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The Classification of Linearly Polarized Transverse Electric Waves

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Recently the author has undertaken the classification of continuous solutions to some common vector PDEs. These include the simplest of Beltrami solutions to hydrodynamic flows and electromagnetic wave equations (which are, in fact, closely related). In this paper, we consider linearly polarized transverse electric wave solutions to the electromagnetic wave equation: $\nabla \times \mathbf{E} = i\omega\mathbf{H}$; $\nabla \times \mathbf{H} = -i\omega\mathbf{E}$. Using Clebsch functions and differential geometric techniques the author is able to give a discussion of the most general possible forms of propagation of linearly polarized waves. A zero curvature condition (which asserts that the Clebsch functions necessary to represent the solutions are expressible in terms of Cartesian variables) reduces the study of vector PDEs to nonlinear PDEs, which may be solved in special cases. The benefit is that the Clebsch representation is the one that is the most useful for analyzing the structure of the flow. This method of analysis is now a feasible solution method due to the tremendous advances in mathematical software allowing one to compute curvatures, given metric functions. For the special case of linearly polarized transverse electric waves the analysis may be applied to a great extent. © 1997 Academic Press

1. INTRODUCTION

Recently Baldwin [1, 2] has undertaken the classification of certain solutions to standard partial differential equations of mathematical physics. These include special cases of solutions to the Beltrami equation and linearly polarized transverse electromagnetic (TEM) solutions to the electromagnetic wave equations.

In this work we consider the more difficult problem of attempting to write down general solutions to the wave equation. This becomes much easier when one assumes a direction of propagation for the wave; this is a standard procedure in electromagnetism texts. In this article we presume

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to determine what possible propagation directions for transverse electric (TE) waves are allowed. The method that is employed is representing the fields in terms of Clebsch functions and using a Lamé analysis to solve for the Clebsch functions.

An arbitrary solution to the wave equation may be expressed by Eq. (2.25), where the functions P, s, u satisfy Eq. (2.27) and are such that the Ricci curvature computed from the inverse metric tensor (where α, t, v are the coordinates) given by Eq. (2.29) vanishes. This curvature sort of analysis owes its heritage to Lamé [5]. The novel feature in the analysis is the use of the Clebsch potentials to describe the vector fields.

In particular, we consider linearly polarized transverse electromagnetic wave solutions. We find the following results: we are able to reproduce familiar solutions to the electromagnetic wave equations, including spherical waves and waves propagating down circular and rectangular waveguides. The author is only able to construct solutions where the propagation direction is in the radial spherical direction or in a Cartesian direction. This is reminiscent of the linearly polarized TEM case.[2]

2. SOLUTION TO THE WAVE EQUATION

In order to study electromagnetic waves propagating in a certain direction, we should first ask how we should define a "wave to be propagating in a certain direction?" Clearly this needs some interpretation. We feel that the following is the most natural interpretation. We need to have the direction of propagation be described by the gradient of a real coordinate function (α), so that we may write

$$\begin{aligned} \mathbf{E} &= e^{i\alpha}\mathbf{E}_0, \\ \mathbf{H} &= e^{i\alpha}\mathbf{H}_0, \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} &\boldsymbol{\nabla}\alpha \times \mathbf{E}_0 \qquad \text{real}, \\ &\boldsymbol{\nabla}\alpha \times \mathbf{H}_0 \qquad \text{real}. \end{aligned} \tag{2.2}$$

That is, the components of the fields restricted to the planes of constant phase are all in phase with one another. This is consistent at least with Rund.[3]

Moreover, the $\mathbf{E}$ and $\mathbf{H}$ fields must satisfy the electromagnetic wave equations, which in appropriate variables, read

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E} &= i\mathbf{H}, \\ \boldsymbol{\nabla} \times \mathbf{H} &= -i\mathbf{E}. \end{aligned} \tag{2.3}$$

