Finite orthodox locally idempotent semigroups having no finite basis of biidentities

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Abstract

A finitely generated existence variety of orthodox locally idempotent semigroups is constructed which has no finite basis of biidentities within the class of all orthodox semigroups. In addition, ordinary semigroup identities valid in this existence variety also cannot be finitely based. This yields examples of finite orthodox locally idempotent semigroups having both no finite basis of biidentities and also no finite basis of identities in the usual sense.

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Introduction

Existence varieties (briefly e-varieties) of regular semigroups, that is, classes of regular semigroups closed under the formation of homomorphic images, regular subsemigroups and direct products, were introduced in [3] and, in the special case of orthodox semigroups, independently under the name of bivarieties in [5]. Furthermore, in the latter paper, the notions of bifree objects, biidentities and biinvariant congruences were introduced for orthodox semigroups in such a way that a theory arose which properly generalizes the theory of varieties of inverse semigroups. Thus, in particular, existence varieties of orthodox semigroups turn out to be just classes determined by sets of biidentities (within the class $O$ of all orthodox semigroups). Besides varieties of inverse semigroups, existence varieties of orthodox semigroups include also all varieties of orthodox completely regular

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This circumstance makes it possible to treat various known results connected with varieties of each of these two types within this common framework.

This remark applies, in particular, to the results on bases of identities of varieties of both inverse semigroups and orthodox completely regular semigroups, and, accordingly, also to the results on bases of identities of individual semigroups of these two types. Thus, for instance, it is known from [7] that all finite strict inverse semigroups have a finite basis of identities. Remember that finite strict inverse semigroups are just finite inverse semigroups all of whose submonoids are at the same time completely regular. On the other hand, from [12] it is known that all finite orthodox completely regular semigroups have a finite basis of identities. Now both these results can be viewed as statements asserting the existence of finite bases of biidentities (within the class $\mathcal{O}$) for the mentioned finite semigroups.

This situation prompts to consider, more generally, finite orthodox semigroups all of whose submonoids are completely regular, that is, finite orthodox locally completely regular semigroups, in this context. In the present paper, however, we show that the two results cited above do not allow for a common generalization even in the case of combinatorial semigroups. We provide finite orthodox semigroups all of whose submonoids are bands, that is, finite orthodox locally idempotent semigroups, which have no finite basis of biidentities (within the class $\mathcal{O}$). More concretely, we construct a finitely generated e-variety of orthodox locally idempotent semigroups which cannot be determined by any finite set of biidentities. Finite generators of this e-variety then represent examples of the semigroups just specified. It is worth noting that this e-variety has the property that its idempotent members are left regular bands.

In addition, our construction is of such a kind that it permits to derive analogous consequences also in the case when orthodox semigroups are considered as ordinary semigroups. This is the viewpoint adopted in §8 of the survey article [14] where analogous issues as above are discussed. It follows that the finite orthodox locally idempotent semigroups mentioned in the previous paragraph also do not have any finite basis of identities in the usual sense. This provides a negative answer to Question 8.2 in [14].

This paper begins in Section 1 with a summary of the theory of existence varieties of orthodox semigroups borrowed from [5]. Then, we are concerned in Section 2 with the e-variety $\mathcal{OLB}$ of all orthodox locally idempotent semigroups and with its sub-e-varieties of the following form. For any variety $\mathcal{A}$ of bands, we consider the e-variety $\mathcal{AOLB}$ of all orthodox locally idempotent semigroups whose idempotents form a band from $\mathcal{A}$. We deal with the word problem for the bifree objects in $\mathcal{AOLB}$, that is, we outline the way how an effective description of the biinvariant congruences corresponding to the e-varieties $\mathcal{AOLB}$ can be obtained.

Central role in the next deliberations of this paper is played by the construction presented in Section 3. In that section, we introduce one particular biinvariant congruence corresponding to a certain sub-e-variety of $\mathcal{OLB}$, which we will denote by $\mathcal{OZ}$. More precisely, $\mathcal{OZ}$ is a sub-e-variety of $\mathcal{AOLB}$ for $\mathcal{A}$ equal to the variety of all left regular bands, and, at the same time, $\mathcal{OZ}$ contains all left regular bands. In Section 4, we show that this e-variety $\mathcal{OZ}$ is finitely generated. In Section 5, we provide an infinite sequence of biidentities valid in $\mathcal{OZ}$, which has the following property. From any set of biidentities valid in $\mathcal{OZ}$ and containing only a finite number of variables, only finitely many members
of this sequence can be deduced (within the class \(\mathcal{OLB}\)). This verifies that biidentities satisfied in \(\mathcal{OZ}\) cannot be finitely based. Moreover, all members of the mentioned sequence are, in fact, only ordinary identities. This implies that for ordinary identities satisfied in \(\mathcal{OZ}\) an analogous conclusion holds.

Thus, as mentioned already, finite generators of the e-variety \(\mathcal{OZ}\) are examples of finite orthodox locally idempotent semigroups having no finite basis of biidentities (within \(\mathcal{O}\)). It should be pointed out, however, that, on the other hand, there exist examples of finite orthodox locally idempotent semigroups which are neither strict nor idempotent and which have a finite basis of biidentities (within \(\mathcal{O}\)). Such semigroups can be obtained in the following way. For every variety \(\mathcal{A}\) of bands, the e-variety \(\mathcal{AO\mathcal{LB}}\) can be determined by a finite set of biidentities, and, for \(\mathcal{A}\) equal to the variety of all left regular bands, at least, the methods of Section 4 make it possible to show that \(\mathcal{AO\mathcal{LB}}\) can also be finitely generated. Finite generators of \(\mathcal{AO\mathcal{LB}}\) then are examples of the semigroups just described. This documents the diversity of the situation in the area of bases of biidentities of finite orthodox locally idempotent semigroups.

Finally, at the end of Section 5, the tools provided in this paper are used to deduce some consequences regarding the lattice of sub-e-varieties of \(\mathcal{O\mathcal{LB}}\) itself. It follows that for any variety \(\mathcal{A}\) of bands containing either all left regular bands or all right regular bands, the interval between \(\mathcal{A}\) and \(\mathcal{AO\mathcal{LB}}\) contains infinitely many e-varieties. In another paper \[4\] it is shown that the uppermost of these intervals, that is, the interval between the variety \(\mathcal{B}\) of all bands and the e-variety \(\mathcal{O\mathcal{LB}}\) is actually uncountable.

1. Existence varieties of orthodox semigroups

The purpose of this section is to recall the fundamentals of the theory of existence varieties of orthodox semigroups from \[5\], to which paper the reader is referred for a more comprehensive exposition. This theory is also included in \[6\] where it is presented within a more general framework. As a supplement, we append a theorem characterizing biidentities that are consequences of a given set of biidentities within a given e-variety of orthodox semigroups. We assume familiarity with the basic facts on orthodox semigroups. At the beginning, we quote from \[2,\,\text{Lemma 1}\] the following observation which will come in handy below.

**Result 1.1** \[2\]. If \(\varphi : S \to T\) is a surjective homomorphism of regular semigroups, then for any mutual inverses \(c, d \in T\), there exist mutual inverses \(a, b \in S\) such that \(\varphi(a) = c\) and \(\varphi(b) = d\).

For any class \(\mathcal{V}\) of regular semigroups, we denote by \(\mathbf{H}(\mathcal{V}),\, \mathbf{S}_r(\mathcal{V})\) and \(\mathbf{P}(\mathcal{V})\), respectively, the classes of all homomorphic images, regular subsemigroups and direct products of semigroups in \(\mathcal{V}\). A class \(\mathcal{V}\) of regular semigroups satisfying \(\mathbf{H}(\mathcal{V}) \subseteq \mathcal{V},\, \mathbf{S}_r(\mathcal{V}) \subseteq \mathcal{V}\) and \(\mathbf{P}(\mathcal{V}) \subseteq \mathcal{V}\) is called an existence variety (or an e-variety). Note that if \(\mathcal{V}\) is any class of orthodox semigroups then the smallest e-variety containing \(\mathcal{V}\) is just \(\mathbf{HS}\), \(\mathbf{P}(\mathcal{V})\), according to \[5,\,\text{Proposition 1.4}\].
We go on with some formal definitions. Let \( X \) be a non-empty set of variables. Often in the subsequent considerations, this set will be assumed infinite. Let

\[
X' = \{ x' : x \in X \}
\]

be a disjoint copy of \( X \). We will work with

\[
F(X) - \text{the free semigroup on } X
\]

and with

\[
F'(X) - \text{the free semigroup on } X \cup X'.
\]

Elements of these semigroups will be called words; in addition, we will use the symbol \( \square \) for the empty word. Note that we will always treat \( F'(X) \) as an ordinary semigroup, even though an involution \( \cdot : F'(X) \to F'(X) \) can be introduced by the formula

\[
(y_1 \cdots y_n)' = y_n' \cdots y_1' \quad (y_1, \ldots, y_n \in X \cup X'),
\]

where, for any \( y \in X \cup X' \), we use the notation

\[
y' = \begin{cases} x' & \text{if } y = x \in X, \\ x & \text{if } y = x' \in X'. \end{cases}
\]

But this involution will be used only as an auxiliary instrument.

We continue with some notions and notations coming from [5]. Let \( X \) be a non-empty set. Given a regular semigroup \( S \), a mapping \( \vartheta : X \cup X' \to S \) such that \( \vartheta(x) \) and \( \vartheta(x') \) are mutual inverses in \( S \) for all \( x \in X \) is called a matched mapping.

By a biidentity over \( X \) we mean any pair \( u \sim v \) of words \( u, v \in F'(X) \). We say that a biidentity \( u \sim v \) is satisfied in an orthodox semigroup \( S \) if, for any matched mapping \( \vartheta : X \cup X' \to S \), we have \( \vartheta(u) = \vartheta(v) \) where \( \vartheta : F'(X) \to S \) is the homomorphism extending \( \vartheta \). A biidentity \( u \sim v \) is satisfied in a class \( V \) of orthodox semigroups if it is satisfied in each member of \( V \).

For any non-empty set \( X \) and for any class \( V \) of orthodox semigroups, we denote by

\[
\rho(V, X) - \text{the set of all biidentities over } X \text{ which are satisfied in all semigroups of } V.
\]

Then \( \rho(V, X) \) can be viewed as a binary relation on \( F'(X) \). (Conversely, binary relations on \( F'(X) \) can be considered as sets of biidentities over \( X \).) Clearly, \( \rho(V, X) \) is a congruence on \( F'(X) \). Often we will write briefly \( \rho(V) \) instead of \( \rho(V, X) \).

Let \( V \) be a class of regular semigroups. A bifree object in \( V \) on a non-empty set \( X \) is defined to be a semigroup \( S \in V \) together with a matched mapping \( \iota : X \cup X' \to S \) having the following property: for any semigroup \( T \in V \) and any matched mapping \( \vartheta : X \cup X' \to T \), there exists a unique homomorphism \( \varphi : S \to T \) which, when composed with \( \iota \), yields \( \vartheta \).

Now, from [5, Theorem 1.9], we know the following fact.
**Result 1.2** [5]. In any class $\mathcal{V}$ of orthodox semigroups satisfying $S_r(\mathcal{V}) \subseteq \mathcal{V}$ and $P(\mathcal{V}) \subseteq \mathcal{V}$, there exists a bifree object on any non-empty set $X$, and it is isomorphic to $F'(X)/\rho(\mathcal{V})$.

**Remark.** In this result, for any $x \in X$, $x\rho(\mathcal{V})$ and $x'\rho(\mathcal{V})$ are mutual inverses in $F'(X)/\rho(\mathcal{V})$.

For any set $\Sigma$ of biidentities and for any $e$-variety $\mathcal{V}$ of orthodox semigroups, we denote by $[\Sigma]_\mathcal{V}$—the class of all semigroups from $\mathcal{V}$ in which all biidentities from $\Sigma$ are satisfied.

Then $[\Sigma]_\mathcal{V}$ is itself an $e$-variety of orthodox semigroups. This can be verified using Result 1.1.

For $e$-varieties of orthodox semigroups, the following Birkhoff-type theorem holds. It was found in [5, Theorem 1.10]. Remember that $O$ stands for the $e$-variety of all orthodox semigroups.

**Result 1.3** [5]. A class $\mathcal{V}$ of orthodox semigroups is an $e$-variety if and only if there exists a set $\Sigma$ of biidentities such that $\mathcal{V} = [\Sigma]_O$. In this connection, it is convenient to introduce also the following terminology. Let $\Sigma$ be a set of biidentities and let $\mathcal{U}$ and $\mathcal{V}$ be $e$-varieties of orthodox semigroups such that $\mathcal{U} \subseteq \mathcal{V}$. We say that $\Sigma$ is a basis of biidentities of $\mathcal{U}$ within $\mathcal{V}$ if $\mathcal{U} = [\Sigma]_\mathcal{V}$. We say that a biidentity $u \equiv v$ is a consequence of the biidentities of $\Sigma$ within $\mathcal{V}$ if $u \equiv v$ is satisfied in the class $[\Sigma]_\mathcal{V}$. Such consequences will be dealt with in the last theorem of this section.

Let $X$ and $Y$ be non-empty sets. By a substitution we mean any mapping $\psi: X \cup X' \to F'(Y)$. Such a mapping extends in a unique way to a homomorphism of the semigroup $F'(X)$ into $F'(Y)$. For the sake of simplicity, we will denote this homomorphism also as $\psi: F'(X) \to F'(Y)$.

Let $\mathcal{V}$ be any $e$-variety of orthodox semigroups, We say that a substitution $\psi: X \cup X' \to F'(Y)$ is regular relatively to $\mathcal{V}$ if, for all $x \in X$, we have

$$\psi(x)\psi(x')\rho(\mathcal{V}) \psi(x) \text{ and } \psi(x')\psi(x)\rho(\mathcal{V}) \psi(x'),$$

that is, if $\psi(x)\rho(\mathcal{V})$ and $\psi(x')\rho(\mathcal{V})$ are mutual inverses in $F'(Y)/\rho(\mathcal{V})$.

Let $X$ be a non-empty set. We say that a congruence $\rho$ on $F'(X)$ is biinvariant if $\rho(O, X) \subseteq \rho$ and $\rho$ is closed with respect to all substitutions $\psi: X \cup X' \to F'(X)$ that are regular relatively to the $e$-variety $O$. The latter condition means that whenever $u, v \in F'(X)$ satisfy $u \rho v$ and $\psi$ is a substitution of the mentioned kind then also $\psi(u) \rho \psi(v)$ holds. But note that biinvariant congruences actually satisfy stronger closure conditions than is the one just named. If $\rho$ is a biinvariant congruence on $F'(X)$ and if $\mathcal{V}$ is any $e$-variety of orthodox semigroups such that $\rho(\mathcal{V}, X) \subseteq \rho$ then $\rho$ is closed in the given sense with respect to all substitutions $\psi: X \cup X' \to F'(X)$ that are regular relatively to $\mathcal{V}$. This follows from Result 1.1 applied to the natural homomorphism $F'(X)/\rho(O) \to F'(X)/\rho(\mathcal{V})$. 

It is easy to see that, for any non-empty set $X$ and for any class $\mathcal{V}$ of orthodox semigroups, the relation $\rho(\mathcal{V}, X)$ is a biinvariant congruence on $F'(X)$. Moreover, from [5, Corollary 1.12], we have the following one-to-one correspondence between e-varieties of orthodox semigroups and biinvariant congruences:

**Result 1.4** [5]. For any infinite set $X$, the rules

$$\mathcal{V} \mapsto \rho(\mathcal{V}, X) \quad \text{and} \quad \rho \mapsto [\rho]_\Box$$

define mutually inverse, order reverting bijections between the lattice of all e-varieties of orthodox semigroups and the lattice of all biinvariant congruences on $F'(X)$.

We conclude this section with the following general form of the Completeness Theorem for Biequational Logic of orthodox semigroups.

**Theorem 1.5.** Let $\mathcal{V}$ be any e-variety of orthodox semigroups. Let $X$ be a non-empty set. Let $\Sigma$ be a set of biidentities over $X$. Then, for any two words $u, v \in F'(X)$, the following holds. The biidentity $u \equiv v$ is a consequence of the biidentities of $\Sigma$ within the class $\mathcal{V}$, that is, we have $u \rho([\Sigma]_{\mathcal{V}}) v$, if and only if either $u \rho(\mathcal{V}) v$ or there exist $\ell \geq 1$ and words $r_1, \ldots, r_\ell, s_1, \ldots, s_\ell \in F'(X)$ such that

$$u \rho(\mathcal{V}) r_1, s_1 \rho(\mathcal{V}) r_2, \ldots, s_{\ell-1} \rho(\mathcal{V}) r_\ell, s_\ell \rho(\mathcal{V}) v,$$

and for each $i = 1, \ldots, \ell$ there exist words $a_i, b_i \in F'(X) \cup \{\Box\}$, a biidentity $c_i \equiv d_i$ in $\Sigma$ and a substitution $\psi_i : X \cup X' \rightarrow F'(X)$ which is regular relatively to $\mathcal{V}$ such that

$$\{r_i, s_i\} = \{a_i \psi_i(c_i) b_i, a_i \psi_i(d_i) b_i\}.$$

**Remark.** If $\mathcal{W}$ is any e-variety of orthodox semigroups satisfying $\mathcal{V} \subseteq \mathcal{W}$ then, applying Result 1.1 to the natural homomorphism $F'(X)/\rho(\mathcal{W}) \rightarrow F'(X)/\rho(\mathcal{V})$, we see that we can replace the requirement that the substitutions $\psi_i$ for $i = 1, \ldots, \ell$ in the above theorem should be regular relatively to $\mathcal{V}$ with the stronger demand that these substitutions should be regular relatively to $\mathcal{W}$. This circumstance will be occasionally exploited later.

**Proof.** All biidentities $u \equiv v$ which can be obtained in the way described in this theorem are obviously consequences of $\Sigma$ within $\mathcal{V}$. Thus, in order to verify the statement of the theorem, it is enough to check that the relation $\kappa$ consisting of all pairs $(u, v)$ of words of $F'(X)$ satisfying the condition in this theorem is a congruence on $F'(X)$ containing $\rho(\mathcal{V}, X)$, and that the semigroup $F'(X)/\kappa$ satisfies all biidentities from $\Sigma$. The verification of the latter requirement may need yet another application of Result 1.1, this time to the natural homomorphism $F'(X)/\rho(\mathcal{V}) \rightarrow F'(X)/\kappa$. $\square$
2. Bifree objects in existence varieties of orthodox locally idempotent semigroups

A regular semigroup $S$ is said to be \textit{locally idempotent} if all submonoids of $S$ are idempotent. Equivalently, this means that for all idempotents $e \in S$, the submonoids $eSe$ are bands.

We have already denoted by $OLB$ the $e$-variety of all orthodox locally idempotent semigroups. In addition, for any variety $A$ of bands, we have denoted by $AOLB$ the class of all orthodox locally idempotent semigroups whose idempotents form a band from $A$. It is readily seen that this class is an $e$-variety of orthodox semigroups.

Moreover, it is easy to verify that the mapping sending each $e$-variety $V$ of orthodox locally idempotent semigroups to its intersection $V \cap B$ with the variety of all bands is a complete lattice homomorphism of the lattice of all sub-$e$-varieties of $OLB$ onto the lattice of all varieties of bands. It induces a complete lattice congruence on the former lattice, and, for any variety $A$ of bands, the interval $[A, AOLB]$ is the congruence class that maps to $A$.

In this section, we indicate in which way the word problem for the bifree objects in the $e$-variety $AOLB$ for every band variety $A$ can be solved. If $A$ is any of the four subvarieties of the variety of rectangular bands then, clearly, $AOLB$ coincides with $A$. Thus we have only to deal with the remaining band varieties $A$, that is, with those varieties which contain all semilattices.

Before doing so, however, we have to introduce a few more notations and we have to recall from [5] some results on the word problem for the bifree objects in varieties of bands.

