal $A_+ = \bigoplus_{i\geqslant 1} A_i$. If A is Koszul, then*

$$\operatorname{depth} \mathcal{E} \geq 2$$

holds, unless $A \cong k[x_1, \ldots, x_e]/(f)$ for some quadratic form f (possibly equal to 0).

Proof. Assuming $\operatorname{depth} \mathcal{E} \leq 1$, we induce on $\dim A$ to prove that A has the desired special form above. When $\dim A = 0$ one has $\operatorname{rank}_k A < \infty$, so 9.6 yields $A_i = 0$ for $i \geq 2$. As A is Gorenstein, this is only possible if $A \cong k$ or $A \cong k[x]/(x^2)$.

Suppose $\dim A = d \geq 1$ and the assertion holds for algebras of dimension $d-1$. Let $k \to k'$ be a field extension. The k'-algebra $A' = k' \otimes_k A$ is clearly Koszul. Let R' be its localization at A'_+. The canonical map $R \to R'$ is flat and $R'/\mathfrak{m}R' \cong k'$, so Lemma 5.3.1 yields $\operatorname{depth} \operatorname{Ext}_{R'}(k', k') = \operatorname{depth} \mathcal{E}$. Thus, we may assume k is infinite, and so find a non-zero-divisor $g \in A_1$. Now $\overline{A} = A/(g)$ is Koszul with $\dim \overline{A} = d-1$. Since $g/1 \in R$ is R-regular and not in $\mathfrak{m}^2$, for $\overline{R} = R/(g/1)$ Proposition 5.2.2 yields $\operatorname{depth} \operatorname{Ext}_{\overline{R}}(k,k) = \operatorname{depth} \mathcal{E}$. As $\overline{R}$ is the localization of $\overline{A}$ at $\bigoplus_{i \geqslant 1} \overline{A}_i$, the induction hypothesis yields $\overline{A} \cong k[\bar{x}_1, \ldots, \bar{x}_{e-1}]/(\bar{f})$ for some quadratic form $\bar{f}$. It follows that A has the desired property. $\square$

Let $\operatorname{mult} R$ denote the multiplicity of R. A Gorenstein ring R has multiplicity 1 if and only if it is regular, and multiplicity 2 if and only if it is a quadratic hypersurface; else, it satisfies $\operatorname{mult} R \geq \operatorname{codim} R + 2$.

The case of minimal multiplicity is covered by the next result.

Proposition 9.8. *If R is Gorenstein and* $\operatorname{mult} R = \operatorname{codim} R + 2$, *then*

$$\operatorname{depth} \mathcal{E} = \begin{cases} 1 & \textit{when } R \textit{ is a hypersurface ring;} \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. One has $\operatorname{codim} R = 1$ if and only if R is a hypersurface, and when this is the case Lemma 8.3 applies. For the rest of the proof assume $\operatorname{codim} R \geq 2$. In view of Lemma 5.3.2 we may further assume $\dim R = 0$. Our hypothesis then implies isomorphisms $\mathfrak{m}^3 = 0$ and $\mathfrak{m}^2 = \operatorname{Ann}_R \mathfrak{m} \cong k$, and non-degeneracy of the map

$$\mu : (\mathfrak{m}/\mathfrak{m}^2) \otimes_k (\mathfrak{m}/\mathfrak{m}^2) \longrightarrow \mathfrak{m}^2 \cong k$$

induced by the product of R. Thus, $A = k \oplus (\mathfrak{m}/\mathfrak{m}^2) \oplus \mathfrak{m}^2$ is a Gorenstein graded k-algebra, with $\operatorname{rank}_k A_1 = \operatorname{codim} R$. By [24, (3.1)] both $\mathcal{E}$ and $\operatorname{Ext}_A(k,k)$ are isomorphic to the tensor algebra of $(A_1)^\vee$ over k, modulo the quadratic form $\operatorname{Hom}_k(\mu, k)(\operatorname{id}^k) \in (A_1)^\vee \otimes_k (A_1)^\vee$. Thus, we may replace R by A. The description of $\operatorname{Ext}_A(k,k)$ shows that as an algebra over $\operatorname{Ext}_A^0(k,k) = k$ it is generated by $\operatorname{Ext}_A^1(k,k)$. This graded vector space has $\operatorname{Ext}_A^1(k,k)^j = 0$ for $j \neq 1$, so $\operatorname{Ext}_A^i(k,k)^j = 0$ for $j \neq i$, so A is Koszul, hence $\operatorname{depth} \mathcal{E} = 2$ by 9.6. $\square$

Recall that if R is Gorenstein with $\operatorname{codim} R \leq 2$, then R is complete intersection, so Lemma 8.3 gives $\operatorname{depth} \mathcal{E} = \operatorname{codim} R$. In the next codimension we prove:

Proposition 9.9. *If R is Gorenstein and* $\operatorname{codim} R = 3$, *then*

$$\operatorname{depth} \mathcal{E} = \begin{cases} 3 & \textit{when } R \textit{ is complete intersection;} \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. By Proposition 8.3 we may assume R is not complete intersection. By Lemma 5.1.6 it suffices to prove $\operatorname{depth} \mathcal{D} = 2$, where $\mathcal{D}$ is the universal enveloping subalgebra of $\pi_R^{\geqslant 2}$. It follows from [1, (3.3)] that $\pi_R^{\geqslant 2}$ is isomorphic to π_A, where $A = \bigoplus_{i=0}^3 \mathrm{H}_i(K)$ and K is the Koszul complex on a minimal set of generators of $\mathfrak{m}$. There exist bases $\{e_1, \dots, e_r\}$ of A_1; $\{f_1, \dots, f_r\}$ of A_2; $\{g\}$ of A_3, such that

$$e_i f_i = g = f_i e_i \quad \text{for } 1 \le i \le r$$

and all other products of basis elements vanish, see [1, (8.4)]. Thus, A is a Gorenstein local ring of minimal multiplicity, so Proposition 9.8 yields $\operatorname{depth} \mathcal{D} = 2$. $\square$

In the balance of this section we deal with artinian rings.

Proposition 9.10. *For an artinian Gorenstein local ring* $(R, \mathfrak{m}, k)$ *with* $\mathfrak{m} \neq 0$ *the following conditions are equivalent.*

(i) $\mathcal{S}$ *is split.*
(ii) $\mathcal{T} = \mathcal{S}^{<0}$.
(iii) $\mathcal{S}^{<0}$ *is a left* $\mathcal{E}$*-submodule of* $\mathcal{S}$.
(iv) $\mathcal{E}$ *is generated as an algebra over* $\mathcal{E}^0 = k$ *by a set* $E \subseteq \mathcal{E}^{\geqslant 1}$, *such that* $\varepsilon \cdot \mathcal{S}^j = 0$ *holds for all pairs* $(\varepsilon, j) \in E \times \mathbb{Z}$ *satisfying* $-\lceil \varepsilon \rceil \le j < 0$.

Proof. From the exact sequence in Theorem 9.2(1), or from Theorem 2.2, one gets $\iota(\mathcal{E}) = \mathcal{S}^{\geqslant 0}$, hence $\iota(\mathcal{E})$ has a unique complement in $\mathcal{S}$ as a graded k-vector space, namely, $\mathcal{S}^{<0}$. Thus, when $\mathcal{S}$ is split one has $\mathcal{T} = \mathcal{S}^{<0}$, and hence (i) implies (ii).

