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On the Finite Completion of Partial Latin Cubes

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INTRODUCTION

Trevor Evans has shown in [1] that any $n \times n$ partial latin square can be embedded in a $2n \times 2n$ latin square. Apart from its combinatorial interest, this result has general algebraic consequences (for example, that finitely presented loops and quasigroups are residually finite, hopfian, and have a solvable word problem [4, 14]). We examine here the higher dimensional analogues of these results.

Unfortunately, any attempt directly to transfer the construction of Evans to the case of latin cubes or hyper-cubes encounters a fundamental combinatorial obstacle: in essence, what one confronts is a special instance of the classic unsolved problem of finding a common transversal for three or more families of sets [6, 9, 10]. However, this combinatorial obstacle can be circumvented by the device of composing latin boxes of various dimensions, and we are able to prove that any finite partial latin cube (or hyper-cube) can be embedded in a finite latin cube (hyper-cube). The algebraic consequences for finitely presented n-quasigroups are the same as those already mentioned for loops and quasigroups. There remains the problem of obtaining a "minimal" embedding corresponding to the result in [1].

LATIN BOXES

Let k be a positive integer, and let $S_0, S_1, \ldots, S_k$ be finite sets. The mapping

$$\alpha\colon \left(\prod_{i=1}^{k} S_i\right) \to S_0$$

will be called a *k-dimensional latin box* if the following condition holds:

Whenever $x_0 = \alpha(x_1, x_2, \ldots, x_k)$ and $x_0' = \alpha(x_1', x_2', \ldots, x_k')$, then any k of the equations

$$x_0 = x_0', \qquad x_1 = x_1', \qquad x_2 = x_2', \ldots, x_k = x_k'$$

imply also the remaining equation.

We shall refer to the above condition as the *latin property*. The maximum cardinality of the sets S_i will be called the *order* of α (and in view of the latin property, this number will always be identical with the cardinality of the set S_0). The k-term sequence of integers $(n_1, n_2, \ldots, n_k)$, where n_i is the cardinality of S_i, will be called the *type* of α. Finally, in the special case that each of the sets S_i is an initial segment of the positive integers, then the latin box α will be called *proper*. Any latin box may be represented by an isotopically equivalent proper latin box.

The usual latin configurations are particular cases of our k-dimensional latin box. For example, an *$r \times s$ latin rectangle* based on the symbols $1, 2, \ldots, n$ is a proper 2-dimensional latin box of order n and type (r, s); and in case $r = s = n$, the latin rectangle is called a *latin square*. A *latin cube* is a proper 3-dimensional latin box of order n and type (n, n, n). Latin hyper-cubes of dimension k are defined similarly. Finally, by a *partial* k-dimensional latin box, we mean a function α: $P \to S_0$ satisfying the latin property, whose domain P is some subset of the cartesian product $(\prod_{i=1}^{k} S_i)$.

In the sequel we shall employ certain known results concerning the extendibility of latin rectangles to latin squares. In fact, some of our theorems may be roughly viewed as higher dimensional analogues of these:

(i) (M. Hall [5]) *If α is a latin rectangle of order n and type (n, r), then there exists a latin square α^* of order n which includes α.*

(ii) (T. Evans [1]) *If α is a latin rectangle of order n and type (r, s), then there exists a latin square α^* of order t which includes α, for any $t \geqslant 2n$.*

For simplicity of exposition in presenting our analogues of these results and their application to the completion problem at hand, we shall follow the historical order in which the two-dimensional versions appeared [5, 8, 1]. Thus we consider the last steps first.

Basic Embedding Results

We now state and prove the two new results which yield our finite embedding theorem for partial latin cubes.

THEOREM 1. *Let α be a 3-dimensional latin box of order n and type (n, n, r). Then α can be embedded in a latin cube of order n^2.*

Proof. We may assume without loss of generality that α is a proper latin box. Then, for each element x in S_1, let α_x be the latin rectangle of order n and type (n, r) which is obtained from α by putting

$$\alpha_x(x', x'') = \alpha(x, x', x'')$$

for each x' in S_2 and each x'' in S_3. By (i) each such function α_x can be completed to a latin square $\alpha_x{}^*$ of order n.