Along with our non-empty (preferably infinite) set $X$ of variables and its disjoint copy $X' = \{x': x \in X\}$, we will need yet two other copies of $X$

$$X^+ = \{x^+: x \in X\} \quad \text{and} \quad X^- = \{x^-: x \in X\}$$

disjoint with each other and disjoint with $X$ and $X'$. We also put

$$X^\pm = X^+ \cup X^-.$$ 

In addition, for any $y \in X \cup X'$, we will use the notation

$$y^+ = \begin{cases} 
  x^+ & \text{if } y = x \in X, \\
  x^- & \text{if } y = x' \in X'.
\end{cases} \quad y^- = \begin{cases} 
  x^- & \text{if } y = x \in X, \\
  x^+ & \text{if } y = x' \in X'.
\end{cases}$$

For any word $u \in F'(X) \cup \{\Box\}$, we denote by

$C(u)$—the content of $u$, that is, the set of all elements $x \in X$ such that $x$ or $x'$ occurs in $u$,

and we put

$$C^\pm(u) = \{x^+: x \in C(u)\} \cup \{x^-: x \in C(u)\}.$$ 

For any word $u \in F'(X) \cup \{\Box\}$, we denote by
\(\Gamma(u)\)—the equivalence relation on the set \(C^\pm(u)\) generated by the relation
\[\{(y_1^-, y_1^+), (y_2^-, y_2^+), \ldots, (y_n^-, y_n^+)\}\]
where \(y_1, y_2, \ldots, y_n \in X \cup X'\) are such that \(u = y_1y_2 \ldots y_n\).

Note that for all words \(u \in F'(X) \cup \{\Box\}\), the equivalence relations \(\Gamma(u)\) on \(C^\pm(u)\) can be considered as subsets of \(X^\pm \times X^\pm\), and viewed in this way, they can be compared by inclusion. Also notice that the relation \(\Gamma(u)\) determines the set \(C(u)\) since \(C(u) = \{x \in X: (x^+, x^-), (x^-, x^-) \in \Gamma(u)\}\). For any word \(u \in F'(X)\), we further put
\[\nu(u) = \left|C^\pm(u)\right| - \left|C^\pm(u)/\Gamma(u)\right|\].

It is not difficult to realize that \(\nu(u)\) is the least number of pairs of distinct elements of \(X^\pm\) needed to generate \(\Gamma(u)\) on \(C^\pm(u)\).

For any word \(u \in F'(X)\), we define
\[0(u)\]—the longest initial segment \(v\) of \(u\) such that \(C(v) \subset C(u)\); it may be empty,
\[\hat{0}(u)\]—the element \(y \in X \cup X'\) for which \(0(u)\) \(y\) is an initial segment of \(u\),
\[\emptyset(u)\]—the longest initial segment \(w\) of \(u\) such that \(\Gamma(w) \subset \Gamma(u)\); it may be empty,
\[\overline{\emptyset}(u)\]—the element \(z \in X \cup X'\) for which \(\emptyset(u)\) \(z\) is an initial segment of \(u\),
\[h(u)\]—the head of \(u\), that is, the element of \(X \cup X'\) occurring first from the left in \(u\),
and we define \(1(u), \overline{1}(u), \overline{\overline{1}}(u), \overline{\emptyset}(u)\) and \(t(u)\) (the tail of \(u\)) dually. Notice that \(0(u)\) is always an initial segment of \(\emptyset(u)\) (sometimes they may be equal) and the dual statement holds for \(1(u)\) and \(\overline{1}(u)\).

Observe that for any word \(u \in F'(X)\), we have \(\nu(u) = 0\) if and only if \(\emptyset(u) = \Box = \overline{\emptyset}(u)\), and if \(\nu(u) \neq 0\), then we have \(\nu(\emptyset(u)) = \nu(u) - 1\) and \(\nu(\overline{\emptyset}(u)) = \nu(u) - 1\). In order to verify the first of these two equalities, for instance, one has to distinguish two cases. Either \(C(\emptyset(u)) = C(u)\), in which case \(C^\pm(\emptyset(u))/\Gamma(\emptyset(u))\) arises from \(C^\pm(u)/\Gamma(u)\) by splitting one of its classes into two subclasses, or \(\left|C(\emptyset(u))\right| = \left|C(u)\right| - 1\), in which case \(\emptyset(u) = 0(u)\) and \(\overline{\emptyset}(u) = \hat{0}(u)\), and \(C^\pm(\emptyset(u))/\Gamma(\emptyset(u))\) arises from \(C^\pm(u)/\Gamma(u)\) by deleting the class \(\hat{0}(u)^-\) and erasing the element \(\hat{0}(u)^+\) from the class containing \(\hat{0}(u)^-\). From the mentioned equalities it easily follows, for instance, that for any \(u \in F'(X)\), we have \(\nu(u) \geq \left|C(u)\right| - 1\). We will frequently use these facts without further remembering them.

Both operators \(0, 1, \emptyset, \hat{0}, \overline{\emptyset}\) can be used repeatedly. Thus, in particular, for any \(u \in F'(X)\) and for any \(i = 1, \ldots, \nu(u) + 1\), we have defined the words \(\emptyset^i(u)\) and \(\hat{0}^i(u)\). Besides, we put \(\emptyset^0(u) = u = \hat{0}^0(u)\). In this connection, note also that \(h(u) = \overline{\emptyset}(\emptyset^\nu(u)(u))\) and \(t(u) = \overline{\hat{0}}(\hat{0}^\nu(u)(u))\).

For any non-empty set \(X\) and for any operator \(Q\) among \(C, h, t, \) we will denote also by \(Q\) the relation
\[Q = \{(u, v) \in F'(X) \times F'(X): Q(u) = Q(v)\}\]
on $F'(X)$. In addition, we will need the relation
$$C' = \{(u, v) \in F'(X) \times F'(X): |C(u)| = |C(v)|, |C(u) - C(v)| \leq 1\}.$$ 

In the subsequent notes, we will assume that $X$ is an infinite set.

For any congruence $\rho$ on $F'(X)$, we define two binary relations $\rho_0, \rho_1$ on $F'(X)$ as follows:
$$\rho_0 = \{(0(u), 0(v)) : u, v \in F'(X), |C(u)|, |C(v)| \geq 2 \text{ and } u \rho v\},$$
and $\rho_1$ is defined dually. Clearly, $\rho \subseteq \rho_0, \rho_1$.

Now we can quote from [5, Proposition 2.11] the following statement. Remember that $B$ stands for the variety of all bands. Note that in [5] these considerations are led, more generally, within the variety of all orthodox completely regular semigroups.

**Result 2.1** [5]. For any biinvariant congruence $\rho$ on $F'(X)$ satisfying $\rho(B) \subseteq \rho \subseteq C$, just one of the following conditions holds:

(i) $\rho_0 \subseteq C$, in which case $\rho_0$ is a biinvariant congruence on $F'(X)$ and $(\rho_0)_0 = \rho_0$,

(ii) $\rho_0 = h \cap C'$,

(iii) $\rho_0 = C'$.

Also the dual statement dealing with $\rho_1$ holds.

Next we remember the following “iterative” solution of the word problem for the bifree objects in all varieties of bands containing semilattices. This result derives from [5, Proposition 2.13], but the original source of these ideas is the trilogy [9–11] on varieties of completely regular semigroups.

**Result 2.2** [5]. Let $\rho$ be a biinvariant congruence on $F'(X)$ such that $\rho(B) \subseteq \rho \subseteq C$. For any $u, v \in F'(X)$, we have $u \rho v$ if and only if the following conditions are satisfied:

(i) $C(u) = C(v)$,

(ii) if $\rho_0 \subseteq C$ then $0(u) = 0(v)$ and, in case $|C(u)| \geq 2$, $0(u) \rho_0 0(v)$,

(iii) if $\rho_1 \subseteq C$ then $1(u) = 1(v)$ and, in case $|C(u)| \geq 2$, $1(u) \rho_1 1(v)$,

(iv) if $\rho_0 = h \cap C'$ then $h(u) = h(v)$,

(v) if $\rho_1 = t \cap C'$ then $t(u) = t(v)$.

We return to the e-varieties $AOLB$ introduced at the beginning of this section. In the next result, we will need the following notation. If $c$ is any word in the free semigroup $F(Z)$ on some non-empty set $Z$, then we will write $c(z_1, \ldots, z_n)$ to indicate that only elements $z_1, \ldots, z_n \in Z$ may occur in $c$. Furthermore, if $X$ is another non-empty set and if $u_1, \ldots, u_n \in F'(X)$ are any words, then by $c(u_1, \ldots, u_n)$ we denote the word of $F'(X)$ obtained by substituting $u_1, \ldots, u_n$ for $z_1, \ldots, z_n$ into $c$. Then we have the following characterization of the biinvariant congruences on $F'(X)$ corresponding to the e-varieties $AOLB$. 


Proposition 2.3. Let $\mathcal{A}$ be any variety of bands. Then, for any non-empty set $X$, $\rho(\mathcal{AOLB}, X)$ is the least congruence on $F'(X)$ containing the following relations:

\[
\begin{align*}
\{(u, uu') & : u \in F'(X)\}, \\
\{(uu'v'w', (uu'v'w')^2) & : u, v, w \in F'(X)\}, \\
\{(yuy', (yy')^2) & : y \in X \cup X', u \in F'(X)\}, \\
\{(c(u_1^2, \ldots, u_n^2), d(u_1^2, \ldots, u_n^2)) & : u_1, \ldots, u_n \in F'(X) \text{ and } c(z_1, \ldots, z_n), d(z_1, \ldots, z_n) \in F(Z) \text{ for some set } Z \text{ with } c\rho(\mathcal{AOLB}, X) d\}. 
\end{align*}
\]

Remark. The second relation in this list can be omitted, after all, since it can be generated from the first and the last relation, where we can take the words $z_1 z_2 z_3$ and $(z_1 z_2 z_3)^2$ for $c$ and $d$, respectively. On the other hand, it will become apparent from the proof that the first three relations alone generate the congruence $\rho(\mathcal{OLB}, X)$ on $F'(X)$.

Proof. Denote by $\kappa$ the least congruence on $F'(X)$ containing the four relations given above. Then $\kappa \subseteq \rho(\mathcal{AOLB}, X)$ readily follows from the fact that the semigroup $F'(X)/\rho(\mathcal{AOLB})$ belongs to $\mathcal{AOLB}$, so that it is orthodox locally idempotent and its idempotents satisfy all identities valid in $\mathcal{A}$. One has only to note that this semigroup clearly satisfies the identity $x^2 \simeq x^3$, and so its idempotents are exactly classes $u^2 \rho(\mathcal{AOLB})$ for $u \in F'(X)$. Thus, in order to complete the proof, it remains to observe that $xx, x'x$ are mutual inverses in $F'(X)/\kappa$ for all $x \in X$ and to show that this semigroup itself is orthodox and locally idempotent. But the first of these properties follows from [5, Lemma 1.6], and the verification of the second property is an easy exercise. Moreover, idempotents in $F'(X)/\kappa$ form a band satisfying all identities valid in $\mathcal{A}$, so that $F'(X)/\kappa$ lies in $\mathcal{AOLB}$. Hence, if $u \simeq v$ is any biidentity over $X$ which is satisfied in $\mathcal{AOLB}$ then $u \kappa v$, which means that $\rho(\mathcal{AOLB}, X) \subseteq \kappa$. Altogether equality prevails.  

Denote by $S$ the variety of all semilattices. Note that $\rho(S) = C$. Return also to our previous assumption that $X$ is an infinite set. For any variety $\mathcal{A}$ of bands such that $S \subseteq \mathcal{A}$, we denote by

$\mathcal{A}_0$—the variety of bands for which $\rho(\mathcal{A}_0, X)$ is $\rho(\mathcal{A}, X)_0$ provided $\rho(\mathcal{A}, X)_0 \subseteq C$,

and we define $\mathcal{A}_1$ dually. These varieties are well defined by Result 2.1(i) and its dual, and they are subvarieties of $\mathcal{A}$.

Now we are ready to state and prove the following “iterative” solution of the word problem for the bifree objects in the e-varieties $\mathcal{AOLB}$. Once again, it is useful to keep in mind Result 2.1 and its dual in this connection.

Proposition 2.4. Let $\mathcal{A}$ be any variety of bands satisfying $S \subseteq \mathcal{A}$. Then, for any $u, v \in F'(X)$, we have $u \rho(\mathcal{AOLB}, X) v$ if and only if the following conditions are satisfied:

(i) $\Gamma(u) = \Gamma(v)$,
(ii) \((h(u)^+, h(v)^+) \in \Gamma(u)\),
(iii) \((t(u)^-, t(v)^-) \in \Gamma(u)\),
(iv) if \(\rho(A)_0 \subseteq C\) then \(\Upsilon(u) = \overline{\Upsilon(v)}\) and, in case \(v(u) \geq 1\), \(\Upsilon(u) \rho(A_0) \Omega LB X \overline{\Upsilon(v)}\),
(v) if \(\rho(A)_1 \subseteq C\) then \(\mathcal{I}(u) = \mathcal{I}(v)\) and, in case \(v(u) \geq 1\), \(\mathcal{I}(u) \rho(A_1) \Omega LB X \mathcal{I}(v)\),
(vi) if \(\rho(A)_0 = h \cap C'\) then \(h(u) = h(v)\),
(vii) if \(\rho(A)_1 = t \cap C'\) then \(t(u) = t(v)\).

**Remark.** Clearly, if \(\rho(A)_0 = h \cap C'\) then (vi) implies (ii). Dually, if \(\rho(A)_1 = t \cap C'\) then (vii) implies (iii). Furthermore, assuming that the direct part of this proposition holds, if \(\rho(A)_0 \subseteq C\) then Result 2.1(i) and induction on \(v(u)\) show that (iv) eventually yields \(h(u) = h(v)\), which again implies (ii). Dually, if \(\rho(A)_1 \subseteq C\) then (v) yields \(t(u) = t(v)\), which implies (iii).

Note that if we take for \(A\) just \(S\), then we get the e-variety \(SOLB\) which is precisely the variety of all inverse locally idempotent semigroups. These semigroups are known rather as combinatorial strict inverse semigroups. The word problem for free semigroups in the mentioned variety has been solved in [13]. A solution of the word problem for free semigroups in \(SOLB\) which becomes identical with the solution arising from the above proposition, if we take \(A = S\) in it (then \(\rho(S)_0 = C = \rho(S)_1\), has been obtained in [1]. The same solution, although expressed in different terms, can also be found in [8].

On the other hand, since \(\rho(B)_0 = \rho(B) = \rho(B)_1\), and so \(B_0 = B = B_1\), if we take \(A = B\) in the above proposition, we hence obtain an inductive description of the bivariant congruence \(\rho(O LB, X)\), that is, we get a solution of the word problem for bifree objects in the e-variety \(OLB\) of all orthodox locally idempotent semigroups.

**Proof of Proposition 2.4 ("only if" part).** The verification of this part of that proposition is a routine based on Proposition 2.3. However, some preparatory considerations may come in handy. If \(u, v, w\) are the three words appearing in the second relation given in Proposition 2.3, then all elements \(h(u)^+, h(v)^+, h(w)^+\) occur in the same class of the equivalence relation \(\Gamma((uu'vv'ww')^2)\), which shows that this relation is equal to \(\Gamma((uu'vv'ww')^2)\). If \(c, d\) and \(u_1, \ldots, u_n\) are the words appearing in the last relation given in Proposition 2.3, then all elements \(h(u_1)^+, t(u_1)^-, \ldots, h(u_n)^+, t(u_n)^-\) occur in the same class of both equivalence relations \(\Gamma(c(u_1^+, \ldots, u_n^+))\) and \(\Gamma(d(u_1^+, \ldots, u_n^+))\) and, consequently, these two relations are obviously the same. These notes make it possible to verify condition (i) and, likewise, conditions (ii), (iii), and also (vi), (vii) can be easily verified. In the course of the verification of the remaining conditions (iv) and (v), only the last relation given in Proposition 2.3 requires special attention. Remember that, by our assumption, \(\rho(A) \subseteq C\), so that, for the words \(c, d\) appearing in the mentioned relation, \(c \rho(A) d\) implies \(C(c) = C(d)\). Let \(u_1, \ldots, u_n\) be the collection of words whose squares appear in that relation, and let \(r, s \in F'(X) \cup \{\Box\}\) be any two additional words. Assume that \(\rho(A)_0 \subseteq C\) and consider the condition (iv). This assumption entails, by Result 2.1(i), that \(\rho(A)_0\) is a bivariant congruence again and that \(\rho(A)_0 = \rho(A)_0\). Then, it is not difficult to realize that one of the following three possibilities must occur. Either \(\bigcup(r c(u_1^+, \ldots, u_n^+) s) = r c(u_1^+, \ldots, u_n^+) s^*\) and \(\bigcup(r d(u_1^+, \ldots, u_n^+) s) = r d(u_1^+, \ldots, u_n^+) s^*\) for some (the same) proper initial segment
Proof. If \( t \) satisfies \( (iv) \) and, dually, also condition \((v)\) can be verified. \( \square \)

Before we can prove the converse part of Proposition 2.4, we have to prepare some tools.

**Lemma 2.5.** Let \( u \in F'(X) \) and let \( y, z \in X \cup X' \) be such that \((y^-, z^+) \in \Gamma(u)\). Then there exist \( v, w \in F'(X) \cup \{\square\} \) such that \( u \rho(\Omega LB) vyzw \).

**Proof.** If \((y^-, z^+) \in \Gamma(u)\) then \( C(y), C(z) \subseteq C(u)\). If \( z = y' \) then the assertion is obvious. Otherwise, by the definition of \( \Gamma(u) \), there exist \( n \geq 1 \) and \( z_0, z_1, \ldots, z_n \in X \cup X' \) such that \( z_0 = y', z_n = z, \) and for each \( i \in \{1, \ldots, n\} \), either \( z_i z_{i-1}^{+}\) or \( z_i z_{i-1}^{-}\) appears as a segment in \( u \). Since \( z_i z_{i-1}^{+} \in \rho(\Omega) \) \( z_i z_{i-1}^{+} \in \rho(\Omega) \), we may assume that it is the first case that always takes place. We proceed by induction on \( n \). If \( n = 1 \) then there is nothing to prove. Thus assume that \( n > 1 \). By the induction hypothesis, we may suppose that there exist \( p, q \in F'(X) \cup \{\square\} \) such that \( u \rho(\Omega LB) p \rho_0 z_{n-1}^{-} q \). In addition, there exist \( r, s \in F'(X) \cup \{\square\} \) such that \( u \rho(\Omega) r z_{n-1}^{+} z_n^{-} s \). Hence we obtain

\[
uu u \rho(\Omega) uu' u \rho(\Omega LB) r z_{n-1}^{+} z_n^{-} z_n u' p \rho_0 z_{n-1}^{-} q \]

\[
uu u \rho(\Omega) uu' u \rho(\Omega LB) r z_{n-1}^{+} z_n^{-} z_n u' p \rho_0 z_{n-1}^{-} z_n^{-} z_n u' p \rho_0 z_{n-1}^{-} z_n^{-} z_n u', \]

having in mind the fact that the semigroups \( F'(X) / \rho(\Omega) \) and \( F'(X) / \rho(\Omega LB) \) are, respectively, orthodox and orthodox locally idempotent. This verifies the assertion. \( \square \)

**Lemma 2.6.** Let \( u \in F'(X) \) and \( v \in F'(X) \cup \{\square\} \) be such that \( \Gamma(v) \subseteq \Gamma(u) \) and \( (t(u)^-, h(v)^+) \in \Gamma(u) \) provided that \( v \neq \square \). Then there exists \( s \in F'(X) \cup \{\square\} \) such that \( u \rho(\Omega LB) v s u s \).
Thus assume that \( v = wy \) for some \( y \in X \cup X' \) and \( w \in F'(X) \cup \{\square\} \). By the induction hypothesis, there exists \( r \in F'(X) \cup \{\square\} \) such that \( u \rho(\mathcal{OLB}) yqu \). By our assumptions, we have \((t(uw)^-, y^+) \in \Gamma(u)\) (notice that \((uw) = t(u)\) and \(t(u) y\) is a segment of \( v \) if \( w \neq \square\), and \((uw) = (u), y = h(v)\) if \( w = \square\)). By Lemma 2.5, there exist \( p, \ q \in F'(X) \cup \{\square\} \) such that \( u \rho(\mathcal{OLB}) p t(uw)yq \). Hence we obtain

\[
\begin{align*}
u & \rho(\mathcal{O}) uu' u \rho(\mathcal{OLB}) p t(uw)yqu'wur \\
& \rho(u) p t(uw)t(uw)yqu'wur t(uw)t(uw)r \\
& \rho(\mathcal{OLB}) p t(uw)(t(uw)'yqu'wur t(uw)'t(uw))r \\
& \rho(u) p t(uw)yqu'wur \\
& \rho(\mathcal{OLB}) uu' uwyqu'wur \rho(\mathcal{O}) wqu'ywur,
\end{align*}
\]

using again the fact that the semigroup \( F'(X)/\rho(\mathcal{OLB}) \) is orthodox locally idempotent. This verifies our claim. \( \square \)

**Corollary 2.7.** If \( u, v \in F'(X) \) are such that \( \Gamma(v) \subseteq \Gamma(u) \) and \( (t(u)^-, h(v)^+) \), \( (t(v)^-, h(u)^+) \) \( \in \Gamma(u) \), then \( u \rho(\mathcal{OLB}) uvu \).