It is clear that (ii) implies (iii). If $\mathcal{S}^{<0}$ is a left $\mathcal{E}$-submodule of $\mathcal{S}$, then it is necessarily a direct complement of $\iota(\mathcal{E})$, so (iii) implies (i) by Theorem 9.2(2).

It is clear that (iii) implies the validity of (iv) for *every* set of generators $E \subseteq \mathcal{E}^{\geqslant 1}$, in particular, for $E = \mathcal{E}^{\geqslant 1}$. Conversely, assume that (iv) holds for some E, pick an arbitrary $\sigma \in \mathcal{S}^{<0}$, and $\varepsilon_1, \dots, \varepsilon_s$ be elements of E. If one has $\lceil \sigma \rceil < -\sum_{i=1}^{s} \lceil \varepsilon_i \rceil$, then $\varepsilon_1 \cdots \varepsilon_s \cdot \sigma \in \mathcal{S}^{<0}$ holds for degree reasons. Else, there is an integer r with

$$1 \le r \le s \quad \text{and} \quad -\sum_{i=r}^{s} \lceil \varepsilon_i \rceil \le \lceil \sigma \rceil < -\sum_{i=r+1}^{s} \lceil \varepsilon_i \rceil.$$

Thus, $\sigma' = \varepsilon_{r+1} \cdots \varepsilon_s \cdot \sigma$ satisfies $-\lceil \varepsilon_r \rceil \le \lceil \sigma' \rceil < 0$, hence

$$\varepsilon_1 \cdots \varepsilon_s \cdot \sigma = (\varepsilon_1 \cdots \varepsilon_{r-1}) \cdot (\varepsilon_r \cdot \sigma') = 0.$$

It follows that $\mathcal{S}^{<0}$ is a left $\mathcal{E}$-submodule of $\mathcal{S}$, that is, (iii) holds. $\square$

When $\dim R = 0$ it is easy to write down a complete resolution of k, see 2.3.

9.11. Let $(R, \mathfrak{m}, k)$ be an artinian Gorenstein ring with $\mathfrak{m} \neq 0$.

If $\epsilon : F \to k$ is a minimal free resolution and $(-)^* = \operatorname{Hom}_R(-, R)$, then

$$T = \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon_0^* \circ \epsilon_0} F_0^* \xrightarrow{\partial_1^*} F_1^* \longrightarrow \cdots$$

where $T_0 = F_0$, is a complete resolution of k satisfying $\partial(T) \subseteq \mathfrak{m}T$.

Indeed, one has $\partial(T) \subseteq \mathfrak{m}T$ by construction. The exactness of T follows easily from the self-injectivity of R, which then implies the exactness of $\mathrm{Hom}_R(T, R)$.

We finish with an example obtained through the dictionary between local algebra and rational homotopy theory; see [5]. For a topological space Y let $\mathrm{H}^n(Y;\mathbb{Q})$ and $\mathrm{H}_n(Y;\mathbb{Q})$ denote its singular (co)homology, ΩY its loop space, and $\mathrm{H}_*(\Omega Y;\mathbb{Q})$ its Pontryagin algebra with multiplication induced by composition of loops.

The construction below was communicated to us by Yves Félix, in answer to our question whether there exists a formal CW complex Y with $\operatorname{depth} \mathrm{H}_*(\Omega Y;\mathbb{Q}) = 1$.

9.12. Let S^2 denote the standard 2-sphere, # a connected sum of smooth manifolds. The following manifolds are formal topological spaces:

$$X = S^2 \times S^2 \times S^2 = X' \quad \text{and} \quad Y = X \# X'.$$

Using a suitable CW decomposition of $X \# X'$, Félix exhibits $\mathrm{H}_*(\Omega Y;\mathbb{Q})$ as a free product of three commutative polynomial algebras:

$$\mathrm{H}_*(\Omega Y;\mathbb{Q}) \cong \mathbb{Q}[\xi_1, \xi_2, \xi_3] * \mathbb{Q}\big[\xi_1', \xi_2', \xi_3'\big] * \mathbb{Q}[\vartheta], \tag{9.12.1}$$

where $\mathrm{H}_*(\Omega X;\mathbb{Q}) = \mathbb{Q}[\xi_1, \xi_2, \xi_3]$, and $\mathrm{H}_*(\Omega X';\mathbb{Q}) = \mathbb{Q}[\xi_1', \xi_2', \xi_3']$ with $\lfloor \xi_i \rfloor = 1 = \lfloor \xi_j' \rfloor$, and $\vartheta \in \mathrm{H}_4(\Omega Y;\mathbb{Q})$ arises from the identification in $Y = X \# X'$ of the orientations of X and X'. None of these algebras equals $\mathbb{Q}$, so [17, (36.e.2)] yields

$$\operatorname{depth} \mathrm{H}_*(\Omega Y;\mathbb{Q}) = 1. \tag{9.12.2}$$

Next we translate Félix's example into commutative algebra.

Example 9.13. Let k be a field of characteristic 0. The ring

$$R = \frac{k[t_1, t_2, t_3, t_1', t_2', t_3']}{(\{t_i^2,\ t_i t_j',\ t_j'^2\}_{1 \le i,j \le 3}, t_1 t_2 t_3 - t_1' t_2' t_3')}$$

is Gorenstein with $\operatorname{codim} R = 6$, $\mathfrak{m}^3 \neq 0 = \mathfrak{m}^4$, and $\operatorname{rank}_k R = 14$.

Its cohomology algebra $\mathcal{E}$ satisfies $\operatorname{depth} \mathcal{E} = 1$ and $\mathcal{E}\zeta \neq \zeta\mathcal{E}$ for every non-zero $\zeta \in \mathcal{E}^{\geqslant 1}$, and its stable cohomology algebra $\mathcal{S}$ is split.

Indeed, the properties of R are clear. By Lemma 5.3.1 we may assume $k = \mathbb{Q}$.

Endow R with an internal grading by assigning (lower) degree -2 to the variables t_i and t_j' one gets an isomorphism $R \cong \bigoplus_{n=0}^{6} \mathrm{H}^n(Y;\mathbb{Q})$ of internally graded rings, where Y is the manifold from (9.12). Since Y is formal, there are isomorphisms

$$\bigoplus_{l+m=-n} \mathrm{Ext}_R^l(\mathbb{Q},\mathbb{Q})^m \cong \mathrm{H}_n(\Omega Y;\mathbb{Q})$$

of $\mathbb{Q}$-vector spaces that combine into an isomorphism of graded $\mathbb{Q}$-algebras

$$\mathrm{Ext}_R(\mathbb{Q},\mathbb{Q}) \cong \mathrm{H}_*(\Omega Y;\mathbb{Q}).$$

Thus, one obtains $\operatorname{depth} \mathcal{E} = 1$ from (9.12.2).

