Regularity criterion of axisymmetric weak solutions to the 3D Navier–Stokes equations

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Abstract

We consider the regularity of axisymmetric weak solutions to the Navier–Stokes equations in $\mathbb{R}^3$. Let $u$ be an axisymmetric weak solution in $\mathbb{R}^3 \times (0, T)$, $w = \text{curl} u$, and $w^\theta$ be the azimuthal component of $w$ in the cylindrical coordinates. Chae–Lee [D. Chae, J. Lee, On the regularity of axisymmetric solutions of the Navier–Stokes equations, Math. Z. 239 (2002) 645–671] proved the regularity of weak solutions under the condition $w^\theta \in L^q(0, T; L^r)$, with $\frac{3}{2} < r < \infty$, $\frac{2}{r} + \frac{3}{q} \leq 2$. We deal with the marginal case $r = \infty$ which they excluded. It is proved that $u$ becomes a regular solution if $w^\theta \in L^1(0, T; \dot{B}^{0}_{\infty, \infty})$.

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1. Introduction

In this paper, we are concerned with the regularity problem of weak solutions to the Navier–Stokes equations in $\mathbb{R}^3 \times [0, T)$:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u &= -\nabla p, & \text{in } \mathbb{R}^3 \times (0, T), \\
\nabla \cdot u &= 0, & \text{in } \mathbb{R}^3 \times (0, T), \\
u(0) &= u_0(x), & \text{in } \mathbb{R}^3,
\end{aligned}
$$

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where \( u = (u^1(x,t), u^2(x,t), u^3(x,t)) \) and \( p = p(x,t) \) denote the unknown velocity vector and the unknown scalar pressure of the fluid at the point \((x,t) \in \mathbb{R}^3 \times (0,T)\), respectively, while \( u_0 = (u_0^1(x), u_0^2(x), u_0^3(x)) \) is a given initial velocity vector satisfying \( \nabla \cdot u_0 = 0 \). Here we use the notation:

\[
\begin{align*}
    u \cdot \nabla v &= \sum_{i=1}^{3} u^i \frac{\partial v}{\partial x_i}, \\
    \nabla \cdot u &= \sum_{i=1}^{3} \frac{\partial u^i}{\partial x_i},
\end{align*}
\]

for vector functions \( u, v \).

Let us recall the definition of Leray–Hopf weak solution.

**Definition 1.1.** Let \( u_0 \in L^2(\mathbb{R}^3) \) with \( \nabla \cdot u_0 = 0 \). The vector field \( u(x,t) \) will be called a Leray–Hopf weak solution of (1.1) in \((0,T)\) if \( u \) satisfies the following properties:

1. \( u \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^1(\mathbb{R}^3)) \) (Leray–Hopf class);
2. \( u_t - \Delta u + u \cdot \nabla u + \nabla p = 0 \) in \( \mathcal{D}'(\mathbb{R}^3 \times [0,T)) \);
3. \( \nabla \cdot u = 0 \) in \( \mathcal{D}'(\mathbb{R}^3 \times [0,T)) \);
4. (The Energy Inequality)

\[
\| u(t) \|_2^2 + 2 \int_0^t \| \nabla u(\tau) \|_2^2 \, d\tau \leq \| u_0 \|_2^2, \quad \forall \ 0 \leq t \leq T.
\]

For given \( u_0 \in L^2(\mathbb{R}^3) \) with \( \nabla \cdot u_0 = 0 \), J. Leray and E. Hopf [11,17] constructed a global Leray–Hopf weak solution \( u(x,t) \). It is well known that the weak solution is unique and regular in two spatial dimensions [27]. In three dimensions, however, the regularity problem of weak solutions is an outstanding open problem in mathematical fluid mechanics. Researchers are interested in the classical problem of finding sufficient conditions for weak solutions of (1.1) such that they become regular. J. Serrin [21,22] is the pioneer in this direction, and later on, Fabes, Jones, and Riviere [8], Giga [10], Sohr [24], Struwe [25] and Takahashi [26] extended Serrin’s regularity criterion: Leray–Hopf weak solutions in Serrin’s class

\[
u \in L^q(0,T; L^r)
\]

