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Stability of abstract nonlinear nonautonomous differential–delay equations with unbounded history-responsive operators ☆

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Abstract

We consider a class of nonautonomous functional–differential equations in a Banach space with unbounded nonlinear history-responsive operators, which have the local Lipshitz property. Conditions for the boundedness of solutions, Lyapunov stability, absolute stability and input–output one are established. Our approach is based on a combined usage of properties of sectorial operators and spectral properties of commuting operators.

Keywords: Abstract nonlinear differential–delay equations; Lyapunov stability; Absolute stability; Input–output stability; Boundedness of solutions

1. Introduction and notation

Stability and boundedness of solutions of parabolic and abstract functional–differential equations were investigated by many specialists, cf. [1,3,5,6,8,9,12–17,23] and references therein. It is mostly assumed that the history-responsive operators are bounded. At the same time equations with unbounded history-responsive operators arise naturally, for instance,

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from problems of heat conduction in materials with thermal memory or of viscoelasticity in materials with shape memory, cf. [19] and references therein. Equations with linear unbounded history-responsive operators were studied in [2, Chapter 5.5], [10,18,21]. However, to the best of our knowledge, the absolute stability and input–output one of abstract differential equations with unbounded history-responsive operators were not investigated in the available literature, although these notions are very important in theory of systems, cf. [22].

In the present paper we consider a class of functional–differential equations in a Banach space with nonlinear history-responsive operators, which have the local Lipshitz property. Conditions for the boundedness of solutions, Lyapunov stability, absolute stability and input–output stability are established.

Our approach is based on a combined usage of properties of sectorial operators and spectral properties of commuting operators. A few words about the contents. The paper consists of 12 sections. In Section 2 we prove the basic lemma of the paper—Lemma 2.1 on solution estimates. In Sections 3 we establish an existence result for mild solutions of the considered equations. In Sections 4, 5 and 6 we specialize Lemma 2.1 in the cases of equations with sectorial, selfadjoint and spectral operators, respectively. The Lyapunov stability, absolute stability and input–output one are investigated in Sections 7, 8 and 9, respectively. Sections 10, 11 and 12 deal with the applications of the main results to parabolic differential–delay equations and integro-differential equations with delay.

Let X be a Banach space with a norm $\|.\|_X$ and Y a Banach subspace with a norm $\|.\|_Y$ continuously imbedded into X. Put $R_+ = [0, \infty)$ and $R_h = [-h, \infty)$ for a finite $h > 0$. As usual, $C(J, X)$ is the space of continuous X-valued functions defined on a set J and equipped with the sup-norm

$$\|v\|_{C(J,X)} = \sup_{t \in J} \|v(t)\|_X \quad \big(v \in C(J, X)\big).$$

For a linear operator A, $D(A)$ is the domain, $\sigma(A)$ is the spectrum,

$$\beta(A) := \inf \operatorname{Re} \sigma(A)$$

and $\lambda_k(A)$ $(k = 1, 2, \ldots)$ are the eigenvalues with their multiplicities.

Now let $A(t)$ $(t \geqslant 0)$ be a linear operator in X with a dense constant domain

$$D\big(A(t)\big) \equiv D_A \subseteq Y, \quad t \geqslant 0.$$

The following equation is the main object of our investigation:

$$\dot{u}(t) = A(t)u(t) + [Fu](t) \quad (t > 0,\ \dot{u} = du/dt), \tag{1.1}$$

where $F : C(R_h, Y) \to C(R_+, X)$ is *a causal nonlinearity* in the sense that

$$[Fu_1](t) = [Fu_2](t) \quad \text{if } u_1(\tau) = u_2(\tau) \quad \text{for all } \tau \in [-h, t] \text{ and } u_1, u_2 \in C(R_h, Y).$$

Take the initial condition

$$u(t) = \phi(t) \quad (-h \leqslant t \leqslant 0), \tag{1.2}$$

where $\phi \in C([-h, 0], Y)$ is a given continuous function.

In the sequel it is assumed that $A(t)$ generates in Y an evolution operator $U(t,s)$. This means, that the Cauchy problem for the "shortened" equation

$$\dot{v} = A(t)v \quad (t \geqslant 0)$$

is well posed in Y [20]. That is, this equation has continuously differentiable solutions with values in D_A, provided the initial vector is in D_A. Besides, $U(t,s)$ acts in Y and defined by the equality $U(t,s)v(s) = v(t)$ for any solution $v(t)$ of the "shortened" equation.

Definition 1.1. A function $u \in C([-h,T], Y)$ $(0 < T < \infty)$, satisfying the equation

$$u(t) = U(t,0)\phi(0) + \int_0^t U(t,s)[Fu](s)\,ds \quad (0 < t < T) \tag{1.3}$$

and condition (1.2) will be called the mild solution of problem (1.1), (1.2) on $(0,T)$.

The existence of the mild solutions for all finite $t > 0$ is assumed. As it was mentioned, below we derive some simple conditions for the existence and uniqueness of the mild solutions.

For a positive number $r \leqslant \infty$, put

$$\Omega_r(J,Y) = \{v \in C(J,Y)\colon \|v\|_{C(J,Y)} \leqslant r\}.$$

Assume that F continuously maps $\Omega_r(R_h, Y)$ into $C(R_+, X)$ and there are nonnegative constants q and l, such that

$$\|(Fv)(t)\|_X \leqslant q \sup_{-h \leqslant s \leqslant t} \|v(s)\|_Y + l \quad (v \in \Omega_r(R_h, Y),\ t \geqslant 0). \tag{1.4}$$

2. The basic lemma

Everywhere below, it is assumed that

$$M_Y := \sup_{t \geqslant 0} \|U(t,0)\|_Y < \infty \quad \text{and} \quad z_Y := \sup_{t \geqslant 0} \int_0^t \|U(t,s)\|_{X \to Y}\,ds < \infty.$$

Put

$$c_0(\phi, l) := M_Y \|\phi(0)\|_Y + z_Y\big(q\|\phi\|_{C([-h,0],Y)} + l\big).$$