To analyze $\mathcal{S}$ we note that a minimal graded free resolution F of $\mathbb{Q}$ starts as

$$\cdots \longrightarrow R(4)^{\binom{3+3}{2}} \oplus R(4)^{3+9+3} \oplus R(6) \longrightarrow R(2)^{3+3} \longrightarrow R \longrightarrow 0$$

where the standard basis of $R(2)^{3+3}$ maps to the generators t_1, t_2, t_3 and t'_1, t'_2, t'_3 of $\mathfrak{m}$, that of $R(4)^{\binom{3+3}{2}}$ to the Koszul relations between these generators, that of $R(4)^{3+9+3}$ to the syzygies defined by the quadratic relations of R, and that of $R(6)$ to the syzygy defined by the cubic relation. The complete resolution T constructed in 9.11 from the minimal resolution F now yields isomorphisms

$$\begin{aligned} &\mathcal{E}^2 = \mathcal{S}^2 \cong \mathbb{Q}(4)^{30} \oplus \mathbb{Q}(6), \qquad && \mathcal{E}^0 = \mathcal{S}^0 \cong \mathbb{Q}, \qquad && \mathcal{S}^{-1} \cong \mathbb{Q}(-6), \\ &\mathcal{E}^1 = \mathcal{S}^1 \cong \mathbb{Q}(2)^6, && && \mathcal{S}^{-2} \cong \mathbb{Q}(-8)^6. \end{aligned}$$

The isomorphism (9.12.1) shows that the algebra $\mathcal{E}$ is generated over $\mathcal{E}^0 = \mathbb{Q}$ by elements from $\mathcal{E}^1$ and $\mathcal{E}^2$. As the action of $\mathcal{E}$ on $\mathcal{S}$ is compatible with internal gradings, see Proposition 3.4, degree considerations yield the following equalities:

$$\mathcal{E}^1 \cdot \mathcal{S}^{-1} = 0, \qquad \mathcal{E}^2 \cdot \mathcal{S}^{-1} = 0, \qquad \mathcal{E}^2 \cdot \mathcal{S}^{-2} = 0.$$

Proposition 9.10 now shows that the stable cohomology algebra $\mathcal{S}$ is split.

10. Stable cohomology algebras of Golod rings

In this section $(R, \mathfrak{m}, k)$ denotes a local ring. The *codepth* of R is the integer $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$. Once again, we consider the graded k-algebras

$$\mathcal{E} = \operatorname{Ext}_R(k, k) \quad \text{and} \quad \mathcal{S} = \widehat{\operatorname{Ext}}_R(k, k)$$

and let $\iota : \mathcal{E} \to \mathcal{S}$ denote the canonical homomorphism of graded algebras.

For all rings R with $\operatorname{codepth} R \geq 2$ analyzed so far in this paper, $\iota(\mathcal{E})$ has a direct complement in $\mathcal{S}$ as a left $\mathcal{E}$-submodule. Here our goal is to produce a ring R for which this fails. We search for it among Golod rings, as their homological properties are in many respects antithetical to those of Gorenstein rings.

Golod rings are usually defined in terms of the series $\sum_{n \geqslant 0} \operatorname{rank}_k \mathcal{E}^n t^n$, see [4, §5] for details and examples. Here it is useful to take as definition their characterization in terms of the structure of the graded k-algebra $\mathcal{E}$.

10.1. The ring R is said to be *Golod* if the universal enveloping algebra $\mathcal{D}$ of the Lie algebra $\pi_R^{\geqslant 2}$ is a free associative k-algebra, [2, Corollary, p. 59].

We will also use a defining homological property of free k-algebras.

10.2. Let $\mathcal{A}$ be a free associative k-algebra on a set $\varXi$ of elements of positive upper degree. Let $\{b_\xi \mid \lceil b_\xi \rceil = \lceil \xi \rceil + 1\}_{\xi \in \varXi}$ be a linearly independent set over $\mathcal{A}$. There is then an exact sequence of graded left $\mathcal{A}$-modules:

$$0 \longrightarrow \coprod_{\xi \in \varXi} \mathcal{A} b_\xi \xrightarrow{\partial} \mathcal{A} \longrightarrow k \longrightarrow 0 \quad \text{with } \partial(b_\xi) = \xi. \tag{10.2.1}$$

It is clear that R is Golod when $\operatorname{codepth} R = 0$ (because then R is regular, so $\mathcal{D} = k$), or when $\operatorname{codepth} R = 1$ (because then R is a singular hypersurface ring, so $\mathcal{D} = k[\vartheta]$, see Example 8.4). When R is Golod with $\operatorname{codepth} R \geq 2$ Theorem 9.2 does not apply, because R is not Gorenstein. Neither does Theorem 7.2(3):

Proposition 10.3. *If R is a Golod ring with* $\operatorname{codepth} R \geq 2$*, then* $\operatorname{depth} \mathcal{E} = 1$.

Proof. If $\mathcal{D}$ is the universal enveloping algebra of $\pi_R^{\geqslant 2}$, then $\operatorname{depth} \mathcal{D} \leq 1$, see 10.1 and 10.2, so $\operatorname{depth} \mathcal{E} \leq 1$ by Lemma 5.1.6; equality holds by Lemma 5.1.7. □

Thus, new tools are needed to study stable cohomology over Golod rings of higher codimension. Local rings with radical square zero are the simplest example of Golod rings. They are the subject of the next theorem, proved at the end of the section; its notation and hypotheses are in force for the rest of the section.

Theorem 10.4. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^2 = 0$ and* $\operatorname{edim} R = e \geq 2$.
The following assertions then hold.

(1) *The exact sequence of Theorem* 7.2(1) *has the form*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\iota} \mathcal{S} \xrightarrow{\eth} (\Sigma\mathcal{I})^e \oplus \coprod_{i=0}^{\infty} \left(\Sigma^{-i}\mathcal{I}\right)^{e^i(e^2-1)} \longrightarrow 0.$$

(2) *The left $\mathcal{E}$-submodule $\iota(\mathcal{E})$ has no direct complement in $\mathcal{S}$.*
(3) *One has* $\operatorname{Ext}^n_{\mathcal{E}}(k, \mathcal{S}) = 0$ *for all $n \in \mathbb{Z}$.*

For the next lemma we introduce shorthand notation: When $\beta, \beta' : M \to N$ are R-linear maps we write $\beta' \equiv \beta \pmod{\mathfrak{m}}$ in place of $(\beta' - \beta)(M) \subseteq \mathfrak{m}N$.

Lemma 10.5. *Let $\delta : U \to R$ be the composition of a projective cover $U \to \mathfrak{m}$ of the R-module $\mathfrak{m}$ with the inclusion $\mathfrak{m} \subseteq R$. For $i \in \mathbb{Z}$, let ∂_{i+1} be the R-linear map*

$$F_{i+1} = U^{\otimes(i+1)} = U \otimes_R U^{\otimes i} \xrightarrow{\delta \otimes_R U^{\otimes i}} R \otimes_R U^{\otimes i} = U^{\otimes i} = F_i$$

where $U^{\otimes i}$ denotes the ith tensor power of U over R, with the conventions $U^{\otimes 0} = R$ and $U^{\otimes i} = 0$ for $i < 0$. The following then hold.

(1) *The pair (F, ∂) is a minimal R-free resolution of k.*
(2) *There are equalities $\partial(F_{i+1}) = \mathfrak{m}F_i$ for all $i \in \mathbb{Z}$.*
(3) *When h and i are integers, such that $i \geq \max\{0, -h\}$, a diagram*

$$\begin{tikzcd}
U \otimes_R F_i \arrow[r, equal] & F_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "\beta'"'] & F_i \arrow[d, "\beta"] \\
U \otimes_R F_{i+h} \arrow[r, equal] & F_{h+i+1} \arrow[r, "\partial_{h+i+1}"] & F_{h+i}
\end{tikzcd}$$

of R-linear maps commutes if and only if $\beta' \equiv U \otimes_R \beta \pmod{\mathfrak{m}}$.