with \( \frac{2}{q} + \frac{3}{r} \leq 1, \ 3 < r \leq \infty \), (1.2)

are necessarily regular. H. Beirão da Veiga [1] extended Serrin’s regularity criterion to the vorticity showing that if

\[
\text{curl } u \in L^q(0,T; L^r)
\]

with \( \frac{2}{q} + \frac{3}{r} = 2, \ 3 \leq r < \infty \), (1.3)

then \( u \) is a regular solution. In the marginal case \( r = \infty \), H. Kozono and Y. Taniuchi [12] proved the regularity of weak solutions under the condition

\[
\text{curl } u \in L^1(0,T; BMO),
\]

where \( BMO \) is the space of the bounded mean oscillation defined by

\[
f \in L^1_{\text{loc}}(\mathbb{R}^3), \quad \sup_{x,R} \frac{1}{|B_R|} \int_{B_R(x)} |f(y) - \bar{f}_{B_R(x)}| \, dy < \infty,
\]
where \( \bar{f}_{BR}(x) \) is the average of \( f \) over \( BR(x) = \{ y \in \mathbb{R}^3; |x - y| < R \} \) (cf. Stein [23]). Recently, by establishing the logarithmic Sobolev inequality in Besov spaces, H. Kozono, T. Ogawa and Y. Taniuchi [13] refined the condition (1.3)–(1.4) to
\[
\text{curl } u \in L^q(0, T; \dot{B}^0_{r, \infty}) \quad \text{with} \quad \frac{2}{q} + \frac{3}{r} = 2, \quad \frac{3}{2} < r \leq \infty. \tag{1.5}
\]
Here and thereafter, \( \dot{B}^s_{p,q} \) stands for the homogeneous Besov space, see Section 2 for the definition. On the other hand, D. Chae and H.J. Choe [4] improved Beirão da Veiga’s regularity criterion by imposing only the two-component vorticity field. More precisely, let \( w = \text{curl } u \), and \( \tilde{w} = (w^1, w^2, 0) \), they proved the regularity of weak solutions in the class
\[
\tilde{w} \in L^q(0, T; L^r) \quad \text{with} \quad \frac{2}{q} + \frac{3}{r} = 2, \quad \frac{3}{2} < r < \infty. \tag{1.6}
\]
In [14], H. Kozono and Y. Yatsu dealt with the marginal case for \( r = \infty \) in (1.6). They showed that if the Leray–Hopf weak solution \( u \) of (1.1) satisfies
\[
\tilde{w} \in L^1(0, T; BMO), \tag{1.7}
\]
then
\[
u \in C([0, T); H^1) \cap C^1((0, T); H^1) \cap C((0, T); H^3),
\]
In particular, this implies that \( u \) is a regular solution in \( \mathbb{R}^3 \times (0, T) \). Very recently, Chen and Zhang [7] proved the regularity of weak solutions in the class
\[
\tilde{w} \in L^q(0, T; \dot{B}^0_{r, \sigma}) \quad \text{with} \quad \frac{2}{q} + \frac{3}{r} = 2, \quad \frac{3}{2} < r \leq \infty, \quad \sigma \leq \frac{2r}{3}. \tag{1.8}
\]
We are concerned with the regularity criteria of axisymmetric weak solutions of the Navier–Stokes equations. Recall the cylindrical coordinates given by
\[
\begin{align*}
x_1 &= r \sin \theta, \\
x_2 &= r \cos \theta, \\
x_3 &= x_3,
\end{align*}
\]
where \( \theta \in [0, 2\pi) \), \( r \in [0, +\infty) \). By an axisymmetric solution of the Navier–Stokes equations we mean a solution of the form:
\[
u(x, t) = u^r(r, x_3, t)e_r + u^\theta(r, x_3, t)e_\theta + u^3(r, x_3, t)e_3,
\]
where we used the basis
\[
e_r = (\sin \theta, \cos \theta, 0), \quad e_\theta = (-\cos \theta, \sin \theta, 0), \quad e_3 = (0, 0, 1).
\]
We shall point out the relation between \( \nabla = (\partial_1, \partial_2, \partial_3) \) and \( (\partial_r, \partial_\theta, \partial_3) \)
\[
\nabla = e_r \partial_r - \frac{1}{r} e_\theta \partial_\theta + e_3 \partial_3. \tag{1.10}
\]
For the axisymmetric vector field \( u \), the vorticity \( w = \text{curl } u \) can be written as
\[
w = w^r e_r + w^\theta e_\theta + w^3 e_3,
\]
where
\[
w^r = -\partial_3 u^\theta, \quad w^\theta = -\partial_r u_3 + \partial_3 u^r, \quad w^3 = \partial_r u^\theta + \frac{u^\theta}{r}.
\]
For the axisymmetric Navier–Stokes equations without swirl (i.e. \( u^\theta = 0 \)), M.R. Ukhovskii and V.I. Yudovich [29], and independently O.A. Ladyzhenskaya [15] proved global existence of regular solution. Recently S. Leonardi, J. Málek, J. Nečas and M. Pokorný [16] gave a refined proof. However, for the axisymmetric Navier–Stokes equations with nonzero swirl component, the regularity problem is still open. There are many studies on the regularity criterion of the axisymmetric weak solutions, see [5,19,20] and reference therein. Let us recall a result proved by D. Chae and J. Lee [5].