Now substituting (2.1) into (2.3), we have

$$\begin{aligned} i\nabla\alpha \times \mathbf{E}_0 + \nabla \times \mathbf{E}_0 &= i\mathbf{H}_0, \\ i\nabla\alpha \times \mathbf{H}_0 + \nabla \times \mathbf{H}_0 &= -i\mathbf{E}_0. \end{aligned} \tag{2.4}$$

Next we can take the dot product of each equation with $\nabla\alpha$ and note that the LHS must be real (due to (2.2)). At once it follows that $\nabla\alpha \cdot \mathbf{E}_0$ and $\nabla\alpha \cdot \mathbf{H}_0$ must both be complex. Therefore we can write

$$\begin{aligned} \mathbf{E}_0 &= iQ\nabla\alpha + \mathbf{e}_0, \\ \mathbf{H}_0 &= iW\nabla\alpha + \mathbf{h}_0, \end{aligned} \tag{2.5}$$

where $\mathbf{e}_0, \mathbf{h}_0, Q, W$ are all real and $\mathbf{e}_0 \cdot \nabla\alpha = 0 = \mathbf{h}_0 \cdot \nabla\alpha$.

Now we substitute (2.5) into (2.4) and equate the real and imaginary parts to find

$$\nabla\alpha \times \mathbf{e}_0 + \nabla Q \times \nabla\alpha = \mathbf{h}_0, \tag{2.6}$$

$$\nabla \times \mathbf{e}_0 = -W\nabla\alpha, \tag{2.7}$$

$$\nabla\alpha \times \mathbf{h}_0 + \nabla W \times \nabla\alpha = -\mathbf{e}_0, \tag{2.8}$$

$$\nabla \times \mathbf{h}_0 = Q\nabla\alpha, \tag{2.9}$$

where we claimed $\mathbf{e}_0$ and $\mathbf{h}_0$ should have no component along $\nabla\alpha$. Notice then from (2.7) and (2.9) that $\mathbf{e}_0 \cdot (\nabla \times \mathbf{e}_0) = 0 = \mathbf{h}_0 \cdot (\nabla \times \mathbf{h}_0)$. Therefore we may represent $\mathbf{e}_0$ and $\mathbf{h}_0$ with just two functions.[4]

In summary, we have derived the system

$$\begin{aligned} e^{-i\alpha}\mathbf{E} = \mathbf{E}_0 &= iQ\nabla\alpha + s\nabla t, \\ e^{-i\alpha}\mathbf{H} = \mathbf{H}_0 &= iW\nabla\alpha + u\nabla v, \end{aligned} \tag{2.10}$$

with

$$s\nabla\alpha \times \nabla t - \nabla\alpha \times \nabla Q = u\nabla v, \tag{2.11}$$

$$\nabla s \times \nabla t = -W\nabla\alpha, \tag{2.12}$$

$$u\nabla\alpha \times \nabla v - \nabla\alpha \times \nabla W = -s\nabla t, \tag{2.13}$$

$$\nabla u \times \nabla v = Q\nabla\alpha. \tag{2.14}$$

These completely code the wave equation and condition (2.2).

The next objective of the Clebsch method is to encode the vector PDEs appearing in (2.11)–(2.14) as relations among the components of a metric tensor. Using α, t and v as coordinates (1, 2, and 3, respectively), we take the inner product of each of these equations with the gradients of each of the coordinate functions. Defining

$$D \equiv \nabla\alpha \cdot \nabla t \times \nabla v, \tag{2.15}$$

there follow that

$$Q_v D = uh^{23} \tag{2.16}$$

$$(s - Q_t) D = uh^{33} \tag{2.17}$$

$$s_v D = Wh^{11} \tag{2.18}$$

$$s_\alpha D = -Wh^{13} \tag{2.19}$$

$$(u - W_v) D = sh^{22} \tag{2.20}$$

$$W_t D = sh^{23} \tag{2.21}$$

$$u_\alpha D = -Qh^{12} \tag{2.22}$$

$$u_t D = Qh^{11}, \tag{2.23}$$

as well as

$$h^{12} = h^{13} = 0. \tag{2.24}$$

Recasting (2.16)–(2.24) in terms of relations of the inverse metric tensor, we find that we may write for (2.10)