Remark. When R is a k-algebra, the minimal resolution F described in part (1) of the theorem coincides with the bar construction of R over k.

Proof. (2) This follows directly from the definition of ∂.

(1) From $\mathfrak{m}^2 = 0$ one gets $\partial^2 = 0$. For $i \geq 1$ set $B_{i-1} = \partial_i(F_i)$ and $Z_i = \operatorname{Ker}(\partial_i)$. From (2) one obtains $\ell(B_{i-1}) = e^i$, where ℓ denotes length over R. The exact sequence $0 \to Z_i \to F_i \to B_{i-1} \to 0$ yields $\ell(Z_i) = (e+1)e^i - e^i = e^{i+1} = \ell(B_i)$. Thus, $\mathrm{H}_i(F) = 0$ for $i \geq 1$ and $\mathrm{H}_0(F) = k$, so F is a minimal free resolution of k.

(3) Pick $u \otimes v \in U \otimes_R F_i = F_{i+1}$. By definition, one has

$$\beta\partial_{i+1}(u \otimes v) = \beta\big(\delta(u)v\big) = \delta(u)\beta(v) = \partial_{h+i+1}\big(u \otimes \beta(v)\big).$$

Therefore, an equality $\beta\partial_{i+1}(u \otimes v) = \partial_{h+i+1}\beta'(u \otimes v)$ holds if and only if one has

$$\beta'(u \otimes v) - u \otimes \beta(v) \in \operatorname{Ker}(\partial_{h+i+1}).$$

As (1) and (2) yield $\operatorname{Ker}(\partial_{h+i+1}) = \partial_{h+i+2}(F_{h+i+2}) = \mathfrak{m}F_{h+i+1}$, the inclusion above is equivalent to the relation $\beta' \equiv U \otimes_R \beta \pmod{\mathfrak{m}}$. $\square$

Lemma 10.6. *Let F denote the minimal free resolution from the preceding lemma, let $\varkappa \in \operatorname{Hom}_R(F,F)_h$ be a homomorphism of complexes, let $\hat{\varkappa}$ denote its image in $\widehat{\operatorname{Hom}}_R(F,F)_h$, and set $m = \max\{0, -h\}$. The following then hold.*

(1) *The map $\varkappa$ (respectively, $\hat{\varkappa}$) is a boundary if and only if for $i = m$ (respectively, for some $i \geq m$) and for all $j \geq 0$ one has*

$$\varkappa_{i+j} \equiv 0 \pmod{\mathfrak{m}}.$$

(2) *The map $\varkappa$ (respectively, $\hat{\varkappa}$) is a cycle if and only if for $i = m$ (respectively, for some $i \geq m$) and for all $j \geq 0$ one has*

$$\varkappa_{i+j} \equiv U^{\otimes j} \otimes_R (-1)^{hj}\varkappa_i \pmod{\mathfrak{m}}.$$

Proof. (1) By definition, $\varkappa$ (respectively, $\hat{\varkappa}$) is a boundary if and only if there exists a homomorphism $\chi \in \operatorname{Hom}_R(F,F)$ of degree $h+1$, such that an equality

$$\varkappa_{i+j} = \partial_{h+i+j+1}\chi_{i+j} + (-1)^h\chi_{i+j-1}\partial_{i+j} \tag{10.6.3}$$

holds for $i = m$ (respectively, for some $i \geq m$) and for all $j \geq 0$.

If χ exists, then $\varkappa_{i+j} \equiv 0 \pmod{\mathfrak{m}}$ for i, j as above, because $\partial \equiv 0 \pmod{\mathfrak{m}}$.

Conversely, assume $\varkappa_{i+j} \equiv 0 \pmod{\mathfrak{m}}$ holds for i, j as above. We construct χ_{i+j} by induction on j. Setting $\chi_{i+j} = 0$ for $j < 0$, we may assume that χ_{i+j-1} has been defined for some $j \geq 0$. One then has the relations

$$\begin{aligned}\big(\varkappa_{i+j} - (-1)^h\chi_{i+j-1}\partial_{i+j}\big)(F_{i+j}) &\subseteq \mathfrak{m}F_{h+i+j} + \chi_{i+j-1}(\mathfrak{m}F_{i+j-1}) \\ &\subseteq \mathfrak{m}F_{h+i+j} \\ &= \partial_{h+i+j+1}(F_{h+i+j+1})\end{aligned}$$

with equality given by Lemma 10.5(2). Since F_{i+j} is free, one can find a homomorphism $\chi_{i+j}: F_{i+j} \to F_{h+i+j+1}$ satisfying (10.6.3).

(2) By definition, $\varkappa$ (respectively, $\hat{\varkappa}$) is a cycle if and only if $\partial_{h+i+j}\varkappa_{i+j} = (-1)^h \varkappa_{i+j-1}\partial_{i+j}$ holds for $i = m$ (respectively, for some $i \geq m$) and for all $j \geq 0$. Iterated applications of 10.5(3) yield the desired assertion. $\square$

It is convenient to describe $\mathcal{E}$ and $\mathcal{S}$ as subrings of infinite matrix rings.

10.7. For every pair $(m,r) \in \mathbb{Z} \times \mathbb{N}$, satisfying $m + r \geq 0$, let $\mathsf{M}_{e^r \times e^{r+m}}(k)$ denote the k-vector space of $e^r \times e^{r+m}$ matrices with elements in k. Let $\mathsf{M}_\infty(k)$ be the k-algebra of all row-and-column-finite matrices with elements in k, under ordinary matrix product. For every matrix $C = (c_{ij}) \in \mathsf{M}_{e^r \times e^{r+m}}(k)$ let $C_\infty = (c^\infty_{pq}) \in \mathsf{M}_\infty(k)$ be the matrix with blocks C along a line of slope e^{-m}:

$$c^\infty_{pq} = \begin{cases} c_{ij} & \text{if } p = i + le^r \text{ and } q = j + le^{r+m} \text{ for some } l \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that for each $m \in \mathbb{Z}$ the following subset of $\mathsf{M}_\infty(k)$ is a k-subspace:

$$\mathcal{C}^m = \bigl\{ C_\infty \in \mathsf{M}_\infty(k) \bigm| C \in \mathsf{M}_{e^r \times e^{r+m}}(k) \text{ for some pair } (m,r) \text{ with } m + r \geq 0 \bigr\}.$$

A key observation is that, furthermore, the following relations hold:

$$\mathcal{C}^m \cdot \mathcal{C}^n \subseteq \mathcal{C}^{m+n} \quad \text{for all } m,n \in \mathbb{Z};$$
$$\mathcal{C}^m \cap \mathcal{C}^n = 0 \quad \text{when } m \neq n.$$

Indeed, let $C_\infty \in \mathcal{C}^m$ be as above, and let $D_\infty \in \mathcal{C}^n$ be obtained from a matrix $D \in \mathsf{M}_{e^s \times e^{s+n}}(k)$, where $(n,s) \in \mathbb{Z} \times \mathbb{N}$ satisfy $n + s \geq 0$. One then has $C_\infty = C'_\infty$, where $C' \in \mathsf{M}_{e^{r+s} \times e^{r+s+m}}(k)$ is the block diagonal matrix with e^s copies of C along a line of slope e^{-m}; also, $D_\infty = D'_\infty$ where $D' \in \mathsf{M}_{e^{r+s+m} \times e^{r+s+m+n}}(k)$ is the block diagonal matrix with e^{r+m} copies of D along a line of slope e^{-n}. Thus, we get

$$C'D' \in \mathsf{M}_{e^{r+s} \times e^{r+s+m+n}}(k) \quad \text{and} \quad C_\infty \cdot D_\infty = (C'D')_\infty \in \mathcal{C}^{m+n}.$$

This proves the inclusion. For the equality, assume $C_\infty = D_\infty \in \mathcal{C}^m \cap \mathcal{C}^n$ with $m \neq n$. Since the lines with slopes e^{-m} and e^{-n} diverge, for $l \gg 0$ the blocks C forming the matrix C_∞ are entirely contained in an area of the matrix D_∞ where every element is equal to 0. Thus, one has $C = 0$, and consequently $C_\infty = 0$.