Lemma 2.1. *Let the conditions* (1.4),

$$q z_Y < 1 \tag{2.1}$$

and

$$c_0(\phi, l) < r(1 - q z_Y) \tag{2.2}$$

hold. Then a mild solution u of problem (1.1), (1.2) *satisfies the inequality*

$$\|u(t)\|_Y \leqslant c_0(\phi, l)(1 - q z_Y)^{-1} \quad (t > 0). \tag{2.3}$$

Proof. Thanks to (2.2), there is a positive T, such that $\|u(t)\|_Y < r$ for $t \leqslant T$. Hence, inequality (1.4) implies

$$\begin{aligned}\|u(t)\|_Y &\leqslant M_Y\|u(0)\|_Y + \int_0^t \|U(t,s)\|_{X\to Y}\Big(q \sup_{-h\leqslant s_1\leqslant s}\|u(s_1)\|_Y + l\Big)\,ds \\ &\leqslant M_Y\|u(0)\|_Y + \sup_{t\geqslant 0}\int_0^t \|U(t,s)\|_{X\to Y}\,ds\Big(q \sup_{-h\leqslant s_1\leqslant t}\|u(s_1)\|_Y + l\Big) \\ &\leqslant M_Y\|u(0)\|_Y + z_Y\Big(l + q\Big(\sup_{0\leqslant s\leqslant t}\|u(s_1)\|_Y + \sup_{-h\leqslant s\leqslant 0}\|\phi(s)\|_Y\Big)\Big) \\ &\leqslant c_0(\phi,l) + z_Y q \sup_{0\leqslant s\leqslant t}\|u(s)\|_Y \quad (t\leqslant T).\end{aligned}$$

Consequently,

$$\sup_{0\leqslant s\leqslant T}\|u(s)\|_Y \leqslant c_0(\phi,l) + z_Y q \sup_{0\leqslant s\leqslant T}\|u(s)\|_Y.$$

Hence, due to (2.1) and (2.2),

$$\sup_{0\leqslant s\leqslant T}\|u(s)\|_Y \leqslant c_0(\phi,l)(1-z_Y q)^{-1} < r.$$

So we can extend this inequality to all $t \geqslant 0$. As claimed. □

Now let $A(t)$ generate an evolution operator $U(t,s)$ in X and S be a constant boundedly invertible linear operator in X with a domain $D(S) \supseteq D_A$ and commuting with $A(t)$:

$$A(t)Sv = SA(t)v \quad (v \in D_A,\ t \geqslant 0).$$

On set $D(S)$, let us introduce the graph norm $\|v\|_S := \|Sv\|_X$ $(v \in D(S))$ and denote the obtained space by X_S. Take $Y = X_S$ and assume that

$$\|(Fv)(t)\|_X \leqslant q \sup_{-h\leqslant s\leqslant t}\|Sv(s)\|_X + l \quad \big(v \in \Omega_r(R_h, X_S);\ t \geqslant 0\big). \tag{2.4}$$

Since

$$SU(t,s)v = U(t,s)Sv \quad \big(v \in D(S);\ t,s \geqslant 0\big),$$

in the considered case $A(t)$ generates an evolution operator in X_S and $M_Y = M_X$, where

$$M_X := \sup_{t\geqslant 0}\|U(t,0)\|_X. \tag{2.5}$$

In addition, we have $z_Y = z_S$, where

$$z_S := \sup_{t\geqslant 0}\int_0^t \|SU(t,s)\|_X\,ds$$

provided z_S and M_X are finite. Set

$$c_S(\phi, l) := M_X \|S\phi(0)\|_X + z_S\big(q\|S\phi\|_{C([-h,0],X)} + l\big).$$

Now Lemma 2.1 yields

Corollary 2.2. *Let a linear operator S commute with $A(t)$ and $D(S) \supseteq D_A$. Let the conditions* (2.4), *$qz_S < 1$ and*

$$c_S(\phi, l) < r(1 - qz_S)$$

hold. Then a mild solution u of problem (1.1), (1.2) *satisfies the inequality*

$$\|Su(t)\|_X \leqslant c_S(\phi, l)(1 - qz_S)^{-1} \quad (t \geqslant 0).$$

3. Existence and uniqueness of solutions

Theorem 3.1. *Let the conditions* (2.1), (2.2),

$$\|(F0)(t)\|_X \leqslant l < \infty \tag{3.1}$$

and

$$\|(Fv_1)(t) - (Fv_2)(t)\|_X \leqslant q \sup_{-h \leqslant s \leqslant t} \|v_1(s) - v_2(s)\|_Y$$
$$\big(t \geqslant 0;\ v_1, v_2 \in \Omega_r(R_h, Y)\big) \tag{3.2}$$

hold. Then problem (1.1), (1.2) *has a unique mild solution. Moreover, that solution satisfies inequality* (2.3).

Proof. We have

$$\|(Fv)(t)\|_X \leqslant \|(F0)(t)\|_X + \|(Fv)(t) - (F0)(t)\|_X \leqslant l + q \sup_{-h \leqslant s \leqslant t} \|v(s)\|_Y$$
$$\big(v \in \Omega_r(R_h, Y)\big).$$

So condition (1.4) holds. Thanks to Lemma 2.1, inequality (2.3) is valid. For arbitrary $x, y \in \Omega_r(R_+, Y)$, define the functions $\tilde{y}, \tilde{x}$ on R_h by

$$\tilde{y}(t) = \tilde{x}(t) = \phi(t) \quad (-h \leqslant t < 0) \quad \text{and} \quad \tilde{x}(t) = x(t), \quad \tilde{y}(t) = y(t) \quad (t \geqslant 0).$$

In addition, define on $\Omega_r(R_+, Y)$ the mapping G_ϕ by

$$(G_\phi x)(t) = (F\tilde{x}_t)(t) \quad (t \geqslant 0).$$

Due to (3.2),

$$\|[G_\phi x](t) - [G_\phi y](t)\|_X = \|F(t, \tilde{x}_t) - F(t, \tilde{y}_t)\|_X \leqslant q \sup_{0 \leqslant s \leqslant t} \|x(s) - y(s)\|_Y.$$

Rewrite Eq. (1.3) under (1.2) as

$$u = \Phi(u), \tag{3.3}$$

where Φ is defined on $\Omega_r(R_+, Y)$ by

$$(\Phi x)(t) = U(t,0)\phi(0) + \int_0^t U(t,t_1)[G_\phi x](t_1)\,dt_1 \quad \bigl(x \in \Omega_r(R_+, Y)\bigr).$$