**Proof.** From the given assumptions it follows that \( \Gamma(vu) \subseteq \Gamma(u) \), and so, by Lemma 2.6, there exists \( s \in F'(X) \cup \{\square\} \) such that \( u \rho(\mathcal{OLB}) uvus \). Hence we get

\[
u \rho(\mathcal{OLB}) uu' uvu'us \rho(\mathcal{OLB}) (uu' uvu')^2us \rho(\mathcal{O}) uvuvus \rho(\mathcal{OLB}) uvu,
\]

as required. \( \square \)

In order to prove the converse part of Proposition 2.4, we need yet the following subsidiary statement.

**Lemma 2.8.** Suppose that \( u, v \in F'(X) \) are words satisfying the conditions (i)–(vii) in Proposition 2.4. Then there exists a word \( w \in F'(X) \) such that \( u \rho(\mathcal{AO}_{\mathcal{LB}}, X) w \) and \( \overrightarrow{(u)} = \overrightarrow{(v)} \), \( \overleftarrow{(w)} = \overleftarrow{(v)} \).

**Proof.** We distinguish three cases according to Result 2.1.

1. Case I: \( \rho(A)_0 \subseteq C \). If \( v(u) = 0 \) then \( \overrightarrow{(u)} = \square = \overrightarrow{(v)} \) and, according to (iv), we have \( \overrightarrow{(u)} = \overrightarrow{(v)} \), so that we can take \( w = u \). If \( v(u) \geq 1 \) then (iv) ensures, in addition, also that \( \overrightarrow{(u)} \rho(A_0 \mathcal{OLB}) \overrightarrow{(v)} \). Therefore, by Proposition 2.3, there exists a finite sequence of words in \( F'(X) \) starting with \( \overrightarrow{(u)} \) and ending with \( \overrightarrow{(v)} \) such that, for any pair \( p, q \) of words next to each other in this sequence, one of the following possibilities occurs for some \( r, s \in F'(X) \cup \{\square\} \):

\[
\begin{align*}
(1) & \ {p, q} = \{rf's, rf'f's\} \text{ for some } f \in F'(X), \\
(2) & \ {p, q} = \{rf'gg'h'h's, r(f'gg'h'h')^2s\} \text{ for some } f, g, h \in F'(X),
\end{align*}
\]
(3) \( \{ p, q \} = \{ ryf'y, r(uyf'y)^2s \} \) for some \( y \in X \cup X' \) and \( f \in F'(X) \),

(4) \( \{ p, q \} = \{ r c(f_1^2, \ldots, f_n^2), r d(f_1^2, \ldots, f_n^2) s \} \) for some \( f_1, \ldots, f_n \in F'(X) \) and for some \( c(x_1, \ldots, x_n), d(x_1, \ldots, x_n) \in F(X) \) satisfying \( c \rho(A_0) d \).

We hence infer the following conclusion. In order to prove the lemma, it suffices to show that for any words \( a, p, q \in F'(X) \) such that \( p = \mathcal{O}(a) \) and one of the possibilities (1)–(4) holds for \( p, q \), there exists some word \( b \in F'(X) \) with \( a \rho(A_{OLB}) b \) and \( \mathcal{O}(b) = \mathcal{O}(a) \), \( \mathcal{O}(b) = q \).

If (1), (2) or (3) hold for \( p, q \) then \( p \rho(OLB) q \) and, clearly, \( \Gamma(p) = \Gamma(q) \). We can write \( a = p \mathcal{O}(a) t \) for some \( t \in F'(X) \cup \{ \} \), whence we get \( a \rho(OLB) q \mathcal{O}(a) t \). Thus we can put \( b = q \mathcal{O}(a) t \), since we easily check that \( \mathcal{O}(b) = q \) and \( \mathcal{O}(b) = \mathcal{O}(a) \), as also \( t(p) = t(q) \).

Now suppose that \( p, q \) are as in (4). Since \( \rho(A_0) \subseteq C \) and \( c \rho(A_0) d \), we may assume that \( C(c) = C(d) = \{ x_1, \ldots, x_n \} \). Moreover, using Result 2.2, we can check that \( c x_{n+1} x_{n+1} \ldots x_1 x_1 \rho(A_0) d x_{n+1} x_{n+1} \ldots x_1 \) where \( x_{n+1} \in X \) is an element distinct from \( x_1, \ldots, x_n \).

Put \( c(x_1, \ldots, x_{n+1}) = c x_{n+1} x_{n+1} \ldots x_1 \) and \( d(x_1, \ldots, x_{n+1}) = d x_{n+1} x_{n+1} \ldots x_1 \). We can write

\[
a = p \mathcal{O}(a) t = r c(f_1^2, \ldots, f_n^2) s \mathcal{O}(a) t
\]

for some \( t \in F'(X) \cup \{ \} \). Furthermore, the words \( r c(f_1^2, \ldots, f_n^2) s \mathcal{O}(a) \) and \( \mathcal{O}(a) s f_n^2 \ldots f_2^2 \) can be easily seen to satisfy the assumptions of Lemma 2.6, whence we obtain that, for some \( \xi \in F'(X) \cup \{ \} \), we have

\[
a \rho(OLB) r c(f_1^2, \ldots, f_n^2) s \mathcal{O}(a) \xi \mathcal{O}(a) s' f_n^2 \ldots f_2^2 \xi \notin t.
\]

This relation can be rewritten as

\[
a \rho(OLB) r \tilde{c}(f_1^2, \ldots, f_n^2, f_{n+1}) t,
\]

where \( f_{n+1} = s \mathcal{O}(a) \mathcal{O}(a) s' \). Since \( \tilde{c} \rho(\mathcal{O}) \tilde{d} \) and \( f_{n+1} \rho(\mathcal{O}) f_2^2 \), we hence deduce that

\[
a \rho(OLB) r \tilde{d}(f_1^2, \ldots, f_n^2, f_{n+1}) t,
\]

using Proposition 2.3 again. This can be rewritten as

\[
a \rho(OLB) r d(f_1^2, \ldots, f_n^2) s \mathcal{O}(a) \xi \mathcal{O}(a) s' f_n^2 \ldots f_2^2 \xi t.
\]

Denote by \( b \) the word on the right in this relation. Then we have \( \Gamma(a) = \Gamma(b) \) by the direct part of Proposition 2.4, whence we readily obtain that \( \mathcal{O}(b) = r d(f_1^2, \ldots, f_n^2) s = q \) and \( \mathcal{O}(b) = \mathcal{O}(a) \), since it is also easy to notice that \( \Gamma(p) = \Gamma(q) \) and \( (t(p)^{-1}, t(q)^{-1}) \in \Gamma(p) \).

Case II: \( \rho(A_0) = h \cap C' \). By Result 2.2, we have \( x_1 x_2 x_3 \rho(A) x_1 x_3 x_1 x_2 x_3 \) for \( x_1, x_2, x_3 \in X \) in this case. Furthermore,

\[
u = \mathcal{O}(a) \mathcal{O}(a) t\]
for some \( t \in F'(X) \cup \{\square\} \). Since \( \Gamma(u) = \Gamma(v) \) by (i) and \( h(u) = h(v) \) by (vi), it can be verified straightforwardly that the words \( \overline{\circ}(u) \overline{\circ}(u) \) and \( (\overline{\circ}(u) \overline{\circ}(u))^t \overline{\circ}(v) \overline{\circ}(v) (\overline{\circ}(v) \overline{\circ}(v))^t \) satisfy the assumptions of Lemma 2.6. Thus, putting 

\[

t = h(u) h(u)^t, \\
r = \overline{\circ}(u) \overline{\circ}(u) (\overline{\circ}(u) \overline{\circ}(u))^t, \\
s = \overline{\circ}(v) \overline{\circ}(v) (\overline{\circ}(v) \overline{\circ}(v))^t,
\]

in view of the previous relation, from the mentioned lemma we deduce that, for some \( \varsigma \in F'(X) \cup \{\square\} \), we have

\[
u \rho(\text{OLB}) rs \varsigma t \rho(\text{O}) frs \varsigma t.
\]

Now put \( w = su \). Since \( h(u) = h(v) \), we get \( w \rho(\text{O}) fsu \), whence, using the last relation, we obtain

\[
w \rho(\text{OLB}) fsfrs \varsigma t.
\]

Now, since \( x_1 x_2 x_3 \rho(A)x_1 x_3 x_1 x_2 x_3 \) and \( f \rho(\text{O}) f^2, r \rho(\text{O}) r^2, s \rho(\text{O}) s^2 \), by Proposition 2.3, we have

\[
frs \rho(\text{ALB}) fsfrs.
\]

This relation together with the previous two relations yields \( u \rho(\text{ALB}) w \). It is also clear that \( \Gamma(w) = \Gamma(u) = \Gamma(v) \), whence \( \overline{\circ}(w) = \overline{\circ}(v) \) and \( \overline{\circ}(w) = \overline{\circ}(v) \) follow.

**Case III:** \( \rho(A)_0 = C' \). By Result 2.2, we have \( x_1 x_2 \rho(A)x_3 x_1 x_2 \) for \( x_1, x_2 \in X \) now. The reasoning in this case is very similar as in Case II; it is, in fact, simpler since the word \( f \) can be omitted from our considerations this time. Note, however, that (ii) must be applied instead of (vi) in this case. The details are left to the reader. \( \Box \)

**Proof of Proposition 2.4** (“if” part). Let \( u, v \in F'(X) \) be words satisfying all conditions (i)–(vii). Then, by Lemma 2.8 and its dual, there exist words \( t, w \in F'(X) \) such that

\[
t \rho(\text{ALB}) u \rho(\text{ALB}) w
\]

and \( \overline{\circ}(t) = \overline{\circ}(v), \overline{\circ}(t) = \overline{\circ}(v), \overline{\circ}(w) = \overline{\circ}(v), \overline{\circ}(w) = \overline{\circ}(v) \). Thus we can write

\[
t = \overline{\circ}(v) \overline{\circ}(v) p \quad \text{and} \quad w = q \overline{\circ}(v) \overline{\circ}(v)
\]

for some \( p, q \in F'(X) \cup \{\square\} \). In addition, by Lemma 2.6 and its dual, there exist \( r, s \in F'(X) \cup \{\square\} \) such that

\[
\overline{\circ}(v) \overline{\circ}(v) \rho(\text{ALB}) vr \quad \text{and} \quad \overline{\circ}(v) \overline{\circ}(v) \rho(\text{ALB}) sv.
\]
Hence we obtain
\[ u \rho(\mathcal{O}) uu' u \rho(\mathcal{AOLB}) tu' w = \mathcal{O}(v) \mathcal{I}(v) pu' q \mathcal{I}(v) \mathcal{I}(v) \rho(\mathcal{OLB}) r v p u' q s v. \]

By the direct part of Proposition 2.4, which has already been proved, and by (i), we hence get
\[ \Gamma(v) = \Gamma(v), \]
which ensures that the words \( v \) and \( r p u' q s v \) satisfy the assumptions of Corollary 2.7, according to which we therefore have \( v \rho(\mathcal{OLB}) vr p u' q s v \).

This fact and the previous relation together yield \( u \rho(\mathcal{AOLB}) v \). \( \square \)

As immediate consequences of Proposition 2.4, we obtain the following pieces of information.

**Corollary 2.9.** Let \( A \) be any variety of bands satisfying \( S \subseteq A \).

(i) For any \( u \in F'(X) \), the element \( u \rho(\mathcal{AOLB}) \) is an idempotent in \( F'(X) / \rho(\mathcal{AOLB}) \) if and only if \( (t(u)^-, h(u)^+) \) \( \in \Gamma(u) \).

(ii) For any \( u, v \in F'(X) \), the elements \( u \rho(\mathcal{AOLB}) \) and \( v \rho(\mathcal{AOLB}) \) are mutual inverses in \( F'(X) / \rho(\mathcal{AOLB}) \) if and only if \( \Gamma(u) = \Gamma(v) \) and \( (t(u)^-, h(v)^+), (t(v)^-, h(u)^+) \) \( \in \Gamma(u) \).

The task to solve the word problem for the bifree objects in the e-varieties \( \mathcal{AOLB} \) would be complete if we convert the “iterative” solution obtained in Proposition 2.4 into a “global” one. This can be done in a similar fashion as in [5] where the “iterative” solution of the word problem for the bifree objects in varieties of bands, which is remembered in our present Result 2.2, has been turned into a “global” one using the machinery developed in [9–11]. (In [5], this procedure has been performed, more generally, for arbitrary varieties of orthodox completely regular semigroups.) But we will not need such a kind of solution in this paper, and so we leave the details to the interested reader.

As another application of Proposition 2.4, we deduce the fact that, for any non-empty family \( \{ A_i : i \in I \} \) of varieties of bands, we have

\[ \bigvee_{i \in I} A_i \mathcal{OLB} = \mathcal{AOLB} \quad \text{where} \quad \mathcal{A} = \bigvee_{i \in I} A_i, \quad \text{and} \]

\[ \bigcap_{i \in I} A_i \mathcal{OLB} = \mathcal{AOLB} \quad \text{where} \quad \mathcal{A} = \bigcap_{i \in I} A_i. \]

Consequently, the e-varieties \( \mathcal{AOLB} \), for all band varieties \( A \), form a complete sublattice in the lattice of all sub-e-varieties of \( \mathcal{OLB} \), which is isomorphic to the lattice of all varieties of bands. Clearly, only the first of the mentioned equalities needs a proof. It amounts to showing that \( \bigcap_{i \in I} \rho(A_i \mathcal{OLB}, X) = \rho(\mathcal{AOLB}, X) \). That is, we have to show that, for any \( u, v \in F'(X) \), we have \( u \rho(\mathcal{AOLB}, X) v \) if and only if \( u \rho(A_i \mathcal{OLB}, X) v \) holds for all \( i \in I \). This statement can be verified using Proposition 2.4 and induction on \( \nu(u) \). In addition, however, also the following facts are needed. It is true that \( \rho(A_0) = \bigvee_{i \in I} \rho(A_i) \), and if \( \rho(A_0) \subseteq C \) then \( (\mathcal{A})_0 = \bigvee_{i \in J} (A_i)_0 \) where \( J = \{ i \in I : \rho(A_i)_0 \subseteq C \} \). This can be proved similarly as in [10, Theorem 1.6(1)]. Also the dual assertions hold. The details are
again left to the reader. Alternatively, a direct proof of the above equality of congruences would follow from the “global” solution of the word problem mentioned in the previous paragraph.

3. The e-variety $OZ$

In this section, for any infinite set $X$ of variables, we construct a special biinvariant congruence $\zeta(X)$ on $F'(X)$ having the following properties. The e-variety $OZ$ corresponding to $\zeta(X)$ contains the variety $LRB$ of all left regular bands and is itself contained in the e-variety $(LRB)OLB$ of all orthodox locally idempotent semigroups whose idempotents form a band in $LRB$. In the next section, we will show that this e-variety $OZ$ is finitely generated. And in the last section of this paper, we will finally see that $OZ$ has no finite basis of biidentities within $(LRB)OLB$.

We begin by drawing from the results of the previous section the following solution of the word problem for the bifree objects in $(LRB)OLB$. Since $\rho(LRB)_0 = \rho(LRB)$ and $\rho(LRB)_1 = C'$, from Proposition 2.4 and the remark after it we get the following description of the corresponding biinvariant congruence on $F'(X)$:

**Corollary 3.1.** For any $u, v \in F'(X)$, we have

$$u \rho((LRB)OLB, X) v \quad \text{if and only if} \quad \begin{align*}
\Gamma(u) &= \Gamma(v), \\
O(u) &= O(v) \quad \text{and, in case } \nu(u) \geq 1, \, O(u) \rho((LRB)OLB, X) O(v). 
\end{align*}$$

In order to define the relation $\zeta(X)$ on $F'(X)$, we have to introduce some notations in advance. For any word $u \in F'(X)$, we denote by

- $\varepsilon(u)$—the positive integer $j$ for which $0(u) = O^j(u)$,
- $\varepsilon(u)$—the equivalence relation on the set $\{1, \ldots, \varepsilon(u) - 1\}$ generated by the relation
  $$\{(i, i+1): i \in \{1, \ldots, \varepsilon(u) - 2\}, \, (O^{i-1}(u))^+, (O^i(u))^+) \in \Gamma(u)\}.$$

Note that the last condition in this definition is equivalent to the requirement that $(t(O^{i}(u))^+, O^{i+1}(u))^+ \in \Gamma(u)$ since $(t(O^{i}(u))^+, (O^{i-1}(u))^+) \in \Gamma(u)$ holds for all $i \in \{1, \ldots, \varepsilon(u) - 1\}$. Clearly, the partition of $\{1, \ldots, \varepsilon(u) - 1\}$ determined by $\varepsilon(u)$ is of the form

$$\{1, \ldots, \varepsilon(u) - 1\} / \varepsilon(u) = \{\{1, \ldots, t_1\}, \{t_1 + 1, \ldots, t_2\}, \ldots, \{t_{x-1} + 1, \ldots, t_x\}\},$$

where $x = x(u)$ is a non-negative integer ($x(u) = 0$ iff $\varepsilon(u) = 1$) and

$$t_1 = t_1(u), \, t_2 = t_2(u), \ldots, t_{x-1} = t_{x-1}(u), \, t_x = t_x(u)$$
are integers satisfying \( 1 \leq t_1 < t_2 < \cdots < t_{k-1} < t_k = \varepsilon(u) - 1 \). In addition, we will sometimes use the convention that \( t_0 = t_0(u) = 0 \). The idea lying behind the above definition is the following. For any \( \ell \in \{1, \ldots, \kappa\} \), the partition \( C^\ell(u)/\Gamma(\Omega^\ell(u)) \) is, of course, finer than \( C^{\ell+1}(u)/\Gamma(\Omega^{\ell+1}(u)) \), but it splits further only such classes of the latter partition which lie within only one class of \( C^\ell(u)/\Gamma(u) \) in all. Furthermore, if \( \ell < \kappa \) then the next partition \( C^{\ell+1}(u)/\Gamma(\Omega^{\ell+1}(u)) \) continues splitting classes of \( C^\ell(u)/\Gamma(\Omega^\ell(u)) \)

lying already within another class of \( C^\ell(u)/\Gamma(u) \).

Now, given an infinite set \( X \), we define the relation \( \zeta(X) \) on \( F'(X) \) as follows. For any \( u, v \in F'(X) \), we put

\[
\zeta(X) \quad \text{if and only if} \quad \Gamma(u) = \Gamma(v), \quad (t(u)^-, t(v)^-) \in \Gamma(u), \quad \varepsilon(u) = \varepsilon(v),
\]

\[
\Gamma(\Omega^i(u)) = \Gamma(\Omega^i(v)) \quad \text{for all } i \in \{1, \ldots, \varepsilon(u)\},
\]

\[
\bar{0}(u) = \bar{0}(v) \quad \text{and, in case } |C(u)| \geq 2, \quad 0(u) \quad \zeta(X) \quad 0(v).
\]

This definition is, of course, inductive on \( |C(u)| \).

