Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of MATHEMATICAL

www.elsevier.com/locate/jmaa

On the nonlinear wave equation $u_{tt} - B(t, \|u\|^2, \|u_x\|^2)u_{xx} = f(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2)$ associated with the mixed homogeneous conditions

Nguyen Thanh Long

Department of Mathematics and Computer Science, University of Natural Science, Viet Nam National University, 227 Nguyen Van Cu Str., Dist. 5, HoChiMinh City, Viet Nam

Received 18 January 2004

Available online 29 January 2005

Submitted by R.E. Showalter

Abstract

In this paper we consider the following nonlinear wave equation:

(1) $u_{tt} - B(t, \|u\|^2, \|u_x\|^2)u_{xx} = f(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2)$, $x \in (0, 1)$, $0 < t < T$,
(2) $u_x(0, t) - h_0 u(0, t) = u_x(1, t) + h_1 u(1, t) = 0$,
(3) $u(x, 0) = \tilde{u}_0(x)$, $u_t(x, 0) = \tilde{u}_1(x)$,

where $h_0 > 0$, $h_1 \geqslant 0$ are given constants and B, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions. In Eq. (1), the nonlinear terms $B(t, \|u\|^2, \|u_x\|^2)$, $f(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2)$ depend on the integrals $\|u\|^2 = \int_\Omega |u(x, t)|^2\,dx$ and $\|u_x\|^2 = \int_0^1 |u_x(x, t)|^2\,dx$. In this paper I associate with problem (1)–(3) a linear recursive scheme for which the existence of a local and unique solution is proved by using standard compactness argument. In case of $B \in C^{N+1}(\mathbb{R}_+^3)$, $B \geqslant b_0 > 0$, $B_1 \in C^N(\mathbb{R}_+^3)$, $B_1 \geqslant 0$, $f \in C^{N+1}([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^2)$ and $f_1 \in C^N([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^2)$ we obtain for the following equation $u_{tt} - [B(t, \|u\|^2, \|u_x\|^2) + \varepsilon B_1(t, \|u\|^2, \|u_x\|^2)]u_{xx} = f(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2) + \varepsilon f_1(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2)$ associated to (2), (3) a weak solution $u_\varepsilon(x, t)$ having an asymptotic expansion of order $N + 1$ in ε, for ε sufficiently small.

E-mail address: longnt@hcmc.netnam.vn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.12.053

Keywords: Kirchhoff–Carrier operator; Galerkin method; Linear recurrent sequence; Asymptotic expansion of order $N+1$

1. Introduction

In this paper I consider a nonlinear wave equation with the Kirchhoff–Carrier operator

$$u_{tt} - B\big(t, \|u\|^2, \|\nabla u\|^2\big)\Delta u = f\big(x, t, u, u_x, u_t, \|u\|^2, \|\nabla u\|^2\big),$$
$$x \in \Omega = (0,1),\ 0 < t < T, \tag{1.1}$$

$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x), \qquad u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where B, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions satisfying conditions specified later and $h_0 > 0$, $h_1 \geqslant 0$ are given constants. In Eq. (1.1), the nonlinear terms $f(x, t, u, u_x, u_t, \|u\|^2, \|\nabla u\|^2)$ and $B(t, \|u\|^2, \|\nabla u\|^2)$ depend on the integrals

$$\|u\|^2 = \int_\Omega \big|u(x,t)\big|^2\,dx \quad \text{and} \quad \|\nabla u\|^2 = \int_\Omega \big|\nabla u(x,t)\big|^2\,dx. \tag{1.4}$$

Equation (1.1) has its origin in the nonlinear vibration of an elastic string [7], for which the associated equation is

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right) u_{xx}, \tag{1.5}$$

here u is the lateral deflection, ρ is the mass density, h is the cross section, L is the length, E is Young's modulus and P_0 is the initial axial tension.

In [2], Carrier also established a model of the type

$$u_{tt} = \left(P_0 + P_1 \int_0^L u^2(y,t)\,dy\right) u_{xx}, \tag{1.6}$$

where P_0 and P_1 are constants.

When $f = 0$ and $B = B(\|\nabla u\|^2)$ is a function depending only on $\|\nabla u\|^2$, the Cauchy or mixed problem for (1.1) has been studied by many authors; see [5,20] and the references cited therein. A survey of the results about the mathematical aspects of Kirchhoff model can be found in [17,18].

In [16] Medeiros has studied the problem (1.1)–(1.3) with $f = f(u) = -bu^2$, where b is a given positive constant, and Ω is a bounded open set of $\mathbb{R}^3$. In [6] Hosoya and Yamada have considered (1.1)–(1.3) with $f = f(u) = -\delta|u|^\alpha u$, where $\delta > 0$, $\alpha \geqslant 0$ are given constants.

In [9,11] the authors have studied the existence and uniqueness of the equation

$$u_{tt} + \lambda\Delta^2 u - B\big(\|\nabla u\|^2\big)\Delta u + \varepsilon|u_t|^{\alpha-1}u_t = F(x,t), \quad x \in \Omega,\ t > 0, \tag{1.7}$$

where $\lambda > 0$, $\varepsilon > 0$, $0 < \alpha < 1$, are given constants, and Ω is a bounded open set of $\mathbb{R}^n$.

In [3], Alain has studied the existence and asymptotic behavior as $\varepsilon \to 0$ of a weak solution of problem (1.1), (1.3) with $B \equiv 1$ associated with the Dirichlet homogeneous boundary condition

$$u(0,t) = u(1,t) = 0, \tag{1.8}$$

where the nonlinear term has the form $f = \varepsilon f_1(t,u)$. By a generalization of [3], Alain and Long [4] have considered problem (1.1), (1.3), (1.8) with $B \equiv 1$ and the nonlinear term having the form

$$f = \varepsilon f_1(t,u,u_t). \tag{1.9}$$

If $B_\varepsilon \equiv 1$ and $f_1 \in C^N(\mathbb{R}_+ \times \mathbb{R}^2)$ satisfies $f_1(t,0,0) = 0$ for all $t \geqslant 0$, an asymptotic expansion of the solution of problem (1.1), (1.3), (1.8), (1.9) up to order $N+1$ in ε is obtained, for ε sufficiently small. This expansion extends to the partial differential equation the results obtained in differential equations [1].

In [10] Long and Diem have studied the linear recursive schemes and asymptotic expansion associated with the nonlinear wave equation

$$u_{tt} - u_{xx} = f(x,t,u,u_x,u_t) + \varepsilon f_1(x,t,u,u_x,u_t), \tag{1.10}$$

associated with (1.2) and (1.3). In the case of $f \in C^2([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$ and $f_1 \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, we have obtained an asymptotic expansion of order 2 in ε, for ε sufficiently small. Afterwards, this result has been extended in [12] to the nonlinear wave equation with the Kirchhoff operator

$$\begin{aligned} &u_{tt} - \big[b_0 + B\big(\|u_x\|^2\big) + \varepsilon B_1\big(\|u_x\|^2\big)\big]u_{xx} \\ &\quad = f(x,t,u,u_x,u_t) + \varepsilon f_1(x,t,u,u_x,u_t), \end{aligned} \tag{1.11}$$

associated with (1.3), (1.8), where $b_0 > 0$ is a given constant and $B \in C^2(\mathbb{R}_+)$, $B_1 \in C^1(\mathbb{R}_+)$, $B \geqslant 0$, $B_1 \geqslant 0$ are given functions.

In this paper I shall first associate with the problem (1.1)–(1.3) a linear recurrent sequence which is bounded in a suitable space of functions. The existence of a local solution is proved by a standard compactness argument. Note that the linearization method in this paper and in the papers [4,10,12,13,19] cannot be used in the papers [5,6,9,11,16]. If $B \in C^{N+1}(\mathbb{R}_+^3)$, $B_1 \in C^N(\mathbb{R}_+^3)$, $B \geqslant b_0 > 0$, $B_1 \geqslant 0$, $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^2)$, and $f_1 \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^2)$, then an asymptotic expansion of order $N+1$ in ε is obtained with a right-hand side of the form $f(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2) + \varepsilon f_1(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2)$ and B stands for $B + \varepsilon B_1$, for ε sufficiently small. This result is a relative generalization of [4,10,12–14,19].

2. Preliminary results, notations

We will omit the definitions of the usual function spaces and denote them by the notation $L^p = L^p(0,1)$, $H^m = H^m(0,1)$.

The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2, or a pair of dual scalar products of continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leqslant p \leqslant \infty$, the Banach space of real functions $u:(0,T)\to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\,dt\right)^{1/p} < +\infty \quad \text{for } 1 \leqslant p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

Let $u(t)$, $u_t(t) = \dot u(t)$, $u_{tt}(t) = \ddot u(t)$, $u_x(t) = \nabla u(t)$, $u_{xx}(t) = \Delta u(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $f = f(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2) = f(x,t,u,v,w,Y,Z)$, we put $D_3 f = \frac{\partial f}{\partial u}$, $D_4 f = \frac{\partial f}{\partial v}$, $D_5 f = \frac{\partial f}{\partial w}$, $D_6 f = \frac{\partial f}{\partial Y}$, $D_7 f = \frac{\partial f}{\partial Z}$. We put

$$a(u,v) = \int_0^1 u_x(x) v_x(x)\,dx + h_0 u(0) v(0) + h_1 u(1) v(1). \tag{2.1}$$

In H^1 we shall use the equivalent norm

$$\|v\|_{H^1} = \left(v^2(0) + \int_0^1 |v'(x)|^2\,dx\right)^{1/2}. \tag{2.2}$$

Then we have the following lemmas.

Lemma 1. *The imbedding $H^1 \hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])} \leqslant \sqrt{2}\|v\|_{H^1} \quad \textit{for all } v \in H^1. \tag{2.3}$$

Lemma 2. *Let $h_0 > 0$ and $h_1 \geqslant 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.1) *is continuous on $H^1 \times H^1$ and coercive on H^1, i.e.,*

(i) $|a(u,v)| \leqslant C_1 \|u\|_{H^1}\|v\|_{H^1}$ *for all $u, v \in H^1$,*
(ii) $a(v,v) \geqslant C_0\|v\|_{H^1}^2$ *for all $v \in H^1$, where $C_0 = \min\{1, h_0\}$, $C_1 = \max\{1, h_0, 2h_1\}$.*

The proofs of these lemmas are straightforward, and we omit the details.

Lemma 3. *There exists the Hilbert orthonormal base $\{\tilde w_j\}$ of L^2 consisting of the eigenfunctions $\tilde w_j$ corresponding to the eigenvalue λ_j such that*

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots, \qquad \lim_{j\to+\infty} \lambda_j = +\infty, \tag{2.4}$$

$$a(\tilde w_j, v) = \lambda_j \langle \tilde w_j, v\rangle \quad \textit{for all } v \in H^1,\ j = 1,2,\ldots. \tag{2.5}$$

Furthermore, the sequence $\{\tilde{w}_j/\sqrt{\lambda_j}\}$ *is also the Hilbert orthonormal base of* H^1 *with respect to the scalar product* $a(\cdot,\cdot)$. *On the other hand, we have also* $\tilde{w}_j$ *satisfying the boundary value problem*

$$\begin{cases} -\Delta\tilde{w}_j = \lambda_j\tilde{w}_j \quad \text{in } (0,1), \\ \tilde{w}_{jx}(0) - h_0\tilde{w}_j(0) = \tilde{w}_{jx}(1) + h_1\tilde{w}_j(1) = 0, \\ \tilde{w}_j \in C^\infty([0,1]). \end{cases} \tag{2.6}$$

The proof of Lemma 3 can be found in [21, Theorem 6.2.1, p. 137], with $H = L^2$, $V = H^1$ and $a(\cdot,\cdot)$ as defined by (2.1).