**Theorem 1.1.** Let \( u \) be an axisymmetric weak solution of the Navier–Stokes equations with \( u_0 \in H^2(\mathbb{R}^3), \nabla \cdot u_0 = 0 \). If \( w^\theta \) satisfies the following condition:

\[
\int_0^T \| w^\theta(t) \|_p^q \, dt < \infty, \quad \frac{2}{q} + \frac{3}{p} \leq 2, \quad \frac{3}{2} < p < \infty, \quad 1 < q \leq \infty, \tag{1.11}
\]

then the solution \( u \) is regular on \( \mathbb{R}^3 \times (0, T) \).

In this paper, we deal with the marginal case \( p = \infty \) in (1.11). The main reason of excluding the marginal case in [5] is that the singular integral operators are not bounded in \( L^\infty \). To get around this difficulty, we will make use of the Littlewood–Paley decomposition to decompose the function into three parts: low frequency, middle frequency and high frequency, and use different estimate for each part. It should be pointed out that the logarithmic Sobolev inequality in [13] is not applicable in our case, since it seems difficult to control the higher derivatives of the solution by the only component \( w^\theta \). Now we state our result as follows.

**Theorem 1.2.** Suppose that \( u(x,t) \) is an axisymmetric weak solution of (1.1) in \( (0, T) \) with \( u_0 \in H^1(\mathbb{R}^3), \nabla \cdot u_0 = 0 \). If \( w^\theta \) satisfies the following condition:

\[
\int_0^T \| w^\theta(t) \|_{\dot{B}^{0}_{\infty, \infty}} \, dt < \infty, \tag{1.12}
\]

then \( u \) is a regular solution in \( \mathbb{R}^3 \times (0, T) \).

**Remark 1.1.** Beale–Kato–Majda [2,18], Kozono–Taniuchi [12], and Kozono–Ogawa–Taniuchi [13] proved similar results for the Navier–Stokes equations without the assumption of any symmetry under the conditions \( w \in L^1(0, T; L^\infty), \ w \in L^1(0, T; BMO), \) and \( w \in L^1(0, T; \dot{B}^{0}_{\infty, \infty}) \), respectively. Our result is an improvement of their results for the axisymmetric case. In addition, the result of Theorem 1.2 holds true for the initial data in \( H^{\frac{1}{2}}(\mathbb{R}^3) \) instead of \( H^1(\mathbb{R}^3) \).

**Remark 1.2.** For the axisymmetric Euler equation

\[
\begin{aligned}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= 0, \\
\nabla \cdot u &= 0, \\
u(0) &= u_0(x),
\end{aligned}
\tag{1.13}
\]
D. Chae [6] showed if $u$ is a smooth solution of (1.13) and satisfies
\begin{equation}
\int_0^T \| w^\theta(t) \|_{\dot{B}^0_{\infty,1}} \, dt < \infty,
\end{equation}
then $u$ can be extended after $t = T$. We do not know whether the condition (1.14) can be replaced by the following condition
\begin{equation}
\int_0^T \| w^\theta(t) \|_{L^\infty} \, dt < \infty,
\end{equation}
or
\begin{equation}
\int_0^T \| w^\theta(t) \|_{\dot{B}^0_{\infty,\infty}} \, dt < \infty.
\end{equation}
Notice that we have the inclusion relation:
\[ \dot{B}^0_{\infty,1} \subset L^\infty \subset \text{BMO} \subset \dot{B}^0_{\infty,\infty}. \]