**Remark.** Note that, in this definition, the conditions \( \Gamma(u) = \Gamma(v) \) and \( \Gamma(\Omega^i(u)) = \Gamma(\Omega^i(v)) \) for all \( i \in \{1, \ldots, \varepsilon(u)\} \) clearly imply \( t_i(u) = t_i(v) \) for all \( i \in \{1, \ldots, \varepsilon(u)\} \).

The last of these equalities (for \( \ell = \varepsilon(u) \)) also implies \( \varepsilon(u) = \varepsilon(v) \). This means that \( \varepsilon(u) = \varepsilon(v) \). Notice also that the mentioned conditions (together with the definition of \( \varepsilon(u) \) and \( \varepsilon(v) \)) imply that \( (\Omega(\Omega^{i-1}(u))^+, \Omega(\Omega^{i-1}(v))^+) \in \Gamma(u) \) holds for all \( i \in \{1, \ldots, \varepsilon(u) - 1\} \).

It is obvious that \( \zeta(X) \) is an equivalence relation on \( F'(X) \). We next observe that \( \rho(\mathcal{L} \mathcal{R} \mathcal{B})\mathcal{O} \mathcal{L} \mathcal{B}, X) \subseteq \zeta(X) \subseteq \rho(\mathcal{L} \mathcal{R} \mathcal{B}, X) \). The first of these inclusions follows from Corollary 3.1 and from the above definition of \( \zeta(X) \) by induction on \( |C(u)| \), if we notice that \( u \rho(\mathcal{L} \mathcal{R} \mathcal{B})\mathcal{O} \mathcal{L} \mathcal{B}, v \) implies, according to Corollary 3.1, \( \Omega(\Omega^{i}(u)) = \Omega(\Omega^{i})(v) \) and \( \Omega^{i}(u) \rho(\mathcal{L} \mathcal{R} \mathcal{B})\mathcal{O} \mathcal{L} \mathcal{B} \Omega^{i}(v) \) and, consequently, \( \Gamma(\Omega^{i}(u)) = \Gamma(\Omega^{i}(v)) \) for all \( i = 0, 1, \ldots, \varepsilon(u) \), which, in turn, yields \( \varepsilon(u) = \varepsilon(v) \), \( \varepsilon(u) = \varepsilon(v) \) and, thereby, also \( \bar{0}(u) = \bar{0}(v) \) and \( 0(u) \rho(\mathcal{L} \mathcal{R} \mathcal{B})\mathcal{O} \mathcal{L} \mathcal{B} 0(v) \). The second of the above-mentioned inclusions follows again by induction on \( |C(u)| \) from the above definition of \( \zeta(X) \) and from Result 2.2 applied to the congruence \( \rho(\mathcal{L} \mathcal{R} \mathcal{B}) \) (remember the corresponding notes on it before Corollary 3.1).

It will be the objective of the further considerations in this section to prove that \( \zeta(X) \) is a biinvariant congruence on \( F'(X) \). When this task is accomplished then we can conclude that \( \zeta(X) \) determines, in the way described in Result 1.4, an e-variety of orthodox semigroups, which we will denote by \( \mathcal{O} \mathcal{Z} \), and that, according to the previous paragraph, this e-variety satisfies \( \mathcal{L} \mathcal{R} \mathcal{B} \subseteq \mathcal{O} \mathcal{Z} \subseteq (\mathcal{L} \mathcal{R} \mathcal{B})\mathcal{O} \mathcal{L} \mathcal{B} \). The fact that we will have the e-variety \( \mathcal{O} \mathcal{Z} \) thus determined by its respective biinvariant congruence, and not merely by a set of biidentities, will prove important in the last two sections of this paper.

In order to prove that \( \zeta(X) \) is a biinvariant congruence on \( F'(X) \), however, we will have to propose yet another somewhat different viewpoint to this relation first, that is to say, we will offer an alternative, but equivalent definition of \( \zeta(X) \). The purpose of this modified
definition will be to facilitate wider employment of inductive arguments in the proofs. First of all, we introduce some more notations.

Let $Y \subseteq X$ be a finite subset, let $Y^\pm = \{x^+: x \in Y\} \cup \{x^-: x \in Y\}$, and let $\Delta$ be an equivalence relation on $Y^\pm$. For any word $u \in F'(X)$ such that $C(u) = Y$, $\Gamma(u) \subseteq \Delta$ and $\varepsilon(u) \geq 2$, we denote by

$$\partial_\Delta(u)$$

—the greatest integer $k \in \{1, \ldots, \varepsilon(u) - 1\}$ such that $\overline{O}(O^{j-1}(u))^+$ for $i = 1, \ldots, k$

and we put

$$\Omega_{\Delta}(u) = \bigcup_{\ell=0}^{\partial_\Delta(u)}(u), \quad \overline{\Omega}_{\Delta}(u) = \overline{\bigcup_{\ell=0}^{\partial_\Delta(u)-1}(u)}.$$\text

Notice that we then have $\partial_{\Gamma}(u) = t_1(u)$, and so $\Omega_{\Gamma}(u) = \Omega_{\Gamma}^{(1)}(u)$. The operator $\Omega_{\Delta}$ can be used repeatedly. Having this in mind, we obtain, more generally, that $\Omega_{\Gamma}(u) = \bigcup_{\ell=0}^{\partial_{\Gamma}(u)}(u)$ holds for all $\ell = 1, \ldots, \varepsilon(u)$.

Note, in passing, that by far not for all equivalence relations $\Delta$ on $Y^\pm$, where $Y \subseteq X$ is a finite subset, there must exist a word $u \in F'(X)$ such that $C(u) = Y$ and $\Gamma(u) \subseteq \Delta$. However, this circumstance does not affect the meaning of the previous and the subsequent definitions in any substantial way.

Thus assume that we are given a finite subset $Y \subseteq X$ and an equivalence relation $\Delta$ on $Y^\pm$. Notice that this relation $\Delta$ determines the subset $Y$ since $Y = \{x \in X: (x^+, x^+), (x^-, x^-) \in \Delta\}$. Then we define a certain relation $\xi_{\Delta}$ on $F'(X)$ as follows. For any $u, v \in F'(X)$, we put

$$u \xi_{\Delta} v \quad \text{if and only if} \quad C(u) = C(v) = Y, \quad \Gamma(u) = \Gamma(v) \subseteq \Delta, \quad \{t(u^+), t(v^-)\} \in \Delta, \quad \varepsilon(u) \geq 2 \iff \varepsilon(v) \geq 2, \quad \text{and}$$

if $\varepsilon(u) = 2$: $\Omega_{\Delta}(u) \xi_{\Delta} \Omega_{\Delta}(v)$,

if $\varepsilon(u) = 1$: $\tilde{\Omega}(u) = \tilde{\Omega}(v)$ and, in case $|C(u)| \geq 2$, $0(u) \xi_{\Gamma(0(u))} 0(v)$.

Note that this definition is inductive on $|C(u)|$, and, for given $|C(u)|$, it is further inductive on $\varepsilon(u)$.

Remark. In the above definition, if $\varepsilon(u) = \varepsilon(v) = 1$ and $|C(u)| \geq 2$, then $\Gamma(u) = \Gamma(v)$ and $\tilde{\Omega}(u) = \tilde{\Omega}(v)$ can be seen to imply $\Gamma(0(u)) = \Gamma(0(v))$ (since $C^\pm(0(u))/\Gamma(0(u))$ then arises from $C^\pm(u)/\Gamma(u)$ by deleting the class $\{\tilde{0}(u^-)\}$ and erasing the element $\tilde{0}(u)^+$ from another of its classes, and the same holds for $v$ in place of $u$). Otherwise, $\Gamma(0(u)) = \Gamma(0(v))$ is also included in the condition $0(u) \xi_{\Gamma(0(u))} 0(v)$ itself, in view of the above definition.

It can also be seen that $u \xi_{\Delta} v$ implies $\varepsilon(u) = \varepsilon(v)$. Really, we have either $\varepsilon(u) = \varepsilon(v) = 1$ or $\varepsilon(u) \geq 2$, $\varepsilon(v) \geq 2$. In the latter case, we have $\Omega_{\Delta}(u) \xi_{\Delta} \Omega_{\Delta}(v)$, which implies, by the inductive nature of the above definition, that $\Omega_{\Delta}(u) \xi_{\Delta} \Omega_{\Delta}(v)$ holds for all $\ell = 1, \ldots, \lambda$ where $\lambda$ is the integer for which $\varepsilon(\Omega_{\Delta}^{\ell}(u)) = \varepsilon(\Omega_{\Delta}^{\ell}(v)) = 1$. Hence we get $\Gamma(\Omega_{\Delta}^{\ell}(u)) = \Gamma(\Omega_{\Delta}^{\ell}(v))$ for all $\ell = 1, \ldots, \lambda$, which together with $\Gamma(u) = \Gamma(v)$ clearly
yields $\partial_{\Delta}(u) = \partial_{\Delta}(v)$ and $\partial_{\Delta}(\Omega^i_{\Delta}(u)) = \partial_{\Delta}(\Omega^i_{\Delta}(v))$ for all $\ell = 1, \ldots, \lambda - 1$. Counting these equalities up, we obtain $\varepsilon(u) = 1 = \varepsilon(v)$, so that $\varepsilon(u) = \varepsilon(v)$ holds, as claimed.

At last, notice that the equalities mentioned inside the previous paragraph and the definition of $\Omega^i_{\Delta}(u)$ and $\Omega^i_{\Delta}(v)$ for $\ell = 1, \ldots, \lambda$ together imply that $(\bar{\Omega}(\Omega^{i-1}_{\Delta}(u))^+, \bar{\Omega}(\Omega^{i-1}_{\Delta}(v))^+) \in \Delta$ holds for all $i \in \{1, \ldots, \varepsilon(u) - 1\}$. In particular, we hence get that $(\bar{\Omega}(\Omega^{\lambda-1}_{\Delta}(u))^+, \bar{\Omega}(\Omega^{\lambda-1}_{\Delta}(v))^+) \in \Delta$ holds for all $\ell = 1, \ldots, \lambda$.

Now we are ready to state the following proposition, which can be regarded as an alternative definition of the relation $\varepsilon(X)$ on $F'(X)$.

**Proposition 3.2.** For any $u, v \in F'(X)$, we have

\[ u \varepsilon(X) v \text{ if and only if } u \varepsilon(\varepsilon(u)) v. \]

**Proof.** Assume first that $u \varepsilon(X) v$ and try to show that then $u \varepsilon(\varepsilon(u)) v$. Using induction on $|C(u)|$, from the definitions of $\varepsilon(X)$ and $\varepsilon(\varepsilon(u))$ it becomes apparent that it remains only to verify the condition $\Omega^i(\varepsilon(u)) \varepsilon(\varepsilon(u)) \Omega^i(\varepsilon(v))$ if $\varepsilon(u) \geq 2$. Unfolding this condition according to the inductive nature of the definition of $\varepsilon(\varepsilon(u))$ (and using $\kappa(u) = \kappa(v)$ along with it), we get that it amounts to checking the conditions $\varepsilon(\Omega^i(u)) = \varepsilon(\Omega^i(v))$ and $(\bar{\Omega}(\Omega^{i-1}(u))^+, \bar{\Omega}(\Omega^{i-1}(v))^+) \in \Delta$ for all $\ell = 1, \ldots, \varepsilon(u)$. (Still the conditions $\varepsilon(\varepsilon(u)) = 0$ and, in case $|C(u)| \geq 2$, $0 \varepsilon(\varepsilon(u)) 0$ have to be checked, in this case, as well, but they are settled in view of the definition of $\varepsilon(X)$, using induction on $|C(u)|$, as above.) But the first of the mentioned two sets of conditions follows from the definition of $\varepsilon(X)$ by the note after the definition of $\Omega^\lambda(u)$, while the second set of these conditions can be simply omitted since it follows from the other conditions as in the last sentence of the remark after the definition of $\varepsilon(X)$.

Conversely, assume that $u \varepsilon(\varepsilon(u)) v$ and check that then $u \varepsilon(X) v$. For this purpose, however, one only has to unfold the condition $\Omega^\lambda(u) \varepsilon(\varepsilon(u)) \Omega^\lambda(v)$ in the definition of $\varepsilon(\varepsilon(u))$ in the case that $\varepsilon(u) \geq 2$, as above. Hence it also follows that $\kappa(u) = \kappa(v)$ is that integer $\kappa$ for which $\varepsilon(\Omega^\kappa(u)) = \varepsilon(\Omega^\kappa(v)) = 1$. Then it is enough only to use the note after the definition of $\Omega^\lambda(u)$ and induction on $|C(u)|$ similarly as before. \[\square\]

Also the following observation will come in handy.

**Lemma 3.3.** Let $Y \subseteq X$ be a finite subset. Let $\Delta$ and $\nabla$ be two equivalence relations on $Y^\pm$ satisfying $\Delta \subseteq \nabla$. Then, for any $u, v \in F'(X)$, it is the case that

\[ u \varepsilon_{\Delta} v \text{ implies } u \varepsilon_{\nabla} v. \]

That is, we then have $\varepsilon_{\Delta} \subseteq \varepsilon_{\nabla}$.

**Proof.** Comparing the definitions of $\varepsilon_{\Delta}$ and $\varepsilon_{\nabla}$, we find out that we have only to show that, if $\varepsilon(u) \geq 2$ and $\varepsilon(v) \geq 2$, then $\Gamma(u) = \Gamma(v) \subseteq \Delta$ and $\Omega_{\Delta}(u) \varepsilon_{\Delta} \Omega_{\Delta}(v)$ imply $\Omega_{\nabla}(u) \varepsilon_{\nabla} \Omega_{\nabla}(v)$. Recall again that $\Omega_{\Delta}(u) \varepsilon_{\Delta} \Omega_{\Delta}(v)$ yields $\Omega^\lambda_{\Delta}(u) \varepsilon_{\Delta} \Omega^\lambda_{\Delta}(u)$ for all $\ell = 1, \ldots, \lambda$ where $\lambda$ is the integer for which $\varepsilon(\Omega^\lambda_{\Delta}(u)) = \varepsilon(\Omega^\lambda_{\Delta}(v)) = 1$. Now, if we
use induction on $\varepsilon(u)$ in the proof of the above displayed implication, we see that, in
order to verify the implication stated in the previous text, we have only to notice that from
the properties just mentioned it follows that $\Omega_{\mathcal{V}}(u) = \Omega_{\mathcal{V}}^k(u)$ and $\Omega_{\mathcal{V}}(v) = \Omega_{\mathcal{V}}^k(v)$ for
some (but the same) integer $k \in \{1, \ldots, \lambda\}$. But this claim can be proved on the basis
of the definitions of these words and the inclusion $\Delta \subseteq \nabla$ in view of the notes in the last
paragraph of the remark after the definition of $\zeta_{\Delta}$. \hfill $\Box$

We start proving that the equivalence relation $\zeta(X)$ is a biinvariant congruence on $F'(X)$
by showing that it is closed with respect to some special kinds of substitutions which
are regular relatively to $\mathcal{OLB}$. We state beforehand a few general facts concerning such
substitutions.

Thus let $Z$ be another non-empty set and let $\psi : X \cup X' \rightarrow F'(Z)$ be a substitution
which is regular relatively to $\mathcal{OLB}$. Along with the set $Z$, we may consider also the set $Z^\pm$
which is of the same form as in the previous section. We introduce the following notation
for elements of $Z^\pm$ which can be viewed as images of elements of $X^\pm$ assigned by the
substitution $\psi$. For any $x \in X$, we put

$$\psi(x^+) = h(\psi(x))^+,$$
$$\psi(x^-) = t(\psi(x))^-. $$

In order to enlighten this idea, we consider also the elements

$$\psi^*(x^+) = t(\psi(x'))^-,$$
$$\psi^*(x^-) = h(\psi(x'))^+. $$

Note that then $\Gamma(\psi(x)) = \Gamma(\psi(x'))$ and $\psi(x^+), \psi^*(x^+), (\psi(x^-), \psi^*(x^-)) \in \Gamma(\psi(x))$
by Corollary 2.9(ii).

Let, in addition, $Y \subseteq X$ be a finite subset and let $\Delta$ be an equivalence relation on $Y^\pm$.
We denote by $C(\psi(Y))$ the union of all sets $C(\psi(x))$ for $x \in Y$, and by $C^\pm(\psi(Y))$ the
union of all sets $C^\pm(\psi(x))$ for $x \in Y$. Furthermore, we denote by

$\psi(\Delta)$—the equivalence relation on $C^\pm(\psi(Y))$ generated by all relations $\Gamma(\psi(x))$
for $x \in Y$ together with the relation

$$\{(\psi(p), \psi(q)) : p, q \in Y^\pm, (p, q) \in \Delta \}. $$

Note that, instead of the last relation, we could use in this definition also the relation
$\{(\psi^*(p), \psi^*(q)) : p, q \in Y^\pm, (p, q) \in \Delta \}$ with the same effect. This follows from the last
note in the previous paragraph.

The following observation is easy to verify and it will prove useful in the subsequent
considerations.

**Lemma 3.4.** For any substitution $\psi : X \cup X' \rightarrow F'(Z)$ which is regular relatively to $\mathcal{OLB}$, and for any word $u \in F'(X)$, we have

$$\Gamma(\psi(u)) = \psi(\Gamma(u)).$$
Also the following note will come in handy in our next proofs.

**Lemma 3.5.** Let \( \psi : X \cup X' \rightarrow F'(Z) \) be any substitution which is regular relatively to \( \text{OLB} \), let \( Y \subseteq X \) be any finite subset, let \( \Delta \) be any equivalence relation on \( Y^\pm \), and let \( u \in F'(X) \) be any word such that \( C(u) = Y \), \( \Gamma(u) \subseteq \Delta \), \( \varepsilon(u) \geq 2 \), and \( \varepsilon(\psi(u)) \geq 2 \). Then the word \( \Omega(\psi(\Delta)(\psi(u))) \) is a proper initial segment of the word \( \psi(\Omega(\Delta)(\psi(u))) \).

**Proof.** We begin with the following note. The words \( \Omega(\psi(u)) \) for all \( i \geq 1 \) having the property that \( \psi(\Omega(\Delta)(u)) \) is an initial segment of \( \Omega(\psi(u)) \), are all of the form \( \psi(\Omega(\psi(u))) \) for suitable \( j \in \{1, \ldots, \partial_\Delta(u)\} \), that is, for some \( j \geq 1 \) such that \( \Omega(\Delta)(u) \) is an initial segment of \( \psi(\Omega(\psi(u))) \). In order to check that no other initial segment of \( \psi(u) \) containing \( \psi(\Omega(\Delta)(u)) \) can appear among the words \( \Omega(\psi(u)) \) for the mentioned \( i \), it is enough to notice that if such segment were of the form \( \psi(u) \) for some proper initial segment \( w \) of \( u \) containing \( \Omega(\Delta)(u) \), but different from \( \Omega(\psi(u)) \) for the given \( j \), then, letting \( y \in X \cup X' \) be the element for which \( wy \) is an initial segment of \( u \), we would come to a contradiction ensuing from the fact that \( \Gamma(\psi(\psi(w))) = \psi(\Gamma(w)) = \psi(\Gamma(x)) = \Gamma'(\psi(wy)) \) by Lemma 3.4, and if we had any proper initial segment \( s \) of \( \psi(u) \) containing \( \psi(\Omega(\Delta)(u)) \) and different from all segments of the form \( \psi(u) \) for all proper initial segments \( w \) of \( u \) containing \( \Omega(\Delta)(u) \), then, denoting by \( t \) the element of \( Z \cup Z' \) for which \( st \) is an initial segment of \( \psi(u) \), we would again come to a contradiction, this time emerging from the fact that \( \Gamma(s) = \Gamma'(st) \), which follows from the fact that \( \Gamma(s) \) contains \( \Gamma'(\psi(x)) \) for all \( x \in C(u) \).