3. The existence and uniqueness theorem

We make the following assumptions:

(A_1) $h_0 > 0$, $h_1 \geqslant 0$;
(A_2) $\tilde{u}_0 \in H^2$, $\tilde{u}_1 \in H^1$;
(A_3) $B \in C^1(\mathbb{R}_+^3)$, $B(t,Y,Z) \geqslant b_0 > 0$;
(A_4) $f \in C^0([0,1]\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+^2)$ satisfies the conditions

$$D_i f \in C^0\big([0,1]\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+^2\big), \quad i \in \{1,3,4,5,6,7\}.$$

(It is not necessary that $f \in C^1([0,1]\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+^2)$.)

With B and f satisfying assumptions (A_3) and (A_4), respectively, we introduce the following constants, for all $M > 0$ and $T > 0$:

$$\begin{aligned} K_0 &= K_0(M,T,f) \\ &= \sup\big\{\big|f(x,t,u,v,w,Y,Z)\big|:\ (x,t,u,v,w,Y,Z) \in A_*(M,T)\big\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} K_1 = K_1(M,T,f) = \sup\Bigg\{&\Bigg(|D_1 f| + \sum_{i=3}^{7}|D_i f|\Bigg)(x,t,u,v,w,Y,Z): \\ &(x,t,u,v,w,Y,Z) \in A_*(M,T)\Bigg\}, \end{aligned} \tag{3.2}$$

$$\tilde{K}_0 = \tilde{K}_0(M,T,B) = \sup\big\{B(t,Y,Z):\ (t,Y,Z) \in \tilde{A}_*(M,T)\big\}, \tag{3.3}$$

$$\begin{aligned} \tilde{K}_1 &= \tilde{K}_1(M,T,B) \\ &= \sup\left\{\left(\left|\frac{\partial B}{\partial t}\right| + \left|\frac{\partial B}{\partial Y}\right| + \left|\frac{\partial B}{\partial Z}\right|\right)(t,Y,Z):\ (t,Y,Z) \in \tilde{A}_*(M,T)\right\}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} A_*(M,T) = \big\{(x,t,u,v,w,Y,Z) \in \mathbb{R}^7:\ &0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant 1, \\ &|u| + |v| + |w| \leqslant M, \\ &0 \leqslant Y \leqslant M^2,\ 0 \leqslant Z \leqslant M^2\big\}, \end{aligned}$$

and

$$\tilde{A}_*(M,T)=\{(t,Y,Z)\in\mathbb{R}^3\colon\ 0\leqslant t\leqslant T,\ 0\leqslant Y\leqslant M^2,\ 0\leqslant Z\leqslant M^2\}.$$

For each $M>0$ and $T>0$, we put

$$W(M,T)=\{v\in L^\infty(0,T;H^2)\colon\ v_t\in L^\infty(0,T;H^1),\ v_{tt}\in L^2(Q_T),\ \|v\|_{L^\infty(0,T;H^2)},\ \|v_t\|_{L^\infty(0,T;H^1)},\ \|v_{tt}\|_{L^2(Q_T)}\leqslant M\},\tag{3.5}$$

$$W_1(M,T)=\{v\in W(M,T)\colon\ v_{tt}\in L^\infty(0,T;L^2)\},\tag{3.6}$$

where $Q_T=\Omega\times(0,T)$. We shall choose the first term $u_0=\tilde{u}_0$. Suppose that

$$u_{m-1}\in W_1(M,T).\tag{3.7}$$

We associate with the problem (1.1)–(1.3) the following variational problem.

Find $u_m\in W_1(M,T)$ which satisfies the linear variational problem

$$\langle\ddot{u}_m(t),v\rangle+b_m(t)a(u_m(t),v)=\langle F_m(t),v\rangle\quad\text{for all } v\in H^1,\tag{3.8}$$

$$u_m(0)=\tilde{u}_0,\qquad \dot{u}_m(0)=\tilde{u}_1,\tag{3.9}$$

where

$$\begin{cases} b_m(t)=B(t,\|u_{m-1}(t)\|^2,\|\nabla u_{m-1}(t)\|^2),\\ F_m(x,t)=f(x,t,u_{m-1}(t),\ \nabla u_{m-1}(t),\ \dot{u}_{m-1}(t),\\ \qquad\qquad \|u_{m-1}(t)\|^2,\ \|\nabla u_{m-1}(t)\|^2).\end{cases}\tag{3.10}$$

Then, we have the following theorem.

Theorem 1. *Let* (A_1)–(A_4) *hold. Then there exist positive constants M, T and the linear recurrent sequence* $\{u_m\}\subset W_1(M,T)$ *defined by* (3.8)–(3.10).

Proof. The proof consists of several steps.

Step 1. *The Galerkin approximation* (introduced by Lions [15]). Consider the basis for H^1 as in Lemma 3 ($w_j=\tilde{w}_j/\sqrt{\lambda_j}$). Put

$$u_m^{(k)}(t)=\sum_{j=1}^{k}c_{mj}^{(k)}(t)w_j,\tag{3.11}$$

where $c_{mj}^{(k)}$ satisfy the system of linear differential equations

$$\langle\ddot{u}_m^{(k)}(t),w_j\rangle+b_m(t)a(u_m^{(k)}(t),w_j)=\langle F_m(t),w_j\rangle,\quad 1\leqslant j\leqslant k,\tag{3.12}$$

$$u_m^{(k)}(0)=\tilde{u}_{0k},\qquad \dot{u}_m^{(k)}(0)=\tilde{u}_{1k},\tag{3.13}$$

where

$$\tilde{u}_{0k}\to\tilde{u}_0\quad\text{strongly in } H^2,\tag{3.14}$$

$$\tilde{u}_{1k}\to\tilde{u}_1\quad\text{strongly in } H^1.\tag{3.15}$$

Let us suppose that u_{m-1} satisfies (3.7). Then it is clear that system (3.12), (3.13) has a unique solution $u_m^{(k)}(t)$ on an interval $0 \leqslant t \leqslant T_m^{(k)} \leqslant T$. The following estimates allow one to take constant $T_m^{(k)} = T$ for all m and k.

Step 2. *A priori estimates*. Put

$$S_m^{(k)}(t) = X_m^{(k)}(t) + Y_m^{(k)}(t) + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds, \tag{3.16}$$

where

$$X_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|^2 + b_m(t) a\left(u_m^{(k)}(t), u_m^{(k)}(t)\right), \tag{3.17}$$

$$Y_m^{(k)}(t) = a\left(\dot{u}_m^{(k)}(t), \dot{u}_m^{(k)}(t)\right) + b_m(t) \left\| \Delta u_m^{(k)}(t) \right\|^2. \tag{3.18}$$

Then, it follows from (3.12), (3.13), and (3.16)–(3.18) that

$$\begin{aligned} S_m^{(k)}(t) &= X_m^{(k)}(0) + Y_m^{(k)}(0) + \int_0^t b_m'(s)\left(a\left(u_m^{(k)}(s), u_m^{(k)}(s)\right) + \left\| \Delta u_m^{(k)}(s) \right\|^2\right) ds \\ &\quad + 2\int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s) \right\rangle ds + 2\int_0^t a\left(F_m(s), \dot{u}_m^{(k)}(s)\right) ds + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds \\ &= X_m^{(k)}(0) + Y_m^{(k)}(0) + I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.19}$$

We shall estimate respectively the following integrals on the right-hand side of (3.19).

First integral. We have

$$\begin{aligned} b_m'(t) &= \frac{\partial B}{\partial t}\left(t, \left\| u_{m-1}(t) \right\|^2, \left\| \nabla u_{m-1}(t) \right\|^2\right) \\ &\quad + 2\frac{\partial B}{\partial Y}\left(t, \left\| u_{m-1}(t) \right\|^2, \left\| \nabla u_{m-1}(t) \right\|^2\right)\left\langle u_{m-1}(t), \dot{u}_{m-1}(t) \right\rangle \\ &\quad + 2\frac{\partial B}{\partial Z}\left(t, \left\| u_{m-1}(t) \right\|^2, \left\| \nabla u_{m-1}(t) \right\|^2\right)\left\langle \nabla u_{m-1}(t), \nabla \dot{u}_{m-1}(t) \right\rangle. \end{aligned} \tag{3.20}$$

By using the assumption (A_3), we obtain from (3.4) and (3.7)

$$\left| b_m'(t) \right| \leqslant (1 + 4M^2)\tilde{K}_1. \tag{3.21}$$

Combining (3.16)–(3.18) and (3.21), we obtain

$$|I_1| \leqslant \frac{(1 + 4M^2)\tilde{K}_1}{b_0} \int_0^t S_m^{(k)}(s)\, ds. \tag{3.22}$$

Second integral. It follows from (3.1), (3.10), (3.16), and (3.17), that

$$|I_2| \leqslant 2\int_0^t \left\| F_m(s) \right\| \left\| \dot{u}_m^{(k)}(s) \right\| ds \leqslant 2K_0 \int_0^t \sqrt{S_m^{(k)}(s)}\, ds. \tag{3.23}$$

Third integral. We have

$$\frac{\partial}{\partial x}F_m = D_1 f[u_{m-1}] + D_3 f[u_{m-1}]\nabla u_{m-1} + D_4 f[u_{m-1}]\Delta u_{m-1} + D_5 f[u_{m-1}]\nabla \dot{u}_{m-1}, \tag{3.24}$$

here we have used the notation

$$D_i f[u_{m-1}] = D_i f\big(x, t, u_{m-1}(t), \nabla u_{m-1}(t), \dot{u}_{m-1}(t), \|u_{m-1}(t)\|^2, \|\nabla u_{m-1}(t)\|^2\big), \quad i = 1, 3, 4, 5.$$

It follows from (3.1), (3.2), (3.7), (3.10), and (3.24) that

$$\|F_m(s)\|_{H^1}^2 = \left\|\frac{\partial}{\partial x}F_m(s)\right\|^2 + F_m^2(0, s) \leqslant K_1^2(1+3M)^2 + K_0^2. \tag{3.25}$$

Then, from (3.16), (3.18), and (3.25), we obtain

$$\begin{aligned}|I_3| &\leqslant 2C_1\int_0^t \|F_m(s)\|_{H^1}\|\dot{u}_m^{(k)}(s)\|_{H^1}\,ds \\ &\leqslant \frac{2C_1}{\sqrt{C_0}}\big[K_1(1+3M)+K_0\big]\int_0^t \sqrt{S_m^{(k)}(s)}\,ds.\end{aligned} \tag{3.26}$$

Fourth integral. Equation (3.12) can be rewritten as follows:

$$\langle \ddot{u}_m^{(k)}(t), w_j\rangle - b_m(t)\langle \Delta u_m^{(k)}(t), w_j\rangle = \langle F_m(t), w_j\rangle, \quad 1 \leqslant j \leqslant k. \tag{3.27}$$