Due to (2.3), under (2.1), (2.2), Φ maps $\Omega_r(R_+, Y)$ into itself. Inequality (3.2) shows that

$$\bigl\|(\Phi x)(t) - (\Phi y)(t)\bigr\|_Y \leqslant z_Y q \sup_{0 \leqslant s \leqslant t} \bigl\|x(s) - y(s)\bigr\|_Y.$$

Now condition (2.2) and the contraction mapping theorem imply the required result. □

4. Equations with sectorial operators

Let A_0 be a constant linear sectorial operator in X, and $-A_0$ generate an asymptotically stable (analytic) semigroup e^{-tA_0}, cf. [11]. Let $B(t)$ $(t \geqslant 0)$ be a variable linear operator in space X with a constant domain D_B, generating in X an evolution operator $U_B(t,s)$, satisfying the inequality

$$\bigl\|U_B(t,s)\bigr\|_X \leqslant C_B \exp\bigl[\alpha(t-s)\bigr] \quad (t \geqslant s \geqslant 0) \tag{4.1}$$

with $C_B \equiv \text{const} > 0$ and $\alpha \equiv \text{const}$. In addition, $D_A \equiv D_B \cap D(A_0)$ is dense and $B(t)$ commutes with A_0:

$$A_0 B(t) v = B(t) A_0 v \quad \bigl(t \geqslant 0,\ v \in D\bigl(A_0 B(t)\bigr) = D\bigl(B(t)A_0\bigr)\bigr). \tag{4.2}$$

Put

$$A(t) = -A_0 + B(t). \tag{4.3}$$

As it is well known, cf. [11], for any $\nu \in [0,1)$ the power A_0^ν is defined. Introduce the space $Y = X^\nu$ with the graph norm

$$\|v\|_\nu := \bigl\|A_0^\nu v\bigr\|_X \quad \bigl(v \in D\bigl(A_0^\nu\bigr)\bigr).$$

Assume that

$$\bigl\|(Fv)(t)\bigr\|_X \leqslant q \sup_{-h \leqslant s \leqslant t} \bigl\|v(s)\bigr\|_\nu + l \quad \bigl(t \geqslant 0;\ v \in \Omega_r(R_h, X^\nu)\bigr). \tag{4.4}$$

Moreover, since A_0 is sectorial, there are positive constants C_δ, C_ν and $\delta \leqslant \beta(A_0)$, such that

$$\|e^{-A_0 t}\|_X \leqslant C_\delta e^{-\delta t} \quad \text{and} \quad \bigl\|A_0^\nu e^{-A_0 t}\bigr\|_X \leqslant \frac{C_\nu e^{-\delta t}}{t^\nu} \quad (t \geqslant 0)$$

provided $\beta(A_0) > 0$, cf. [11]. Due to (4.2) and (4.3), $A(t)$ generates an evolution operator in X defined by

$$U(t,s) = e^{-A_0(t-s)} U_B(t,s).$$

Moreover,

$$A_0^\nu U(t,s) v = U(t,s) A_0^\nu v \quad (v \in X^\nu,\ t,s \geqslant 0).$$

So due to (4.1),

$$\|U(t,s)\|_X \leqslant C_\delta C_B e^{-(\delta-\alpha)(t-s)}$$

and with $\delta > \alpha$ we have $M_X = C_B C_\delta$, and

$$z_Y = \sup_{t\geqslant 0} \int_0^t \|A_0^\nu U(t,s)\|_X \, ds \leqslant \sup_{t\geqslant 0} \int_0^t \|A_0^\nu e^{-A_0(t-s)}\|_X \|U_B(t,s)\|_X \, ds \leqslant z_\nu,$$

where

$$z_\nu := C_\delta C_B \int_0^\infty \frac{e^{-(\delta-\alpha)t}\, dt}{t^\nu}.$$

In addition, $c_Y(\phi, l) = c_\nu(\phi, l)$, where

$$c_\nu(\phi, l) := C_\delta C_B \|\phi(0)\|_\nu + z_\nu \bigl(q\|\phi\|_{C([-h,0],X^\nu)} + l\bigr).$$

Now Corollary 2.2 yields

Theorem 4.1. *Let A_0 be a linear sectorial operator in X and $-A_0$ generate an asymptotically stable semigroup. Let the conditions* (4.1)–(4.4),

$$q z_\nu < 1 \tag{4.5}$$

and

$$c_\nu(\phi, l) < r(1 - q z_\nu)$$

hold. Then a mild solution u of problem (1.1), (1.2) *satisfies the inequality*

$$\|u(t)\|_\nu \leqslant c_\nu(\phi, l)(1 - q z_\nu)^{-1} \quad (t \geqslant 0). \tag{4.6}$$

5. Equations with selfadjoint operators

In this section X is a Hilbert space. With applications in mind, let us consider the operator $A(t)$ defined by (4.3), where A_0 is a positive definite selfadjoint operator in X. Again $B(t)$ is a linear operator generating an evolution operator in space X. In addition, assume that the conditions (4.1)–(4.3) and

$$\beta(A_0) > 0 \quad \text{and} \quad -\beta(A_0) + \alpha < 0 \tag{5.1}$$

hold. Clearly,

$$\|A_0^\nu e^{-A_0 t}\|_X \leqslant \sup_{s\in\sigma(A)} \bigl\{s^\nu e^{-st}\colon\ s \geqslant \beta(A_0)\bigr\} = \psi(A_0, t) \quad (t \geqslant 0),$$

where

$$\psi(A_0, t) := \begin{cases} \frac{\nu^\nu}{t^\nu} e^{-\nu} & \text{if } t \leqslant \frac{\nu}{\beta(A_0)}, \\ \beta(A_0)^\nu e^{-\beta(A_0)t} & \text{if } t > \frac{\nu}{\beta(A_0)}. \end{cases} \tag{5.2}$$

Since A_0 and $B(t)$ commute, we have $U(t,s) = e^{-A_0(t-s)}U_B(t,s)$,

$$\|U(t,0)\|_X \leqslant C_B e^{-(\beta(A_0)-\alpha)t} \leqslant C_B \tag{5.3}$$

and

$$\|A_0^\nu U(t,s)\|_X \leqslant \|A_0^\nu e^{-A_0(t-s)}\|_X \|U_B(t,s)\|_X \leqslant C_B e^{\alpha(t-s)}\psi(A_0, t-s) \quad (t \geqslant s \geqslant 0).$$