Now let β be any latin cube of order n, and let γ be any latin square of order n. Let $N = \{1, 2,..., n\}$ and define the 3-dimensional function $\lambda\colon N^2 \times N^2 \times N^2 \to N^2$ as follows:

$$\lambda((x_1, y_1), (x_2, y_2), (x_3, y_3)) = \begin{cases} (\gamma(y_1, \alpha(x_1, x_2, x_3)), \beta(y_1, y_2, y_3)), & \text{if } x_3 \leqslant r \\ (\gamma(y_1, \alpha^*_{x_1}(x_2, x_3)), \beta(x_1, y_2, y_3)), & \text{otherwise.} \end{cases}$$

To check that λ is a latin box, we suppose that we have the equation

$$\lambda((x_1, y_1), (x_2, y_2), (x_3, y_3)) = \lambda((x_1', y_1'), (x_2', y_2'), (x_3', y_3'))$$

and, further, that we have at least two of the following three equations:

$$(x_1, y_1) = (x_1', y_1'), \tag{1}$$

$$(x_2, y_2) = (x_2', y_2'), \tag{2}$$

$$(x_3, y_3) = (x_3', y_3'). \tag{3}$$

We then show that in fact all three of these equations must hold. There are three cases to consider and the verification is routine.

Case 1. Equations (1) and (2) hold. Remembering the relation of the $\alpha_x{}^*$'s to α, we have by our initial hypothesis and the definition of λ the equation

$$\gamma(y_1, \alpha^*_{x_1}(x_2, x_3)) = \gamma(y_1', \alpha^*_{x_1'}(x_2', x_3')).$$

By (1) we have $y_1 = y_1'$, so by the latin property for γ we get $\alpha^*_{x_1}(x_2, x_3) = \alpha^*_{x_1'}(x_2', x_3')$. But (1) and (2) imply $x_1 = x_1'$ and $x_2 = x_2'$, so by the latin property for the map $\alpha^*_{x_1} = \alpha^*_{x_1'}$, we get also $x_3 = x_3'$. Now this shows we need consider only the two possibilities: either $x_3 = x_3' \leqslant r$ or $x_3 = x_3' > r$. Thus, by definition of λ, we will have either $\beta(y_1, y_2, y_3) = \beta(y_1', y_2', y_3')$ or $\beta(x_1, y_2, y_3) = \beta(x_1', y_2', y_3')$. But since (1) and (2) imply the equalities $x_1 = x_1'$, $y_1 = y_1'$, and $y_2 = y_2'$, we will get in either case, by the latin property for β, that the equality $y_3 = y_3'$ holds also. Finally, the two results $x_3 = x_3'$ and $y_3 = y_3'$ combine to give Eq. (3).

Case 2. Equations (2) and (3) hold. Since (3) implies $x_3 = x_3'$, we need only consider the two possibilities: either $x_3 = x_3' \leqslant r$ or $x_3 = x_3' > r$. We examine these separately. If $x_3 = x_3' \leqslant r$ holds, then our initial hypothesis and the definition of λ give us the pair of equations:

$$\gamma(y_1, \alpha(x_1, x_2, x_3)) = \gamma(y_1', \alpha(x_1', x_2', x_3')),$$
$$\beta(y_1, y_2, y_3) = \beta(y_1', y_2', y_3').$$

Since (2) and (3) imply $y_2 = y_2'$ and $y_3 = y_3'$, the latin property of β implies $y_1 = y_1'$ also. But then the latin property of γ implies that $\alpha(x_1, x_2, x_3) = \alpha(x_1', x_2', x_3')$ holds. Since (2) and (3) also imply $x_2 = x_2'$ and $x_3 = x_3'$, the latin property of α then gives $x_1 = x_1'$. Finally, the two results $x_1 = x_1'$ and $y_1 = y_1'$ combine to give Eq. (1) in this case. On the other hand, if $x_3 = x_3' > r$ holds, then our initial hypothesis and the definition of λ give us the pair of equations

$$\gamma(y_1, \alpha_{x_1}^*(x_2, x_3)) = \gamma(y_1', \alpha_{x_1'}^*(x_2', x_3')),$$
$$\beta(x_1, y_2, y_3) = \beta(x_1', y_2', y_3').$$

Since (2) and (3) imply $y_2 = y_2'$ and $y_3 = y_3'$, the latin property of β implies also $x_1 = x_1'$. Thus $\alpha_{x_1}^* = \alpha_{x_1'}^*$, and since (2) and (3) imply $x_2 = x_2'$ and $x_3 = x_3'$, we must have $\alpha_{x_1}^*(x_2, x_3) = \alpha_{x_1'}^*(x_2', x_3')$. But the latin property of γ then gives $y_1 = y_1'$. Finally, the two results $x_1 = x_1'$ and $y_1 = y_1'$ combine to give Eq. (1) in this case, too.