Hence, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating that

$$\int_0^t \|\ddot{u}_m^{(k)}(s)\|^2\,ds \leqslant 2\int_0^t b_m^2(s)\|\Delta u_m^{(k)}(s)\|^2\,ds + 2\int_0^t \|F_m(s)\|^2\,ds. \tag{3.28}$$

From (3.1), (3.3), (3.7), (3.10), (3.16), (3.18), and (3.28) we deduce that

$$I_4 \leqslant 2\tilde{K}_0\int_0^t S_m^{(k)}(s)\,ds + 2TK_0^2. \tag{3.29}$$

Combining (3.19), (3.22), (3.23), (3.26), and (3.29), we then have

$$\begin{aligned}S_m^{(k)}(t) &\leqslant X_m^{(k)}(0) + Y_m^{(k)}(0) + 2TK_0^2 + 2\left[K_0 + \frac{C_1}{\sqrt{C_0}}\big(K_0 + K_1(1+3M)\big)\right] \\ &\quad\times \int_0^t \sqrt{S_m^{(k)}(s)}\,ds + \left(2\tilde{K}_0 + \frac{(1+4M^2)\tilde{K}_1}{b_0}\right)\int_0^t S_m^{(k)}(s)\,ds \\ &\leqslant X_m^{(k)}(0) + Y_m^{(k)}(0) + d_1(M,T) + d_2(M,T)\int_0^t S_m^{(k)}(s)\,ds,\end{aligned} \tag{3.30}$$

where

$$d_1(M,T) = 2TK_0^2 + T\left[K_0 + \frac{C_1}{\sqrt{C_0}}\big(K_0 + K_1(1+3M)\big)\right]^2, \tag{3.31}$$

$$d_2(M,T) = 1 + 2\tilde{K}_0 + \frac{(1+4M^2)\tilde{K}_1}{b_0}. \tag{3.32}$$

Now, we need an estimate on the term $X_m^{(k)}(0) + Y_m^{(k)}(0)$. We have

$$\begin{aligned} X_m^{(k)}(0) + Y_m^{(k)}(0) &= \|\tilde{u}_{1k}\|^2 + a(\tilde{u}_{1k}, \tilde{u}_{1k}) \\ &\quad + B\big(0, \|\tilde{u}_0\|^2, \|\nabla\tilde{u}_0\|^2\big)\big[a(\tilde{u}_{0k}, \tilde{u}_{0k}) + \|\Delta\tilde{u}_{0k}\|^2\big]. \end{aligned} \tag{3.33}$$

By using the assumption (A_2) we can deduce from (3.14), (3.15), and (3.33) that there exists a constant $M > 0$, independent of k and m, such that

$$X_m^{(k)}(0) + Y_m^{(k)}(0) \leqslant M^2/2 \quad \text{for all } k \text{ and } m. \tag{3.34}$$

Notice that, from the assumptions (A_3), (A_4) we have

$$\lim_{T\to 0_+} \sqrt{T} K_i(M,T,f) = \lim_{T\to 0_+} \sqrt{T}\tilde{K}_i(M,T,B) = 0, \quad i = 0, 1. \tag{3.35}$$

Then, from (3.31), (3.32), and (3.35), we can always choose the constant $T > 0$ such that

$$\left(\frac{M^2}{2} + d_1(M,T)\right)\exp\big(Td_2(M,T)\big) \leqslant M^2, \tag{3.36}$$

and

$$\begin{aligned} k_T &= 2(1+\sqrt{2})\sqrt{1 + \frac{1}{b_0C_0}}\big[2M^2\sqrt{T}\tilde{K}_1 + (1+2M)\sqrt{T}K_1\big] \\ &\quad \times \exp\left(\frac{1}{2}\left(1 + \frac{1}{b_0C_0}\right)\big[1 + (1+4M^2)C_1T\tilde{K}_1\big]\right) < 1. \end{aligned} \tag{3.37}$$

Finally, it follows from (3.30), (3.34), and (3.36) that

$$S_m^{(k)}(t) \leqslant M^2\exp\big(-Td_2(M,T)\big) + d_2(M,T)\int_0^t S_m^{(k)}(s)\,ds,$$
$$0 \leqslant t \leqslant T_m^{(k)} \leqslant T. \tag{3.38}$$

By using Gronwall's lemma we deduce from (3.38) that

$$S_m^{(k)}(t) \leqslant M^2\exp\big(-Td_2(M,T)\big)\exp\big(td_2(M,T)\big) \leqslant M^2 \quad \text{for all } t \in \big[0, T_m^{(k)}\big]. \tag{3.39}$$

So we can take constant $T_m^{(k)} = T$ for all m and k. Therefore, we have

$$u_m^{(k)} \in W_1(M,T) \quad \text{for all } m \text{ and } k. \tag{3.40}$$

From (3.40) we can extract from $\{u_m^{(k)}\}$ a subsequence $\{u_m^{(k_i)}\}$ such that

$$u_m^{(k_i)} \to u_m \quad \text{in } L^\infty(0,T;H^2) \text{ weak } *, \tag{3.41}$$

$$\dot u_m^{(k_i)} \to \dot u_m \quad \text{in } L^\infty(0,T;H^1) \text{ weak } *, \tag{3.42}$$

$$\ddot u_m^{(k_i)} \to \ddot u_m \quad \text{in } L^2(Q_T) \text{ weak}, \tag{3.43}$$

where

$$u_m \in W(M,T). \tag{3.44}$$

Then we can take limits in (3.12), (3.13) with $k = k_i \to +\infty$, by (3.40)–(3.43), we have u_m satisfying (3.8)–(3.10) in $L^2(0,T)$, weak.

On the other hand, it follows from (3.7), (3.8), and (3.44) that $\ddot u_m = b_m(t)\Delta u_m + F_m \in L^\infty(0,T;L^2)$, hence $u_m \in W_1(M,T)$. The proof of Theorem 1 is complete. □

Theorem 2. *Let* (A_1)–(A_4) *hold. Then there exist positive constants* M, T *satisfying* (3.34), (3.36), *and* (3.37) *such that the problem* (1.1)–(1.3) *has a unique weak solution* $u \in W_1(M,T)$.
On the other hand, the linear recurrent sequence $\{u_m\}$ *defined by* (3.8)–(3.10) *converges to the solution* u *strongly in the space* $W_1(T) = \{v \in L^\infty(0,T;H^1):\ \dot v \in L^\infty(0,T;L^2)\}$.
Furthermore, we have also the estimate

$$\|u_m - u\|_{L^\infty(0,T;H^1)} + \|\dot u_m - \dot u\|_{L^\infty(0,T;L^2)} \leqslant C k_T^m \quad \textit{for all } m, \tag{3.45}$$

where the constant $k_T < 1$ *is defined by* (3.37) *and* C *is a constant depending only on* T, u_0, u_1, *and* k_T.

Proof. (a) *Existence of the solution.* First, we note that $W_1(T)$ is a Banach space with respect to the norm (see [15])

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;H^1)} + \|\dot v\|_{L^\infty(0,T;L^2)}. \tag{3.46}$$

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $v_m = u_{m+1} - u_m$. Then v_m satisfies the variational problem

$$\begin{aligned} &\langle \ddot v_m(t), v\rangle + b_{m+1}(t) a\big(v_m(t), v\big) - \big(b_{m+1}(t) - b_m(t)\big)\langle \Delta u_m(t), v\rangle \\ &\quad = \langle F_{m+1}(t) - F_m(t), v\rangle \quad \text{for all } v \in H^1, \end{aligned} \tag{3.47}$$

$$v_m(0) = \dot v_m(0) = 0. \tag{3.48}$$

We take $v = \dot v_m$ in (3.47), after integrating in t

$$\begin{aligned} \big\|\dot v_m(t)\big\|^2 + b_{m+1}(t) a\big(v_m(t), v_m(t)\big) &= \int_0^t b'_{m+1}(s) a\big(v_m(s), v_m(s)\big)\,ds \\ &\quad + 2\int_0^t \big(b_{m+1}(s) - b_m(s)\big)\langle \Delta u_m(s), \dot v_m(s)\rangle\,ds \\ &\quad + 2\int_0^t \langle F_{m+1}(s) - F_m(s), \dot v_m(s)\rangle\,ds. \end{aligned} \tag{3.49}$$

On the other hand, from (3.2), (3.4), (3.7), and (3.21) we get

$$\left|b'_{m+1}(t)\right| \leqslant (1+4M^2)\tilde{K}_1, \tag{3.50}$$

$$\left|b_{m+1}(t)-b_m(t)\right| \leqslant 2(1+\sqrt{2})M\tilde{K}_1\|v_{m-1}\|_{W_1(T)}, \tag{3.51}$$

$$\left\|F_{m+1}(t)-F_m(t)\right\| \leqslant (1+\sqrt{2})(1+2M)K_1\|v_{m-1}\|_{W_1(T)}. \tag{3.52}$$

It follows from (3.49)–(3.52) that

$$\begin{aligned}
&\left\|\dot{v}_m(t)\right\|^2+\left\|v_m(t)\right\|_{H^1}^2 \\
&\quad\leqslant \left(1+\frac{1}{b_0C_0}\right)(1+\sqrt{2})^2T\left[2M^2\tilde{K}_1+(1+2M)K_1\right]^2\|v_{m-1}\|_{W_1(T)}^2 \\
&\qquad+\left(1+\frac{1}{b_0C_0}\right)\left[1+(1+4M^2)C_1\tilde{K}_1\right]\int_0^t\left(\left\|\dot{v}_m(s)\right\|^2+\left\|v_m(s)\right\|_{H^1}^2\right)ds.
\end{aligned} \tag{3.53}$$

By using Gronwall's lemma we deduce from (3.53) that

$$\|v_m\|_{W_1(T)} \leqslant k_T\|v_{m-1}\|_{W_1(T)} \quad \text{for all } m, \tag{3.54}$$

where

$$\begin{aligned}
k_T &= 2(1+\sqrt{2})\sqrt{1+\frac{1}{b_0C_0}}\left[2M^2\sqrt{T}\tilde{K}_1+(1+2M)\sqrt{T}K_1\right] \\
&\quad\times\exp\left(\frac{1}{2}\left(1+\frac{1}{b_0C_0}\right)\left[1+(1+4M^2)C_1T\tilde{K}_1\right]\right)<1.
\end{aligned}$$

Hence

$$\|u_{m+p}-u_m\|_{W_1(T)} \leqslant \|u_1-u_0\|_{W_1(T)}\frac{k_T^m}{1-k_T} \quad \text{for all } m, p. \tag{3.55}$$

It follows from (3.55) that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Therefore there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strongly in } W_1(T). \tag{3.56}$$

We also note that $u_m \in W_1(M,T)$, then from the sequence $\{u_m\}$ we can deduce a subsequence $\{u_{m_j}\}$ such that

$$u_{m_j} \to u \quad \text{in } L^\infty(0,T;H^2) \text{ weak } *, \tag{3.57}$$

$$\dot{u}_{m_j} \to \dot{u} \quad \text{in } L^\infty(0,T;H^1) \text{ weak } *, \tag{3.58}$$

$$\ddot{u}_{m_j} \to \ddot{u} \quad \text{in } L^2(Q_T) \text{ weak}, \tag{3.59}$$

$$u \in W(M,T). \tag{3.60}$$

We notice that

$$\left| \int_0^T b_m(t) a\big(u_m(t), v(t)\big)\,dt - \int_0^T B\big(t, \|u(t)\|^2, \|u_x(t)\|^2\big) a(u,v)\,dt \right|$$
$$\leqslant C_1\big(\tilde{K}_0 \|u_m - u\|_{W_1(T)} + 2(1+\sqrt{2}) M^2 \tilde{K}_1 \|u_{m-1} - u\|_{W_1(T)}\big)\|v\|_{L^1(0,T;H^1)}$$
$$\text{for all } v \in L^1(0,T;H^1). \tag{3.61}$$