In view of the discussion above, $\mathcal{C} = (\mathcal{C}^m)_{m \in \mathbb{Z}}$ is a graded k-algebra with unit 1_∞.

We define a graded subalgebra $\mathcal{A}$ of $\mathcal{C}$ as follows. Set

$$A^{(m)}_i = \bigl(E^{(m)}_i\bigr)_\infty \in \mathcal{C}^m \quad \text{for each pair } (m,i) \in \mathbb{N} \times [1,e], \text{ where}$$
$$E^{(m)}_i = [0,\dots,0,1,0,\dots,0] \in \mathsf{M}_{1 \times e^m}(k) \quad \text{with 1 in } i\text{th position.}$$

Let $\mathcal{A}$ be the subalgebra generated over $k1_\infty$ by $A_1^{(1)}, \dots, A_e^{(1)} \in \mathcal{C}^1$. It is easy to see:

$$A_i^{(m)} A_j^{(n)} = A_{(i-1)e^n+j}^{(m+n)}.$$

Thus, $\mathcal{A}$ has a k-basis consisting of matrices with precisely one non-zero entry in every row, and in distinct basis elements this entry occurs in a different column.

Part (2) of the next lemma is known: For a k-algebra R it is obtained by computing $\operatorname{Ext}_R(k,k)$ as the cohomology of the cobar construction, which in this case is the tensor algebra on $\operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ with zero differential; the general case can be found in [27, Theorem 1, Corollary 3]. A proof is included for completeness.

Lemma 10.8. *In the notation of* 10.7 *the following hold.*

(1) *There is an isomorphism of graded k-algebras $\mathcal{S} \cong \mathcal{C}$, inducing $\iota(\mathcal{E}) \cong \mathcal{A}$.*
(2) *The associative k-algebra $\mathcal{E}$ is freely generated over k by a k-basis of $\mathcal{E}^1$.*
(3) *One has $\operatorname{Ext}^n_{\mathcal{E}}(k, \mathcal{E}) = 0$ if $n \neq 1$ and*

$$\operatorname{rank}_k \operatorname{Ext}^1_{\mathcal{E}}(k, \mathcal{E})^i = \begin{cases} 0 & \textit{for } i \le -2; \\ e & \textit{for } i = -1; \\ e^i(e^2-1) & \textit{for } i \ge 0. \end{cases}$$

Proof. Let $F \to k$ be the minimal free resolution from Lemma 10.5. We fix a basis $X_1 = \{x_1, \dots, x_e\}$ of U over R. For each $i \ge 0$ it canonically provides a basis X_i of $F_i = U^{\otimes i}$ over R, and thus a basis $\overline{X}_i$ of $F_i/\mathfrak{m}F_i$ over k.

(1) Each $\sigma \in \mathcal{S}^h$ is the class of a cycle $\hat{\varkappa} \in \widehat{\operatorname{Hom}}_R(F, F)_{-h}$, where $\varkappa : F \to F$ is a homomorphism of complexes of R-modules of degree $-h$. Thus, $\varkappa_{\geqslant s} : F_{\geqslant s} \to F_{\geqslant s-h}$ is a chain map for some integer s. It induces a k-linear map

$$\mathrm{H}_s(\varkappa) = \bar{\varkappa}_s : U^{\otimes(s+h)}/\mathfrak{m}U^{\otimes(s+h)} \longrightarrow U^{\otimes s}/\mathfrak{m}U^{\otimes s}.$$

Let $S \in \mathsf{M}_{e^s \times e^{s+h}}(k)$ be the matrix of $\bar{\varkappa}_s$ in the bases $\overline{X}_{s+h}$ and $\overline{X}_s$, and form the matrix $S_\infty \in \mathcal{C}^h$. Lemma 10.6 shows that S_∞ does not depend on the choices of $\varkappa$ or s, so setting $\alpha(\sigma) = S_\infty$ one obtains a map $\alpha : \mathcal{S} \to \mathcal{C}$. The definitions of the products in $\mathcal{S}$ and $\mathcal{C}$ imply that α is a homomorphism of algebras. Part (1) of Lemma 10.6 shows that α is injective and part (2) that it is surjective.

Let $\{\xi_1, \dots, \xi_e\}$ be the basis of $\mathcal{E}^1$ dual to the basis $\overline{X}_1$ of $U/\mathfrak{m}U$. By definition, $\alpha(\iota(\xi_j)) = A_j^{(1)}$ for $j = 1, \dots, e$, so α maps the subalgebra of $\mathcal{E}$ generated by $\{\xi_1, \dots, \xi_e\}$ surjectively onto the subalgebra $\mathcal{A}$ of $\mathcal{C}$. Thus, for each $i \in \mathbb{Z}$ one has

$$e^i = \operatorname{rank}_k \mathcal{E}^i \ge \operatorname{rank}_k \mathcal{A}^i = e^i,$$

with equalities given by the constructions in Lemma 10.5 and in 10.7, respectively. Thus, α restricts to an isomorphism $\iota(\mathcal{E}) \cong \mathcal{A}$.

(2) As shown above, $\operatorname{rank}_k \mathcal{E}^i = e^i$ and $\mathcal{E}$ is generated by e elements of degree 1. It follows that there are no relations between the generators of $\mathcal{E}$.

(3) In view of (2), the exact sequence of 10.2 yields $\operatorname{Ext}^n_{\mathcal{E}}(k,\mathcal{E})=0$ for $n\neq 0,1$ and an exact sequence of graded vector spaces

$$0\longrightarrow \operatorname{Hom}_{\mathcal{E}}(k,\mathcal{E})\longrightarrow \mathcal{E}\xrightarrow{\partial^*}\Sigma\mathcal{E}^e\longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(k,\mathcal{E})\longrightarrow 0$$

where $\partial^*(\rho)=(\xi_1\rho,\dots,\xi_e\rho)$. We see that $\operatorname{Hom}_{\mathcal{E}}(k,\mathcal{E})=0$. Counting k-ranks in the sequence above we now get the desired expressions for $\operatorname{Ext}^1_{\mathcal{E}}(k,\mathcal{E})^i$. □

Proof of Theorem 10.4. The notation from the preceding proof stays in force.