Hence,

$$\int_0^t \|A_0^\nu U(t,s)\|_X \, ds \leqslant z(A_0, B) \quad (t \geqslant 0),$$

where

$$z(A_0, B) := C_B \int_0^\infty e^{-\alpha t}\psi(t, A_0)\, dt. \tag{5.4}$$

Clearly, this integral is simple calculated. As above, take the space X^ν with the graph norm $\|v\|_\nu := \|A_0^\nu v\|_X$ $(v \in D(A_0^\nu))$. Under (4.4) put

$$c(A_0, B, \phi, l) := C_B \|A_0^\nu \phi(0)\|_X + z(A_0, B)\big(q\|A_0^\nu \phi\|_{C([-h,0],X)} + l\big).$$

Now Corollary 2.2 implies

Theorem 5.1. *Let X be a Hilbert space and A_0 a selfadjoint operator in X. In addition, under conditions* (4.1)–(4.4) *and* (5.1)*, let*

$$qz(A_0, B) < 1 \tag{5.5}$$

and

$$c(A_0, B, \phi, l) < r\big(1 - qz(A_0, B)\big).$$

Then a mild solution u of problem (1.1), (1.2) *satisfies the inequality*

$$\|A_0^\nu u(t)\|_X \leqslant c(A_0, B, \phi, l)\big(1 - qz(A_0, B)\big)^{-1} \quad (t \geqslant 0).$$

In particular, let $L^2(\omega)$ be a separable Hilbert space of functions defined on a bounded closed set $\omega \subseteq \mathbf{R}^n$ with the scalar product

$$(f, g) = \int_\omega f(x)\bar{g}(x)\, dx.$$

Then the space $C(\omega)$ of continuous functions with the sup-norm $\|.\|_{C(\omega)}$ is continuously imbedded in $L^2(\omega)$. Assume now that operator $A(t)$ in $X = L^2(\omega)$ has the form (4.3) and with $m_1, m_2 \in [0, 1)$,

$$\|Fv\|_{L^2(\omega)} \leqslant q_1 \|A_0^{m_1} v\|_{C(\omega)} \quad \big(\|A_0^{m_1} v\|_{C(\omega)} \leqslant r_1,\ v \in D\big(A_0^{m_1}\big)\big) \tag{5.6}$$

and

$$\|v\|_{C(\omega)} \leqslant q_2 \| A_0^{m_2} v \|_{L^2(\omega)} \quad \big(v \in D\big(A_0^{m_2}\big)\big). \tag{5.7}$$

Then with $\nu = m_1 + m_2 < 1$, we have

$$\| A_0^{m_1} v \|_{C(\omega)} \leqslant q_2 \| A_0^{m_2} A_0^{m_1} v \|_{L^2(\omega)} = q_2 \| A_0^{\nu} v \|_{L^2(\omega)}. \tag{5.8}$$

Now (5.6) implies condition (4.5) with $r = r_1 q_2$ and $q = q_1 q_2$. Thus, we can apply Theorem 5.1. For instance, let A_0 be a positive definite selfadjoint operator in $L^2(\omega)$ with the discrete spectrum:

$$A_0 = \sum_{k=1}^{\infty} \lambda_k P_k,$$

where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ are the eigenvalues of A_0 with their multiplicities, $P = (.\,, e_k)e_k$, and e_k are the eigenvectors with $\|e_k\|_{L^2(\omega)} = 1$. Let

$$\sum_{k=1}^{\infty} \lambda_k^{-2m_2} < \infty$$

and

$$c_e := \sup_k \|e_k\|_{C(\omega)} < \infty.$$

Then

$$\|v\|_{C(\omega)} = \sup_x \left| \sum_{k=1}^{\infty} \lambda_k^{-m_2} \big(A_0^{m_2} v, e_k\big) e_k(x) \right| \leqslant c_e \left| \sum_{k=1}^{\infty} \lambda_k^{-m_2} \big(A_0^{m_2} v, e_k\big) \right|.$$

Hence, by the Schwarz inequality and Parseval equality, we have

$$\begin{aligned} \|v\|_{C(\omega)}^2 &= \sup_x \left| \sum_{k=1}^{\infty} \lambda_k^{-m_2} \big(A_0^{m_2} v, e_k\big) e_k(x) \right|^2 \\ &\leqslant c_e^2 \sum_{k=1}^{\infty} \lambda_k^{-2m_2} \sum_{k=1}^{\infty} \big| \big(A_0^{m_2} v, e_k\big) \big|^2 = c_e^2 \sum_{k=1}^{\infty} \lambda_k^{-2m_2} \| A_0^{m_2} v \|_{L^2(\omega)}^2. \end{aligned} \tag{5.9}$$

Consequently, condition (5.7) holds with

$$q_2 = c_e \left[\sum_{k=1}^{\infty} \lambda_k^{-2m_2} \right]^{1/2}. \tag{5.10}$$

6. Equations with spectral operators

Let A_0 be a spectral operator of the scalar type with a positive spectrum in a Banach space X, cf. [4]. That is, there exists a spectral measure E_s, such that

$$A_0 = \int_{\beta(A_0)}^{\infty} s \, dE_s \quad \big(\beta(A_0) > 0\big). \tag{6.1}$$