Now it remains to remember that, by the definition of the operator \( \Omega(\Delta) \), all elements \( \psi(\Omega(\psi(u))) \) for all \( i \geq 1 \) lie in the same class of the partition \( Y^\pm /\Delta \), and hence, according to the note in the previous paragraph, and in compliance with the definition of \( \psi(\Delta) \), all elements \( \psi(\Omega(\psi(u))) \) for all \( i \) specified above lie in the same class of the partition \( C^\pm(\psi(Y))/\psi(\Delta) \), since we have \( \psi(\Omega(\Omega(\psi(u)))) \) \( \in \psi(\Delta) \) for those \( i, j \) which correspond to each other in the way described above. But this means that the word \( \Omega(\psi(\Delta)(\psi(u))) \) is contained in all segments \( \Omega(\psi(u)) \) for all \( i \) as above. This finding confirms the claim of this lemma. \( \square \)

We continue by introducing the first of the above-mentioned special kinds of substitutions. Let \( a, b \in X \) and \( z \not\in X \) be three distinct elements. Let

\[
\mu_{z}^{a,b} : X \cup X' \rightarrow F'(\{X - \{a, b\}\} \cup \{z\})
\]

be the substitution given by the rules

- \( a \mapsto z \), \( a' \mapsto z' \),
- \( b \mapsto z \), \( b' \mapsto z' \),
- \( x \mapsto x \), \( x' \mapsto x' \) for all \( x \in X - \{a, b\} \).

Then, obviously, \( \mu_{z}^{a,b} \) is regular relatively to the whole e-variety \( \Omega \).

Notice that if \( Y \subseteq X \) is any (finite) subset then, of course, the set \( C^\pm(\mu_{z}^{a,b}(Y)) \) is just \( \mu_{z}^{a,b}(Y)^\pm \), and it is, clearly, the same set as \( \mu_{z}^{a,b}(Y)^\pm \) (we utilise here in the common way
the notation previously introduced). If, in addition, $\Delta$ is any equivalence relation on $Y^{\pm}$, and if $a, b \in Y$, then the partition $\mu_{z}^{a,b}(Y^{\pm})/\mu_{z}^{a,b}(\Delta)$ can be obtained from the partition $Y^{\pm}/\Delta$ by the following procedure: identify the elements $a^+$ and $b^+$ and change the name of the element thus arising to $z^+$, identify the elements $a^-$ and $b^-$ and change the name of the element thus arising to $z^-$; this identification may put together some classes of $Y^{\pm}/\Delta$, and it is in this way that $\mu_{z}^{a,b}(Y^{\pm})/\mu_{z}^{a,b}(\Delta)$ comes up.

We intend to show that the family of relations $\xi_{\Delta}$ introduced earlier in this section, for all finite non-empty sets $Y$ and for all equivalence relations $\Delta$ on $Y^{\pm}$, is closed with respect to all substitutions $\mu_{z}^{a,b}$ described above. That is, we are about to prove the following assertion.

**Lemma 3.6.** If $Y \subseteq X$ is a finite subset, if $\Delta$ is an equivalence relation on $Y^{\pm}$, and if $u, v \in F^{*}(X)$ are any words such that $u \xi_{\Delta} v$, then, for any $a, b \in X, a \neq b$, and any $z \notin X$, we have

$$\mu_{z}^{a,b}(u) \xi_{\mu_{z}^{a,b}(\Delta)} \mu_{z}^{a,b}(v).$$

**Proof.** Remember that $u \xi_{\Delta} v$ implies $C(u) = C(v) = Y$. If $a \notin Y$ or $b \notin Y$ then there is essentially nothing to prove. Thus assume further that $a, b \in Y$. Incidentally this means that $|C(u)| \geq 2$. Furthermore, $u \xi_{\Delta} v$ also implies $\Gamma(u) = \Gamma(v) \subseteq \Delta$ and $(t(u)^-, t(v)^-) \in \Delta$. Hence we obtain $C(\mu_{z}^{a,b}(u)) = C(\mu_{z}^{a,b}(v)) = \mu_{z}^{a,b}(Y), \Gamma(\mu_{z}^{a,b}(u)) = \Gamma(\mu_{z}^{a,b}(v)) \subseteq \mu_{z}^{a,b}(\Delta)$ in view of Lemma 3.4, and $(t(\mu_{z}^{a,b}(u))^-, t(\mu_{z}^{a,b}(v))^-) \in \mu_{z}^{a,b}(\Delta)$. The verification of the remaining conditions ensuring that $\mu_{z}^{a,b}(u) \xi_{\mu_{z}^{a,b}(\Delta)} \mu_{z}^{a,b}(v)$ holds will proceed by induction on $|C(u)|$, and, for given $|C(u)|$, by another induction on $|u|$. According to the definition of $\xi_{\Delta}$, there are two basic cases to distinguish:

**Case 1.** Assume that $|u| = |v| = 1$. Then $\Gamma(u) = \Gamma(0(u)0(u))$ and $\Gamma(v) = \Gamma(0(v)0(v))$, whence we obtain

$$\Gamma(\mu_{z}^{a,b}(u)) = \Gamma(\mu_{z}^{a,b}(0(u)0(u))) \quad \text{and} \quad \Gamma(\mu_{z}^{a,b}(v)) = \Gamma(\mu_{z}^{a,b}(0(v)0(v))). \quad (*)$$

In addition, $u \xi_{\Delta} v$ then implies $\hat{0}(u) = \hat{0}(v)$ and $0(u) \xi_{\Gamma(0(u))} 0(v)$. Then we have to distinguish two subcases:

**Subcase 1.1.** Assume that the element $\hat{0}(u) = \hat{0}(v)$ does not occur in $\{a, a', b, b'\}$. Then $\hat{0}(\mu_{z}^{a,b}(u)) = 0(u) = \hat{0}(v) = \hat{0}(\mu_{z}^{a,b}(v))$ and $0(\mu_{z}^{a,b}(u)) = \mu_{z}^{a,b}(0(u)), 0(\mu_{z}^{a,b}(v)) = \mu_{z}^{a,b}(0(v))$. Hence, having $(*)$ in view, we obtain that $\varepsilon(\mu_{z}^{a,b}(u)) = \varepsilon(\mu_{z}^{a,b}(v)) = 1$. Moreover, our induction on $|C(u)|$ makes it possible to infer from $0(u) \xi_{\Gamma(0(u))} 0(v)$ the relation

$$0(\mu_{z}^{a,b}(u)) \xi_{\Gamma(0(\mu_{z}^{a,b}(u)))} 0(\mu_{z}^{a,b}(v)), \quad (***)$$

if we have also the previous equalities and Lemma 3.4 in mind. This verifies the required conditions.

**Subcase 1.2.** Assume that the element $\hat{0}(u) = \hat{0}(v)$ appears in $\{a, a', b, b'\}$. As a consequence, we get that $C(\mu_{z}^{a,b}(0(u))) = C(\mu_{z}^{a,b}(u))$ and $C(\mu_{z}^{a,b}(0(v))) = C(\mu_{z}^{a,b}(v))$. This
implies that \( \tilde{O}(\mu_{a,b}^{c}(0(u))) = \tilde{O}(\mu_{a,b}^{c}(u)), \) \( \tilde{O}(\mu_{a,b}^{c}(0(v))) = \tilde{O}(\mu_{a,b}^{c}(v)) \), and \( 0(\mu_{a,b}^{c}(0(u))) = 0(\mu_{a,b}^{c}(u)), 0(\mu_{a,b}^{c}(0(v))) = 0(\mu_{a,b}^{c}(v)) \). Furthermore, our induction on \( |C(u)| \) and Lemma 3.4 make it possible to deduce from \( 0(u) \equiv \Omega(0(v)) \) the relation

\[
\mu_{a,b}^{c}(0(u)) \equiv \Omega(0(v)) \equiv \mu_{a,b}^{c}(0(v)). \quad (***)
\]

Then we again have to distinguish two subcases:

Subcase I.2.i. Assume that \( \Gamma(\mu_{a,b}^{c}(0(u))) = \Gamma(\mu_{a,b}^{c}(u)) \). Equivalently, this means that \( \Gamma(\mu_{a,b}^{c}(0(v))) = \Gamma(\mu_{a,b}^{c}(v)) \), since we have seen above that \( \Gamma(\mu_{a,b}^{c}(u)) = \Gamma(\mu_{a,b}^{c}(v)) \) and, in addition, \( 0(u) \equiv \Omega(0(v)) \) implies \( \Gamma(0(u)) = \Gamma(0(v)) \), whence we get \( \Gamma(\mu_{a,b}^{c}(0(u))) = \Gamma(\mu_{a,b}^{c}(0(v))) \) by Lemma 3.4. From these equalities it follows that \( \Omega(\mu_{a,b}^{c}(0(u))) = \Omega(\mu_{a,b}^{c}(0(v))) \) and \( \Omega(\mu_{a,b}^{c}(v)) = \Omega(\mu_{a,b}^{c}(0(v))) \), and hence also \( \epsilon(\mu_{a,b}^{c}(u)) = \epsilon(\mu_{a,b}^{c}(0(v))) \) and \( \epsilon(\mu_{a,b}^{c}(v)) = \epsilon(\mu_{a,b}^{c}(0(v))) \). From (***), we know that \( \epsilon(\mu_{a,b}^{c}(0(u))) \leq 2 \) if \( \epsilon(\mu_{a,b}^{c}(0(v))) \leq 2 \). Hence, according to the previous equalities, we get that \( \epsilon(\mu_{a,b}^{c}(u)) \geq 2 \) if \( \epsilon(\mu_{a,b}^{c}(v)) \geq 2 \). This verifies the required conditions.

Subcase I.2.ii. Assume that \( \epsilon(\mu_{a,b}^{c}(u)) \geq 2 \) and \( \epsilon(\mu_{a,b}^{c}(v)) \geq 2 \). Then the equalities in the last paragraph but one show that \( \Omega(\mu_{a,b}^{c}(\Delta)) = \Omega(\mu_{a,b}^{c}(0(u))) \) and \( \Omega(\mu_{a,b}^{c}(\Delta)) = \Omega(\mu_{a,b}^{c}(0(v))) \). Further on, since \( \Gamma(\mu_{a,b}^{c}(0(u))) = \Gamma(\mu_{a,b}^{c}(u)) \) is a subcase, we can obtain \( \Omega(\mu_{a,b}^{c}(u)) = \Omega(\mu_{a,b}^{c}(0(v))) \), and, in case \( |C(u)| = 2 \), also the relation (**). This verifies the remaining condition which we had to verify.

Subcase I.2.iii. Assume that \( \Gamma(\mu_{a,b}^{c}(0(u))) \leq \Gamma(\mu_{a,b}^{c}(u)) \). We have seen above that it is equivalent to the assumption that \( \mu_{a,b}^{c}(\Omega(0(v))) \leq \mu_{a,b}^{c}(0(v)) \). Then, in view of (**), we have \( \Omega(\mu_{a,b}^{c}(u)) = \mu_{a,b}^{c}(0(u)) \) and \( \Omega(\mu_{a,b}^{c}(v)) = \mu_{a,b}^{c}(0(v)) \), whence, according to our previous assumptions, we obtain that \( \epsilon(\mu_{a,b}^{c}(u)) \geq 2 \) and \( \epsilon(\mu_{a,b}^{c}(v)) \geq 2 \). Once again, from (**), we also know that \( \epsilon(\mu_{a,b}^{c}(0(u))) \geq 2 \) if \( \epsilon(\mu_{a,b}^{c}(0(v))) \geq 2 \). Then we have to distinguish yet the following three subcases:

Subcase I.2.iii. Assume that \( \epsilon(\mu_{a,b}^{c}(0(u))) \geq 2 \) and \( \epsilon(\mu_{a,b}^{c}(0(v))) \geq 2 \), and that \( \Omega(\mu_{a,b}^{c}(\Delta)) = \mu_{a,b}^{c}(0(u)) \). Then, using the definition of \( \xi_{\mu_{a,b}^{c}(\Delta)} \), in view of the previous equalities, we get the relation

\[
\Omega(\mu_{a,b}^{c}(\Delta)) \equiv \xi_{\mu_{a,b}^{c}(\Delta)} \Omega(\mu_{a,b}^{c}(\Delta)) \equiv \mu_{a,b}^{c}(0(u)). \quad (\dagger)
\]
requirement that $\Omega_{\mu_z^{a,b}(\Delta)}(\mu_z^{a,b}(v)) = \mu_z^{a,b}(0(v))$. Namely, these two conditions are equivalent to $(\overline{\Omega}(\mu_z^{a,b}(0(u))))^+, \mu_z^{a,b}(\overline{0(u)})^+ \notin \mu_z^{a,b}(\Delta)$ and $(\overline{\Omega}(\mu_z^{a,b}(0(v))))^+, \mu_z^{a,b}(\overline{0(v)})^+ \notin \mu_z^{a,b}(\Delta)$, respectively. The equivalence of these last two conditions follows from the facts that $\overline{0(u)} = \overline{0(v)}$ and also $(\overline{\Omega}(\mu_z^{a,b}(0(u))))^+, \overline{\Omega}(\mu_z^{a,b}(0(v)))^+ \in \Gamma(\mu_z^{a,b}(0(u))) \subset \Gamma(\mu_z^{a,b}(u)) \subseteq \mu_z^{a,b}(\Delta)$. This can be obtained from (***) in the same way as in the last paragraph of the remark after the definition of $\zeta_\Delta$. Having this cleared up, we can arrive from (****) at the relation (?) in the same way as in the previous subcase.

**Subcase 1.2 ii′′′.** Assume that $\epsilon(\mu_z^{a,b}(0(u))) \geq 2$, $\epsilon(\mu_z^{a,b}(0(v))) \geq 2$, and that $\Omega_{\mu_z^{a,b}(\Delta)}(\mu_z^{a,b}(u))$ is a proper initial segment of $\mu_z^{a,b}(0(u))$. According to the considerations in the previous subcase, the last assumption is equivalent to the condition that $\Omega_{\mu_z^{a,b}(\Delta)}(\mu_z^{a,b}(v))$ is a proper initial segment of $\mu_z^{a,b}(0(v))$. Hence we derive the same equalities as in the earlier Subcase 1.2 ii′′, and as we have $\Gamma(\mu_z^{a,b}(0(u))) \subset \Gamma(\mu_z^{a,b}(u)) \subseteq \mu_z^{a,b}(\Delta)$ all along, in the same way as in that former subcase, we can come from (****) to the relation (?). And this is the last condition we had to verify.

**Case II.** Assume that $\epsilon(u) \geq 2$ and $\epsilon(v) \geq 2$. Then $u \xi_\Delta v$ implies $\Omega_\Delta(u) \xi_\Delta \Omega_\Delta(v)$. whence, using induction on $\epsilon(u)$, as explained in the first paragraph of this proof, we obtain the relation

$$\mu_z^{a,b}(\Omega_\Delta(u)) \zeta_{\mu_z^{a,b}(\Delta)} \mu_z^{a,b}(\Omega_\Delta(v)).$$

Notice also that $C(\Omega_\Delta(u)) = C(u)$ and $C(\Omega_\Delta(v)) = C(v)$, whence we get $C(\mu_z^{a,b}(\Omega_\Delta(u))) = C(\mu_z^{a,b}(u))$ and $C(\mu_z^{a,b}(\Omega_\Delta(v))) = C(\mu_z^{a,b}(v))$. This also means that $\overline{0}(\mu_z^{a,b}(\Omega_\Delta(u))) = \overline{0}(\mu_z^{a,b}(u))$, $\overline{0}(\mu_z^{a,b}(\Omega_\Delta(v))) = \overline{0}(\mu_z^{a,b}(v))$, and $0(\mu_z^{a,b}(\Omega_\Delta(u))) = 0(\mu_z^{a,b}(u))$, $0(\mu_z^{a,b}(\Omega_\Delta(v))) = 0(\mu_z^{a,b}(v))$. Then we have to distinguish two subcases:

**Subcase II.1.** Assume that $\Gamma(\mu_z^{a,b}(\Omega_\Delta(u))) = \Gamma(\mu_z^{a,b}(u))$. Equivalently, this means that $\Gamma(\mu_z^{a,b}(\Omega_\Delta(u))) = \Gamma(\mu_z^{a,b}(v))$, since we have already seen that $\Gamma(\mu_z^{a,b}(u)) = \Gamma(\mu_z^{a,b}(v))$ and, in addition, $\Omega_\Delta(u) \xi_\Delta \Omega_\Delta(v)$ implies $\Gamma(\Omega_\Delta(u)) = \Gamma(\Omega_\Delta(v))$, whence we get $\Gamma(\mu_z^{a,b}(\Omega_\Delta(u))) = \Gamma(\mu_z^{a,b}(\Omega_\Delta(v)))$ by Lemma 3.4. From the mentioned equalities we see that $\overline{0}(\mu_z^{a,b}(u)) = \overline{0}(\mu_z^{a,b}(\Omega_\Delta(u)))$ and $\overline{0}(\mu_z^{a,b}(v)) = \overline{0}(\mu_z^{a,b}(\Omega_\Delta(v)))$, so that also $\epsilon(\mu_z^{a,b}(u)) = \epsilon(\mu_z^{a,b}(\Omega_\Delta(u)))$ and $\epsilon(\mu_z^{a,b}(v)) = \epsilon(\mu_z^{a,b}(\Omega_\Delta(v)))$. From (***′′′) we know that $\epsilon(\mu_z^{a,b}(\Omega_\Delta(u))) \geq 2$ if $\epsilon(\mu_z^{a,b}(\Omega_\Delta(v))) \geq 2$. Thus, in view of the previous equalities, we obtain that $\epsilon(\mu_z^{a,b}(u)) \geq 2$ if $\epsilon(\mu_z^{a,b}(v)) \geq 2$. Then we have to distinguish yet two subcases:

**Subcase II.1.i.** Assume that $\epsilon(\mu_z^{a,b}(u)) = \epsilon(\mu_z^{a,b}(v)) = 1$. Then, using the definition of $\zeta_{\mu_z^{a,b}(\Delta)}$ and the equalities given in the previous two paragraphs, from (***′′′) we obtain $\overline{0}(\mu_z^{a,b}(u)) = \overline{0}(\mu_z^{a,b}(v))$ and, in case $|C(\mu_z^{a,b}(u))| \geq 2$, also the relation (**). This verifies the required conditions.