It follows from (3.56) and (3.61) that

$$\int_0^T b_m(t) a\big(u_m(t), v(t)\big)\,dt \to \int_0^T B\big(t, \|u(t)\|^2, \|u_x(t)\|^2\big) a(u,v)\,dt$$
$$\text{for all } v \in L^1(0,T;H^1). \tag{3.62}$$

Similarly

$$\big\| F_m - f\big(x,t,u,u_x,\dot{u},\|u\|^2,\|u_x\|^2\big)\big\|_{L^\infty(0,T;L^2)}$$
$$\leqslant (1+\sqrt{2})(1+2M)K_1 \|u_{m-1} - u\|_{W_1(T)}. \tag{3.63}$$

Hence, from (3.56) and (3.63), we obtain

$$F_m \to f\big(x,t,u,u_x,\dot{u},\|u\|^2,\|u_x\|^2\big) \quad \text{strongly in } L^\infty(0,T;L^2). \tag{3.64}$$

Then we can take limits in (3.8)–(3.10) with $m = m_j \to +\infty$, we then can deduce from (3.57)–(3.59), (3.62), and (3.64) that there exists $u \in W(M,T)$ satisfying the equation

$$\langle \ddot{u}(t), v\rangle + B\big(t, \|u(t)\|^2, \|u_x(t)\|^2\big) a\big(u(t), v\big)$$
$$= \big\langle f\big(x,t,u,u_x,\dot{u},\|u(t)\|^2,\|u_x(t)\|^2\big), v\big\rangle \quad \text{for all } v \in H^1, \tag{3.65}$$

and the initial conditions

$$u(0) = \tilde{u}_0, \qquad \dot{u}(0) = \tilde{u}_1. \tag{3.66}$$

On the other hand, we have from (3.62), (3.64), and (3.65) that

$$\ddot{u} = B\big(t, \|u\|^2, \|u_x\|^2\big) u_{xx} + f\big(x,t,u,u_x,\dot{u},\|u\|^2,\|u_x\|^2\big) \in L^\infty(0,T;L^2). \tag{3.67}$$

Hence, we obtain $u \in W_1(M,T)$. The existence proof is completed.

(b) *Uniqueness of the solution*. Let u_1, u_2 both be weak solutions of the problem (1.1)–(1.3) such that $u_i \in W_1(M,T)$, $i = 1,2$. Then $u = u_1 - u_2$ satisfies the following variational problem:

$$\langle \ddot{u}(t), v\rangle + \tilde{B}_1(t) a\big(u(t), v\big) - \big(\tilde{B}_1(t) - \tilde{B}_2(t)\big)\langle \Delta u_2(t), v\rangle$$
$$= \big\langle \tilde{F}_1(t) - \tilde{F}_2(t), v\big\rangle \quad \text{for all } v \in H^1, \tag{3.68}$$

and the initial conditions

$$u(0) = \dot{u}(0) = 0, \tag{3.69}$$

where

$$\begin{cases} \tilde{B}_i(t) = B(t, \|u_i(t)\|^2, \|\nabla u_i(t)\|^2), \\ \tilde{F}_i(t) = \tilde{f}(t,x,u_i,\nabla u_i,\dot{u}_i,\|u_i(t)\|^2,\|\nabla u_i(t)\|^2), \quad i = 1,2. \end{cases} \tag{3.70}$$

Take $v = \dot{u}$ in (3.68), we then obtain after integrating by parts

$$\|\dot{u}(t)\|^2 + b_0 C_0 \|u(t)\|_{H^1}^2 \leqslant \int_0^t \tilde{B}_1'(s) a\big(u(s), u(s)\big)\, ds$$
$$+ 2\int_0^t \big(\tilde{B}_1(s) - \tilde{B}_2(s)\big)\big\langle \Delta u_2(s), \dot{u}(s)\big\rangle\, ds + 2\int_0^t \big\langle \tilde{F}_1(s) - \tilde{F}_2(s), \dot{u}(s)\big\rangle\, ds. \quad (3.71)$$

Put

$$Z(t) = \|\dot{u}(t)\|^2 + \|u(t)\|_{H^1}^2 \quad (3.72)$$

and

$$\tilde{K}_M = \left(1 + \frac{1}{b_0 C_0}\right)\Big[\big((1 + 4M^2)C_1 + 2(1 + \sqrt{2})M^2\big)\tilde{K}_1$$
$$+ \big(1 + (1 + \sqrt{2})(1 + 2M)\big)K_1\Big], \quad (3.73)$$

then it follows from (3.71)–(3.73) that

$$Z(t) \leqslant \tilde{K}_M \int_0^t Z(s)\, ds \quad \text{for all } t \in [0, T]. \quad (3.74)$$

Using Gronwall's lemma we deduce $Z(t) = 0$, i.e., $u_1 = u_2$. The proof of Theorem 2 is complete. □

Remark 1. • In the case of $B \equiv 1$, $f = f(t, u, u_t)$, $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^2)$, $f(t, 0, 0) = 0$, $\forall t \geqslant 0$, and the Dirichlet homogeneous condition (1.8) standing for (1.2), we have obtained some results in the paper [4].

• In the case of the function $f \in C^1([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, $B \equiv 1$, we have also obtained some results in [10]. However, the result above does not use the assumption $f \in C^1([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^2)$.

4. Asymptotic expansion of solutions

In this part, let (A_1)–(A_4) hold. We also make the following assumptions:

(A_5) $B_1 \in C^1(\mathbb{R}_+^3)$, $B_1(t, Y, Z) \geqslant 0$;
(A_6) f_1 satisfy the assumption (A_4).

We consider the following perturbed problem, where ε is a small parameter, $|\varepsilon| \leqslant 1$:

$$(\mathrm{P}_\varepsilon)\quad \begin{cases} u_{tt} - B_\varepsilon(t, \|u\|^2, \|u_x\|^2)\Delta u = F_\varepsilon(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2), \\ \quad 0 < x < 1,\ 0 < t < T, \\ u_x(0, t) - h_0 u(0, t) = u_x(1, t) + h_1 u(1, t) = 0, \\ u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x), \\ F_\varepsilon(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2) = f(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2) \\ \quad + \varepsilon f_1(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2), \\ B_\varepsilon(t, \|u\|^2, \|u_x\|^2) = B(t, \|u\|^2, \|u_x\|^2) + \varepsilon B_1(t, \|u\|^2, \|u_x\|^2). \end{cases}$$

First, we note that if the functions $\tilde{u}_0$, $\tilde{u}_1$, B, B_1, f, f_1 satisfy the assumptions (A_1)–(A_6), then the a priori estimates of the Galerkin approximation sequence $\{u_m^{(k)}\}$ in the proof of Theorem 1 for the problem (1.1)–(1.3) corresponding to $B = B_\varepsilon$, $f = F_\varepsilon$, $|\varepsilon| \leqslant 1$, satisfy

$$u_m^{(k)} \in W_1(M, T), \tag{4.1}$$

where M, T are constants independent of ε. Indeed, in the processing we choose the positive constants M and T as in (3.34), (3.36), (3.37), wherein $K_i(M, T, f)$ and $\tilde{K}_i(M, T, B)$, $i = 0, 1$, stand for $K_i(M, T, f) + K_i(M, T, f_1)$ and $\tilde{K}_i(M, T, B) + \tilde{K}_i(M, T, B_1)$, $i = 0, 1$, respectively. Hence, the limit u_ε in suitable function spaces of the sequence $\{u_m^{(k)}\}$ as $k \to +\infty$, afterwards $m \to +\infty$, is a unique weak solution of the problem (P_ε) satisfying

$$u_\varepsilon \in W_1(M, T). \tag{4.2}$$

Then we can prove, in a manner similar to the proof of Theorem 2, that the limit u_0 in suitable function spaces of the family $\{u_\varepsilon\}$ as $\varepsilon \to 0$ is a unique weak solution of the problem (P_0) corresponding to $\varepsilon = 0$ satisfying

$$u_0 \in W_1(M, T). \tag{4.3}$$

Then, we have the following theorem.

Theorem 3. *Let* (A_1)–(A_6) *hold. Then there exist constants* $M > 0$ *and* $T > 0$ *such that, for every* ε *with* $|\varepsilon| \leqslant 1$, *problem* (P_ε) *has a unique weak solution* $u_\varepsilon \in W_1(M, T)$ *satisfying the asymptotic estimation*

$$\|u_\varepsilon - u_0\|_{L^\infty(0,T;H^1)} + \|\dot{u}_\varepsilon - \dot{u}_0\|_{L^\infty(0,T;L^2)} \leqslant C|\varepsilon|, \tag{4.4}$$

where C *is a constant depending only on* b_0, h_0, T, M, $K_1(M, T, f)$, $\tilde{K}_1(M, T, B)$, $K_0(M, T, f_1)$, *and* $\tilde{K}_0(M, T, B_1)$.

Proof. Put $v = u_\varepsilon - u_0$. Then v satisfies the variational problem

$$\begin{cases} \langle \ddot{v}(t), w\rangle + b_\varepsilon(t) a(v(t), w) = (b_\varepsilon(t) - b_0(t))\langle \Delta u_0(t), w\rangle + \langle f_\varepsilon(t) - f_0(t), w\rangle \\ \quad + \varepsilon\langle f_{1\varepsilon}(t), w\rangle + \varepsilon b_{1\varepsilon}(t)\langle \Delta u_\varepsilon(t), w\rangle \quad \text{for all } w \in H^1, \\ v(0) = \dot{v}(0) = 0, \end{cases} \tag{4.5}$$

where

$$\begin{cases} b_\varepsilon(t) = B(t, \|u_\varepsilon(t)\|^2, \|\nabla u_\varepsilon(t)\|^2), \\ b_{1\varepsilon}(t) = B_1(t, \|u_\varepsilon(t)\|^2, \|\nabla u_\varepsilon(t)\|^2), \\ f_\varepsilon(t) = f_\varepsilon(x,t) = f(x,t,u_\varepsilon, \nabla u_\varepsilon, \dot{u}_\varepsilon, \|u_\varepsilon\|^2, \|\nabla u_\varepsilon\|^2), \\ f_{1\varepsilon}(t) = f_{1\varepsilon}(x,t) = f_1(x,t,u_\varepsilon, \nabla u_\varepsilon, \dot{u}_\varepsilon, \|u_\varepsilon\|^2, \|\nabla u_\varepsilon\|^2). \end{cases} \tag{4.6}$$