$$\begin{aligned} e^{-i\alpha}\mathbf{E} = \mathbf{E}_0 &= iP(\alpha, t, v) u_t \nabla\alpha + s(t, v)\nabla t, \\ e^{-i\alpha}\mathbf{H} = \mathbf{H}_0 &= iP(\alpha, t, v) s_v \nabla\alpha + u(t, v)\nabla v, \end{aligned} \tag{2.25}$$

where $P \equiv Q/u_t = W/s_v$ and the last equality follows from (2.18) and (2.23). Also,

$$h^{ij} = \begin{bmatrix} D/P & 0 & 0 \\ 0 & \dfrac{u - (Ps_v)_v}{s} D & \dfrac{(Ps_v)_t}{s} D \\ 0 & \dfrac{(Pu_t)_v}{u} D & \dfrac{s - (Pu_t)_t}{u} D \end{bmatrix} \tag{2.26}$$

and

$$\times \frac{(Pu_t)_v}{u} = \frac{(Ps_v)_t}{s}. \tag{2.27}$$

Considerable simplifications take place in the case of TE or TM (transverse magnetic) waves, where $u(v, t)_t$ or $s(v, t)_v$ will be zero. Notice that generically for linearly polarized electromagnetic waves the component of the E field orthogonal to the direction of propagation is not necessarily orthogonal to the H field ($\nabla t \cdot \nabla v \equiv h^{23} \neq 0$).

As is well known, the determinant of the matrix given by the contravariant metric tensor must be equal to D^2, where D is given by (2.15). This implies

$$D^{-1} = (Pus)^{-1}\big(us - s(Ps_v)_v - u(Pu_t)_t + (Ps_v)_v(Pu_t)_t - (Ps_v)_t(Pu_t)_v\big). \qquad (2.28)$$

Thus the contravariant metric for the linearly polarized electromagnetic waves may be expressed by

$$h^{ij} = \big(us - s(Ps_v)_v - u(Pu_t)_t + (Ps_v)_v(Pu_t)_t - (Ps_v)_t(Pu_t)_v\big)^{-1} \times \begin{bmatrix} us & 0 & 0 \\ 0 & (u - (Ps_v)_v)uP & (Ps_v)_t uP \\ 0 & (Pu_t)_v sP & (s - (Pu_t)_t)sP \end{bmatrix}, \qquad (2.29)$$

where $P = P(\alpha, t, v)$, $u = u(t, v)$, $s = s(t, v)$, the relation (2.27) holds, and the above metric yields zero curvature. The fields are then given by (2.25). We next restrict the set of solutions to (2.25)–(2.29) to the case of TE waves.

3. SOLUTIONS TO THE PLANE-POLARIZED TE PROBLEM

The vanishing of the function Q (equivalently u_t) is precisely the case of TE waves (see (2.10); $W = 0$ for TM waves). Then (2.10) and (2.16)–(2.24) simplify to

$$e^{-i\alpha}\mathbf{E} = \mathbf{E}_0 = s(t, v)\nabla t, \qquad (3.1)$$

$$e^{-i\alpha}\mathbf{H} = \mathbf{H}_0 = iW(\alpha, v)\nabla\alpha + \nabla v, \qquad (3.2)$$

with

$$(1 - W_v)D = sh^{22} \qquad (3.3)$$

$$s_v D = Wh^{11} \qquad (3.4)$$

$$sD = uh^{33}, \qquad (3.5)$$

and

$$h^{23} = h^{12} = h^{13} = 0. \qquad (3.6)$$

Equations (3.3)–(3.6) correspond in turn to (2.16), (2.19), (2.20), and (2.24). Since $u = u(v)$ from (2.22) and (2.23), then u in turn may be taken to be simply unity, as may be seen from (2.25) and (3.2).