**Subcase II.1.ii.** Assume that $\epsilon(\mu_z^{a,b}(u)) \geq 2$ and $\epsilon(\mu_z^{a,b}(v)) \geq 2$. Then the equalities in the last paragraph but one show that $\Omega_{\mu_z^{a,b}(\Delta)}(\mu_z^{a,b}(u)) = \Omega_{\mu_z^{a,b}(\Delta)}(\mu_z^{a,b}(\Omega_\Delta(u)))$ and $\Omega_{\mu_z^{a,b}(\Delta)}(\mu_z^{a,b}(v)) = \Omega_{\mu_z^{a,b}(\Delta)}(\mu_z^{a,b}(\Omega_\Delta(v)))$. Having these equalities in mind and using
Subcase II.2. Assume that $\Gamma(\mu_{z}^{a,b}(\Omega_{\Delta}(u))) \subset \Gamma(\mu_{z}^{a,b}(u))$. We have seen above that this condition is equivalent to the demand that $\Gamma(\mu_{z}^{a,b}(\Omega_{\Delta}(v))) \subset \Gamma(\mu_{z}^{a,b}(v))$. Then it is obvious from the above-mentioned properties that $\varepsilon(\mu_{z}^{a,b}(u)) \geq 2$ and $\varepsilon(\mu_{z}^{a,b}(v)) \geq 2$. In addition, once again, from ($\dagger\dagger$) we know that $\varepsilon(\mu_{z}^{a,b}(\Omega_{\Delta}(u))) \geq 2$ iff $\varepsilon(\mu_{z}^{a,b}(\Omega_{\Delta}(v))) \geq 2$. Then, as before, we again have to distinguish two subcases:

Subcase II.2.i. Assume that $\varepsilon(\mu_{z}^{a,b}(\Omega_{\Delta}(u))) = \varepsilon(\mu_{z}^{a,b}(\Omega_{\Delta}(v))) = 1$. This assumption means that the word $\mu_{z}^{a,b}(\Omega_{\Delta}(u))$ must be an initial segment of $\Omega\mu_{z}^{a,b}(\Delta)(\mu_{z}^{a,b}(u))$ and the word $\mu_{z}^{a,b}(\Omega_{\Delta}(v))$ must be an initial segment of $\Omega\mu_{z}^{a,b}(\Delta)(\mu_{z}^{a,b}(v))$. On the other hand, from Lemma 3.5 we know that the word $0(\Omega\mu_{z}^{a,b}(\Delta)(\mu_{z}^{a,b}(u)))$ is a proper initial segment of $\mu_{z}^{a,b}(\Omega_{\Delta}(u))$ and the word $0(\Omega\mu_{z}^{a,b}(\Delta)(\mu_{z}^{a,b}(v)))$ is a proper initial segment of $\mu_{z}^{a,b}(\Omega_{\Delta}(v))$. In view of our previous assumption, it hence readily follows that

$$\varepsilon(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u))) = \varepsilon(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v))) = 1.$$ 

At the same time, we hence also obtain the equalities

$$\bar{0}(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u))) = \bar{0}(\mu_{z}^{a,b}(\Omega_{\Delta}(u))), \quad \bar{0}(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v))) = \bar{0}(\mu_{z}^{a,b}(\Omega_{\Delta}(v))),$$

and, in the same way,

$$0(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u))) = 0(\mu_{z}^{a,b}(\Omega_{\Delta}(u))), \quad 0(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v))) = 0(\mu_{z}^{a,b}(\Omega_{\Delta}(v))).$$

Further on, comparing the two facts on initial segments mentioned previously, we come to the conclusion that $\Gamma(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u))) = \Gamma(\mu_{z}^{a,b}(\Omega_{\Delta}(u)))$ and $\Gamma(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v))) = \Gamma(\mu_{z}^{a,b}(\Omega_{\Delta}(v)))$. Since we have already seen above that $\Gamma(\mu_{z}^{a,b}(\Omega_{\Delta}(u))) = \Gamma(\mu_{z}^{a,b}(\Omega_{\Delta}(v)))$, we hence get that

$$\Gamma(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u))) = \Gamma(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v))).$$

Notice also, in passing, that we have $\Gamma(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u))) = \Gamma(\mu_{z}^{a,b}(\Omega_{\Delta}(u))) \subset \Gamma(\mu_{z}^{a,b}(u)) = \mu_{z}^{a,b}(\Gamma(u)) \subseteq \mu_{z}^{a,b}(\Delta)$. Since $\Gamma(\mu_{z}^{a,b}(u)) = \Gamma(\mu_{z}^{a,b}(v))$, using the definition of the operator $\Omega_{\mu_{z}^{a,b}(\Delta)}$, from the above equality we also infer that

$$\langle t(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u))), t(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v))) \rangle \in \mu_{z}^{a,b}(\Delta).$$

Moreover, using the definition of $\zeta_{\mu_{z}^{a,b}(\Delta)}$ and the equalities inside the above text, from ($\dagger\dagger$) we deduce that

$$\bar{0}(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u))) = \bar{0}(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v))).$$
and, in case $|C(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u))))| = |C(\mu_{z}^{a,b}(\Omega_{\Delta}(u))))| > 2$, also that

$$0(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u)))) \leq |C(\mu_{z}^{a,b}(\Omega_{\Delta}(u))))|.$$ 

Now, according to the definition of $\xi_{\mu_{z}^{a,b}(\Delta)}$, all properties displayed above together ensure that the relation $(\dag)$ holds. And this is the remaining condition which we had to verify.

**Subcase II.2.ii.** Assume that $\varepsilon(\mu_{z}^{a,b}(\Omega_{\Delta}(u)))) > 2$ and $\varepsilon(\mu_{z}^{a,b}(\Omega_{\Delta}(v)))) > 2$. Then we have to distinguish yet the following two subcases:

**Subcase II.2.i.** Assume that the word $\mu_{z}^{a,b}(\Omega_{\Delta}(u)))$ is an initial segment of the word $\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u)))$. This assumption can be seen to be equivalent to the requirement that the word $\mu_{z}^{a,b}(\Omega_{\Delta}(u)))$ is an initial segment of the word $\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v)))$. Really, Lemma 3.5 shows that these two conditions are equivalent to $(\overline{O}(\mu_{z}^{a,b}(\Omega_{\Delta}(u))))$, $\overline{O}(\mu_{z}^{a,b}(\Delta))$, and $(\overline{O}(\mu_{z}^{a,b}(\Omega_{\Delta}(v))))$, $\overline{O}(\mu_{z}^{a,b}(\Delta))$, respectively.

The equivalence of these last two conditions follows on the one hand from the fact that $\overline{O}(\mu_{z}^{a,b}(\Omega_{\Delta}(u))))$, $\overline{O}(\mu_{z}^{a,b}(\Omega_{\Delta}(v)))) \in \mu_{z}^{a,b}(\Delta)$, which can be deduced from $(\dag)$ in the same way as in the last paragraph of the remark after the definition of $\xi_{\Delta}$, and on the other hand from the fact that $\mu_{z}^{a,b}(\Omega_{\Delta}(u)))$, $\mu_{z}^{a,b}(\Omega_{\Delta}(v))) \in \mu_{z}^{a,b}(\Delta)$, which can be justified as follows. Since, according to the assumptions delimiting Subcase II.2, the word $\mu_{z}^{a,b}(\Omega_{\Delta}(u)))$ is an initial segment of $\Omega(\mu_{z}^{a,b}(u)))$ and the word $\mu_{z}^{a,b}(\Omega_{\Delta}(v)))$ is an initial segment of $\Omega(\mu_{z}^{a,b}(v)))$, as in the proof of Lemma 3.5, we have $\overline{C}(\mu_{z}^{a,b}(u))) = \mu_{z}^{a,b}(\Omega(\mu_{z}^{a,b}(v)))$ for some $i \in \{1, \ldots, \delta_{\Delta}(u)))$ and $\overline{C}(\mu_{z}^{a,b}(v))) = \mu_{z}^{a,b}(\Omega(\mu_{z}^{a,b}(v)))$ for some $j \in \{1, \ldots, \delta_{\Delta}(v)))$. Hence we get that the element $\overline{O}(\mu_{z}^{a,b}(u)))$ lies in the same $\mu_{z}^{a,b}(\Delta)$-class as $\mu_{z}^{a,b}(\overline{O}(\mu_{z}^{a,b}(u)))$ and the element $\overline{O}(\mu_{z}^{a,b}(v)))$ lies in the same $\mu_{z}^{a,b}(\Delta)$-class as $\mu_{z}^{a,b}(\overline{O}(\mu_{z}^{a,b}(v)))$, whence the above claim follows in view of the fact that $\overline{O}(\mu_{z}^{a,b}(u)))$, $\overline{O}(\mu_{z}^{a,b}(v))) \in \Delta$. This last fact can be derived from the equalities $\Gamma(u)) = \Gamma(v))$ and $\Gamma(\Omega_{\Delta}(u))) = \Gamma(\Omega_{\Delta}(v)))$, reasoning in the same way as in the last paragraph of the remark after the definition of $\xi_{\Delta}$. Having thus clarified the assumptions of this subcase, we can continue by noting that these assumption (together with the previous ones) yield that now we have

$$\varepsilon(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u)))) > 2 \quad \text{and} \quad \varepsilon(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v)))) > 2.$$ 

Then we can go on to deduce similar consequences as in Subcase II.2.i. Thus, in the same way as in that subcase, we obtain the equality $(\#)$ and the condition $(\#\#)$, including the notes accompanying these conditions in the text around. Further on, from our present assumptions and Lemma 3.5 we see that we have $\Omega_{\mu_{z}^{a,b}(\Delta)}(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u)))) = \Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(\Omega_{\Delta}(u))))$ and $\Omega_{\mu_{z}^{a,b}(\Delta)}(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v)))) = \Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(\Omega_{\Delta}(v))))$. Having these equalities in view and using the definition of $\xi_{\mu_{z}^{a,b}(\Delta)}$, from $(\dag)$ we now deduce the relation

$$\Omega_{\mu_{z}^{a,b}(\Delta)}(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(u)))) \xi_{\mu_{z}^{a,b}(\Delta)} \Omega_{\mu_{z}^{a,b}(\Delta)}(\Omega_{\mu_{z}^{a,b}(\Delta)}(\mu_{z}^{a,b}(v))))$$.
which together with the previous properties ensures, according to the definition of $\zeta_{\mu^{a,b}}$, that the relation $(\dagger)$ holds. As before, this is the remaining condition that we had to verify.

Subcase II.2.ii′. Assume that the word $\Omega_{\mu^{a,b}}(\Delta)(\mu^{a,b}Z(u))$ is a proper initial segment of the word $\mu^{a,b}(\Omega_{\Delta}(u))$. We have seen in the previous subcase that this assumption is equivalent to the demand that the word $\Omega_{\mu^{a,b}}(\Delta)(\mu^{a,b}(v))$ is a proper initial segment of the word $\mu^{a,b}(\Omega_{\Delta}(v))$. Then, using the definition of the operator $\Omega_{\mu^{a,b}}(\Delta)$, from these conditions we immediately conclude that we have $\Omega_{\mu^{a,b}}(\Delta)(\mu^{a,b}Z(u)) = \Omega_{\mu^{a,b}}(\Delta)(\mu^{a,b}(\Omega_{\Delta}(u)))$ and $\Omega_{\mu^{a,b}}(\Delta)(\mu^{a,b}(v)) = \Omega_{\mu^{a,b}}(\Delta)(\mu^{a,b}(\Omega_{\Delta}(v)))$. Having these equalities in mind and using the definition of $\zeta_{\mu^{a,b}}$, from (††) we obtain the relation $(\dagger)$, which is again the remaining condition that we had to verify. ✷

Corollary 3.7. If $u, v \in F'(X)$ are any words satisfying $u \zeta(X) v$, then
\[
\mu^{a,b}(u) \zeta(\mu^{a,b}(X)) \mu^{a,b}(v)
\]
holds for any $a, b \in X, a \neq b$, and any $z \notin X$. (Remember that $\mu^{a,b}(X) = (X - \{a, b\}) \cup \{z\}$.)

Proof. From Proposition 3.2 we know that $u \zeta(X) v$ holds if and only if $u \zeta_{\Gamma(u)} v$ is true. By Lemma 3.6, this condition yields $\mu^{a,b}(u) \zeta(\mu^{a,b}(\Gamma(u))) \mu^{a,b}(v)$, that is, $\mu^{a,b}(u) \zeta(\mu^{a,b}(\Gamma(u))) \mu^{a,b}(v)$, in view of Lemma 3.4. Using Proposition 3.2 once again, we see this is true if and only if $\mu^{a,b}(u) \zeta(\mu^{a,b}(X)) \mu^{a,b}(v)$. ✷

Now we are able to prove the following fact.

Corollary 3.8. The equivalence relation $\zeta(X)$ is a congruence relation on $F'(X)$.

Proof. It is clearly enough to show that whenever $u, v \in F'(X)$ satisfy $u \zeta(X) v$ and $y \in X \cup X'$ then $uy \zeta(X) vy$ and $yu \zeta(X) yv$ hold. Moreover, applying an appropriate substitution of the form $\mu^{a,b}Z(\text{and then renaming the variable } z)$, from Corollary 3.7 we see that it is enough to consider only the case when $C(y) \not\subseteq C(u) = C(v)$ in the two conditions just mentioned.

We check the first of these two conditions. Clearly, $\Gamma(u) = \Gamma(v)$ and $(t(uy)^{-}, t(vy)^{-}) \in \Gamma(u)$ yield $\Gamma(uy) = \Gamma(vy)$. Also $(t(uy)^{-} = y^{-} = t(vy)^{-}$. Moreover, the additional assumption named in the previous paragraph means that $0(uy) = y = 0(vy)$ and $0(u) = u, 0(v) = v$, so that $u \zeta(X) v$ yields $0(uy) \zeta(X) 0(vy)$. Hence we also get $\epsilon(uy) = 1 = \epsilon(vy)$, so that $x(uy) = 0 = x(vy)$, and so all conditions required to assure that $uy \zeta(X) vy$ are fulfilled.

We next check the second of the two conditions mentioned above. Applying induction on $C(u)$, we see that $u \zeta(X) v$ eventually yields $h(u) = h(v)$. Hence, from $\Gamma(u) = \Gamma(v)$ we obtain $\Gamma(yu) = \Gamma(yv)$. Further on, $t(yu) = t(u)$ and $t(yv) = t(v)$, and so from $(t(u)^{-}, t(v)^{-}) \in \Gamma(u)$ we get $(t(yu)^{-}, t(yv)^{-}) \in \Gamma(yu)$. Moreover, from our additional
assumption that \( C(\gamma) \not
subseteq C(u) = C(v) \) it follows that \( \tilde{0}(yu) = \tilde{0}(u) = \tilde{0}(v) = \tilde{0}(yv) \) and 
\( 0(yu) = y0(u), 0(yv) = y0(v). \) Thus, if \( |C(u)| = 1 \), we have \( 0(yu) = y = 0(yv), \) and if \( |C(u)| \geq 2 \), using induction on \(|C(u)|\) in the course of the verification of the condition in question, from \( 0(u) \zeta(X) 0(v) \) we may conclude that \( y0(u) \zeta(X) y0(v), \) which yields 
\( 0(yu) \zeta(X) 0(yv). \) Furthermore, from the mentioned assumption it is also obvious that the partition \( C^\pm(yu)/\Gamma(yu) \) arises from \( C^\pm(u)/\Gamma(u) \) by appending the element \( y^- \) to 
its class containing \( h(u)^+ \) and by adding the new class \( \{ y^+ \} \), and that \( C^\pm(yv)/\Gamma(yv) \) arises in the same way from \( C^\pm(v)/\Gamma(v) \). Similar remarks apply, more generally, to the partitions \( C^\pm(yu)/\Gamma(\mathcal{O}^i(yu)) \) and \( C^\pm(yv)/\Gamma(\mathcal{O}^i(yv)) \) with respect to \( C^\pm(u)/\Gamma(\mathcal{O}^i(u)) \) and \( C^\pm(v)/\Gamma(\mathcal{O}^i(v)) \), respectively, for all \( i = 0, 1, \ldots, \varepsilon(u) - 1 \). Hence it becomes obvious that \( \mathcal{O}^i(yu) = y\mathcal{O}^i(u) \) and \( \mathcal{O}^i(yv) = y\mathcal{O}^i(v) \) hold for all \( i \) as before, and that \( \varepsilon(yu) = \varepsilon(u) = \varepsilon(v) = \varepsilon(yv) \) and \( \iota(yu) = \iota(u) = \iota(v) = \iota(yv) \). Consequently, all remaining conditions which must be verified in order to ensure that \( yu \zeta(X) yv \) holds 
follow from the respective conditions included in \( u \zeta(X) v \), and therefore they are fulfilled, 
as well. \( \square \)

Next we introduce yet another special kind of substitutions. Let \( c \in X \) be any element, 
let \( Z \) be another (finite) set disjoint with \( X \), and let \( r, s \in F'(Z) \) be any words satisfying 
\( rsr \rho(\mathcal{OLB}) r, srs \rho(\mathcal{OLB}) s \). Equivalently, these words satisfy the conditions \( \Gamma'(r) = \Gamma'(s) \) and \( (t(t(r)^-, h(s)^+), (t(s)^-, h(r)^+)) \in \Gamma'(r) \), according to Corollary 2.9(ii). Let 
\[ \chi^c_{r,s} : X \cup X' \rightarrow F'(X \cup Z) \]
be the substitution given by the rules
\[
c \mapsto r, \quad c' \mapsto s, \\
x \mapsto x, \quad x' \mapsto x' \quad \text{for all } x \in X - \{ c \}.
\]
Then, clearly, \( \chi^c_{r,s} \) is regular relatively to \( \mathcal{OLB} \).

As before, we now intend to show that the family of relations \( \zeta_\Delta \), for all finite sets \( Y \) 
and for all equivalence relations \( \Delta \) on \( Y^\pm \), is closed with respect to all substitutions \( \chi^c_{r,s} \)
just described. That is, we are going to prove the following claim.

**Lemma 3.9.** If \( Y \subseteq X \) is a finite subset, if \( \Delta \) is an equivalence relation on \( Y^\pm \), and if 
\( u, v \in F'(X) \) are any words such that \( u \zeta_\Delta v \), then, for any \( c \in X \), for any set \( Z \) disjoint 
with \( X \), and for any \( r, s \in F'(Z) \) satisfying \( rsr \rho(\mathcal{OLB}) r, srs \rho(\mathcal{OLB}) s \), we have 
\[ \chi^c_{r,s}(u) \zeta_{\chi^c_{r,s}(\Delta)} \chi^c_{r,s}(v). \]

**Proof.** Recall that \( u \zeta_\Delta v \) implies \( C(u) = C(v) = Y \). If \( c \notin Y \) then there is nothing to prove. 
Thus assume further that \( c \in Y \). As in the proof of Lemma 3.6, from \( u \zeta_\Delta v \) implying also 
\( \Gamma'(u) = \Gamma'(v) \subseteq \Delta \) and \( (t(t(u)^-, t(v)^-)) \in \Delta \) we deduce \( C(\chi^c_{r,s}(u)) = C(\chi^c_{r,s}(v)) = C(\chi^c_{r,s}(Y)), \) \( \Gamma(\chi^c_{r,s}(u)) = \Gamma(\chi^c_{r,s}(v)) \subseteq \chi^c_{r,s}(\Delta) \) and \( (t(\chi^c_{r,s}(u))^-, t(\chi^c_{r,s}(v))^-) \in \chi^c_{r,s}(\Delta). \)
As before, the verification of the remaining conditions ensuring the validity of \( \chi^c_{r,s}(u) \zeta_{\chi^c_{r,s}(\Delta)} \chi^c_{r,s}(v) \) will proceed by induction on \(|C(u)|\), and, for given \(|C(u)|\), by another
induction on \( \varepsilon(u) \). According to the definition of \( \zeta_A \), there are again two basic cases to distinguish:

**Case I.** Assume that \( \varepsilon(u) = \varepsilon(v) = 1 \). As before, we hence obtain

\[
\Gamma(\chi^c_{r,s}(u)) = \Gamma(\chi^c_{r,s}(0u\bar{0}(u))) \quad \text{and} \quad \Gamma(\chi^c_{r,s}(v)) = \Gamma(\chi^c_{r,s}(0v\bar{0}(v))).
\]

Furthermore, \( u \bar{\in} A \) then implies \( \bar{0}(u) = \bar{0}(v) \) and, in case \( |C(u)| \geq 2 \), also \( 0(u) \bar{\in} \Gamma(0u) \), \( 0(v) \). Then we have to distinguish two subcases:

**Subcase I.1.** Assume that the element \( \bar{0}(u) = \bar{0}(v) \) is not equal to either of the elements \( c, c' \). By the way, note that this means that \( |C(u)| \geq 2 \). Then \( \bar{0}(\chi^c_{r,s}(u)) = \bar{0}(u) = \bar{0}(v) = \bar{0}(\chi^c_{r,s}(v)) \) and \( 0(\chi^c_{r,s}(u)) = 0(\chi^c_{r,s}(v)) \). Hence, having \( (\dagger) \) in view, we get that \( \varepsilon(\chi^c_{r,s}(u)) = \varepsilon(\chi^c_{r,s}(v)) = 1 \). As before, our induction on \( |C(u)| \) makes it possible to infer from \( 0(u) \bar{\in} \Gamma(0u) \), \( 0(v) \) the relation

\[
0(\chi^c_{r,s}(u)) \bar{\in} \Gamma(0(\chi^c_{r,s}(u))) \quad \text{if we take the previous equalities and Lemma 3.4 into consideration. This verifies the required conditions.}
\]

**Subcase I.2.** Assume that the element \( \bar{0}(u) = \bar{0}(v) \) is equal either to \( c \) or to \( c' \). This means that \( \chi^c_{r,s}(\bar{0}(u)) \) is either \( r \) or \( s \). Then we have to distinguish two more subcases:

**Subcase I.2.i.** Assume that \( \varepsilon(\chi^c_{r,s}(\bar{0}(u))) = 1 \). Since obviously

\[
\bar{0}(\chi^c_{r,s}(u)) = \bar{0}(\chi^c_{r,s}(\bar{0}(u))) = \bar{0}(\chi^c_{r,s}(\bar{0}(v))) = \bar{0}(\chi^c_{r,s}(v)) \quad \text{and}
\]

\[
0(\chi^c_{r,s}(u)) = 0(u)0(\chi^c_{r,s}(\bar{0}(u))), \quad 0(\chi^c_{r,s}(v)) = 0(v)0(\chi^c_{r,s}(\bar{0}(v)));
\]

from \( (\dagger) \) it can be seen that then \( \varepsilon(\chi^c_{r,s}(u)) = \varepsilon(\chi^c_{r,s}(v)) = 1 \). If \( |C(u)| = 1 \), that is, if \( C(u) = \{c\} \), then \( 0(u) = 0(v) = \emptyset \), and so \( 0(\chi^c_{r,s}(u)) = 0(\chi^c_{r,s}(\bar{0}(u))) = 0(\chi^c_{r,s}(\bar{0}(v))) = 0(\chi^c_{r,s}(v)) \) (note that these words are empty if \( |C(r)| = |C(s)| = 1 \). If \( |C(u)| \geq 2 \) then, of course, \( |C(\chi^c_{r,s}(u))| \geq 2 \). Furthermore, it then follows that the relation \( 0(u) \bar{\in} \Gamma(0u) \) \( 0(v) \bar{\in} \Gamma(0v) \) by Proposition 3.2, yields \( 0(u)0(\chi^c_{r,s}(\bar{0}(u))) = \chi^c(\chi^c_{r,s}(\bar{0}(u))) \) \( 0(v)0(\chi^c_{r,s}(\bar{0}(v))) \) accordingly to Corollary 3.8, and this is, in turn, the same relation as \( (\dagger\dagger) \), in view of the previous equalities and Proposition 3.2. This verifies the required conditions.