Taking $w = \dot{v}$ in (4.5), after integration by parts in t, we get

$$\begin{aligned} \|\dot{v}(t)\|^2 + b_\varepsilon(t)a\big(v(t), v(t)\big) &= \int_0^t b'_\varepsilon(s)a\big(v(s), v(s)\big)\,ds \\ &\quad + 2\int_0^t \big(b_\varepsilon(s) - b_0(s)\big)\big\langle \Delta u_0(s), \dot{v}(s)\big\rangle\,ds + 2\varepsilon \int_0^t b_{1\varepsilon}(s)\big\langle \Delta u_\varepsilon(s), \dot{v}(s)\big\rangle\,ds \\ &\quad + 2\int_0^t \big\langle f_\varepsilon(s) - f_0(s), \dot{v}(s)\big\rangle\,ds + 2\varepsilon \int_0^t \big\langle f_{1\varepsilon}(s), \dot{v}(s)\big\rangle\,ds. \end{aligned} \tag{4.7}$$

Let $\sigma(t) = \|\dot{v}(t)\|^2 + \|v(t)\|^2_{H^1}$, then, we can prove the following inequality in a similar manner:

$$\|\dot{v}(t)\|^2 + b_0C_0\|v(t)\|^2_{H^1} \leqslant \varepsilon^2 T\gamma_2 + (\gamma_1 + \gamma_2)\int_0^t \sigma(s)\,ds, \quad 0 \leqslant t \leqslant T, \tag{4.8}$$

where

$$\begin{cases} \gamma_1 = (2(1+\sqrt{2})M^2 + (1+4M^2)C_1)\tilde{K}_1(M,T,B) \\ \qquad + (2 + (1+\sqrt{2})(1+2M))K_1(M,T,f), \\ \gamma_2 = K_0(M,T,f_1) + M\tilde{K}_0(M,T,B_1). \end{cases} \tag{4.9}$$

Next, by (4.8) and Gronwall's lemma, we obtain

$$\sigma(t) \leqslant \left(1 + \frac{1}{b_0C_0}\right)\varepsilon^2 T\gamma_2 \exp\left(\left(1 + \frac{1}{b_0C_0}\right)(\gamma_1+\gamma_2)T\right) \quad \text{for all } t \in [0,T]. \tag{4.10}$$

Hence

$$\|v\|_{L^\infty(0,T;H^1)} + \|\dot{v}\|_{L^\infty(0,T;L^2)} \leqslant C|\varepsilon|, \tag{4.11}$$

where $C = 2\sqrt{(1+1/(b_0C_0))T\gamma_2}\exp(1/2(1+1/(b_0C_0))(\gamma_1+\gamma_2)T)$.

The proof of Theorem 3 is complete. □

The next result gives an asymptotic expansion of the weak solution u_ε of order $N+1$ in ε, for ε sufficiently small. We use the following notations:

$$f[u] = f\big(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2\big), \qquad B[u] = B\big(t,\|u\|^2,\|u_x\|^2\big).$$

Now, we assume that

(A_7) $B \in C^{N+1}(\mathbb{R}_+^3)$, $B_1 \in C^N(\mathbb{R}_+^3)$, $B(t, Y, Z) \geqslant b_0 > 0$, $B_1(t, Y, Z) \geqslant 0$;
(A_8) $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^2)$, $f_1 \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^2)$.

Let $u_0 \in W_1(M,T)$ be a weak solution of the problem (P_0) corresponding to $\varepsilon = 0$.

Let us consider the weak solutions $u_1, u_2, \ldots, u_N \in W_1(M,T)$ (with suitable constants $M > 0$ and $T > 0$) defined by the following problems:

$$(Q_1)\quad \begin{cases} \ddot{u}_1 - B[u_0]\Delta u_1 = \tilde{F}_1[u_1], \quad 0 < x < 1,\ 0 < t < T, \\ \nabla u_1(0,t) - h_0 u_1(0,t) = \nabla u_1(1,t) + h_1 u_1(1,t) = 0, \\ u_1(x,0) = \dot{u}_1(x,0) = 0, \end{cases}$$

where

$$\tilde{F}_1[u_1] = \pi_1[f] + \pi_0[f_1] + \big(\rho_1[B] + \rho_0[B_1]\big)\Delta u_0 \tag{4.12}$$

with $\pi_0[f]$, $\pi_1[f]$, $\rho_0[B]$, $\rho_1[B]$ are defined as follows:

$$\pi_0[f] = f[u_0] \equiv f\big(x, t, u_0, \nabla u_0, \dot{u}_0, \|u_0\|^2, \|\nabla u_0\|^2\big), \tag{4.13}$$

$$\pi_1[f] = \pi_0[D_3 f]u_1 + \pi_0[D_4 f]\nabla u_1 + \pi_0[D_5 f]\dot{u}_1 + 2\pi_0[D_6 f]\langle u_0, u_1\rangle + 2\pi_0[D_7 f]\langle \nabla u_0, \nabla u_1\rangle, \tag{4.14}$$

$$\rho_0[B] = B[u_0] \equiv B\big(t, \|u_0\|^2, \|\nabla u_0\|^2\big), \tag{4.15}$$

and

$$\rho_1[B] = 2\rho_0[D_6 B]\langle u_0, u_1\rangle + 2\rho_0[D_7 B]\langle \nabla u_0, \nabla u_1\rangle, \tag{4.16}$$

with $2 \leqslant i \leqslant N$,

$$(Q_i)\quad \begin{cases} \ddot{u}_i - B[u_0]\Delta u_i = \tilde{F}_i[u_i], \quad 0 < x < 1,\ 0 < t < T, \\ \nabla u_i(0,t) - h_0 u_i(0,t) = \nabla u_i(1,t) + h_1 u_i(1,t) = 0, \\ u_i(x,0) = \dot{u}_i(x,0) = 0, \quad i = 1, 2, \ldots, N, \end{cases}$$

where

$$\tilde{F}_i[u_i] = \pi_i[f] + \pi_{i-1}[f_1] + \sum_{k=1}^{i} \big(\rho_k[B] + \rho_{k-1}[B_1]\big)\Delta u_{i-k}, \tag{4.17}$$

with $\pi_i[f] = \pi_i[f, u_0, u_1, \ldots, u_i]$, $\rho_i[B] = \rho_i[B, u_0, u_1, \ldots, u_i]$, $2 \leqslant i \leqslant N$ defined by the recurrence formulas

$$\pi_i[f] = \sum_{k=0}^{i-1} \frac{i-k}{i} \Bigg\{ \pi_k[D_3 f]u_{i-k} + \pi_k[D_4 f]\nabla u_{i-k} + \pi_k[D_5 f]\dot{u}_{i-k} + 2\sum_{j=0}^{i-k-1} \frac{i-k-j}{i-k}\big(\pi_k[D_6 f]\langle u_j, u_{i-k-j}\rangle + \pi_k[D_7 f]\langle \nabla u_j, \nabla u_{i-k-j}\rangle\big)\Bigg\},$$
$$2 \leqslant i \leqslant N, \tag{4.18}$$

$$\rho_i[B] = \frac{2}{i}\sum_{k=0}^{i-1}\sum_{j=0}^{i-k-1} (i-k-j)\big(\rho_k[D_6 B]\langle u_j, u_{i-k-j}\rangle + \rho_k[D_7 B]\langle \nabla u_j, \nabla u_{i-k-j}\rangle\big), \quad 2 \leqslant i \leqslant N. \tag{4.19}$$

We also note that $\pi_i[f]$ is the first-order function with respect to u_i, ∇u_i, $\dot{u}_i$. In fact,

$$\begin{aligned}\pi_i[f] &= \pi_0[D_3 f]u_i + \pi_0[D_4 f]\nabla u_i + \pi_0[D_5 f]\dot{u}_i + 2\pi_0[D_6 f]\langle u_0, u_i\rangle \\ &\quad + 2\pi_0[D_7 f]\langle \nabla u_0, \nabla u_i\rangle + \text{terms depending on } \big(i, \pi_k[D_\nu f], u_k, \nabla u_k, \dot{u}_k\big), \\ &\nu = 3,4,5,6,7,\ k = 0,1,\ldots,i-1. \end{aligned} \tag{4.20}$$

Similarly

$$\begin{aligned}\rho_i[B] &= 2\rho_0[D_6 B]\langle u_0, u_i\rangle + 2\rho_0[D_7 B]\langle \nabla u_0, \nabla u_i\rangle \\ &\quad + \text{terms depending on } \big(i, \rho_k[D_6 B], \rho_k[D_7 B], \nabla u_k\big), \\ &k = 0,1,\ldots,i-1. \end{aligned} \tag{4.21}$$

Let $u_\varepsilon \in W_1(M,T)$ be a unique weak solution of the problem (P_ε). Then $v = u_\varepsilon - \sum_{i=0}^N u_i \varepsilon^i \equiv u_\varepsilon - h$ satisfies the problem

$$\begin{aligned}&\ddot{v} - B_\varepsilon[v+h]\Delta v = F_\varepsilon[v+h] - F_\varepsilon[h] + \big(B_\varepsilon[v+h] - B_\varepsilon[h]\big)\Delta h + E_\varepsilon(x,t), \\ &\quad 0 < x < 1,\ 0 < t < T, \\ &\nabla v(0,t) - h_0 v(0,t) = \nabla v(1,t) + h_1 v(1,t) = 0, \\ &v(x,0) = \dot{v}(x,0) = 0, \end{aligned} \tag{4.22}$$

where

$$E_\varepsilon(x,t) = F_\varepsilon[h] - f[u_0] + \big(B_\varepsilon[h] - B[u_0]\big)\Delta h - \sum_{i=1}^N \tilde{F}_i[u_i]\varepsilon^i. \tag{4.23}$$

Then, we have the following lemmas.

Lemma 4. *The functions $\pi_i[f]$, $\rho_i[B]$, $0 \leqslant i \leqslant N$ above are defined by the following formulas*:

$$\pi_i[f] = \frac{1}{i!}\frac{\partial^i}{\partial \varepsilon^i}\big(f[h]\big)\Big|_{\varepsilon=0}, \quad 0 \leqslant i \leqslant N, \tag{4.24}$$

$$\rho_i[B] = \frac{1}{i!}\frac{\partial^i}{\partial \varepsilon^i}\big(B[h]\big)\Big|_{\varepsilon=0}, \quad 0 \leqslant i \leqslant N. \tag{4.25}$$

Proof. (i) It is easy to see that

$$\pi_0[f] = f[h]|_{\varepsilon=0} = f[u_0] \equiv f\big(x, t, u_0, \nabla u_0, \dot{u}_0, \|u_0\|^2, \|\nabla u_0\|^2\big).$$

With $i = 1$, we have

$$\pi_1[f] = \frac{\partial}{\partial \varepsilon}\big(f[h]\big)\Big|_{\varepsilon=0}. \tag{4.26}$$

But

$$\begin{aligned}\frac{\partial}{\partial \varepsilon}\big(f[h]\big) &= D_3 f[h]\frac{\partial}{\partial \varepsilon}h + D_4 f[h]\frac{\partial}{\partial \varepsilon}\nabla h + D_5 f[h]\frac{\partial}{\partial \varepsilon}\dot{h} + D_6 f[h]\frac{\partial}{\partial \varepsilon}\big(\|h\|^2\big) \\ &\quad + D_7 f[h]\frac{\partial}{\partial \varepsilon}\big(\|\nabla h\|^2\big). \end{aligned} \tag{4.27}$$

On the other hand, from the formulas

$$h=\sum_{i=0}^{N}u_i\varepsilon^i,\qquad \frac{\partial}{\partial\varepsilon}h=\sum_{i=0}^{N}iu_i\varepsilon^{i-1},$$
$$\frac{\partial}{\partial\varepsilon}\big(\|h\|^2\big)=2\left\langle h,\frac{\partial}{\partial\varepsilon}h\right\rangle,\qquad \frac{\partial}{\partial\varepsilon}\big(\|\nabla h\|^2\big)=2\left\langle \nabla h,\frac{\partial}{\partial\varepsilon}\nabla h\right\rangle,$$

we have

$$\frac{\partial}{\partial\varepsilon}h\Big|_{\varepsilon=0}=u_1,\qquad \frac{\partial}{\partial\varepsilon}\nabla h\Big|_{\varepsilon=0}=\nabla u_1,\qquad \frac{\partial}{\partial\varepsilon}\dot h\Big|_{\varepsilon=0}=\dot u_1,$$
$$\frac{\partial}{\partial\varepsilon}\big(\|h\|^2\big)\Big|_{\varepsilon=0}=2\langle u_0,u_1\rangle,\qquad \frac{\partial}{\partial\varepsilon}\big(\|\nabla h\|^2\big)\Big|_{\varepsilon=0}=2\langle \nabla u_0,\nabla u_1\rangle. \tag{4.28}$$

Hence, it follows from (4.26)–(4.28), that

$$\pi_1[f]=\pi_0[D_3f]u_1+\pi_0[D_4f]\nabla u_1+\pi_0[D_5f]\dot u_1+2\pi_0[D_6f]\langle u_0,u_1\rangle$$
$$+2\pi_0[D_7f]\langle\nabla u_0,\nabla u_1\rangle. \tag{4.29}$$

Thus, (4.14) holds.