For instance, let $\{e_k\}$ be a Schauder basis in a Hilbert space with a scalar product $(.,.)$ and $\{d_k\}$ a basis biorthogonal to $\{e_k\}$. Put $Q = (.,d_k)e_k$ and consider the operator

$$A_0 = \sum_{k=1}^{\infty} \lambda_k Q_k \quad (\lambda_k > 0).$$

Then A_0 can be written as (6.1). Furthermore, let conditions (4.1)–(4.3) hold. Take into account that

$$A_0^{\nu} e^{-A_0 t} = \int_{\beta(A_0)}^{\infty} s^{\nu} \exp[-ts]\, dE_s$$

and

$$\sup_{s \in \sigma(A_0)} s^{\nu} \exp[-ts] \leqslant \psi(A_0, t),$$

where ψ is defined by (5.2). Due to formula (ii) from [4, p. 2189],

$$\|e^{-A_0 t}\|_X \leqslant \theta_E e^{-\beta(A)t} \quad \text{and} \quad \left\|A_0^{\nu} e^{-A_0 t}\right\|_X \leqslant \theta_E \psi(A_0, t),$$

where

$$\theta_E := 4 \sup_{\delta \in \Sigma(A_0)} E(\delta).$$

Here $\Sigma(A_0)$ is the sigma-algebra of the Borel subsets of $[0, \infty)$. Since A_0 and $B(t)$ commute, according to (4.1),

$$\left\|U(t,0)\right\|_X \leqslant \|e^{-A_0(t-s)}\|_X \left\|U_B(t,s)\right\|_X \leqslant \theta_E C_B e^{-(\beta(A_0)-\alpha)t} \leqslant \theta_E C_B \quad (t \geqslant 0)$$

and

$$\left\|A_0^{\nu} U(t,s)\right\|_X \leqslant \left\|A_0^{\nu} e^{-A_0(t-s)}\right\|_X \left\|U_B(t,s)\right\|_X \leqslant \theta_E C_B e^{\alpha(t-s)} \psi(A_0, t-s).$$

Hence,

$$\int_0^t \left\|A_0^{\nu} U(t,s)\right\| ds \leqslant \theta_E z(A_0, B) \quad (t \geqslant 0),$$

where $z(A_0, B)$ is defined by (5.4). Define the space X^{ν} as in the previous section. Put

$$c_E(\phi, l) := C_B \theta_E \left[\left\|A_0^{\nu} \phi(0)\right\|_X + z(A_0, B)\left(q \left\|A_0^{\nu} \phi\right\|_{C([-h,0],X)} + l\right)\right].$$

Now Corollary 2.2 implies

Theorem 6.1. *Let A_0 be a spectral operator of the scalar type. In addition, under conditions* (4.1)–(4.4) *and* (5.1)*, let $q\theta_E z_E(A_0, B) < 1$ and*

$$c_E(\phi, l) < r\left(1 - q\theta_E z(A_0, B)\right).$$

Then a mild solution u of problem (1.1), (1.2) *satisfies the inequality*

$$\left\|A_0^{\nu} u(t)\right\|_X \leqslant c_E(\phi, l)\left(1 - q\theta_E z(A_0, B)\right)^{-1} \quad (t \geqslant 0).$$

Remark 6.2. If X is a Hilbert space, then there is a selfadjoint invertible operator, such that $Q_s = TE_sT^{-1}$ is an orthogonal spectral measure, cf. [4, Lemma XV.6.1, p. 1945]. In this case one can take

$$\theta_E := \|T\|_X\|T^{-1}\|_X.$$

7. Lyapunov stability

Definition 7.1. The zero solution to Eq. (1.1) is said to be stable in space Y in the Lyapunov sense, if for any $\epsilon > 0$, there is a $\delta > 0$, such that the inequality

$$\|\phi\|_{C([-h,0],Y)} \leqslant \delta$$

implies $\|u\|_{C(R_+,Y)} \leqslant \epsilon$ for any mild solution u of problem (1.1), (1.2).

Theorem 7.2. *Let conditions* (1.4) *and* (2.1) *hold with* $l = 0$. *Then the zero solution to Eq.* (1.1) *is stable in* Y *in the Lyapunov sense.*

Indeed, this result immediately follows from Lemma 2.1 when $l = 0$.

Now let A_0 be a constant linear sectorial operator in X. Again take space $Y = X^\nu$ with norm $\|v\|_\nu = \|A_0^\nu v\|_X$ ($v \in D(A_0^\nu)$) and let $B(t)$ be a linear operator in X with a constant domain D_B, generating in X an evolution operator $U_B(t,s)$. Theorem 4.1 yields

Corollary 7.3. *Let* A_0 *be a linear sectorial operator in* X *and* $-A_0$ *generate an asymptotically stable semigroup. Let conditions* (4.1)–(4.5) *hold with* $l = 0$. *Then the zero solution to Eq.* (1.1) *is stable in* X^ν *in the Lyapunov sense.*

In addition, Theorem 5.1 implies

Corollary 7.4. *Let* X *be a Hilbert space and* A_0 *a positive definite selfadjoint operator. In addition, let conditions* (4.1)–(4.4) *with* $l = 0$, (5.1) *and* (5.5) *hold. Then the zero solution to Eq.* (1.1) *is stable in* X^ν *in the Lyapunov sense.*

Note that Theorems 4.1 and 5.1 give us a possibility to estimate the domain of attraction of the zero solution.

To consider the stability of equations with spectral operators one can apply Theorem 6.1.

8. Absolute stability

Assume that F continuously maps $C(R_h, Y)$ into $C(R_+, X)$ and there is a constant $q > 0$, such that

$$\|(Fv)(t)\|_X \leqslant q \sup_{-h\leqslant s\leqslant t}\|v(s)\|_Y \quad \big(v \in C(R_h, Y),\ t \geqslant 0\big). \tag{8.1}$$

Definition 8.1. The zero solution of Eq. (1.1) is said to be absolutely stable in Y in the class of nonlinearities (8.1), if under (8.1), there exists a positive constant c_0 independent of the specific form of function F (but dependent on q), such that the inequality

$$\|u\|_{C(R_+,Y)} \leqslant c_0\|\phi\|_{C([-h,0],Y)}$$

holds for any mild solution u of problem (1.1), (1.2).

Theorem 8.2. *Let condition* (2.1) *hold. Then the zero solution to Eq.* (1.1) *is absolutely stable in Y in the class of nonlinearities* (8.1).

Indeed, this result follows from Lemma 2.1 when $l = 0$.

Now let A_0 be a constant linear sectorial operator in X and $B(t)$ be a linear operator in X with a constant domain D_B, generating in X an evolution operator $U_B(t,s)$. Again take space $Y = X^\nu$ and assume that

$$\big\|(Fv)(t)\big\|_X \leqslant q \sup_{-h\leqslant s\leqslant t}\big\|v(s)\big\|_\nu \quad \big(v \in C(R_h, X^\nu),\ t \geqslant 0\big). \tag{8.2}$$

Theorem 4.1 yields

Corollary 8.3. *Let A_0 be a linear sectorial operator in X and $-A_0$ generate an asymptotically stable semigroup. In addition, let conditions* (4.1)–(4.3) *and* (4.5) *hold. Then the zero solution to Eq.* (1.1) *is absolutely stable in X^ν in the class of nonlinearities* (8.2).