**Subcase I.2.ii.** Assume that \( \varepsilon(\chi^c_{r,s}(\bar{0}(u))) \geq 2 \). Then clearly \( \varepsilon(\chi^c_{r,s}(u)) \geq 2 \) and \( \varepsilon(\chi^c_{r,s}(v)) \geq 2 \). Denote by \( \Gamma \) the restriction of \( \chi^c_{r,s}(\Delta) \) to the set \( C^=\{r\} = C^=\{s\} \). Then it is easy to see from \( (\dagger) \) that \( \Omega(\chi^c_{r,s}(\Delta))(\chi^c_{r,s}(u)) = 0(u) \Omega(\chi^c_{r,s}(\bar{0}(u))) \) and \( \Omega(\chi^c_{r,s}(\Delta))(\chi^c_{r,s}(v)) = 0(v) \Omega(\chi^c_{r,s}(\bar{0}(v))) \). If \( |C(u)| = 1 \) then \( 0(u) = 0(v) = \emptyset \), and so we hence get \( \Omega(\chi^c_{r,s}(\Delta))(\chi^c_{r,s}(u)) = \Omega(\chi^c_{r,s}(\Delta))(\chi^c_{r,s}(v)) \). If \( |C(u)| \geq 2 \) then, as in the previous subcase, the relation \( 0(u) \bar{\in} \Gamma(0u) \), \( 0(v) \bar{\in} \Gamma(0v) \) accordingly to Proposition 3.2, entails \( 0(u)0(\chi^c_{r,s}(\bar{0}(u))) \) \( 0(v)0(\chi^c_{r,s}(\bar{0}(v))) \) in view of Corollary 3.8. As previously, we can again rewrite this relation by means of Proposition 3.2 and we can also simplify it by making use of the above equalities. Since, in addition, we
have \( \Gamma(\Omega_{\chi_{c,r,s}(\Delta)}(\chi_{c,r,s}(u))) \subseteq \Gamma(\chi_{c,r,s}(\Gamma(u))) \subseteq \chi_{c,r,s}(\Delta) \), using Lemma 3.3, we hence finally get the relation

\[
\Omega_{\chi_{c,r,s}(\Delta)}(\chi_{c,r,s}(u)) \subseteq \chi_{c,r,s}(\Omega_{\chi_{c,r,s}(\Delta)}(\chi_{c,r,s}(v)),
\]

which is the remaining condition that we had to verify.

**Case II.** Assume that \( \varepsilon(u) \geq 2 \) and \( \varepsilon(v) \geq 2 \). In this case, however, all considerations and arguments are literally the same as in the respective case in the proof of Lemma 3.6. \( \square \)

**Corollary 3.10.** If \( u, v \in F'(X) \) are any words satisfying \( u \xi(X) v \), then

\[
\chi_{c,r,s}(u) \xi(X \cup Z) \chi_{c,r,s}(v)
\]

holds for any \( c \in X \), for any set \( Z \) disjoint with \( X \), and for any \( r, s \in F'(Z) \) satisfying \( rsr\rho(\mathcal{OLB})r \), \( srs\rho(\mathcal{OLB})s \).

**Proof.** This follows straightforwardly from Proposition 3.2 and Lemma 3.9 by means of Lemma 3.4, just as in the proof of Corollary 3.7. \( \square \)

Finally we are ready to prove the following statement.

**Corollary 3.11.** The equivalence relation \( \xi(X) \) is a biinvariant congruence on \( F'(X) \).

**Proof.** We have shown in Corollary 3.8 that \( \xi(X) \) is a congruence on \( F'(X) \). Thus it remains to accomplish the check-up of the fact that this congruence is biinvariant. We have already seen that \( \rho((\mathcal{CRB})\mathcal{OLB}, X) \subseteq \xi(X) \). Thus, in view of the notes after the definition of biinvariant congruences, it is enough to prove that \( \xi(X) \) is closed with respect to all substitutions \( \psi : X \cup X' \rightarrow F'(X) \) that are regular relatively to \( \mathcal{OLB} \). That is, we have to show, for any such substitution \( \psi \), that \( u \xi(X) v \) implies \( \psi(u) \xi(X) \psi(v) \) for all \( u, v \in F'(X) \). However, it is obvious that for given words \( u, v \in F'(X) \), there exists a sequence of substitutions \( \sigma_1, \ldots, \sigma_m \), each of which is of the form \( \chi_{c,r,s} \), for various elements \( c \in C(u) = C(v) \) and suitable words \( r, s \), such that the sets \( C(r) = C(s) \) are pairwise disjoint for different substitutions, and there exists another sequence of substitutions \( \tau_1, \ldots, \tau_n \), each of which is of the form \( \mu_{c}^{a,b} \) for suitable elements \( a, b \) and \( z \), such that the following is true. The word \( \psi(u) \) is \( \tau_1(\ldots \tau_n(\sigma_1(\ldots \sigma_m(u) \ldots)) \ldots) \) and the word \( \psi(v) \) is \( \tau_1(\ldots \tau_n(\sigma_1(\ldots \sigma_m(v) \ldots)) \ldots) \) (maybe up to some common renaming of variables). Therefore, \( \psi(u) \xi(X) \psi(v) \) follows from \( u \xi(X) v \) by Corollaries 3.10 and 3.7. \( \square \)

Thus, in accordance with Result 1.4, \( \xi(X) \) corresponds to an e-variety of orthodox semigroups, which we have agreed to denote by \( \mathcal{OZ} \), and all conclusions concerning this e-variety which we have deduced in the previous parts of this section hold true.
4. Finite generators of \( \mathcal{O} \mathcal{Z} \)

In this section, we will prove that the e-variety \( \mathcal{O} \mathcal{Z} \) introduced in the previous section is finitely generated. More concretely, we will show that \( \mathcal{O} \mathcal{Z} \) is generated by any of its finitely generated bifree objects on sufficiently many generators, that is, by any semigroup \( F'(Z)/\zeta(Z) \) where \( Z \) is a finite set having sufficiently many elements. We have seen that \( \rho((L\mathcal{R}\mathcal{B})\mathcal{O} \mathcal{L} \mathcal{B}, Z) \subset \zeta(Z) \). Moreover, from the inductive description of \( \rho((L\mathcal{R}\mathcal{B})\mathcal{O} \mathcal{L} \mathcal{B}, Z) \) in Corollary 3.1 it is apparent that, for finite \( Z \), the semigroups \( F'(Z)/\rho((L\mathcal{R}\mathcal{B})\mathcal{O} \mathcal{L} \mathcal{B}) \) are finite. Hence we see that, for finite \( Z \), the semigroups \( F'(Z)/\zeta(Z) \) are finite, as well. Thus, for any finite but sufficiently large \( Z \), the semigroup \( F'(Z)/\zeta(Z) \) will represent a finite semigroup generating the e-variety \( \mathcal{O} \mathcal{Z} \).

We begin our considerations by introducing another particular family of substitutions. Let \( Y \subseteq X \) be any finite non-empty subset of our infinite set \( X \), let \( \Pi = \{\mathcal{S}_1, \ldots, \mathcal{S}_k\} \) be any partition of the set \( Y^\pm \) having at least two classes, and let \( Z \) be any other (possibly finite, but sufficiently large) set. We choose distinct elements \( z_{ij} \) in \( Z \) for all integers \( i, j \) satisfying \( 1 \leq i < j \leq k \), and we consider the substitution

\[
\eta_\Pi : Y \cup Y' \to F'(Z)
\]

which we define as follows. For any \( x \in Y \), we take those indices \( h, \ell \in \{1, \ldots, k\} \) for which \( x^+ \in \mathcal{S}_h, x^- \in \mathcal{S}_\ell \), and we put

\[
\eta_\Pi(x) = \begin{cases} 
  z_{1h} z_{1l} \cdots z_{h-1h} z_{h-1h} z_{h-1h+1} \cdots z_{kk} z_{k\ell} z_{k\ell} \\
  (z_{1\ell} z_{1l} \cdots z_{\ell-1\ell} z_{\ell-1\ell} z_{\ell+1\ell} \cdots z_{kk} z_{k\ell})^\prime \quad \text{if } h < \ell, \\
  z_{1h} z_{1l} \cdots z_{h-1h} z_{h-1h+1} \cdots z_{kk} z_{k\ell} \\
  (z_{1\ell} z_{1l} \cdots z_{\ell-1\ell} z_{\ell-1\ell} z_{\ell+1\ell} \cdots z_{kk} z_{k\ell})^\prime \quad \text{if } h = \ell, \\
  z_{1h} z_{1l} \cdots z_{h-1h} z_{h-1h+1} \cdots z_{kk} z_{k\ell} z_{k\ell} \\
  (z_{1\ell} z_{1l} \cdots z_{\ell-1\ell} z_{\ell-1\ell} z_{\ell+1\ell} \cdots z_{kk} z_{k\ell})^\prime \quad \text{if } h > \ell,
\end{cases}
\]

\[
\eta_\Pi(x') = (\eta_\Pi(x))'.
\]

Note that, in these formulas, we have used the involution \( ' \) on \( F'(Z) \) introduced earlier in this paper. Clearly, the substitution \( \eta_\Pi \) thus defined is regular relatively to \( \mathcal{O} \). It is obvious that then, for every \( x \in Y \), we have

\[
C^\pm(\eta_\Pi(x))/\Gamma(\eta_\Pi(x)) = C^\pm(\eta_\Pi(x'))/\Gamma(\eta_\Pi(x'))
\]

\[
= \left\{ \left\{ z_{1h}, \ldots, z_{h-1h}, z_{h+1h}, \ldots, z_{kk} \right\}, \left\{ z_{1\ell}, \ldots, z_{\ell-1\ell}, z_{\ell+1\ell}, \ldots, z_{kk} \right\} \right. \\
\cup \left\{ z_{jh} \right\}; j = 1, \ldots, h - 1, j \neq \ell \}
\]

\[
\cup \left\{ z_{jk} \right\}; j = h + 1, \ldots, k, j \neq \ell \}
\]

\[
\cup \left\{ z_{\ell j} \right\}; j = 1, \ldots, \ell - 1, j \neq h \}
\]

\[
\cup \left\{ z_{\ell j} \right\}; j = \ell + 1, \ldots, k, j \neq h \}, \text{ and}
\]
Proof. We will see in the following proof that it is possible to take \( \xi(Y) \) finite.

Remark. Consider the corresponding substitution \( \psi : Y \cup Y' \rightarrow F'(Z) \) which is regular relatively to \( \Omega \) such that \( \psi(u), \psi(v) \not\in \xi(Z) \). This shows that the biidentity \( u \equiv v \) does not hold in the semigroup \( F'(Z)/\xi(Z) \), which is the bifree semigroup in \( \Omega Z \) on \( Z \). Taking it all in all, this means that the semigroup \( F'(Z)/\xi(Z) \) generates the e-variety \( \Omega Z \). We have seen that this semigroup is finite.

Proposition 4.1. There exists a positive integer \( m \) such that the following holds. Let \( Z \) be a finite set satisfying \( |Z| = m \). Consider any words \( u, v \in F'(X) \) such that \( \psi(u), \psi(v) \not\in \xi(X) \), that is, take any biidentity \( u \equiv v \) over \( X \) which does not hold in \( \Omega Z \). Put \( Y = C(u) \cup C(v) \). Then there exists a substitution \( \psi : Y \cup Y' \rightarrow F'(Z) \) which is regular relatively to \( \Omega \) such that \( \psi(u), \psi(v) \not\in \xi(Z) \). This shows that the biidentity \( u \equiv v \) does not hold in the semigroup \( F'(Z)/\xi(Z) \), which is the bifree semigroup in \( \Omega Z \) on \( Z \). Taking it all in all, this means that the semigroup \( F'(Z)/\xi(Z) \) generates the e-variety \( \Omega Z \). We have seen that this semigroup is finite.

Remark. We will see in the following proof that it is possible to take \( m = 7 \) in the above proposition.

Proof. Let \( u, v \in F'(X) \) be any words such that \( \psi(u), \psi(v) \not\in \xi(X) \) and let \( Y = C(u) \cup C(v) \). We distinguish several cases according to the definition of \( \xi(X) \) given in the previous section.

Case 1. Assume that \( C(u) \not= C(v) \). Suppose, for instance, that, for some \( x \in Y \), we have \( x \in C(u) \), but \( x \notin C(v) \). Choose any \( y \in X \), \( z \notin x \), and consider the substitution \( \psi : Y \cup Y' \rightarrow F'(Z) \) given by \( x \mapsto x, x' \mapsto x', \) and \( y \mapsto z, y' \mapsto z' \) for all \( y \in Y \setminus \{x\} \). Then \( x \in C(\psi(u)), \) but \( x \notin C(\psi(v)) \). Let \( \xi(Z) \) be the class in \( C^2(v)/\Gamma(v) \) containing \( p \) and let \( \xi(Z) \) be the union of the remaining classes in \( C^2(v)/\Gamma(v) \), so that \( \xi(Z) \) contains \( q \). Put \( \Pi = \{\xi(Z), \xi(Z)\} \) and consider the corresponding substitution \( \psi : Y \cup Y' \rightarrow F'(Z) \) defined above. Then it follows from the above remarks that \( C^2(\psi(v))/\Gamma(\psi(v)) = \{\xi(Z), \xi(Z)\} \), while we have \( C^2(\psi(u))/\Gamma(\psi(u)) = \{\xi(Z), \xi(Z)\} \), since \( \psi(u) = \psi(v) = \psi(Z) \) and

\[
\begin{align*}
h(\eta(\xi(x))) & = t(\eta(\xi(x))) = \begin{cases} z_{12}^+ & \text{if } h = 1, \\
z_{1h} & \text{if } h > 1, \end{cases} \\
t(\eta(\xi(x))) & = h(\eta(\xi(x))) = \begin{cases} z_{12}^+ & \text{if } \ell = 1, \\
z_{1\ell} & \text{if } \ell > 1. \end{cases}
\end{align*}
\]

Notice that the last two formulas correspond in a sense to our initial assumptions that \( x^+ = \xi_x, x^- = \xi_x \). Now suppose that \( u \in F'(X) \) is any word and \( \Pi = \{\xi_1, \ldots, \xi_k\} \) is any partition of the set \( C^2(u) \) having at least two classes such that the partition \( C^2(u)/\Gamma(u) \) is a refinement of \( \Pi \). Then it can be seen that, from the first of the above three formulas, from the definitions of \( \Gamma(u) \) and \( \Gamma(\eta(u)) \), and from the previous notice, it follows that the partition \( C^2(\eta(u))/\Gamma(\eta(u)) \) consists precisely of the classes

\[
\{z_{i1}^-, z_{i1}^+, z_{i2}^-, z_{i2}^+, \ldots, z_{ik}^-, z_{ik}^+\} \quad \text{for } i = 1, \ldots, k.
\]

Also point out that then \( |C(\eta(u))| = \xi(Z) \).

Finally, we are ready to state and prove the following proposition saying that the e-variety \( \Omega Z \) is finitely generated.
(\eta_\Pi(p), \eta_\Pi(q)) \in \eta_\Pi(\Gamma(u)) = \Gamma(\eta_\Pi(u)) \text{ (remember Lemma 3.4 and the definitions preceding it in this connection). Thus } \Gamma(\eta_\Pi(u)) \neq \Gamma(\eta_\Pi(v)), \text{ and so } (\eta_\Pi(u), \eta_\Pi(v)) \notin \zeta(\{z_{12}\}).

Case 3. Assume that \( \Gamma(u) = \Gamma(v) \), but \((t(u)^-, t(v)^-) \neq \Gamma(u) \). Let \( S_1 \) be the class in \( C^+(u)/\Gamma(u) \) containing \( t(u)^- \) and let \( S_2 \) be the union of the remaining classes in \( C^+(u)/\Gamma(u) \), so that \( S_2 \) contains \( t(v)^- \). Put \( P = \{ S_1, S_2 \} \) and consider the same substitution \( \eta_\Pi \) as in the previous case. Then, in the same way as before, we get that, in this case, we have \( C^+(\eta_\Pi(u))/\Gamma(\eta_\Pi(u)) = C^+(\eta_\Pi(v))/\Gamma(\eta_\Pi(v)) = \{ [x_1^+], [z_{12}] \} \), while \((t(\eta_\Pi(u))^- = z_{12}^+ \) and \( t(\eta_\Pi(v))^- = z_{12}^- \). Thus \( \Gamma(\eta_\Pi(u)) = \Gamma(\eta_\Pi(v)) \), but \((t(\eta_\Pi(u))^-, t(\eta_\Pi(v))^-) \neq \Gamma(\eta_\Pi(u)), \) and so \((\eta_\Pi(u), \eta_\Pi(v)) \notin \zeta(\{z_{12}\}). \)

Case 4. Assume that \( \Gamma(u) = \Gamma(v) \) and \((t(u)^-, t(v)^-) \in \Gamma(u) \), but \( \tilde{0}(u) \neq \tilde{0}(v) \).
Choose three distinct elements \( a, b, c \in X \). If \( \tilde{0}(v) \neq \tilde{0}(u)^- \), consider the substitution \( \psi: Y \cup Y' \to F'(\{a, b, c\}) \) given by \( 0(u) \mapsto a, O(u)^- \mapsto a', \tilde{0}(v) \mapsto b, \tilde{0}(v)^- \mapsto b', \tilde{0}(v) \mapsto c, \tilde{0}(v)^- \mapsto c' \) for all \( y \in Y - C(0(u)\tilde{0}(v)) \). Then \( \tilde{0}(\psi(u)) = a \) and \( \tilde{0}(\psi(v)) = b \), whence we see that \( (\psi(u), \psi(v)) \notin \zeta(\{a, b, c\}) \).
If \( \tilde{0}(v) = \tilde{0}(u)^- \), consider instead the substitution \( \psi: Y \cup Y' \to F'(\{a, b, c\}) \) given by \( \tilde{0}(u) \mapsto b', \tilde{0}(u) \mapsto a' \) and \( y \mapsto c, \tilde{0}(v) \mapsto a' \) for all \( y \in Y - C(0(u)) \). Then \( \tilde{0}(\psi^*(u)) = a \) and \( \tilde{0}(\psi^*(v)) = b \), so that, this time, we similarly get \( (\psi(u), \psi(v)) \notin \zeta(\{a, b, c\}) \).