Suppose that we have defined the functions $\pi_k[f]$, $\pi_k[D_jf]$, $j=3,4,5,6,7$; $k=0,1,\dots,i-1$ from formulas (4.13), (4.14), and (4.24). Therefore, it follows from (4.27) that

$$\frac{\partial^i}{\partial\varepsilon^i}\big(f[h]\big)=\frac{\partial^{i-1}}{\partial\varepsilon^{i-1}}\frac{\partial}{\partial\varepsilon}\big(f[h]\big)$$
$$=\sum_{k=0}^{i-1}C_{i-1}^k\Big\{\frac{\partial^k}{\partial\varepsilon^k}\big(D_3f[h]\big)\frac{\partial^{i-k}}{\partial\varepsilon^{i-k}}(h)+\frac{\partial^k}{\partial\varepsilon^k}\big(D_4f[h]\big)\frac{\partial^{i-k}}{\partial\varepsilon^{i-k}}(\nabla h)$$
$$+\frac{\partial^k}{\partial\varepsilon^k}\big(D_5f[h]\big)\frac{\partial^{i-k}}{\partial\varepsilon^{i-k}}(\dot h)+\frac{\partial^k}{\partial\varepsilon^k}\big(D_6f[h]\big)\frac{\partial^{i-k}}{\partial\varepsilon^{i-k}}\big(\|h\|^2\big)$$
$$+\frac{\partial^k}{\partial\varepsilon^k}\big(D_7f[h]\big)\frac{\partial^{i-k}}{\partial\varepsilon^{i-k}}\big(\|\nabla h\|^2\big)\Big\}. \tag{4.30}$$

We also note that

$$\frac{\partial^i}{\partial\varepsilon^i}h\Big|_{\varepsilon=0}=i!u_i,\quad 0\leqslant i\leqslant N. \tag{4.31}$$

On the other hand,

$$\frac{\partial^m}{\partial\varepsilon^m}\big(\|h\|^2\big)=2\frac{\partial^{m-1}}{\partial\varepsilon^{m-1}}\left\langle h,\frac{\partial}{\partial\varepsilon}h\right\rangle=2\sum_{j=0}^{m-1}C_{m-1}^j\left\langle\frac{\partial^j}{\partial\varepsilon^j}(h),\frac{\partial^{m-j}}{\partial\varepsilon^{m-j}}(h)\right\rangle. \tag{4.32}$$

Hence

$$\frac{\partial^m}{\partial\varepsilon^m}\big(\|h\|^2\big)\Big|_{\varepsilon=0}=2\sum_{j=0}^{m-1}C_{m-1}^j\big\langle j!u_j,(m-j)!u_{m-j}\big\rangle$$
$$=2\sum_{j=0}^{m-1}j!(m-j)!C_{m-1}^j\langle u_j,u_{m-j}\rangle. \tag{4.33}$$

Similarly

$$\frac{\partial^m}{\partial\varepsilon^m}\big(\|\nabla h\|^2\big)\Big|_{\varepsilon=0} = 2\sum_{j=0}^{m-1} j!(m-j)!C_{m-1}^j \langle\nabla u_j, \nabla u_{m-j}\rangle. \tag{4.34}$$

It follows from (4.30), (4.31), (4.33), and (4.34) that

$$\begin{aligned}
&\frac{\partial^i}{\partial\varepsilon^i}\big(f[h]\big)\Big|_{\varepsilon=0}\\
&\quad= \sum_{k=0}^{i-1} k!C_{i-1}^k \Bigg\{\pi_k[D_3 f]\frac{\partial^{i-k}}{\partial\varepsilon^{i-k}}(h)\Big|_{\varepsilon=0} + \pi_k[D_4 f]\frac{\partial^{i-k}}{\partial\varepsilon^{i-k}}(\nabla h)\Big|_{\varepsilon=0}\\
&\qquad + \pi_k[D_5 f]\frac{\partial^{i-k}}{\partial\varepsilon^{i-k}}(\dot h)\Big|_{\varepsilon=0} + \pi_k[D_6 f]\frac{\partial^{i-k}}{\partial\varepsilon^{i-k}}\big(\|h\|^2\big)\Big|_{\varepsilon=0}\\
&\qquad + \pi_k[D_7 f]\frac{\partial^{i-k}}{\partial\varepsilon^{i-k}}\big(\|\nabla h\|^2\big)\Big|_{\varepsilon=0}\Bigg\}\\
&\quad= \sum_{k=0}^{i-1}(i-k)(i-1)!\Bigg\{\pi_k[D_3 f]u_{i-k} + \pi_k[D_4 f]\nabla u_{i-k} + \pi_k[D_5 f]\dot u_{i-k}\\
&\qquad + \frac{2}{i-k}\sum_{j=0}^{i-k-1}(i-k-j)\big(\pi_k[D_6 f]\langle u_j, u_{i-k-j}\rangle\\
&\qquad + \pi_k[D_7 f]\langle\nabla u_j, \nabla u_{i-k-j}\rangle\big)\Bigg\}.
\end{aligned} \tag{4.35}$$

Hence

$$\begin{aligned}
\pi_i[f] &= \frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}\big(f[h]\big)\Big|_{\varepsilon=0}\\
&= \sum_{k=0}^{i-1}\frac{i-k}{i}\Bigg\{\pi_k[D_3 f]u_{i-k} + \pi_k[D_4 f]\nabla u_{i-k} + \pi_k[D_5 f]\dot u_{i-k}\\
&\quad + \frac{2}{i-k}\sum_{j=0}^{i-k-1}(i-k-j)\big(\pi_k[D_6 f]\langle u_j, u_{i-k-j}\rangle\\
&\quad + \pi_k[D_7 f]\langle\nabla u_j, \nabla u_{i-k-j}\rangle\big)\Bigg\}.
\end{aligned} \tag{4.36}$$

Hence, the first part of Lemma 4 is proved.

(ii) In the case of $B = B[h] = B(t, \|h\|^2, \|\nabla h\|^2)$. Applying the formulas (4.13), (4.14), (4.18) with $f = f(t, Y, Z)$, $D_i f = 0$, $i = 1, 3, 4, 5$, $D_6 f = D_6 B = \frac{\partial B}{\partial Y}$, $D_7 f = D_7 B = \frac{\partial B}{\partial Z}$ and $\pi_i[f] = \rho_i[B]$, we obtain formulas (4.15), (4.16), (4.19) and second part of Lemma 4 is proved. □

Lemma 5. *Let* (A_1), (A_2), (A_7), *and* (A_8) *hold. Then there exists a constant* $\tilde{K}$ *such that*

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \leqslant \tilde{K}|\varepsilon|^{N+1}, \tag{4.37}$$

where $\tilde{K}$ *is a constant depending only on* M, T, N *and the constants*

$$\tilde{K}_i(M,T,B) = \sup_{0\leqslant t\leqslant T,\ 0\leqslant Y,\ Z\leqslant M^2} \sum_{\alpha_2+\alpha_6+\alpha_7=i} \left|D_2^{\alpha_2} D_6^{\alpha_6} D_7^{\alpha_7} B(t,Y,Z)\right|,$$
$$i = 1,2,\dots,N+1,$$
$$\tilde{K}_i(M,T,B_1) = \sup_{0\leqslant t\leqslant T,\ 0\leqslant Y,\ Z\leqslant M^2} \sum_{\alpha_2+\alpha_6+\alpha_7=i} \left|D_2^{\alpha_2} D_6^{\alpha_6} D_7^{\alpha_7} B_1(t,Y,Z)\right|,$$
$$i = 1,2,\dots,N,$$
$$K_i(M,T,f) = \sup \sum_\beta \left|D_1^{\beta_1} D_3^{\beta_3} D_4^{\beta_4} D_5^{\beta_5} D_6^{\beta_6} D_7^{\alpha_7} f[u]\right|, \quad i = 1,2,\dots,N+1,$$
$$K_i(M,T,f_1) = \sup \sum_\beta \left|D_1^{\beta_1} D_3^{\beta_3} D_4^{\beta_4} D_5^{\beta_5} D_6^{\beta_6} D_7^{\alpha_7} f_1[u]\right|, \quad i = 1,2,\dots,N,$$

where, in each case, sup *is taken over* $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$, $|u|, |u_x|, |\dot{u}| \leqslant M$, $0 \leqslant Y$, $Z \leqslant M^2$ *and the sum* $\sum_\beta$ *is taken over* $\beta = (\beta_1, \beta_3, \dots, \beta_7) \in \mathbb{Z}_+^6$ *satisfying* $|\beta| = \beta_1 + \beta_3 + \cdots + \beta_7 = i$.

Proof. In the case of $N = 1$, the proof of Lemma 5 is easy, hence we omit the details, which we only prove with $N \geqslant 2$.

By using Maclaurin's expansion of the functions $f[h]$ and $f_1[h]$ around the point $\varepsilon = 0$ up to order $N+1$ and order N, respectively, we obtain from (4.24), that

$$f[h] - f[u_0] = \sum_{i=1}^{N} \pi_i[f]\varepsilon^i + \varepsilon^{N+1} R_{N+1}[f,\varepsilon,\theta_1], \tag{4.38}$$

and

$$f_1[h] = \sum_{i=0}^{N-1} \pi_i[f_1]\varepsilon^i + \varepsilon^N R_N[f_1,\varepsilon,\theta_2], \tag{4.39}$$

where $\pi_i[f]$, $0 \leqslant i \leqslant N$ are defined by (4.13), (4.14), (4.18); $R_{N+1}[f,\varepsilon,\theta_1]$ and $R_N[f_1,\varepsilon,\theta_2]$ are defined as follows:

$$R_{N+1}[f,\varepsilon,\theta_1] = \frac{1}{(N+1)!} \frac{\partial^{N+1}}{\partial\varepsilon^{N+1}}\big(f[h]\big)\Big|_{\varepsilon=\theta_1\varepsilon}, \tag{4.40}$$

and

$$R_N[f_1,\varepsilon,\theta_2] = \frac{1}{N!} \frac{\partial^N}{\partial\varepsilon^N}\big(f_1[h]\big)\Big|_{\varepsilon=\theta_2\varepsilon}, \tag{4.41}$$

with $0 < \theta_i < 1$, $i = 1, 2$.