(1) Theorem 7.2(1) yields an exact sequence of graded left $\mathcal{E}$-modules

$$0\longrightarrow \mathcal{E}\xrightarrow{\iota}\mathcal{S}\xrightarrow{\eth}\coprod_{i=-1}^{\infty}\left(\Sigma^{-i}\mathcal{I}\right)^{\mu^{i+1}}\longrightarrow 0 \tag{10.8.1}$$

where $\mathcal{I}=\operatorname{Hom}_k(\mathcal{E},k)$ and $\mu^i=\operatorname{rank}_k\operatorname{Ext}^i_R(k,R)$. These numbers are given by:

$$\mu^i=\begin{cases}0 & \text{for } i\le -1;\\ e & \text{for } i=0;\\ e^{i-1}(e^2-1) & \text{for } i\ge 1.\end{cases} \tag{10.8.2}$$

Indeed, the hypothesis $\mathfrak{m}^2=0$ yields an exact sequence of R-modules

$$0\longrightarrow k^e\longrightarrow R\longrightarrow k\longrightarrow 0.$$

It induces an exact sequence of homomorphisms of k-vector spaces

$$0\longrightarrow \operatorname{Hom}_R(k,R)\longrightarrow R\longrightarrow \operatorname{Hom}_R\left(k^e,R\right)\longrightarrow \operatorname{Ext}^1_R(k,R)\longrightarrow 0$$

giving the values of μ^i in (10.8.2) for $i\le 1$. The same sequence yields isomorphisms

$$\operatorname{Ext}^{i-1}_R\left(k^e,R\right)\cong \operatorname{Ext}^i_R(k,R)\quad \text{for all } i\ge 2,$$

which imply $\mu^i=e\mu^{i-1}$ for all $i\ge 2$; the last equality in (10.8.2) follows.

(3) The k-algebra $\mathcal{E}$ is free by Lemma 10.8(2), so the resolution of k displayed in 10.2 gives $\operatorname{Ext}^n_{\mathcal{E}}(k,\mathcal{S})=0$ for all $n\neq 0,1$.

Next we prove $\operatorname{Hom}_{\mathcal{E}}(k,\mathcal{S})=0$. By A.5, this is equivalent to the following assertion: If $\varkappa\in\operatorname{Hom}_R(F,F)_h$ is such that $\hat{\varkappa}\in\widehat{\operatorname{Hom}}_R(F,F)_h$ is a cycle and $\operatorname{cl}(\hat{\varkappa})\in\widehat{\operatorname{Ext}}{}^{-h}_R(k,k)$ satisfies $\operatorname{Ext}^{\geqslant s}_R(k,k)\cdot\operatorname{cl}(\hat{\varkappa})=0$ for some $s\ge 0$, then $\operatorname{cl}(\hat{\varkappa})=0$.

Set $m=\max\{0,-h\}$. As $\hat{\varkappa}$ is a cycle, for some integer i with $i\ge m$ one has

$$\partial_{h+i+j}\varkappa_{i+j}=(-1)^h\varkappa_{i+j-1}\partial_{i+j}\quad \text{for all } j\ge 0.$$

After increasing i or s (if necessary) we may assume $s=i+h\ge 0$.

Let $\xi\in\operatorname{Hom}_R(F,F)^s$ be a chain map. As $\operatorname{cl}(\xi)\cdot\operatorname{cl}(\hat{\varkappa})=0$, Lemma 10.6(1) yields

$$(\xi\varkappa)_{i+j}(F_{i+j})\subseteq \mathfrak{m}F_j\quad \text{for all } j\gg 0.$$

On the other hand, one has $\eth_j \circ (\xi\varkappa)_{i+j} = (-1)^{h+s}(\xi\varkappa)_{i+j-1} \circ \eth_{i+j}$ for all $j \geq 0$, so Lemma 10.5(3) implies that these inclusions hold, in fact, for all $j \geq 0$.

Recall that X_i denotes the standard basis of $F_i = U^{\otimes i}$. For each $u \in X_i$ one has $\varkappa_i(u) = \sum_{v \in X_{i+h}} a_{uv} v$ with uniquely defined $a_{uv} \in R$. On the other hand, for each $v \in X_{h+i}$ the R-linear map $F_{h+i} \to F_0 = R$ sending v to 1 and every $v' \in X_{h+i} \setminus \{v\}$ to 0 extends to a chain map $\xi_v : F \to F$ of degree $-h-i$. We get

$$a_{uv} = (\xi_v)_{h+i}\left(\sum_{v \in X_{h+i}} a_{uv} v\right) = (\xi_v)_{h+i}\big(\varkappa_i(u)\big) = (\xi_v \varkappa)_i(u) \in \mathfrak{m}$$

hence $\varkappa_i(F_i) \subseteq \mathfrak{m}F_{h+i}$ holds. Lemma 10.5(3) now yields $\varkappa_i(F_i) \subseteq \mathfrak{m}F_{h+i}$ for all $i \geq 0$, from where we conclude $\operatorname{cl}(\hat{\varkappa}) = 0$, see Lemma 10.6(1).

Finally, we prove $\operatorname{Ext}^1_{\mathcal{E}}(k, \mathcal{S}) = 0$. As $\operatorname{Hom}_{\mathcal{E}}(k, \mathcal{S}) = 0$, the exact sequence (10.8.1) of graded left $\mathcal{E}$-modules induces an exact sequence of graded vector spaces

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_{\mathcal{E}}\left(k, \coprod_{i=-1}^{\infty} \left(\Sigma^{-i}\mathcal{I}\right)^{\mu^{i+1}}\right) &\longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(k, \mathcal{E}) \xrightarrow{\iota_*} \operatorname{Ext}^1_{\mathcal{E}}(k, \mathcal{S}) \\ &\xrightarrow{\eth_*} \operatorname{Ext}^1_{\mathcal{E}}\left(k, \coprod_{i=-1}^{\infty} \left(\Sigma^{-i}\mathcal{I}\right)^{\mu^{i+1}}\right). \end{aligned} \tag{10.8.3}$$

Since k has a resolution by free $\mathcal{E}$-modules of finite rank, see 10.2, the functors $\operatorname{Ext}^n_{\mathcal{E}}(k, -)$ commute with direct sums. The graded left $\mathcal{E}$-module $\mathcal{I} = \operatorname{Hom}_k(\mathcal{E}, k)$ satisfies $\operatorname{Hom}_{\mathcal{E}}(k, \mathcal{I}) \cong k$ and is injective, so we obtain

$$\operatorname{Hom}_{\mathcal{E}}\left(k, \coprod_{i=-1}^{\infty} \left(\Sigma^{-i}\mathcal{I}\right)^{\mu^{i+1}}\right) \cong \coprod_{i=-1}^{\infty} \Sigma^{-i} k^{\mu^{i+1}}; \tag{10.8.4}$$

$$\operatorname{Ext}^1_{\mathcal{E}}\left(k, \coprod_{i=-1}^{\infty} \left(\Sigma^{-i}\mathcal{I}\right)^{\mu^{i+1}}\right) = 0. \tag{10.8.5}$$

Comparing (10.8.4), (10.8.2), and Lemma 10.8(3) we see that in (10.8.3) the map ι_* is bijective. In view of (10.8.5) this implies $\operatorname{Ext}^1_{\mathcal{E}}(k, \mathcal{S}) = 0$.