Case 3. Equations (1) and (3) hold. Here the argument that (2) must hold also is similar to Case 1 above.

Having verified that λ is a 3-dimensional latin box of order n^2 and type (n^2, n^2, n^2), we next observe that for any fixed choice of y_1, y_2, y_3, the correspondence

$$\alpha(x_1, x_2, x_3) \leftrightarrow \lambda((x_1, y_1), (x_2, y_2), (x_3, y_3))$$

will describe an isomorphic embedding of the original latin box α in the latin box λ. Thus if λ^* is any proper latin box isomorphic to λ, then we readily obtain the desired embedding of α in a latin cube of order n^2. This completes the proof of Theorem 1.

THEOREM 2. *Let α be a 3-dimensional latin box of order n and type (r, r, r). Then α can be embedded in a latin cube of order t^4, for any $t \geqslant 2n$.*

Proof. Let t be a fixed positive integer satisfying $t \geqslant 2n$. We show α can be embedded in a 3-dimensional latin box of order t^2 and type (t^2, t^2, r), then apply Theorem 1. As before, we may assume at the outset, without loss of generality, that the given latin box α is proper. Then, for each element x in S_3, we let α_x be the latin rectangle of order n and type (r, r) which is obtained from α by putting

$$\alpha_x(x', x'') = \alpha(x', x'', x)$$

for each x' in S_1 and each x'' in S_2. By (ii) each such function α_x can be completed to a latin square $\alpha_x{}^*$ of order t.

Now let β be any latin cube of order t, and let γ be any latin square of order t. Let $T = \{1, 2,..., t\}$, let $R = \{1, 2,..., r\}$, and define the 3-dimensional function μ: $T^2 \times T^2 \times R \to T^2$ as follows:

$$\mu(x_1, y_1), (x_2, y_2), x_3)$$
$$= \begin{cases} ((\alpha(x_1, x_2, x_3), \gamma(y_1, y_2)), & \text{if } x_1 \leqslant r \text{ and } x_2 \leqslant r \\ ((\alpha_{x_3}^*(x_1, x_2), \beta(y_1, y_2, x_3)), & \text{if } x_1 > r \text{ or } x_2 > r. \end{cases}$$

The verification that this function μ is a latin box of order t^2 and type (t^2, t^2, r) is straightforward (and quite similar to the proof for Theorem 1). It remains to observe that for any fixed choice of y_1, y_2, the correspondence

$$\alpha(x_1, x_2, x_3) \leftrightarrow \mu((x_1, y_1), (x_2, y_2), x_3)$$

will provide an isomorphic embedding of the given latin box α in the latin box μ. Applying Theorem 1 to μ then completes the proof of Theorem 2.

Higher Dimensions

By using essentially the same techniques as in the two preceeding proofs, we obtain k-dimensional analogues of Theorems 1 and 2, for each integer $k > 3$.

THEOREM 3. *Let α be a k-dimensional latin box of order n and type $(n, n,..., n, r)$. Then α can be embedded in a k-dimensional latin hyper-cube of order n^2.*

Proof. Although the notation becomes cumbersome with the increase in dimension, the idea of the argument here is the same as for Theorem 1, but with a few rather obvious modifications in detail. For example, when

$k = 4$, the role of the α_x's in Theorem 1 would here be played by functions α_{x_1,x_2} defined by $\alpha_{x_1,x_2}(x_3, x_4) = \alpha(x_1, x_2, x_3, x_4)$. Accordingly, the k-dimensional function λ would then be defined as:

$$\lambda((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4))$$
$$= \begin{cases} (\gamma(y_1, y_2, \alpha(x_1, x_2, x_3, x_4)), \beta(y_1, y_2, y_3, y_4)), & \text{if } x_3, x_4 \leqslant r, \\ (\gamma(y_1, y_2, \alpha^*_{x_1,x_2}(x_3, x_4)), \beta(x_1, x_2, y_3, y_4)), & \text{otherwise,} \end{cases}$$

where β is a 4-dimensional latin hyper-cube, γ is a latin cube, and $\alpha^*_{x_1,x_2}$ is the completion of α_{x_1,x_2} to a latin square obtained via (i). We omit further details.