Combining (4.38)–(4.41), we then obtain

$$F_\varepsilon[h] - f[u_0] = f[h] - f[u_0] + \varepsilon f_1[h] = \sum_{i=1}^{N} \big(\pi_i[f] + \pi_{i-1}[f_1]\big)\varepsilon^i + \varepsilon^{N+1} R_N[f, f_1, \varepsilon, \theta_1, \theta_2], \tag{4.42}$$

with

$$R_N[f, f_1, \varepsilon, \theta_1, \theta_2] = R_{N+1}[f, \varepsilon, \theta_1] + R_N[f_1, \varepsilon, \theta_2]. \tag{4.43}$$

Similarly, we use Maclaurin's expansion around the point $\varepsilon = 0$ up to order $N + 1$ of the functions $B[h]$ and the function $B_1[h]$ up to order N, we obtain from (4.25), that

$$B[h] - B[u_0] = \sum_{i=1}^{N} \rho_i[B]\varepsilon^i + \varepsilon^{N+1} \tilde{R}_{N+1}[B, \varepsilon, \theta_3], \tag{4.44}$$

and

$$B_1[h] = \sum_{i=0}^{N-1} \rho_i[B_1]\varepsilon^i + \varepsilon^N \tilde{R}_N[B_1, \varepsilon, \theta_4], \tag{4.45}$$

where

$$\tilde{R}_{N+1}[B, \varepsilon, \theta_3] = \frac{1}{(N+1)!} \frac{\partial^{N+1}}{\partial \varepsilon^{N+1}} \big(B[h]\big)\Big|_{\varepsilon = \theta_3 \varepsilon}, \tag{4.46}$$

and

$$\tilde{R}_N[B_1, \varepsilon, \theta_4] = \frac{1}{N!} \frac{\partial^N}{\partial \varepsilon^N} \big(B_1[h]\big)\Big|_{\varepsilon = \theta_4 \varepsilon}, \tag{4.47}$$

with $0 < \theta_i < 1$, $i = 3, 4$.

Combining (4.44)–(4.47), we then obtain

$$B_\varepsilon[h] - B[u_0] = B[h] - B[u_0] + \varepsilon B_1[h] = \sum_{i=1}^{N} \big(\rho_i[B] + \rho_{i-1}[B_1]\big)\varepsilon^i + \varepsilon^{N+1} \tilde{R}_N[B, B_1, \varepsilon, \theta_3, \theta_4], \tag{4.48}$$

with

$$\tilde{R}_N[B, B_1, \varepsilon, \theta_3, \theta_4] = \tilde{R}_{N+1}[B, \varepsilon, \theta_3] + \tilde{R}_N[B_1, \varepsilon, \theta_4]. \tag{4.49}$$

Hence

$$\begin{aligned}
&\big(B_\varepsilon[h] - B[u_0]\big)\Delta h \\
&\quad = \left[\sum_{i=1}^{N} \big(\rho_i[B] + \rho_{i-1}[B_1]\big)\varepsilon^i\right]\left(\sum_{j=0}^{N} \varepsilon^j \Delta u_j\right) + \varepsilon^{N+1} \Delta h \tilde{R}_N[B, B_1, \varepsilon, \theta_3, \theta_4] \\
&\quad = \sum_{i=1}^{N^2} \left[\sum_{k=1}^{i} \big(\rho_k[B] + \rho_{k-1}[B_1]\big)\Delta u_{i-k}\right]\varepsilon^i + \varepsilon^{N+1} \Delta h \tilde{R}_N[B, B_1, \varepsilon, \theta_3, \theta_4] \\
&\quad = \sum_{i=1}^{N} \left[\sum_{k=1}^{i} \big(\rho_k[B] + \rho_{k-1}[B_1]\big)\Delta u_{i-k}\right]\varepsilon^i + \varepsilon^{N+1} R_N^{(1)}[B, B_1, h, \varepsilon, \theta_3, \theta_4],
\end{aligned} \tag{4.50}$$

with

$$R_N^{(1)}[B, B_1, h, \varepsilon, \theta_3, \theta_4] = \Delta h \tilde{R}_N[B, B_1, \varepsilon, \theta_3, \theta_4] + \sum_{i=N+1}^{N^2} \left[\sum_{k=1}^{i} \big(d_k[B] + d_{k-1}[B_1]\big) \Delta u_{i-k} \right] \varepsilon^{i-N-1}. \tag{4.51}$$

Combining (4.12)–(4.17), (4.23), (4.42), (4.43), (4.50), and (4.51), we then obtain

$$\begin{aligned} E_\varepsilon(x,t) &= F_\varepsilon[h] - f[u_0] + \big(B_\varepsilon[h] - B[u_0]\big)\Delta h - \sum_{i=1}^{N} \varepsilon^i \tilde{F}_i[u_i] \\ &= \sum_{i=1}^{N} \left[\pi_i[f] + \pi_{i-1}[f_1] + \sum_{k=1}^{i} \big(\rho_k[B] + \rho_{k-1}[B_1]\big)\Delta u_{i-k} - \tilde{F}_i[u_i] \right] \varepsilon^i \\ &\quad + \varepsilon^{N+1}\big(R_N[f, f_1, \varepsilon, \theta_1, \theta_2] + R_N^{(1)}[B, B_1, h, \varepsilon, \theta_3, \theta_4]\big) \\ &= \varepsilon^{N+1}\big(R_N[f, f_1, \varepsilon, \theta_1, \theta_2] + R_N^{(1)}[B, B_1, h, \varepsilon, \theta_3, \theta_4]\big). \end{aligned} \tag{4.52}$$

By the boundedness of the functions u_i, ∇u_i, $\dot{u}_i$, $i = 0, 1, 2, \ldots, N$ in the function space $L^\infty(0, T; H^1)$, we obtain from (4.40), (4.41), (4.43), (4.46), (4.47), (4.49), (4.51), and (4.52) that

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \leqslant \tilde{K} |\varepsilon|^{N+1}, \tag{4.53}$$

where $\tilde{K}$ is a constant depending only on M, T, N and the constants $\tilde{K}_i(M, T, B)$, $K_i(M, T, f)$, $i = 1, 2, \ldots, N+1$, $\tilde{K}_i(M, T, B_1)$, $K_i(M, T, f_1)$, $i = 1, 2, \ldots, N$.

The proof of Lemma 5 is completed. □

Now, we consider the sequence of functions $\{v_m\}$ defined by

$$\begin{cases} v_0 \equiv 0, \\ \ddot{v}_m - B_\varepsilon[v_{m-1} + h]\Delta v_m = F_\varepsilon[v_{m-1} + h] - F_\varepsilon[h] + (B_\varepsilon[v_{m-1} + h] - B_\varepsilon[h])\Delta h \\ \qquad + E_\varepsilon(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ \nabla v_m(0,t) - h_0 v_m(0,t) = \nabla v_m(1,t) + h_1 v_m(1,t) = 0, \\ v_m(x,0) = \dot{v}_m(x,0) = 0, \quad m \geqslant 1. \end{cases} \tag{4.54}$$

With $m = 1$, we have the problem

$$\begin{cases} \ddot{v}_1 - B_\varepsilon[h]\Delta v_1 = E_\varepsilon(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ \nabla v_1(0,t) - h_0 v_1(0,t) = \nabla v_1(1,t) + h_1 v_1(1,t) = 0, \\ v_1(x,0) = \dot{v}_1(x,0) = 0. \end{cases} \tag{4.55}$$

By multiplying the two sides of (4.55) by $\dot{v}_1$, we find without difficulty from (4.37) that

$$\big\|\dot{v}_1(t)\big\|^2 + b_{1,\varepsilon}(t) a\big(v_1(t), v_1(t)\big) \leqslant 2\tilde{K}|\varepsilon|^{N+1} T \|\dot{v}_1\|_{L^\infty(0,T;L^2)} + \int_0^t \big|b'_{1,\varepsilon}(s)\big| a\big(v_1(s), v_1(s)\big)\, ds, \tag{4.56}$$

where

$$b_{1,\varepsilon}(t) = B_\varepsilon[h] = B\big(t, \|h(t)\|^2, \|\nabla h(t)\|^2\big) + \varepsilon B_1\big(t, \|h(t)\|^2, \|\nabla h(t)\|^2\big).$$

We have

$$\begin{aligned} b'_{1,\varepsilon}(t) &= D_2 B[h] + \varepsilon D_2 B_1[h] + 2\big(D_6 B[h] + \varepsilon D_6 B_1[h]\big)\langle h(t), \dot h(t)\rangle \\ &\quad + 2\big(D_7 B[h] + \varepsilon D_7 B_1[h]\big)\langle \nabla h(t), \nabla \dot h(t)\rangle, \end{aligned} \tag{4.57}$$

hence

$$\big|b'_{1,\varepsilon}(t)\big| \leqslant \big[1 + 4(N+1)^2 M^2\big]\big(\tilde K_1(M,T,B) + \tilde K_1(M,T,B_1)\big) \equiv \eta_1. \tag{4.58}$$

It follows from (4.56), (4.58) that

$$\begin{aligned} &\|\dot v_1(t)\|^2 + b_0 C_0 \|v_1(t)\|_{H^1}^2 \\ &\quad \leqslant 2\tilde K |\varepsilon|^{N+1} T \|\dot v_1\|_{L^\infty(0,T;L^2)} + C_1 \eta_1 \int_0^t \|v_1(s)\|_{H^1}^2\, ds. \end{aligned} \tag{4.59}$$

Using Gronwall's lemma we obtain

$$\begin{aligned} &\|\dot v_1\|_{L^\infty(0,T;L^2)} + \|v_1\|_{L^\infty(0,T;H^1)} \\ &\quad \leqslant 2\left(1 + \frac{1}{\sqrt{b_0 C_0}}\right) T \tilde K |\varepsilon|^{N+1} \exp\left(\frac{C_1 \eta_1 T}{b_0 C_0}\right). \end{aligned} \tag{4.60}$$

We shall prove that there exists a constant C_T, independent of m and ε, such that

$$\|\dot v_m\|_{L^\infty(0,T;L^2)} + \|v_m\|_{L^\infty(0,T;H^1)} \leqslant C_T |\varepsilon|^{N+1}, \quad |\varepsilon| \leqslant 1 \text{ for all } m. \tag{4.61}$$