Recall that $\psi(A_0,t)$ and $z(A_0,B)$ are defined by (5.2) and (5.4), respectively. Theorem 5.1 implies

Corollary 8.4. *Let X be a Hilbert space and A_0 a positive definite selfadjoint operator. In addition, let conditions* (4.1)–(4.3), (5.1) *and* (5.5) *hold. Then the zero solution to Eq.* (1.1) *is absolutely stable in X^ν in the class of nonlinearities* (8.2).

Note that Theorem 6.1 allows us to consider the absolute stability of equations with spectral operators.

9. Input–output stability

Let us consider the equation

$$\dot u(t) = A(t)u(t) + [Fu](t) + \psi(t) \quad (t > 0), \tag{9.1}$$

where $\psi \in C(R_+, Y)$ is a given function (input).

Definition 9.1. We will say that Eq. (9.1) is input–output stable in space Y, if for any $\epsilon > 0$, there is a $\delta > 0$, such that the inequality

$$\|\psi\|_{C(R_+,Y)} \leqslant \delta$$

implies $\|u\|_{C(R_+,Y)} \leqslant \epsilon$ for any solution u of (9.1) under the zero initial condition $u(t) = 0$ $(t \leqslant 0)$.

Theorem 9.2. *Let conditions* (2.1) *and* (8.1) *hold. Then Eq.* (9.1) *is input–output stable in space Y.*

This result is due to Lemma 2.1.

Again let A_0 be a constant linear sectorial operator in X and $B(t)$ a linear operator in X with a constant domain D_B, generating in X an evolution operator. Theorem 4.1 yields

Corollary 9.3. *Let A_0 be a linear sectorial operator in X and $-A_0$ generate an asymptotically stable semigroup. Let conditions* (4.1)–(4.3), (4.5) *and* (8.2) *hold. Then Eq.* (9.1) *is input–output stable in space X^ν.*

Moreover, Theorem 5.1 implies

Corollary 9.4. *Let X be a Hilbert space and A_0 a a selfadjoint operator. In addition, let conditions* (4.1)–(4.3), (5.1), (5.5) *and* (8.2) *hold. Then Eq.* (9.1) *is input–output stable in space X^ν.*

Note that Theorem 6.1 allows us to consider the input–output stability of equations with spectral operators.

10. Absolute stability of parabolic equations with delay

Consider the problem

$$\frac{\partial u(t,x)}{\partial t} = a(t)\frac{\partial^2 u(t,x)}{\partial x^2} - c(t)u(t) + F_1\big(u_x(t-h,x)\big)$$
$$(-\pi < x < \pi,\ t > 0) \tag{10.1}$$

with the periodic boundary conditions

$$u(t,-\pi) = u(t,\pi), \quad u_x(t,-\pi) = u_x(t,\pi) \quad (t > 0). \tag{10.2}$$

Here F_1 continuously maps $\mathbf{R}$ into itself with the property

$$\big|F_1(y)\big| \leqslant q_1|y| \quad (y \in \mathbf{R}), \tag{10.3}$$

and $a(t), c(t)$ are positive scalar functions. Assume that

$$\inf_{t\geqslant 0} a(t) = 1. \tag{10.4}$$

In addition, let

$$\inf_{t\geqslant 0} c(t) > 1/2. \tag{10.5}$$

Take $X = L^2[-\pi,\pi]$ and $\omega = [-\pi,\pi]$. Problem (10.1), (10.2) can be written as (1.1) with

$$A(t)v \equiv a(t)\frac{d^2 v}{dx^2} - c(t)v(t), \quad v \in D_A,$$

where

$$D_A = \left\{ v \in L^2 = L^2[-\pi,\pi]:\ \frac{d^2 v}{dx^2} \in L^2;\ v(-\pi) = v(\pi),\ v'(-\pi) = v'(\pi) \right\}.$$

Put $b(t) = a(t) - 1$,

$$A_0 v(x) = -\frac{d^2 v(x)}{dx^2} + v(x)/2 \quad (v \in D_A)$$

and

$$B(t)v(x) = b(t)\frac{d^2 v(x)}{dx^2} - \big(c(t) - 1/2\big)v(x) \quad (v \in D_B)$$

with $D_B = D_A$. Then the eigenvalues and normed eigenfunctions of A_0 are

$$\lambda_k(A_0) = k^2 + \frac{1}{2} \quad \text{and} \quad e_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \quad (k = 0, \pm 1, \pm 2, \ldots),$$

respectively. So $\beta(A_0) = 1/2$. Moreover, for any

$$v = \sum_{k=-\infty}^{\infty} c_k e_k \in D_A,$$

where c_k are the Fourier coefficients of v, we can write out

$$A_0 v = \sum_{k=-\infty}^{\infty} (k^2 + 1/2) c_k e_k.$$

Define A_0^ν by

$$A_0^\nu v = \sum_{k=-\infty}^{\infty} (k^2 + 1/2)^\nu c_k e_k \quad \big(v \in D(A^\nu)\big)$$

with

$$D(A^\nu) := \left\{ v \in L^2[-\pi,\pi]:\ \sum_{k=-\infty}^{\infty} (k^2 + 1/2)^{2\nu} |c_k|^2 < \infty \right\}.$$

But for any $v \in D_A$,

$$\begin{aligned}(v_x, v_x) &= -(v_{xx}, v) = \big((A_0 - 1/2)v, v\big) = (A_0 v, v) - 1/2(v, v) \\ &\leqslant \big\| A_0^{1/2} v \big\|^2_{L^2(\omega)}. \end{aligned} \tag{10.6}$$

Here $(.\,,.)$ is the scalar product. So due to (10.3) we have

$$\big\| F_1\big(v_x(t-h,.)\big) \big\|_{L^2(\omega)} \leqslant q \big\| v_x(t-h,.) \big\|_{L^2(\omega)} \leqslant q \big\| A_0^{1/2} v(t-h,.) \big\|_{L^2(\omega)} \quad \big(v \in D\big(A_0^{1/2}\big)\big). \tag{10.7}$$