Therefore the contravariant metric tensor reads

$$h^{ij} = \begin{bmatrix} s_v D/W & 0 & 0 \\ 0 & (1 - W_v)D/s & 0 \\ 0 & 0 & sD \end{bmatrix}_{ij}. \tag{3.7}$$

However, the determinant of this matrix must itself be D^2 (see (2.15)). Therefore, we have

$$D = W/\left(s_v(1 - W_v)\right) \tag{3.8}$$

and the contravariant metric tensor must read

$$h^{ij} = \begin{bmatrix} 1/(1 - W_v) & 0 & 0 \\ 0 & W/(ss_v) & 0 \\ 0 & 0 & sW/((1 - W_v)s_v) \end{bmatrix}_{ij}, \tag{3.9}$$

where $W = W(\alpha, v)$, $s = s(t, v)$, and α, t, v are the coordinates. The **E** and **H** fields of linearly polarized TE waves are given from the above functions, as in (3.1) and (3.2). It remains to calculate the curvature of this matrix and to set it equal to zero.

The 12 and 23 components of the Ricci tensor formed from h^{ij} are particularly simple. They become

$$\begin{aligned} R_{12} &= -\frac{1}{4} W_{\alpha v} \frac{s_t s_v - s s_{tv}}{s s_v (W_v - 1)}, \\ R_{23} &= \frac{1}{4} W_{vv} \frac{s_t s_v - s s_{tv}}{s s_v (W_v - 1)}. \end{aligned} \tag{3.10}$$

Case 1. If we assume that $s(t, v)$ may not be decomposed into the product of a function of t times a function of v then it must necessarily follow from (3.10) that $\partial W(\alpha, v)/\partial v = a$ constant, so

$$W(\alpha, v) = W0(\alpha) + Lv. \tag{3.11}$$

Case 2. Otherwise $s(t, v)$ may be taken to be simply a function of $v : s(t, v) = s(v)$. This may be shown in the following fashion: one possible solution to Eqs. (3.10) implies that $s(t, v) = s(v)f(t)$. By examining (3.1), however, we see that $f(t)$ may be absorbed in such a way that $f(t) = 1$, as has been done above. We discuss Case 2 at greater length below.

3.1. Case 1

If $s(t,v)$ may not be decomposed as a product of a function of t and a function of v, then it necessarily follows that $W = W0(\alpha) + Lv$, in order that the 12 and 23 components of the Ricci tensor vanish. The 13 component of the Ricci tensor now reads

$$R_{13} = \frac{1}{2} \frac{W0_{\alpha} L}{(W0(\alpha) + Lv)^2}. \tag{3.12}$$

Thus Case 1 decomposes into two cases, each of which ensures the vanishing of the 13 component of the curvature tensor.

Case 1.1 $(L = 0)$. We shall see in a moment that this is sometimes related to the case of spherical coordinates. The situation $L = 0$ implies

$$W = W0(\alpha). \tag{3.13}$$

Under this assumption the 11 component of the Ricci tensor reads

$$R_{11} = -\frac{1}{2}\left(\frac{2W0_{\alpha\alpha} - 3(W0_{\alpha})^2/W0}{W0}\right). \tag{3.14}$$

Setting the R_{11} component of the curvature tensor to zero implies that

$$W = 1/(k_1 + k_2\alpha)^2. \tag{3.15}$$

The 22 and 33 components of the Ricci tensor are then proportional (they will vanish together):

$$R_{33} = R_{22}/s^2 \tag{3.16}$$

and

$$R_{22} = \frac{1}{2}\left(\frac{\begin{array}{r} 2s^3 s_v^2 s_{vv} + s^4 s_v s_{vvv} + 2s_v^2 s_t^2 - s s_v s_t s_{tv} - s s_v^2 s_{tt} \\ + s^2 s_{ttv} s_v + 2k_2^2 s^3 s_v^3 - s^4 s_{vv}^2 - s^2 s_{tv}^2 \end{array}}{s^2 s_v^2}\right). \tag{3.17}$$

If $s = s(v)$, see Case 2.1.1.