(2) If $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}'$ for some left graded $\mathcal{E}$-submodule $\mathcal{T}'$ of $\mathcal{S}$, then Theorem 7.2(2) yields $\mathcal{T}' = \Gamma\mathcal{S}$, hence $\Gamma\mathcal{S}^i = \mathcal{S}^i$ for $i < 0$. One has $\mathcal{S}^i \neq 0$, see Theorem 6.5, so A.2 and A.5 imply $\operatorname{depth}_{\mathcal{E}} \mathcal{S} = \operatorname{depth}_{\mathcal{E}} \Gamma\mathcal{S} = 0$. This contradict (3). □

We finish by applying lemmas used in the proof of Theorem 10.4 to show that the action of absolute cohomology on bounded cohomology from the *right* may be far from nilpotent—in contrast to the action from the *left*, cf. Lemma 1.3.2.

Example 10.9. If $\mathfrak{m}^2 = 0$ and $\operatorname{edim} R = e \geq 2$, then for every $n < 0$ there exists $\beta_n \in \overline{\operatorname{Ext}}^n_R(k, k)$, such that $\{\varepsilon \in \operatorname{Ext}_R(k, k) \mid \beta_n \cdot \varepsilon = 0\}$ is equal to 0.

By Theorem 7.2(1) and Lemma 10.8(1), it suffices to show that each matrix

$$C_n = \left(C^{(n)}\right)_\infty \in \mathcal{C}^n, \quad \text{where } n < 0 \text{ and } C^{(n)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathsf{M}_{e^{-n}\times 1}(k),$$

satisfies $(C_n\mathcal{A}) \cap \mathcal{A} = 0$. Indeed, for each $A \in \mathsf{M}_\infty(k)$ the matrix C_nA is obtained by inserting $(e^{-n}-1)$ rows of zeroes between every pair of adjacent rows of A. On the other hand, the k-basis of $\mathcal{A}$ described in 10.7 shows that each non-zero matrix in $\mathcal{A}$ has a non-zero entry in every row.

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Appendix A. Depth over graded algebras

In this appendix k is a field, $\mathcal{A}$ is a graded k-algebra with $\mathcal{A}^0 = k$ and $\mathcal{A}^i = 0$ for all $i < 0$. Throughout, $\mathcal{M}$ denotes a graded left $\mathcal{A}$-module. By a customary abuse of notation, we let k denote also the graded $\mathcal{A}$-module $\mathcal{A}/\mathcal{A}^{\geqslant 1}$.

A.1. The *depth* of $\mathcal{M}$ over $\mathcal{A}$ is the number

$$\operatorname{depth}_{\mathcal{A}} \mathcal{M} = \inf\left\{n \in \mathbb{N} \mid \operatorname{Ext}^n_{\mathcal{A}}(k, \mathcal{M}) \neq 0\right\},$$

see [16]. Clearly, one has $0 \leq \operatorname{depth}_{\mathcal{A}} \mathcal{M} \leq \infty$, and $\operatorname{depth}_{\mathcal{A}} \mathcal{M} = \infty$ holds if and only if $\operatorname{Ext}_{\mathcal{A}}(k, \mathcal{M}) = 0$. We systematically write $\operatorname{depth}_{\mathcal{A}} \mathcal{A}$ in place of $\operatorname{depth} \mathcal{A}$.

Here we collect general facts about depth, for use in the body of the paper.

The long exact sequence of functors $\operatorname{Ext}^n_{\mathcal{A}}(k, -)$ yields a familiar formula:

A.2. The depths of the graded left $\mathcal{A}$-modules appearing in an exact sequence $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ are linked by an inequality

$$\operatorname{depth}_{\mathcal{A}} \mathcal{M} \geq \inf\{\operatorname{depth}_{\mathcal{A}} \mathcal{L}, \operatorname{depth}_{\mathcal{A}} \mathcal{N}\}.$$

Equality holds when the sequence splits, or when $\operatorname{depth}_{\mathcal{A}} \mathcal{L} \neq \operatorname{depth}_{\mathcal{A}} \mathcal{N} + 1$.

For finite modules over finitely generated commutative algebras depth measures lengths of regular sequences. In general, only a weaker statement holds.

A.3. Assume $\mathcal{A} = \mathcal{A}'/(\vartheta')$ and $\mathcal{M} = \mathcal{M}'/\vartheta'\mathcal{M}'$ for some graded k-algebra $\mathcal{A}'$, a central element $\vartheta' \in \mathcal{A}'^{\geqslant 1}$, and a graded left $\mathcal{A}'$-module $\mathcal{M}'$.

When ϑ' is a non-zero-divisor on $\mathcal{M}'$ the following hold.

(1) $\operatorname{depth}_{\mathcal{A}'} \mathcal{M}' = \operatorname{depth}_{\mathcal{A}'} \mathcal{M} + 1$.
(2) If ϑ' is a non-zero-divisor on $\mathcal{A}'$, then $\operatorname{depth}_{\mathcal{A}} \mathcal{M} = \operatorname{depth}_{\mathcal{A}'} \mathcal{M}' - 1$.

Indeed, for each $n \geq 0$ there are isomorphisms of graded k-vector spaces

$$\operatorname{Ext}^{n-1}_{\mathcal{A}'}(k, \mathcal{M}) \cong \operatorname{Ext}^{n-1}_{\mathcal{A}'}(k, \mathcal{M}') \oplus \operatorname{Ext}^{n}_{\mathcal{A}'}(k, \mathcal{M}'),$$
$$\operatorname{Ext}^{n-1}_{\mathcal{A}}(k, \mathcal{M}) \cong \operatorname{Ext}^{n}_{\mathcal{A}'}(k, \mathcal{M}')$$

obtained by transcribing Rees' classical argument for commutative algebras.

Pursuing the analogy with commutative algebra, we define (left) section functors.

A.4. For each $i \geq 0$ the graded subspace $\mathcal{A}^{\geqslant i}$ of $\mathcal{A}$ is a two-sided ideal, so the following subspaces of $\mathcal{M}$ are graded left $\mathcal{A}$-submodules:

$$\Gamma^i\mathcal{M} = \{\mu \in \mathcal{M} \mid \mathcal{A}^{\geqslant i} \cdot \mu = 0\} \quad \text{and} \quad \Gamma\mathcal{M} = \bigcup_{i=0}^{\infty} \Gamma^i\mathcal{M}.$$

Section functors carry information on the vanishing of depth.

A.5. $\operatorname{depth}_{\mathcal{A}} \mathcal{M} = 0$ if and only if $\Gamma^1\mathcal{M} \neq 0$, if and only if $\Gamma\mathcal{M} \neq 0$.

Indeed, the equivalence of the first two conditions comes from the isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(k, \mathcal{M}) \cong \{\mu \in \mathcal{M} \mid \mathcal{A}^{\geqslant 1} \cdot \mu = 0\} = \Gamma^1 M.$$

It is clear that the second condition implies the third one. Conversely, if $\Gamma\mathcal{M} \neq 0$, then $\mathcal{A}^{\geqslant i} \cdot \mu = 0$ for some $\mu \in \mathcal{M} \setminus \{0\}$ and some integer $i \geq 1$. Choosing i minimal with this property, for $\mathcal{N} = \mathcal{A}^{\geqslant i-1} \cdot \mu$ we get $0 \neq \mathcal{N} \subseteq \Gamma^1\mathcal{M} \neq 0$.