**2. Preliminaries**

We first introduce the Littlewood–Paley decomposition. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F} f = \hat{f}$ is defined by
\[ \hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) \, dx, \]
and its inverse Fourier transform $\mathcal{F}^{-1} f = f^\vee$ is defined by
\[ f^\vee(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} \hat{f}(\xi) \, d\xi. \]

Let us choose a nonnegative radial function $\phi \in \mathcal{S}(\mathbb{R}^3)$ such that
\[ 0 \leq \hat{\phi}(\xi) \leq 1, \quad \hat{\phi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2, \end{cases} \]
and let
\[ \psi(x) = \phi(x) - 2^{-3} \phi(x/2), \quad \phi_j(x) = 2^{3j} \phi(2^j x), \quad \psi_j(x) = 2^{3j} \psi(2^j x), \quad j \in \mathbb{Z}. \]

For $j \in \mathbb{Z}$, the Littlewood–Paley projection operators $S_j$ and $\Delta_j$ are respectively defined by
\begin{align}
S_j f &= \phi_j * f, \\
\Delta_j f &= \psi_j * f.
\end{align}
Informally, $\Delta_j$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$, while $S_j$ is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. Observe that $\Delta_j = S_j - S_{j-1}$. Also, if $f$ is an $L^2$ function then
\[ S_j f \to 0 \text{ in } L^2 \text{ as } j \to -\infty \text{ and } S_j f \to f \text{ in } L^2 \text{ as } j \to +\infty \] (this is an easy consequence of Parseval’s theorem). By telescoping the series, we thus have the Littlewood–Paley decomposition

\[
f = \sum_{j=-\infty}^{+\infty} \Delta_j f, \tag{2.3}
\]
for all \( f \in L^2 \), where the summation is in the \( L^2 \) sense. Notice that

\[
\Delta_j f = \sum_{l=j-2}^{j+2} \Delta_l(\Delta_j f) = \sum_{l=j-2}^{j+2} \psi_l * \psi_j * f,
\]
then from the Young inequality, it follows that

\[
\|\Delta_j f\|_q \leq C 2^{3j\left(\frac{1}{p} - \frac{1}{q}\right)} \|\Delta_j f\|_p, \tag{2.4}
\]
where \( 1 \leq p \leq q \leq \infty \), \( C \) is a constant independent of \( f, j \).

If \( T \) is a singular integral operator of convolution type, and its kernel \( K(y) \) satisfies

\[
K(y) \in C^\infty(\mathbb{R}^3 \setminus \{0\}), \quad \int_{\mathbb{S}^2} K(y) \, d\sigma(y) = 0,
\]
then we also have

\[
\|T(\Delta_j f)\|_q \leq C 2^{3j\left(\frac{1}{p} - \frac{1}{q}\right)} \|\Delta_j f\|_p, \tag{2.5}
\]
for \( 1 \leq p \leq q \leq \infty \). We can refer to [23] for the proof and more properties of \( T \).

Let \( s \in \mathbb{R}, p, q \in [1, \infty) \), the homogeneous Besov space \( \dot{B}^s_{p,q} \) is defined by the full-dyadic decomposition such as

\[
\dot{B}^s_{p,q} = \{ f \in Z'(\mathbb{R}^3) : \|f\|_{\dot{B}^s_{p,q}} < \infty \},
\]
where \( \|f\|_{\dot{B}^s_{p,q}} = (\sum_{j=-\infty}^{\infty} 2^{jsq} \|\Delta_j f\|^q_p)^{\frac{1}{q}} \) and \( Z'(\mathbb{R}^3) \) denotes the dual space of \( Z(\mathbb{R}^3) = \{ f \in S(\mathbb{R}^3) : D^\alpha \hat{f}(0) = 0; \ \forall \alpha \in \mathbb{N}^3 \text{ multi-index} \} \) and can be identified by the quotient space of \( S'/\mathcal{P} \) with the polynomials space \( \mathcal{P} \). We refer to [3,28] for more details.