Case 1.2. In this case, $W0$ is a constant which may be taken to be zero, due to a shift in v. That is, $W = Lv$. Again R_{22} and R_{33} will vanish together:

$$R_{33} = (1 - L)R_{22}/s^2 \tag{3.18}$$

and

$$R_{22} = \frac{s}{2(1-L)}\left\{\left[s(s_v)^2 - v(s_v)^3 + 2v^2 s_{vv}(s_v)^2 - v^2 s(s_{vv})^2\right]/v^2 - \frac{1-L}{s^3}\left[s^2(s_{tv})^2 - s_v s^2 s_{ttv} + s s_t s_v s_{tv} - 2(s_t)^2(s_v)^2 + s(s_v)^2 s_{tt}\right]\right\}. \tag{3.19}$$

Any solution $s(t,v)$ to (3.19) then yields a set of electromagnetic fields,

$$e^{-i\alpha}\mathbf{E} = \mathbf{E}_0 = s(t,v)\nabla t, \tag{3.20}$$

$$e^{-i\alpha}\mathbf{H} = \mathbf{H}_0 = iLv\nabla\alpha + \nabla v, \tag{3.21}$$

that represent linearly polarized electromagnetic waves travelling in the direction given by α. Moreover, the intrinsic coordinates t, v, α are expressible in terms of the usual Cartesian variables. One of the standard cases that belongs to Case 1.2 is the propagation of TEM waves in a rectangular waveguide.

One may well ask what is the advantage of the method we develop here in discussing TE waves, as opposed to a standard discussion say as given in Landau and Lifschitz for example. In the latter, a direction of propagation is assumed. Here no such assumption is made, we attempt to derive all the possible forms of the propagation directions.

3.2. Case 2

If $s = s(v)$ the 12 and 23 components of the Ricci tensor are zero; however, the rest of the elements of the Ricci curvature tensor are still quite complicated. The author has been unable to find general solutions from the partial differential equations obtained by setting to zero the 13 component of the Ricci curvature tensor. For completeness we name those solutions that do not fall into the later classification as exotic solutions.

Case 2.0 (Exotic solutions). We list the 11 and 13 components of the curvature tensor below for which the zero sets should be the simplest to solve:

$$R_{11} = \frac{-1}{2\left(s_v R^2\left(1-R_v^2\right)\right.}\Big((1-R_v)^2 s_v\left(2RR_{\alpha\alpha} - 3R_\alpha^2\right) + (1-R_v)R^2(s_v RR_{vv} + sRR_{vvv} + s_v R_{v\alpha\alpha}) + s_v(R_{\alpha v})^2 R^2 + sR^3(R_{vv})^2\Big), \tag{3.22}$$

$$R_{13} = \frac{1}{4}\,\frac{ss_v\left(2R_\alpha R_v - 2R_\alpha R_v^2 + R_{\alpha v}R(R_v - 2) - RR_\alpha R_{vv}\right)}{R^2 ss_v(1-R_v)}. \tag{3.23}$$

Case 2.1. (Special cases). Let us assume that the criteria of Case 1 is also satisfied: $W(\alpha, v) = W0(\alpha) + Lv$ (see (3.11)). As before, the analyses of Case 1 must hold true. Therefore, it must follow that either $W(\alpha, v) = W0(\alpha)$ (Case 2.1.1) or $W(\alpha, v) = Lv$ (Case 2.1.2).

Case 2.1.1 (See also Case 1.1). A further case may be reduced to spherical coordinates. Under the assumptions of Case 2.1 with $W(\alpha, v) = W0(\alpha)$ we see, by comparing to Case 1.1 (Eq. (3.15)), that the vanishing of the 13 component of the curvature tensor must yield