In addition, since $b(t) \geqslant 0$, $c(t) \geqslant 1/2$, and $B(t)$ is selfadjoint, by virtue of simple calculations we get

$$\big(B(t)v, v\big) \leqslant 0 \quad (v \in D_A,\ t \geqslant 0)$$

and therefore

$$\left\|U_B(t,s)\right\|_{L^2(\omega)} \leqslant 1 \quad (t \geqslant s \geqslant 0).$$

Thus, conditions (4.1) hold with $C_B = 1$ and $\alpha = 0$. According to (5.2), we can write out

$$\psi(A_0,t) \equiv \psi_{1/2}(t) := \begin{cases} \frac{1}{\sqrt{2te}} & \text{if } t \leqslant 1, \\ \sqrt{1/2}e^{-t/2} & \text{if } t > 1. \end{cases} \tag{10.8}$$

Thus

$$z(A_0,B) = z_{1/2} = \int_0^\infty \psi_{1/2}(t)\,dt.$$

Due to Corollary 8.4, *problem* (10.1), (10.2) *is absolutely stable in* $X^{1/2}$ *in the class of nonlinearities* (10.3), *provided* $qz_{1/2} < 1$.

11. Lyapunov stability of parabolic equations with delay

Again consider problem (10.1), (10.2) assuming now that instead of (10.3), the condition

$$\left|F_1(y)\right| \leqslant q_1|y| \quad \left(y \in \mathbf{R}\colon\ |y| \leqslant r_1\right) \tag{11.1}$$

holds with a finite $r_1 > 0$. Again take $X = L^2[-\pi,\pi]$ and $\omega = [-\pi,\pi]$. Define A_0 and $B(t)$ as in the previous section. Put

$$\|v\|_{C(\omega)} = \sup_{-\pi \leqslant x \leqslant \pi} \left|v(x)\right|.$$

According to (5.9),

$$\|v\|_{C(\omega)} \leqslant q_2 \left\|A_0^{1/3} v\right\|_{L^2(\omega)}^2 \tag{11.2}$$

with

$$q_2 = \frac{1}{\sqrt{2\pi}} \left[\sum_{k=-\infty}^{\infty} \left(k^2 + \frac{1}{2}\right)^{-2/3}\right]^{1/2},$$

since $\|e_k\|_{C(\omega)} = 1/\sqrt{2\pi}$. Then (10.6) implies

$$\|u_x\|_{C(\omega)} \leqslant q_2 \left\|A_0^{1/3} u_x\right\|_{L^2(\omega)} \leqslant q_2 \left\|A_0^{1/3} A_0^{1/2} u\right\|_{L^2(\omega)} = q_2 \left\|A_0^{5/6} u\right\|_{L^2(\omega)} \quad \left(u \in D\left(A_0^{5/6}\right)\right).$$

Thus (11.1) yields

$$\left\|F_1\left(u_x(t-h,.)\right)\right\|_{L^2(\omega)} \leqslant q_1 q_2 \left\|A_0^{5/6} u(t-h,.)\right\|_{L^2(\omega)},$$

provided

$$\left\|A_0^{5/6} u(t-h,.)\right\|_{L^2(\omega)} \leqslant r \equiv q_2 r_1. \tag{11.3}$$

According to (5.2),

$$\psi(A_0,t)=\psi_{5/6}(t):=\begin{cases}\frac{(5/6)^{5/6}}{t^{5/6}}e^{-5/6} & \text{if } t\leqslant 5/3,\\ (1/2)^{-5/6}e^{-t/2} & \text{if } t>5/3,\end{cases}$$

since $\beta(A_0)=1/2$. Thus,

$$z(A_0,B)=z_{5/6}:=\int_0^\infty \psi_{5/6}(t)\,dt.$$

Due to Corollary 7.4, problem (10.1), (10.2), under (11.1) is stable in the Lyapunov sense in space $X^{5/6}$, provided

$$q_1q_2z_{5/6}<1.$$

Note that Theorem 5.1 gives us a possibility to estimate the region of attraction of a stationary solution.

12. Integro-differential equations with delay

In this section we take $\omega=[-\pi,\pi]\times[0,1]$ and space $X=L^2(\omega)$. Consider the equation

$$\frac{\partial u(t,x,y)}{\partial t}=u_{xx}(t,x,y)-u(t,x,y)+\int_0^y Q(y,y_1)u(t,x,y_1)\,dy_1 + F_1\bigl(u_x(t-h,x,y)\bigr)+\psi(t,x,y)$$
$$\bigl(-\pi<x<\pi,\ y\in[0,1],\ t>0\bigr) \tag{12.1}$$

with a given scalar function $\psi(.,.,.)$ defined on $R_+\times[-\pi,\pi]\times[0,1]$, and the boundary conditions

$$u(t,-\pi,y)=u(t,\pi,y),\quad u_x(t,-\pi,y)=u_x(t,\pi,y)\quad \bigl(y\in[0,1],\ t>0\bigr). \tag{12.2}$$

Here Q is a scalar Hilbert–Schmidt kernel defined on $0\leqslant y_1\leqslant y\leqslant 1$ and $F_1:\mathbf{R}\to\mathbf{R}$ is a continuous function, satisfying condition (10.3).

Equations of the type (12.1) arise in various applications, cf. [15]. Problem (12.1), (12.2) can be written as (9.1) with

$$A(t)v(x,y)\equiv v_{xx}(x,y)-v(x,y)+\int_0^y Q(y,y_1)v(x,y_1)\,dy_1\quad (v\in D_A),$$

where

$$D_A=\left\{v\in L^2(\omega):\ \frac{\partial^2 v}{\partial x^2}\in L^2(\omega);\ v(-\pi,y)=v(\pi,y),\ v_x(-\pi,y)=v_x(\pi,y);\ y\in[0,1]\right\}.$$

In addition, $[Fv](t) = F_1(v_x(t-h,.,.))$. Put

$$A_0 v(x,y) = -\frac{\partial^2 v(x,y)}{\partial x^2} + v(x,y)/2 \quad (v \in D_A)$$

and

$$B(t)v(x,y) = B_0 v(x,y) - v(x,y)/2,$$

where B_0 is defined by

$$(B_0 v)(x,y) := \int_0^y Q(y,y_1)v(x,y_1)\,dy_1 \quad \bigl(v \in D_B \equiv L^2(\omega)\bigr).$$