Several applications of depth in the body of the paper hinge on the next result.

Proposition A.6. *If a graded left $\mathcal{A}$-module $\mathcal{K}$ satisfies $\mathcal{K} = \Gamma\mathcal{K}$, then*

$$\operatorname{depth}_{\mathcal{A}} \mathcal{M} \leq \inf\{n \in \mathbb{N} \mid \operatorname{Ext}^n_{\mathcal{A}}(\mathcal{K}, \mathcal{M}) \neq 0\}.$$

Equality holds if $\mathcal{K} = \Gamma^i\mathcal{K} \neq 0$ for some integer $i \geq 1$.

Proof. Set $m = \operatorname{depth}_{\mathcal{A}} \mathcal{M}$. We prove the last assertion by induction on i. If $i = 1$, then $\mathcal{K}$ is a direct sum of shifts of copies of k, so $\operatorname{Ext}^n_{\mathcal{A}}(\mathcal{K}, \mathcal{M})$ is a direct product of shifts of $\operatorname{Ext}^n_{\mathcal{A}}(k, \mathcal{M})$, and the assertion is clear. When $i > 1$ the exact sequence

$$0 \longrightarrow \Gamma^{i-1}\mathcal{K} \longrightarrow \Gamma^i\mathcal{K} \longrightarrow \mathcal{L} \longrightarrow 0$$

of graded left $\mathcal{A}$-modules, where $\mathcal{L} = \Gamma^i\mathcal{K}/\Gamma^{i-1}\mathcal{K}$, induces an exact sequence

$$\operatorname{Ext}^{n-1}_{\mathcal{A}}(\Gamma^{i-1}\mathcal{K}, \mathcal{M}) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}}(\mathcal{L}, \mathcal{M}) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}}(\Gamma^i\mathcal{K}, \mathcal{M}) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}}(\Gamma^{i-1}\mathcal{K}, \mathcal{M})$$

for each n. By the base of the induction, $\operatorname{Ext}^n_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ vanishes for $n < m$ and does not for $n = m$. For $n < m$ the first and last terms vanish by the induction assumption. Thus, $\operatorname{Ext}^n_{\mathcal{A}}(\Gamma^i\mathcal{K}, \mathcal{M})$ vanishes for $n < m$ and does not for $n = m$.

In general, each $\Gamma^i\mathcal{K}$ is a graded left submodule of $\mathcal{K}$, so there are exact sequences

$$0 \longrightarrow \varprojlim_i{}^1 \operatorname{Ext}_{\mathcal{A}}^{n-1}\big(\Gamma^i\mathcal{K}, \mathcal{M}\big) \longrightarrow \operatorname{Ext}_{\mathcal{A}}^n(\mathcal{K}, \mathcal{M}) \longrightarrow \varprojlim_i \operatorname{Ext}_{\mathcal{A}}^n\big(\Gamma^i\mathcal{K}, \mathcal{M}\big) \longrightarrow 0$$

for all $n \geq 0$, where $\varprojlim^1$ is the first right derived functor of $\varprojlim$, see [34, (3.5.10)]. In view of the finite case, these sequences yield $\operatorname{Ext}_{\mathcal{A}}^n(\mathcal{K}, \mathcal{M}) = 0$ for $n < m$. □

Corollary A.7. *If $\mathcal{B}$ is a graded k-subalgebra of $\mathcal{A}$, such that $\mathcal{A}$ is free as a graded right $\mathcal{B}$-module and one has $\mathcal{A}^{\geqslant i} \subseteq \mathcal{A} \cdot \mathcal{B}^{\geqslant 1}$ for some $i \geqslant 1$, then*

$$\operatorname{depth}_{\mathcal{B}} \mathcal{M} = \operatorname{depth}_{\mathcal{A}} \mathcal{M}.$$

Proof. For $\overline{\mathcal{A}} = \mathcal{A}/(\mathcal{A} \cdot \mathcal{B}^{\geqslant 1})$ standard arguments yield $\operatorname{Ext}_{\mathcal{A}}^n(\overline{\mathcal{A}}, \mathcal{M}) \cong \operatorname{Ext}_{\mathcal{B}}^n(k, \mathcal{M})$ for each $n \in \mathbb{Z}$. Since $\overline{\mathcal{A}} = \Gamma^i\overline{\mathcal{A}}$, the proposition yields the desired equality. □

A graded k-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is said to be *normal* if $\mathcal{B}^{\geqslant 1} \cdot \mathcal{A} = \mathcal{A} \cdot \mathcal{B}^{\geqslant 1}$.

Corollary A.8. *Assume* $\operatorname{rank}_k \mathcal{A}^i$ *is finite for each i and $\mathcal{B}$ is a normal subalgebra of $\mathcal{A}$. A finite subset $E \subseteq \mathcal{A}$ is a basis of $\mathcal{A}$ as a graded right $\mathcal{B}$-module if and only if it is a basis of $\mathcal{A}$ as a graded left $\mathcal{B}$-module. When such a set E exists, one has*

$$\operatorname{depth} \mathcal{B} = \operatorname{depth} \mathcal{A}.$$

Proof. By symmetry, to prove the first assertion it suffices to show that if E is a basis of $\mathcal{A}$ as a graded right $\mathcal{B}$-module, then it is one as a left $\mathcal{B}$-module. The image $\overline{E}$ of E in $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{A} \cdot \mathcal{B}^{\geqslant 1}$ is a k-basis of $\overline{\mathcal{A}}$. The map

$$\gamma : \overline{\mathcal{A}} \otimes_k \mathcal{B} \longrightarrow \mathcal{A} \quad \text{given by } \gamma\left(\sum_{e \in E} a_e \bar{e} \otimes \delta_e\right) = \sum_{e \in E} (-1)^{|e||\delta_e|} a_e \delta_e e$$

is a morphism of graded left $\mathcal{B}$-modules. Since $\mathcal{B}$ is normal in $\mathcal{A}$ one has

$$k \otimes_{\mathcal{B}} \mathcal{A} \cong \mathcal{A}/\big(\mathcal{B}^{\geqslant 1} \cdot \mathcal{A}\big) = \mathcal{A}/\big(\mathcal{A} \cdot \mathcal{B}^{\geqslant 1}\big) = \overline{\mathcal{A}},$$

so $k \otimes_{\mathcal{B}} \gamma$ is bijective. By (a graded version of) Nakayama's Lemma the map γ is then surjective. Comparison of k-ranks shows that it is bijective. Thus, E is a basis of $\mathcal{A}$ as left $\mathcal{B}$-module. When a basis E as above exists one has $\operatorname{rank}_k \overline{\mathcal{A}} < \infty$, whence the first isomorphism below; the isomorphism γ induces the second one:

$$\overline{\mathcal{A}} \otimes_k \operatorname{Ext}_{\mathcal{B}}^n(k, \mathcal{B}) \cong \operatorname{Ext}_{\mathcal{B}}^n(k, \overline{\mathcal{A}} \otimes_k \mathcal{B}) \cong \operatorname{Ext}_{\mathcal{B}}^n(k, \mathcal{A}).$$

They yield $\operatorname{depth} \mathcal{B} = \operatorname{depth}_{\mathcal{B}} \mathcal{A}$, and Corollary A.7 gives $\operatorname{depth}_{\mathcal{B}} \mathcal{A} = \operatorname{depth} \mathcal{A}$. □

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