$$W = \frac{1}{\alpha^2} \tag{3.24}$$

(up to a shift and rescaling). Thus,

$$h_{11} = 1, \tag{3.25}$$

$$h_{22} = \alpha^2 s(v) \frac{ds}{dv}, \tag{3.26}$$

$$h_{33} = g_{22}/(s^2). \tag{3.27}$$

Setting the R_{22} component of the curvature to zero implies

$$2(s_v)^2 s_{vv} + s s_v s_{vvv} + 2 s_v^3 - s(s_{vv})^2. \tag{3.28}$$

A particular solution is

$$s(v) = \sqrt{1 - e^{-2v}}. \tag{3.29}$$

Then substituting

$$t = \phi, \tag{3.30}$$

$$v = -\ln \sin \theta, \tag{3.31}$$

$$\alpha = r \tag{3.32}$$

implies

$$e^{-i\alpha}\mathbf{E} = \mathbf{E}_0 = \frac{\cos\theta}{r \sin\theta}\hat{\phi}, \tag{3.33}$$

$$e^{-i\alpha}\mathbf{H} = \mathbf{H}_0 = i\frac{\hat{r}}{r^2} - \frac{\cos\theta}{r\sin\theta}\hat{\theta}. \tag{3.34}$$

Case 2.1.2. Now we take

$$W \equiv Lv; \qquad s(t, v) = s(v). \tag{3.35}$$

All the components of the Ricci curvature tensor are found to vanish except for the 22 and 33 components. These will vanish when

$$s(s')^2 - v(s')^3 + 2v^2 s''(s')^2 + v^2 s s' s''' - v^2 s(s'')^2 = 0 \tag{3.36}$$

(compare to (3.17); Eq. (3.36) may be obtained from (3.17) by setting $s_t = 0$, etc., where $'$ denotes a derivative with respect to v. This may be seen to be related to a Bessel equation by a change of variables from v to ρ by means of the transformation

$$\frac{d\rho}{\rho} = \frac{dv}{s(v)}. \tag{3.37}$$

This leads to the covariant metric tensor, where α, t, and ρ are the variables, whose nonvanishing components are

$$g_{11} = 1 - L, \tag{3.38}$$

$$g_{22} = \frac{\rho(\dot{v} + \rho\ddot{v})}{Lv}, \tag{3.39}$$

$$g_{33} = (L - 1)g_{22}/\rho^2. \tag{3.40}$$

The dots refer to derivatives with respect to ρ. It is now clear, that upon substituting

$$v(\rho) = J_0(\rho), \tag{3.41}$$

we find

$$g_{11} = 1 - L, \tag{3.42}$$

$$g_{22} = \rho^2/L, \tag{3.43}$$

$$g_{33} = (1 - L)/L, \tag{3.44}$$

with all other components of the covariant metric tensor vanishing. This is immediately recognized as the metric of cylindrical coordinates. We then may identify