Then the eigenvalues of A_0 (with infinite dimensional subspaces) are

$$\lambda_k(A_0) = k^2 + 1/2 \quad (k = 0, \pm 1, \pm 2, \ldots).$$

So $\beta(A_0) = 1/2$. Again put

$$e_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Any function $e_k(x)f(y)$ with $f \in L^2[0,1]$ is an eigenfunction for A_0. Let d_j $(j = 1, 2, \ldots)$ be a normal orthogonal basis in $L^2[0,1]$. Then for any

$$v = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} d_j e_k \in D_A,$$

where c_{jk} are the Fourier coefficients of v, we can write out

$$A_0 v = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} (k^2 + 1/2) c_{jk} d_j e_k.$$

So we can define A_0^ν as

$$A_0^\nu v = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} (k^2 + 1/2)^\nu c_{jk} d_j e_k \quad \bigl(v \in D(A^\nu)\bigr)$$

with

$$D(A^\nu) := \left\{ v \in L^2(\omega)\colon \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} (k^2 + 1/2)^{2\nu} |c_{jk}|^2 < \infty \right\}.$$

Furthermore, let $(.,.)$ be the scalar product in $L^2([-\pi,\pi] \times [0,1])$. Then (10.6) holds. So due to (10.3) we have inequality (10.7). Simple calculations show that

$$\bigl\| U_B(t,s) \bigr\|_{L^2(\omega)} \leqslant e^{-(t-s)/2} \| e^{B_0(t-s)} \|_{L^2(\omega)}.$$

Moreover, B_0 is a quasinilpotent Hilbert–Schmidt operator. So due to Theorem 6.9.1 from [7]

$$\| e^{B_0 t} \|_{L^2(\omega)} \leqslant \sum_{k=0}^{\infty} \frac{N^k(B_0) t^k}{(k!)^{3/2}} \quad (t \geqslant 0),$$

where

$$N(B_0) := \left[\operatorname{Trace} B_0 B_0^*\right]^{1/2}$$

is the Hilbert–Schmidt norm. The asterisk means the adjointness. Hence, for any $\epsilon \in (0, 1/2)$, we can easily calculate the constant M_ϵ, such that

$$\left\| U_B(t,s) \right\|_{L^2(\omega)} \leqslant M_\epsilon e^{(\epsilon - 1/2)(t-s)} \quad (t \geqslant s \geqslant 0).$$

Furthermore, according to (5.2), $\psi(A_0, t) = \psi_{1/2}(t)$, where $\psi_{1/2}(t)$ is defined by (10.8). Thus

$$z(A_0, B) \leqslant z_{1/2,\epsilon} := M_\epsilon \int_0^\infty e^{(\epsilon - 1/2)t} \psi_{1/2}(t)\, dt.$$

Due to Corollary 9.4 *problem* (12.1), (12.2) *is input–output stable* in $X^{1/2}$, *provided*

$$q z_{1/2,\epsilon} < 1 \quad \text{for some } \epsilon \in (0, 1/2).$$

References

[1] T.A. Burton, B. Zhang, A priory bounds and stability in the retarded nonlinear wave equation, J. Math. Anal. Appl. 166 (1992) 456–485.
[2] C. Corduneanu, Integral Equations and Applications, Cambridge Univ. Press, Cambridge, 1991.
[3] J.M. Cushing, Integrodifferential Equations and Delay Models in Population Dynamics, Lecture Notes in Biomathematics, vol. 20, Springer, Berlin, 1977.
[4] N. Dunford, J.T. Schwartz, Linear Operators III. Spectral Operators, Wiley–Interscience, New York, 1971.
[5] E. Făsangová, J. Prüss, Asymptotic behavior of a semilinear viscoelastic beam model, Arch. Math. 77 (2001) 488–497.
[6] G. Friecskue, Convergence to equilibrium for delay–diffusion equations with small delay, J. Dynam. Differential Equations 5 (1993) 89–103.
[7] M.I. Gil', Operator Functions and Localization of Spectra, Lecture Notes in Math., vol. 1830, Springer, Berlin, 2003.
[8] M.I. Gil', On global stability of parabolic systems with delay, Appl. Anal. 69 (1998) 57–71.
[9] M.I. Gil', Solution estimates for abstract nonlinear time-variant differential delay equations, J. Math. Anal. Appl. 270 (2002) 51–65.
[10] G. Gripenberg, S.-O. Londen, O. Staffans, Volterra Integral and Functional Equations, Cambridge Univ. Press, Cambridge, 1990.
[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lectures Notes in Math., vol. 840, Springer, New York, 1981.
[12] Y. Kuang, H.L. Smith, Global stability in diffusive delay Lotka–Volterra systems, Differential Integral Equations 4 (1991) 117–128.
[13] S.M. Lenhart, C.C. Travis, Global stability of a biological model with time delay, Proc. Amer. Math. Soc. 96 (1986) 75–78.
[14] S. Murakami, Stable equilibrium points of some diffusive functional differential equations, Nonlinear Anal. 25 (1995) 1037–1043.
[15] C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum, New York, 1992.
[16] C.V. Pao, Systems of parabolic equations with continuous and discrete time delays, J. Math. Anal. Appl. 205 (1997) 157–185.
[17] C.V. Pao, Parabolic systems in unbounded domains II, Equations with time delays, J. Math. Anal. Appl. 225 (1998) 557–586.
[18] J. Prüss, Evolutionary Integral Equations and Applications, Birkhäuser, Basel, 1993.

[19] W.M. Ruess, Existence of solutions of partial functional-differential equations with delay, in: Theory and Applications Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, 1996, pp. 259–288.
[20] H. Tanabe, Functional Analytic Methods for Partial Differential Equations, Dekker, New York, 1997.
[21] H. Tanabe, Fundamental solutions for linear retarded functional differential equations in Banach space, Funkcial. Ekvac. 35 (1992) 149–177.
[22] M. Vidyasagar, Nonlinear Systems Analysis, second ed., Prentice Hall, Englewood Cliffs, NJ, 1993.
[23] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer, New York, 1996.