We will use notation for the axisymmetric vector field \( u \),

\[
\tilde{u} = u^r e_r + u^3 e_3, \quad \tilde{w} = w^r e_r + w^3 e_3,
\]
and

\[
\tilde{\nabla} = (\partial_r, \partial_3).
\]

**Lemma 2.1.** Let \( u \) be an axisymmetric vector field. Then the following equalities hold:

\[
|\nabla \tilde{u}|^2 = \left|\frac{u^r}{r}\right|^2 + |\tilde{\nabla} u^r|^2 + |\tilde{\nabla} u^3|^2, \tag{2.6}
\]

\[
|\nabla (u^\theta e_\theta)|^2 = \left|\frac{u^\theta}{r}\right|^2 + |\tilde{\nabla} u^\theta|^2. \tag{2.7}
\]
Proof. Applying the relationship (1.10), we have
\[
\nabla \tilde{u} = \nabla \left( u^r e_r + u^3 e_3 \right) = \nabla \left( u^r \sin \theta, u^r \cos \theta, u^3 \right)
\]
\[
= \begin{pmatrix}
\sin \theta (\partial_r u^r + \cos \theta \frac{w_r^r}{r}), & \sin \theta \cos \theta \partial_r u^r - \sin \theta \cos \theta \frac{w_r^r}{r}, & \sin \theta \partial_3 u^r \\
\sin \theta \cos \theta \partial_r u^r - \sin \theta \cos \theta \frac{w_r^r}{r}, & (\cos \theta)^2 \partial_r u^r + (\sin \theta)^2 \frac{w_r^r}{r}, & \cos \theta \partial_3 u^r \\
\sin \theta \partial_r u^3, & \cos \theta \partial_r u^3, & \partial_3 u^3
\end{pmatrix}.
\]
Then by a direct computation, one can prove (2.6). Similarly, we can prove (2.7).

Lemma 2.2. (Chae and Lee [5, Lemma 2]) Let \( u \) be an axisymmetric vector field with \( \text{div} \ u = 0 \) and \( w = \text{curl} \ u \) vanish sufficiently fast near infinity in \( \mathbb{R}^3 \), then \( \nabla \tilde{u} \) and \( \nabla (u^\theta e_\theta) \) can be represented as the singular integral form:
\[
\nabla \tilde{u}(x) = C w^\theta e_\theta(x) + \left[ K \ast (w^\theta e_\theta) \right](x),
\]
\[
\nabla \left( u^\theta e_\theta(x) \right) = C \tilde{w}(x) + \left[ H \ast (\tilde{w}) \right](x),
\]
where the kernels \( (K(x)) \) and \( (H(x)) \) are matrix valued functions homogeneous of degree \(-3\), defining a singular integral operator by convolution, and \( f \ast g(x) = \int_{\mathbb{R}^3} f(x - y)g(y) \, dy \) denotes the standard convolution operator. The matrices \( C \) and \( \tilde{C} \) are constant.

From Lemma 2.2 and the \( L^p \) boundedness of Calderon–Zygmund singular integral operators, we can deduce that

Lemma 2.3. Let \( 1 < p < \infty \). Then we have
\[
\||\nabla \tilde{u}||_p \leq C \| w^\theta \|_p, \quad (2.10)
\]
\[
\||\nabla (u^\theta e_\theta)||_p \leq C \| \tilde{w} ||_p, \quad (2.11)
\]
where \( C \) is a constant depending only on \( p \).

Finally let us recall the well-known Biot–Savart law \[18\]. Let \( u \) be a smooth vector function, and \( w = \text{curl} \ u \). If \( \nabla \cdot u = 0 \), then \( \nabla u \) can be written in terms of \( w \):
\[
\nabla u(x) = C w(x) + K \ast w(x),
\]
where \( C \) is a constant matrix, and \( K \) is a matrix valued function homogeneous of degree \(-3\). Hence we also have
\[
\||\nabla u||_p \leq C \| w \|_p, \quad 1 < p < \infty, \quad (2.13)
\]