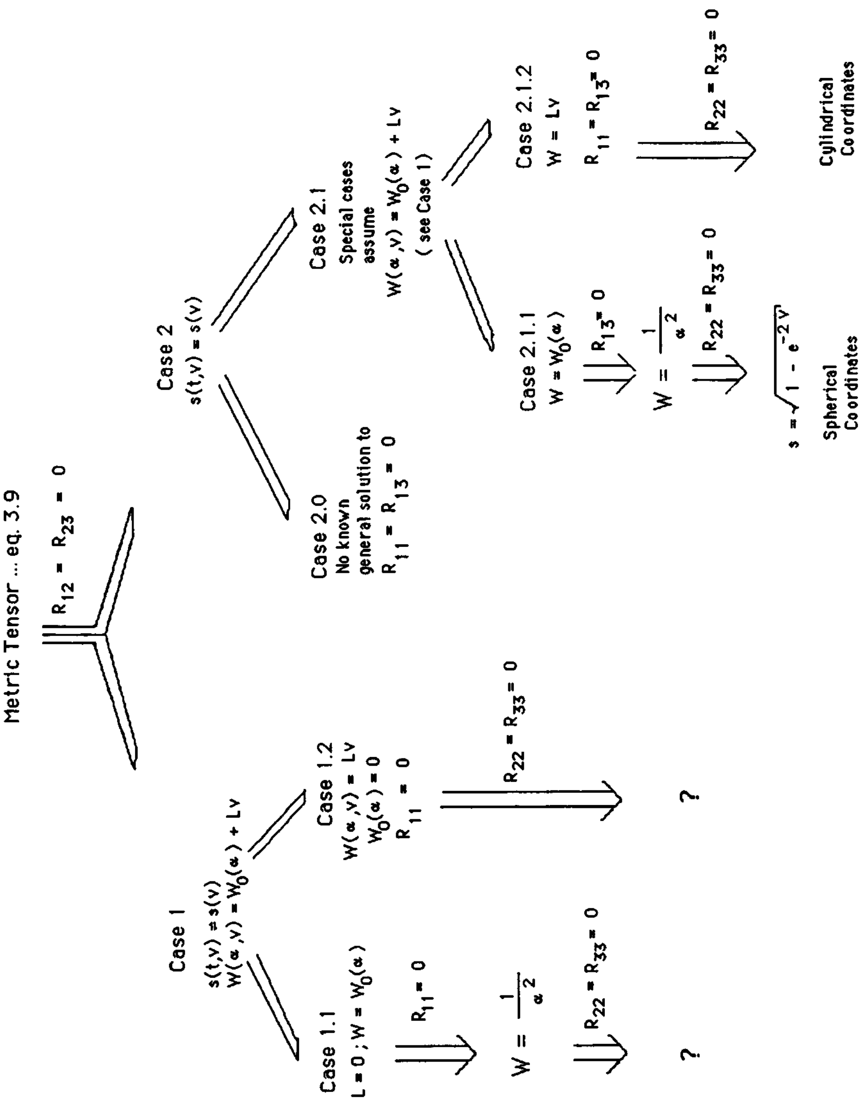
$$\alpha = z/\sqrt{1 - L}, \tag{3.45}$$

$$t = \sqrt{L}\,\theta, \tag{3.46}$$

$$\rho = \sqrt{L/(1 - L)}\,r, \tag{3.47}$$

where r, θ, z are the usual cylindrical coordinates.

FIG. 1. A schematic of the solutions to the linearly polarized TE waves via the Clebsch method. The **E** and **H** fields are given by Eqs. (3.1) and (3.2). The contravariant metric tensor for the coordinates α, t, and v is given by Eq. (3.9). The partial differential equations that are solved in this work, are those resulting from setting the Ricci curvature computed from the metric tensor to zero. Transformations between α, t, and v and standard orthogonal coordinate systems are then sought.



Metric Tensor ... eq. 3.9
R12 = R23 = 0
Case 1
s(t,v) = s(v)
W(α,v) = W0(α) + Lv
Case 1.1
L = 0 ; W = W0(α)
R11 = 0
W = 1/α²
R22 = R33 = 0
?
Case 1.2
W(α,v) = Lv
W0(α) = 0
R11 = 0
R22 = R33 = 0
?
Case 2
s(t,v) = s(v)
Case 2.0
No known
general solution to
R11 = R13 = 0
Case 2.1
Special cases
assume
W(α,v) = W0(α) + Lv
(see Case 1)
Case 2.1.1
W = W0(α)
R13 = 0
W = 1/α²
R22 = R33 = 0
s = √(1 - e^-2v)
Spherical
Coordinates
Case 2.1.2
W = Lv
R11 = R13 = 0
R22 = R33 = 0
Cylindrical
Coordinates

Combining everything, we find that

$$\mathbf{E}_0 = \sqrt{L}\,\frac{dJ_0\left(r\sqrt{L/(1-L)}\,\right)}{dr}\,\hat{\theta}, \tag{3.48}$$

$$\mathbf{H}_0 = i\frac{L}{\sqrt{1-L}}J_0\left(r\sqrt{L/(1-L)}\,\right)\hat{z} + \frac{dJ_0\left(r\sqrt{L/(1-L)}\,\right)}{dr}\hat{r}. \tag{3.49}$$

To make this solution look like a standard solution for TE waves travelling down a cylindrical waveguide as in Landau and Lifschitz, one may substitute $L = k_z^2/\kappa^2$.

Although there are ‘‘exotic solutions’’ (Case 2.0), all the solutions which the author has constructed belong to either Case 1 or Case 2.1.

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