THEOREM 4. *There exists a positive integer p such that, if α is any k-dimensional latin box of order n and type $(r, r,..., r, r)$, then α can be embedded in a k-dimensional latin hyper-cube of order p.*

Proof. Here we argue by induction on k, where $k \geqslant 4$, but otherwise the proof is the same as for Theorem 2. From the given k-dimensional function α (which we may assume is proper), we form truncated functions α_x, for each $x \leqslant r$, defined by $\alpha_x(x_1, x_2,..., x_{k-1}) = \alpha(x_1, x_2,..., x_{k-1}, x)$. By the inductive hypothesis (or by Theorem 2 in the initial case when $k = 4$), there exists a positive integer q such that each of the functions α_x can be completed to a proper $(k-1)$-dimensional latin box of order q and type $(q, q,..., q)$. Letting $Q = \{1, 2,..., q\}$ and $R = \{1, 2,..., r\}$, we then define the k-dimensional function μ: $Q^2 \times Q^2 \times \cdots \times Q^2 \times R \to Q^2$ as follows:

$$\mu((x_1, y_1), (x_2, y_2),..., (x_{k-1}, y_{k-1}), x_k)$$
$$= \begin{cases} (\alpha(x_1, x_2,..., x_{k-1}, x_k), \gamma(y_1, y_2,..., y_{k-1})), & \text{if } x_1, x_2,..., x_{k-1} \leqslant r, \\ (\alpha^*_{x_k}(x_1, x_2,..., x_{k-1}), \beta(y_1, y_2,..., y_{k-1}, x_k)), & \text{otherwise,} \end{cases}$$

where β and γ are latin hyper-cubes of dimensions k and $k-1$, respectively. The verification that this μ satisfies the latin property is again straightforward, and for any fixed choice of the elements $y_1, y_2,..., y_{k-1}$, the correspondence

$$\alpha(x_1, x_2,..., x_k) \leftrightarrow \mu((x_1, y_1), (x_2, y_2),..., (x_{k-1}, y_{k-1}), x_k)$$

provides an embedding of the original latin box α in the latin box μ. Finally, applying Theorem 3 to μ, we obtain an embedding of α in a k-dimensional latin hyper-cube of order $p = q^4$, which completes the inductive step in the proof.

Embedding Partial Latin Cubes

Let α be an arbitrary partial latin cube of order r. We wish to show that α can be finitely completed. To accomplish this, we first construct a 3-dimensional latin box α^* of order $2r$ and type (r, r, r) which includes α, and then we apply Theorem 2. In this manner we obtain

THEOREM 5. *Any finite partial latin cube can be finitely completed.*

Proof. The required function α^* is readily obtained by selecting an arbitrary latin cube β of order r and defining α^* so that $\alpha^*(x, x', x'') = \alpha(x, x', x'')$ if α assigns a value to the cell (x, x', x''); otherwise, put $\alpha^*(x, x', x'') = r + \beta(x, x', x'')$. Because the ranges of the functions α and β contain only the numbers $1, 2,..., r$ by hypothesis, the device of adding r to the values of β serves to insure that the latin property for α^* is not violated. Therefore, we are provided by Theorem 2 with a latin cube of order $(2r)^4$ which includes α^* and hence also α.

In the same fashion, as a corollary to Theorem 4, we also obtain the corresponding result for higher dimensions:

THEOREM 6. *Any finite partial k-dimensional latin hyper-cube can be finitely completed.*

The proof is a direct extension of the preceeding argument and is omitted.

Algebraic Consequences

An n-ary algebra (A, π) with the property that, in the equation $\pi(x_1, x_2, ..., x_n) = x_0$, any n elements uniquely determine the $(n + 1)$st, is called an *n-quasigroup* (also called a *n-skein* by Evans in [2]). Such algebras have been studied by Radó and Hosszú [11], Sandik [12], Belousov [13], and others. Note that if an n-quasigroup (A, π) has finite order, then its operation π is an n-dimensional latin box.

As a consequence of the procedure for solving the word problem for finitely presented n-quasigroups (analogous to the algorithm for loops described in [3]), T. Evans has shown that any finite partial n-quasigroup can be embedded in a complete n-quasigroup, though the completion obtained by this procedure will be of infinite order. Our results (Theorems 5 and 6) provide a finite completion for any finite partial n-quasigroup. Thus, besides giving an alternative solution to the word problem, our results imply that finitely presented n-quasigroups are both residually finite and hopfian (see [4, 14] for details).

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