3. Proof of Theorem 1.2

First we derive a priori estimate for the smooth solution of (1.1). More precisely, we will show the following a priori estimate
\[
\sup_{0 \leq t \leq T} \| u(t) \|_{H^1} \leq C \left( \| u_0 \|_{H^1} + \sqrt{C T} + e \right) \exp \left( C \int_0^T \| w^\theta(t) \|_{\ell^2_{\infty, \infty}} \, dt \right). \quad (3.1)
\]
Taking the curl on (1.1), we obtain
\[
w_t + \Delta w + u \cdot \nabla w - w \cdot \nabla u = 0. \quad (3.2)
Multiplying (3.2) by $w$ and integrating by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|^2_2 + \|\nabla w(t)\|^2_2 = \int_{\mathbb{R}^3} (w \cdot \nabla u) \cdot w \, dx.
\] (3.3)
Here we used the fact $(u \cdot \nabla w, w) = 0$ since $\text{div} \, u = 0$. Now we use
\[
w = w^r e_r + w^\theta e_\theta + w^3 e_3, \quad \nabla = e_r \partial_r - \frac{1}{r} e_\theta \partial_\theta + e_3 \partial_3.
\]
Then the right-hand side of (3.3) can be written as
\[
\int_{\mathbb{R}^3} (w \cdot \nabla u) \cdot w \, dx = \int_{\mathbb{R}^3} w^r \partial_r u^r w^r + w^r \partial_r u^\theta w^\theta + w^r \partial_r u^3 w^3 + \frac{1}{r} w^\theta u^r w^\theta - \frac{1}{r} w^\theta u^3 w^r
\]
\[
+ w^3 \partial_3 u^r w^r + w^3 \partial_3 u^\theta w^\theta + w^3 \partial_3 u^3 w^3 \, dx
\]
\[
\triangleq I_1(t) + \cdots + I_8(t),
\] (3.4)
where we have used the fact
\[
\partial_\theta e_r = -e_\theta, \quad \partial_\theta e_\theta = e_r.
\]
In the following, we will estimate each term on right-hand side of (3.4) separately. We first consider $I_1(t)$. Lemma 2.1 implies that
\[
|\partial_r u^r| \leq \|\nabla \tilde{u}\|, \quad \text{as well as by virtue of Lemma 2.2, we have}
\]
\[
\|I_1(t)\| \leq C \int_{\mathbb{R}^3} \|w^r\|^2 \|\partial_r u^r\| \, dx \leq C \int_{\mathbb{R}^3} \|w^r\|^2 \left(\|\nabla w^\theta\| + \|K \ast (w^\theta e_\theta)\|\right) \, dx.
\]
Then using the Littlewood–Paley decomposition (2.3), we decompose $w^\theta$ as follows:
\[
w^\theta = \sum_{j=-\infty}^{+\infty} \Delta_j w^\theta = \sum_{j<-N} \Delta_j w^\theta + \sum_{j=-N}^{N} \Delta_j w^\theta + \sum_{j>N} \Delta_j w^\theta,
\] (3.5)
here $N$ is a positive integer to be chosen later. Substituting this into $I_1(t)$, we have
\[
I_1(t) = \sum_{j<-N} \int_{\mathbb{R}^3} \|w^r\|^2 \left(\|\Delta_j w^\theta\| + \|K \ast (\Delta_j w^\theta e_\theta)\|\right) \, dx
\]
\[
+ \sum_{j=-N}^{N} \int_{\mathbb{R}^3} \|w^r\|^2 \left(\|\Delta_j w^\theta\| + \|K \ast (\Delta_j w^\theta e_\theta)\|\right) \, dx
\]
\[
+ \sum_{j>N} \int_{\mathbb{R}^3} \|w^r\|^2 \left(\|\Delta_j w^\theta\| + \|K \ast (\Delta_j w^\theta e_\theta)\|\right) \, dx
\]
\[
\triangleq I_{1,1}(t) + I_{1,2}(t) + I_{1,3}(t).
\] (3.6)
For $I_{1,1}(t)$, from the Hölder inequality, (2.4), and (2.5), it follows that
\[
|I_{1,1}(t)| \leq \|w^r\|^2_2 \sum_{j<-N} \left(\|\Delta_j w^\theta\|_\infty + \|K \ast (\Delta_j w^\theta e_\theta)\|_\infty\right)
\]
\[
\leq C \|w^r\|^2_2 \sum_{j<-N} 2^{\frac{3}{2}j} \|\Delta_j w\|_2
\]
\[
\leq C 2^{-\frac{3}{2}N} \|w\|_2^3.
\] (3.7)
For $I_{1,2}(t)$, by the Hölder inequality and (2.5), we have