By multiplying the two sides of (4.54) with $\dot v_m$ and after integration in t, we obtain

$$\begin{aligned} \|\dot v_m(t)\|^2 + b_0 C_0 \|v_m(t)\|_{H^1}^2 &\leqslant \int_0^t \big|b'_{m,\varepsilon}(s)\big| a\big(v_m(s), v_m(s)\big)\, ds \\ &\quad + 2\int_0^t \big(\|f[v_{m-1}+h] - f[h]\| + \|f_1[v_{m-1}+h] - f_1[h]\|\big)\|\dot v_m\|\, ds \\ &\quad + 2\int_0^t \big|B[v_{m-1}+h] - B[h]\big| \|\Delta h\| \|\dot v_m\|\, ds \\ &\quad + 2\int_0^t \big|B_1[v_{m-1}+h] - B_1[h]\big| \|\Delta h\| \|\dot v_m\|\, ds + 2\tilde K |\varepsilon|^{N+1} \int_0^t \|\dot v_m\|\, ds, \end{aligned} \tag{4.62}$$

where

$$
\begin{aligned}
b_{m,\varepsilon}(t) &= B_\varepsilon[v_{m-1}+h] = B[v_{m-1}+h] + \varepsilon B_1[v_{m-1}+h],\\
b'_{m,\varepsilon}(t) &= D_2B[v_{m-1}+h] + \varepsilon D_2 B_1[v_{m-1}+h]\\
&\quad + 2\big(D_6B[v_{m-1}+h] + \varepsilon D_6B_1[v_{m-1}+h]\big)\langle v_{m-1}+h, \dot v_{m-1}+\dot h\rangle\\
&\quad + 2\big(D_7B[v_{m-1}+h] + \varepsilon D_7B_1[v_{m-1}+h]\big)\langle \nabla v_{m-1}+\nabla h, \nabla\dot v_{m-1}+\nabla\dot h\rangle.
\end{aligned}
$$

Hence

$$
\left|b'_{m,\varepsilon}(t)\right| \leqslant \left[1+4(N+2)^2M^2\right]\left(\tilde K_1(M,T,B)+\tilde K_1(M,T,B_1)\right) \equiv \eta_2. \tag{4.63}
$$

By (4.62) and (4.63), after some lengthy calculations we can prove the following inequality:

$$
\|v_m\|_{W_1(T)} \leqslant \sigma\|v_{m-1}\|_{W_1(T)} + \delta \quad \text{for all } m \geqslant 1, \tag{4.64}
$$

where

$$
\begin{aligned}
&\sigma = \sigma_T\eta_3, \qquad \delta = \sigma_T\tilde K|\varepsilon|^{N+1},\\
&\sigma_T = \left(1+\frac{1}{\sqrt{b_0C_0}}\right)\sqrt{T}\exp\left(\frac{1}{2}T(1+C_1\eta_2)\right),\\
&\eta_3 = (1+\sqrt{2})(1+2M)\left[K_1(M,T,f)+K_1(M,T,f_1)\right]\\
&\qquad + 2(1+\sqrt{2})M^2\left[\tilde K_1(M,T,B)+\tilde K_1(M,T,B_1)\right].
\end{aligned}
$$

We assume that

$$
\sigma < 1 \quad \text{with the suitable constant } T > 0. \tag{4.65}
$$

We shall now require the following lemma whose proof is immediate.

Lemma 6. *Let the sequence* $\{\Psi_m\}$ *satisfy*

$$
\Psi_m \leqslant \sigma\Psi_{m-1} + \delta \quad \textit{for all } m \geqslant 1,\ \Psi_0 = 0, \tag{4.66}
$$

where $0 \leqslant \sigma < 1$, $\delta \geqslant 0$ *are the given constants. Then*

$$
\Psi_m \leqslant \frac{\delta}{1-\sigma} \quad \textit{for all } m \geqslant 1. \tag{4.67}
$$

Applying Lemma 6 with $\Psi_m = \|v_m\|_{W_1(T)}$, it follows from (4.64), that

$$
\|\dot v_m\|_{L^\infty(0,T;L^2)} + \|v_m\|_{L^\infty(0,T;H^1)} = \|v_m\|_{W_1(T)} \leqslant \frac{\delta}{1-\sigma} = C_T|\varepsilon|^{N+1}, \tag{4.68}
$$

where $C_T = \sigma_T\tilde K/(1-\sigma_T\eta_3)$.

On the other hand, the linear recurrent sequence $\{v_m\}$ defined by (4.54) converges strongly in the space $W_1(T)$ to the solution v of problem (4.22). Hence, letting $m \to +\infty$ in (4.68) gives

$$
\|\dot v\|_{L^\infty(0,T;L^2)} + \|v\|_{L^\infty(0,T;H^1)} \leqslant C_T|\varepsilon|^{N+1}
$$

or

$$
\left\|\dot u_\varepsilon - \sum_{i=0}^N \dot u_i\varepsilon^i\right\|_{L^\infty(0,T;L^2)} + \left\|u_\varepsilon - \sum_{i=0}^N u_i\varepsilon^i\right\|_{L^\infty(0,T;H^1)} \leqslant C_T|\varepsilon|^{N+1}. \tag{4.69}
$$

Thus, we have the following theorem.

Theorem 4. *Let* (A_1), (A_2), (A_7), *and* (A_8) *hold. Then there exist constants* $M > 0$ *and* $T > 0$ *such that, for every* ε, *with* $|\varepsilon| \leqslant 1$, *the problem* (P_ε) *has a unique weak solution* $u_\varepsilon \in W_1(M, T)$ *satisfying an asymptotic estimation up to order* $N + 1$ *as in* (4.69), *the functions* $u_0, u_1, \ldots, u_N$ *being the weak solutions of problems* $(P_0), (Q_1), \ldots, (Q_N)$, *respectively.*

Remark 2. • In the case of $B \equiv 1$, $B_1 \equiv 0$, $f_1 \equiv 0$, $f = f(t, u, u_t)$, $f \in C^{N+1}(\mathbb{R}_+ \times \mathbb{R}^2)$ and the Dirichlet homogeneous condition (1.8) standing for (1.2), we have obtained the results above in the paper [4].

• In the case of functions $f \in C^2([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, $f_1 \in C^1([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3)$ and $N = 1$, we have also obtained some results concerning in the papers [10,12,13] in the cases as follows:

(a) $B \equiv 1$, $B_1 \equiv 0$ (see [10]).
(b) $B_\varepsilon = b_0 + B(z) + \varepsilon B_1(z)$, where $b_0 > 0$ is a given constant and $B \in C^2(\mathbb{R}_+)$, $B_1 \in C^1(\mathbb{R}_+)$, $B \geqslant 0$, $B_1 \geqslant 0$ and (1.2) standing for the Dirichlet homogeneous condition (1.8) (see [12]).
(c) $B_\varepsilon = B(t, z) + \varepsilon B_1(t, z)$, $B \in C^2(\mathbb{R}_+^2)$, $B_1 \in C^1(\mathbb{R}_+^2)$, $B \geqslant b_0 > 0$, and $B_1 \geqslant 0$ (see [13]).

• In the case of $B_\varepsilon \equiv 1$, $F_\varepsilon = f(\varepsilon, x, t, u, u_x, u_t) + \varepsilon f_1(\varepsilon, x, t, u, u_x, u_t)$ with $f \in C^{N+1}([0, 1] \times [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, $f_1 \in C^N([0, 1] \times [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, we have also obtained some results above in the paper [14].

Remark 3. If the nonlinear term B is of the form $B(t, x, \|u_x\|^2)$, then this problem is open. However, with $B = B(t, x, \|u\|^2)$, Larkin [8] has studied problem $u_{tt} - B(t, x, \|u\|^2)u_{xx} + f(x, t, u_t) = f_1(x, t)$ associated with the Dirichlet homogeneous condition and initial condition.

Acknowledgments

The author sincerely thanks the referee who has contributed many exact opinions in order to perfect manuscript of this paper. Particularly, the referee has given to the author a remark about changing term B by $B(t, x, \|u_x\|^2)$.

References

[1] J. Boujot, P.N.D. Alain, J.P. Veyrier, Oscillateurs harmoniques faiblement perturbés: L'algorithme numérique des "par de géants," RAIRO Anal. Numér. 14 (1980) 3–23.
[2] G.F. Carrier, On the nonlinear vibrations problem of elastic string, Quart. J. Appl. Math. 3 (1945) 157–165.
[3] P.N.D. Alain, Sur un problème hyperbolique faiblement non-linéaire en dimension 1, Demonstratio Math. 16 (1983) 269–289.
[4] P.N.D. Alain, N.T. Long, Linear approximation and asymptotic expansion associated to the nonlinear wave equation in one dimension, Demonstratio Math. 19 (1986) 45–63.
[5] Y. Ebihara, L.A. Medeiros, M.M. Minranda, Local solutions for a nonlinear degenerate hyperbolic equation, Nonlinear Anal. 10 (1986) 27–40.

[6] M. Hosoya, Y. Yamada, On some nonlinear wave equation I: Local existence and regularity of solutions, J. Fac. Sci. Univ. Tokyo, Sect. I A, Math. 38 (1991) 225–238.
[7] G.R. Kirchhoff, Vorlesungen über Mathematiche Physik: Mechanik, Teuber, Leipzig, 1876, Section 29.7.
[8] N.A. Larkin, Global regular solutions for the nonhomogeneous Carrier equation, Math. Prob. Engrg. 8 (2002) 15–31.
[9] N.T. Long, et al., On the nonlinear vibrations equation with a coefficient containing an integral, Comp. Math. Math. Phys. 33 (1993) 1171–1178.
[10] N.T. Long, T.N. Diem, On the nonlinear wave equation $u_{tt} - u_{xx} = f(x, t, u, u_x, u_t)$ associated with the mixed homogeneous conditions, Nonlinear Anal. 29 (1997) 1217–1230.
[11] N.T. Long, T.M. Thuyet, On the existence, uniqueness of solution of the nonlinear vibrations equation, Demonstratio Math. 32 (1999) 749–758.
[12] N.T. Long, P.N.D. Alain, T.N. Diem, Linear recursive schemes and asymptotic expansion associated the Kirchhoff–Carrier operator, J. Math. Anal. Appl. 267 (2002) 116–134.
[13] N.T. Long, On the nonlinear wave equation $u_{tt} - B(t, \|u_x\|^2)u_{xx} = f(x, t, u, u_x, u_t)$ associated with the mixed homogeneous conditions, J. Math. Anal. Appl. 274 (2002) 102–123.
[14] N.T. Long, Asymptotic expansion of the solution for nonlinear wave equation with the mixed homogeneous conditions, Nonlinear Anal. 45 (2001) 261–272.
[15] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non-linéaires, Dunod/Gauthier–Villars, Paris, 1969.
[16] L.A. Medeiros, On some nonlinear perturbation of Kirchhoff–Carrier operator, Comp. Appl. Math. 13 (1994) 225–233.
[17] L.A. Medeiros, J. Limaco, S.B. Menezes, Vibrations of elastic strings: Mathematical aspects, Part one, J. Comput. Anal. Appl. 4 (2) (2002) 91–127.
[18] L.A. Medeiros, J. Limaco, S.B. Menezes, Vibrations of elastic strings: Mathematical aspects, Part two, J. Comput. Anal. Appl. 4 (3) (2002) 211–263.
[19] E.L. Ortiz, P.N.D. Alain, Linear recursive schemes associated with some nonlinear partial differential equations in one dimension and the Tau method, SIAM J. Math. Anal. 18 (1987) 452–464.
[20] S.I. Pohozaev, On a class of quasilinear hyperbolic equation, Math. USSR Sb. 25 (1975) 145–158.
[21] P.A. Raviart, J.M. Thomas, Introduction à l'analyse numérique des equations aux dérivées partielles, Masson, Paris, 1983.