$$|I_{1,2}(t)| \leq \|w^r\|_2^2 \sum_{j=-N}^{N} \|\Delta_j w^\theta\|_\infty \leq CN \|w\|_2^2 \|w^\theta\|_{\dot{B}^0_{\infty,\infty}}. \quad (3.8)$$

For $I_{1,3}(t)$, from the Hölder inequality, (2.4), (2.5), and the Gagliardo–Nirenberg inequality, it follows that

$$|I_{1,3}(t)| \leq \|w^r\|_3^2 \sum_{j>N} \|\Delta_j w^\theta\|_2$$

$$\leq C \|w^r\|_3^2 \left( \sum_{j>N} 2^{-j} \right)^{\frac{1}{2}} \left( \sum_{j>N} 2^{2j} \|\Delta_j w^\theta\|_2^2 \right)^{\frac{1}{2}}$$

$$\leq C 2^{-N} \|w\|_2 \|\nabla w\|_2.$$  

Thus, summing up (3.7)–(3.9), we obtain

$$|I_1(t)| \leq C \left( 2^{-\frac{3}{2}N} \|w\|_2^3 + N \|w\|_2^2 \|w^\theta\|_{\dot{B}^0_{\infty,\infty}} + 2^{-\frac{N}{2}} \|w\|_2 \|\nabla w\|_2^2 \right). \quad (3.10)$$

Similarly, since

$$|\partial_r u^3|, \left| \frac{u^r}{r} \right|, |\partial_3 u^r|, |\partial_3 u^3| \leq |\nabla \tilde{u}| \leq C \left( |w^\theta| + |K \ast (w^\theta e_\theta)| \right),$$

we get

$$|I_3(t)|, |I_4(t)|, |I_6(t)|, |I_8(t)| \leq C \left( 2^{-\frac{3}{2}N} \|w\|_2^3 + N \|w\|_2^2 \|w^\theta\|_{\dot{B}^0_{\infty,\infty}} + 2^{-\frac{N}{2}} \|w\|_2 \|\nabla w\|_2^2 \right). \quad (3.11)$$

Now we turn to estimate $I_2(t)$. Using the decomposition (3.5) again, $I_2(t)$ can be written as

$$I_2(t) = \sum_{j<-N} \int_{\mathbb{R}^3} w^r \partial_r u^\theta \Delta_j w^\theta \, dx + \sum_{j=-N}^{N} \int_{\mathbb{R}^3} w^r \partial_r u^\theta \Delta_j w^\theta \, dx$$

$$+ \sum_{j>N} \int_{\mathbb{R}^3} w^r \partial_r u^\theta \Delta_j w^\theta \, dx$$

$$\triangleq I_{2,1}(t) + I_{2,2}(t) + I_{2,3}(t). \quad (3.12)$$

Lemmas 2.1 and 2.2 imply

$$|\partial_r u^\theta| \leq |\nabla (u^\theta e_\theta)| \leq |\tilde{u}| + |H \ast \tilde{u}|.$$  

Hence noting that the singular operator is bounded on $L^p$ for $p \in (1, \infty)$, then by the same procedure leading to (3.11),

$$|I_2(t)| \leq C \left( 2^{-\frac{3}{2}N} \|w\|_2^3 + N \|w\|_2^2 \|w^\theta\|_{\dot{B}^0_{\infty,\infty}} + 2^{-\frac{N}{2}} \|w\|_2 \|\nabla w\|_2^2 \right). \quad (3.13)$$
Since Lemmas 2.1 and 2.2 also tell us that
\[ |u_\theta r|, |\partial_3 u_\theta| \leq |\nabla (u_\theta e_\theta)| \leq |\tilde{w}| + |H * \tilde{w}|, \]
the terms $I_5$, $I_7$ can be similarly estimated as the term $I_2$.

Combining all above estimates, we finally obtain
\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 \leq C \left( 2^{-\frac{N}{2}} \|w\|_2^3 + N \|w\|_2^2 \|w_\theta\|_{\dot{\mathcal{B}}^0_{\infty, \infty}} + 2^{-\frac{N}{2}} \|w\|_2 \|\nabla w\|_2^2 \right).
\]
(3.14)

Now we choose $N$ in (3.14) so that
\[ C 2^{-\frac{N}{2}} \|w\|_2 \leq \frac{1}{2}, \]
i.e.
\[ N \geq \frac{2 \log^+ (C \|w\|_2)}{\log 2} + 2, \]
where $\log^+ t = \log t$ for $1 \leq t$ and $\log^+ t = 0$ for $0 < t < 1$. Then the inequality (3.14) implies that
\[
\frac{d}{dt} \|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 \leq C \|w(t)\|_2^2 \|w_\theta(t)\|_{\dot{\mathcal{B}}^0_{\infty, \infty}} \log (\|w(t)\|_2 + e) + C,
\]
for all $0 < t < T$. By the Gronwall inequality, we have
\[
\|w(t)\|_2^2 \leq (\|w(0)\|_2^2 + CT) \exp \left( C \int_0^t \|w_\theta(s)\|_{\dot{\mathcal{B}}^0_{\infty, \infty}} \log (\|w(s)\|_2 + e) \, ds \right).
\]
(3.16)

Defining $Z(t) = \log (\|w(t)\|_2 + e)$, the inequality (3.16) implies that
\[
Z(t) \leq \log (\|w(0)\|_2 + \sqrt{CT} + e) + C \int_0^t \|w_\theta(s)\|_{\dot{\mathcal{B}}^0_{\infty, \infty}} Z(s) \, ds.
\]
(3.17)

Applying the Gronwall inequality to $Z(t)$ again, we have
\[
Z(t) \leq \log (\|w(0)\|_2 + \sqrt{CT} + e) \exp \left( C \int_0^t \|w_\theta(s)\|_{\dot{\mathcal{B}}^0_{\infty, \infty}} \, ds \right),
\]
(3.18)

which together with (2.13) implies that
\[
\sup_{0 \leq t \leq T} \|\nabla u(t)\|_2 \leq C \left( \|\nabla u_0\|_2 + \sqrt{CT} + e \right) \exp (C \int_0^T \|w_\theta(t)\|_{\dot{\mathcal{B}}^0_{\infty, \infty}} \, dt).
\]
(3.19)

On the other hand, $u$ satisfies the energy inequality, i.e.
\[
\|u(t)\|_2^2 + 2 \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \leq \|u_0\|_2^2, \quad \forall \ 0 \leq t \leq T.
\]
(3.20)

From (3.19) and (3.20), we obtain the desired estimate (3.1).
Now we are in a position to complete the proof of Theorem 1.2. Since \( u_0 \in H^1(\mathbb{R}^3) \) with \( \nabla \cdot u_0 = 0 \), from the local existence theorem for the strong solution [9], it follows that there exist \( T_* > 0 \) and an axisymmetric solution \( v \) of (1.1) satisfying
\[
  v(t) \in C([0, T_*); H^1(\mathbb{R}^3)) \cap C^1((0, T_*); H^1) \cap C((0, T_*); H^3), \quad v(0) = u_0.
\]
Since the weak solution \( u \) satisfies the energy inequality (3.20), we may apply Serrin’s uniqueness criterion [22] to conclude that
\[
  u \equiv v \quad \text{on } [0, T_*).
\]
Thus it is sufficient to show that \( T_* = T \). Suppose that \( T_* < T \). Without loss of generality, we may assume that \( T_* \) is the maximal existence time for \( v(t) \). Since \( u(t) = v(t) \) on \( [0, T_*) \), by the assumption (1.12), we have
\[
  \int_0^{T_*} \| (\text{curl } v)^\theta \|_{B^{0,0}_{\infty,\infty}} \, dt < \infty.
\]
Then it follows from (3.1) that the existence time of \( v(t) \) can be extended after \( t = T_* \) which contradicts the maximality of \( t = T_* \). This completes the proof of Theorem 1.2.

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References