In the further cases, we will assume that \( \Gamma(u) = \Gamma(v), \) \((t(u)^-, t(v)^-) \in \Gamma(u) \), and \( \tilde{0}(u) = \tilde{0}(v) \). Then, clearly, \( C(0(u)) = C(0(v)) \), and we cannot have \( |C(u)| = |C(v)| = 1 \), since this would obviously entail the relation \( u \xi \) \( v \) \( X \). Thus we may further assume that \( |C(u)| = |C(v)| \geq 2 \), so that \( 0(u) \neq \tilde{0}(v) \).

Case 5. Assume all conditions given in the previous paragraph. In addition, however, assume that \( \Gamma(0(u)) \neq \Gamma(0(v)). \) We have seen in Case 2 that it is possible to provide a substitution \( \psi: C(0(u)) \cup C(0(u)^-) \to F'(\{z_{12}\}) \) such that \( \Gamma(\psi(0(u))) \neq \Gamma(\psi(0(v))) \).
Choose a new element \( s \) distinct from \( z_{12} \) and extend this substitution to get a substitution \( \psi: Y \cup Y' \to F'(\{z_{12}, s\}) \) by the rules \( 0(u) \mapsto s \) and \( 0(u)^- \mapsto s' \) for all \( x \in Y - C(0(u)\tilde{0}(v)) \). Then \( 0(\psi(u)) = \tilde{0}(\psi(v)) = s \) and \( 0(\psi(u)^-) = \psi(0(u)) \). \( 0(\psi(v)) = \psi(0(v)) \).
Hence we get that \( \gamma(0(\psi(u))) \neq \Gamma(0(\psi(v))) \), and so \( 0(\psi(u)), 0(\psi(v)) \notin \zeta(\{z_{12}, s\}) \). Hence it follows that \( (\psi(u), \psi(v)) \notin \zeta(\{z_{12}, s\}) \).

Case 6. Assume all conditions given in the last paragraph but one. In addition, assume also that \( \Gamma(0(u)) \neq \Gamma(0(v)) \), but \((t(0(u))^- \neq \Gamma(0(u)) \). We have seen in Case 3 that it is possible to construct a substitution \( \psi: C(0(u)) \cup C(0(u)^-) \to F'(\{z_{12}\}) \) such that \( \Gamma(\psi(0(u))) = \Gamma(\psi(0(u)^-)) \), but \((t(0(u))^- \neq \Gamma(0(u)) \).
Extending this substitution in the same way as in the previous case, we get a substitution \( \psi: Y \cup Y' \to F'(\{z_{12}, s\}) \) which, by the same arguments as in that case, has the properties that \( \gamma(0(\psi(u))) \neq \Gamma(0(\psi(u))) \), but \( (t(0(\psi(u))^- \neq \Gamma(0(\psi(u))) \). This means that \( 0(\psi(u)), 0(\psi(v)) \notin \zeta(\{z_{12}, s\}) \), and hence we again conclude that \( (\psi(u), \psi(v)) \notin \zeta(\{z_{12}, s\}) \).

Thus we may further assume that \( \Gamma(u) = \Gamma(v), \) \((t(u)^-, t(v)^-) \in \Gamma(u) \), \( \tilde{0}(u) = \tilde{0}(v) \), we may also suppose that \( |C(u)| \geq 2 \), and, in addition, we may assume that \( \Gamma(0(u)) = \Gamma(0(v)) \) and \((t(0(u))^- \neq \Gamma(0(u))) \in \Gamma(0(u)) \). Hence it clearly follows that then we also have \( \Gamma(0(u)\tilde{0}(u)) \neq \Gamma(0(v)\tilde{0}(v)) \), or, equivalently, \( \Gamma(\psi^{(0(u))^-}(u)) \neq \Gamma(\psi^{(0(v))^-}(v)) \).
This equality together with \( \Gamma(u) = \Gamma(v) \), in turn, entails that \( u \xi v \). Now suppose that \( x(u) \neq x(v) \). We claim that then there exists \( \varphi \in \{1, \ldots, \min\{x(u), x(v)\} \} \) such that
\[ \Gamma(O^{[\psi]}(u)) \neq \Gamma(O^{[\psi]}(v)) \]. Really, if we had \( \Gamma(O^{[\psi]}(u)) = \Gamma(O^{[\psi]}(v)) \) for all \( \ell \in \{1, \ldots, \min\{\kappa(u), \kappa(v)\}\} \), just as in the remark after the definition of \( \xi(X) \), we would get \( t_\ell(u) = t_\ell(v) \) for all \( \ell \in \{1, \ldots, \min\{\kappa(u), \kappa(v)\}\} \), which together with \( \kappa(u) \neq \kappa(v) \) would contradict the equality \( \varepsilon(u) = \varepsilon(v) \). Therefore the required index \( \varphi \) exists, which shows that we need not discuss the possibility that \( \kappa(u) \neq \kappa(v) \) separately, but we may directly assume next that \( \Gamma(O^{[\psi]}(u)) \neq \Gamma(O^{[\psi]}(v)) \) for some \( \varphi \in \{1, \ldots, \min\{\kappa(u), \kappa(v)\}\} \).

Moreover, assume that \( \varphi \) is the smallest number with this property. Then, as we shall see at once, we necessarily have \( \varphi \neq \kappa(u), \varphi \neq \kappa(v) \). Suppose, on the contrary, that \( \varphi = \min\{\kappa(u), \kappa(v)\} \). Since, as we have seen above, \( \Gamma(O^{[\psi]}(u)) = \Gamma(O^{[\psi]}(v)) \), and \( \varepsilon(u) - 1 = t_\ell(u), \varepsilon(v) - 1 = t_\ell(v), \) we hence get that, in such a case, we must have \( \kappa(u) \neq \kappa(v) \). Assume, for instance, that \( \kappa(u) < \kappa(v) \). Then \( \varphi = \kappa(v) \), and so \( \varphi + 1 \leq \kappa(v) \).

By the definition of \( \varphi \), we have \( \Gamma(O^{[\psi]}(u)) = \Gamma(O^{[\psi]}(v)) \), and, by the previous inequality and by the definition of \( t_\ell(v) \) for \( \ell = 1, \ldots, \kappa(v) \), we also know that \( (O^{[\psi]}(v))^{[1]}(u), (O^{[\psi]}(v))^{[1]}(v)) \neq \Gamma(v) \), where \( t_{\varphi+1}(v) < \varepsilon(v) \).

But this means that in the partition \( C^{\pm}(v)/\Gamma(O^{[\psi]}(v)) \), when it is compared with \( C^{\pm}(v)/\Gamma(O^{[\psi]}(v)) \), at least two different classes of \( C^{\pm}(v)/\Gamma(v) \) are further split, while in the partition \( C^{\pm}(u)/\Gamma(O^{[\psi]}(u)) \) this happens with only one class of \( C^{\pm}(u)/\Gamma(u) \) (keep also the last equality given above in mind here). The last claim is true in view of the previous condition saying that \( \varphi = \kappa(u) \). But this contradicts our earlier conclusion, according to which the two partitions in question should be the same. Hence we see that, indeed, \( \varphi < \kappa(u), \varphi < \kappa(v) \).

We summarize once again what we have found in the previous paragraph. It turns out that we have to deal with the case when \( \Gamma(u) = \Gamma(v), (t(u)^-, t(v)^-) \in \Gamma(u), \) \( \hat{0}(u) = \hat{0}(v) \)—then we may suppose that \( |C(u)| \geq 2 \), moreover, we assume that \( \Gamma(0(v)) = \Gamma(0(v)), (t(0(u))^-, t(0(v))^-) \in \Gamma(0(u)) \)—hence we also get that \( \Gamma(O^{[\psi]}(u)) = \Gamma(O^{[\psi]}(v)) \) and \( \varepsilon(u) = \varepsilon(v) \), and we also assume that there exists some \( \varphi \in \{1, \ldots, \min\{\kappa(u), \kappa(v)\}\} \) such that \( \Gamma(O^{[\psi]}(u)) \neq \Gamma(O^{[\psi]}(v)) \)—we take the smallest number \( \varphi \) with this property and we know that then we have \( \varphi < \kappa(u), \varphi < \kappa(v) \).

As we shall see, in this case there are still two possibilities to distinguish whose analysis will be the content of the next two cases appearing below. (From our previous discussion it follows that then there is still one more case to consider—see the quite last case appearing at the end of this proof.) In order to indicate the object of our discussion in the next two cases, observe first that if \( Z \) is any non-empty set and if \( \psi : Y \cup Y' \rightarrow F(Z) \) is any substitution which is regular relatively to \( O \) and \( O' \), then from \( \Gamma(u) = \Gamma(v) \) and \( (t(u)^-, t(v)^-) \in \Gamma(u) \) we get \( \Gamma(\psi(u)) = \psi(\Gamma(u)) = \psi(\Gamma(v)) = \psi(\Gamma(u)) = \Gamma(\psi(u)) \) and \( (t(\psi(u))^-, \psi(\psi(v))^-) \in \psi(\Gamma(u)) \).

Now, in order to prove the statement of our proposition, it is enough to provide a finite non-empty set \( Z \) such that \( |Z| \) does not depend on \( u \) and \( v \), and a substitution \( \psi : Y \cup Y' \rightarrow F(Z) \) which is regular relatively to \( O \) such that \( C(\psi(\psi(u))) = C(\psi(u)), C(\psi(\psi(v))) = C(\psi(v)) \) (this ensures that \( \varepsilon(\psi(u)) > 1, \varepsilon(\psi(v)) > 1 \), and hence \( \kappa(\psi(u)) > 0, \kappa(\psi(v)) > 0 \), and such that \( (C(\psi(u))^+, C(\psi(v))^+) \neq \Gamma(\psi(u)) \). This last condition will guarantee that \( \Gamma(O^{[\psi]}(\psi(u))) \neq \Gamma(O^{[\psi]}(\psi(v))) \), since, in the partitions \( C^{\pm}(\psi(u))/\Gamma(O^{[\psi]}(\psi(u))) \) and \( C^{\pm}(\psi(v))/\Gamma(O^{[\psi]}(\psi(v))) \), different classes of \( C^{\pm}(\psi(u))/\Gamma(\psi(u)) \) will be further split. But this will mean that \( (\psi(u), \psi(v)) \notin \xi(Z) \), which we need to obtain.
Case 7. Assume all conditions given at the beginning of the previous paragraph. In addition, assume that $G(Q_{\varphi}(u)-1)(u)$, $G(Q_{\varphi}(v)-1)(v)$ $\not\in G(u)$. Remember that, by the definition of $\varphi$, the partitions $C^\pm(u)/G(Q_{\varphi}(u)(u))$ and $C^\pm(v)/G(Q_{\varphi}(v)(v))$, when compared with the two partitions mentioned previously, split further subclasses of two different classes of $C^\pm(u)/G(u)$. Let $p,q \in C^\pm(u)$ be any two elements such that $(p,q) \in G(Q_{\varphi}(u)-1)(u)$, but $(p,q) \not\in G(Q_{\varphi}(v)(v))$. Then, of course, $p,q$ both lie in the class of $C^\pm(u)/G(u)$ containing $G(Q_{\varphi}(u)-1)(u)+$. Next, let $r,s \in C^\pm(v)$ be any two elements such that $(r,s) \in G(Q_{\varphi}(v)-1)(v)$, but $(r,s) \not\in G(Q_{\varphi}(v)(v))$. Then, again, $r,s$ both lie in the class of $C^\pm(v)/G(v)$ containing $G(Q_{\varphi}(v)-1)(v)+$. Let $G_1$ be the class of $C^\pm(u)/G(Q_{\varphi}(u)(u))$ containing $p$, and let $G_2$ be the union of all other classes of that partition included in that class of $C^\pm(u)/G(u)$ which contain $p$ and $q$—then $G_2$ contains $q$. Similarly, let $G_3$ be the class of $C^\pm(v)/G(Q_{\varphi}(v)(v))$ containing $r$, and let $G_4$ be the union of all other classes of that partition included in that class of $C^\pm(v)/G(v)$ which contain $r$ and $s$—then $G_4$ contains $s$. At last, let $G_5$ be the union of all other classes of $C^\pm(u)/G(u)$ $[=C^\pm(v)/G(v)]$, that is, the union of all classes of that partition which do not contain any of the elements $p,q,r,s$. Put $G_1 = G_5 \cup G_4$ and $G = \{G_1,G_2,G_3,G_4\}$. Then $G$ is a partition of the set $Y^\pm$. Put further $Z = \{z_1, z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34}\}$ and consider the corresponding substitution $\eta_G: Y \to Y'$ defined prior to this proposition. In addition, put $G_1 = \{z_1, z_{12}, z_{13}, z_{14}\}$, $G_2 = \{z_{12}, z_{13}, z_{14}\}$, $G_3 = \{z_{13}, z_{23}, z_{24}\}$, $G_4 = \{z_{14}, z_{24}, z_{34}\}$. Then considerations similar to those appearing before this proposition show that the partition $[G_1, G_2, G_3, G_4]$ of the set $Z^\pm$ is a refinement of $C^\pm(\eta_G)(u)/G(\eta_G)(u))$ $[=C^\pm(\eta_G)(v)/G(\eta_G)(v))$], which, in turn, is a refinement of $[G_1 \cup G_2, G_3 \cup G_4]$, since $C^\pm(u)/G(u)$ is a refinement of $[G_1 \cup G_2, G_3 \cup G_4]$. However, since $(p,q)$, $(r,s) \in G(u)$, we have $(\eta_G(p), \eta_G(q)), (\eta_G(r), \eta_G(s)) \in G(u)$, and as $p \in G_1$, $q \in G_2$, $r \in G_3$, $s \in G_4$, and hence $\eta_G(p) = z_1$, $\eta_G(q) = z_{12}$, $\eta_G(r) = z_{13}$, $\eta_G(s) = z_{14}$, we see that altogether this means that $C^\pm(\eta_G)(u)/G(\eta_G)(u)) = \{z_1 \cup z_{12}, z_{13}, z_{14}\}$. Further on, as above, $[G_1, G_2, G_3, G_4]$ is a refinement of $C^\pm(\eta_G)(u)/G(\eta_G)(Q_{\varphi}(u)(u))$, which itself is a refinement of $[G_1 \cup G_2, G_3 \cup G_4]$, as $C^\pm(u)/G(Q_{\varphi}(u)(u))$ is a refinement of $[G_1 \cup G_2, G_3 \cup G_4]$. By the way, this shows that $\eta_G(Q_{\varphi}(u)(u))$ is an initial segment of $\eta_G(Q_{\varphi}(v)(v))$, and so $C(\eta_G(Q_{\varphi}(u)(u))) = C(\eta_G(u))$. Furthermore, from the considerations at the beginning of this discussion preceding the choice of the elements $r,s$, it follows that $(r,s) \in G(Q_{\varphi}(u)(u))$, and hence $(\eta_G(r), \eta_G(s)) \in G(\eta_G)(Q_{\varphi}(u)(u))$. But this entails the equality $C^\pm(\eta_G(u))/G(\eta_G)(Q_{\varphi}(u)(u))) = \{z_1 \cup z_{13}, z_{14}\}$. Using analogous arguments, we obtain that $\eta_G(Q_{\varphi}(v)(v))$ is an initial segment of $\eta_G(Q_{\varphi}(v)(v))$, so that $C(\eta_G(Q_{\varphi}(v)(v))) = C(\eta_G(v))$, and we likewise deduce the equality $C^\pm(\eta_G(v))/G(\eta_G)(Q_{\varphi}(v)(v))) = \{z_1 \cup z_{12}, z_{14}\}$. From these and the previous findings we conclude that $C^\pm(\eta_G)(u)/G(\eta_G)(Q_{\varphi}(u)(u)) = \{z_1 \cup z_{13}, z_{14}\}$ and $C^\pm(\eta_G(v))/G(\eta_G)(Q_{\varphi}(v)(v))) = \{z_1 \cup z_{12}, z_{13}\}$, which difference together with the previous expression for $C^\pm(\eta_G(u))/G(\eta_G)(u)$ means that $(\eta_G(u))^+, (\eta_G(v))^+) \not\in G(u)$, as required.

Case 8. Assume all conditions given at the beginning of the last paragraph but one. In addition, assume that $G(Q_{\varphi}(u)-1)(u)$, $G(Q_{\varphi}(v)-1)(v) \in G(u)$ holds this time. Once again, remember that, by the definition of $\varphi$, the partitions $C^\pm(u)/G(Q_{\varphi}(u)(u))$ and $C^\pm(v)/G(Q_{\varphi}(v)(v))$ are the same. In the present case, the meaning of the given ad-
ditional assumption is that the partitions $C^\pm(u)/\Gamma(Q^{(v)}(u))$ and $C^\pm(v)/\Gamma(Q^{(u)}(v))$, when compared with the two partitions mentioned previously, split further subclasses of the same class of $C^\pm(u)/\Gamma(u)$. Nevertheless, since $\Gamma(Q^{(v)}(u)) \neq \Gamma(Q^{(u)}(v))$, we may assume, for instance, that there exist elements $p,q \in C^\pm(u)$ such that $(p,q) \in \Gamma(Q^{(v)}(v))$, but $(p,q) \notin \Gamma(Q^{(v)}(u))$. Yet we hence get that $(p,q) \in \Gamma(Q^{(v)}(u))$, which shows that $p,q$ must both lie in the class of $C^\pm(u)/\Gamma(u)$ containing $\Gamma(Q^{(v)}(u))$. Let $\mathcal{E}_1$ be the class of $C^\pm(u)/\Gamma(Q^{(v)}(u))$ containing $p$, and let $\mathcal{E}_2$ be the union of all other classes of this partition included in that class of $C^\pm(u)/\Gamma(u)$ which contains $p$ and $q$—then $\mathcal{E}_2$ contains $q$. Furthermore, since we have $\varphi < \chi(v)$, we may next consider the relation $\Gamma(Q^{(v)}(u))$. Let $r,s \in C^\pm(u)$ be any two elements such that $(r,s) \in \Gamma(Q^{(v)}(v))$, but $(r,s) \notin \Gamma(Q^{(v)}(u))$. Then $r,s$ both lie in the class of $C^\pm(u)/\Gamma(v)$ containing $\Gamma(Q^{(v)}(v))$, and as $(\Gamma(Q^{(v)}(v)))^+ \neq \Gamma(v)$, this shows that $p,q$ and $r,s$ are in distinct classes of $C^\pm(u)/\Gamma(u)$, all $\neq C^\pm(u)/\Gamma(v)$). Let $\mathcal{E}_3$ be the class of $C^\pm(u)/\Gamma(Q^{(v)}(v))$ containing $r$, and let $\mathcal{E}_4$ be the union of all other classes of this partition included in that class of $C^\pm(u)/\Gamma(v)$ which contains $r$ and $s$—then $\mathcal{E}_4$ contains $s$. At last, let $\mathcal{E}_5^*$ be the union of all other classes of $C^\pm(u)/\Gamma(u)$, that is, the union of all classes of this partition which do not contain any of the elements $p,q,r,s$. Put $\mathcal{E}_6 = \mathcal{E}_2 \cup \mathcal{E}_4^*$ and $\Pi = [\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4]$. Then $\Pi$ is a partition of the set $Y^\pm$. As in the previous case, put further $Z = \{z_{12}, z_{13}, z_{24}, z_{23}, z_{34}\}$ and consider the corresponding substitution $\eta_{\Pi} : Y \cup Y' \to F'(Z)$. In addition, put $\mathcal{E}_1 = \{z_{12}, z_{13}, z_{14}\}, \mathcal{E}_2 = \{z_{12}, z_{23}, z_{24}\}, \mathcal{E}_3 = \{z_{13}, z_{23}, z_{34}\}, \mathcal{E}_4 = \{z_{14}, z_{24}, z_{34}\}$. Then the same considerations as in the previous case show that we have $C^\pm(\eta_{\Pi}(u))/\Gamma(\eta_{\Pi}(u)) = [\mathcal{E}_1 \cup \mathcal{E}_2, \mathcal{E}_3 \cup \mathcal{E}_4] = C^\pm(\eta_{\Pi}(v))/\Gamma(\eta_{\Pi}(v))$. Likewise, similar reasoning as in the previous case, but this time based on the above notes, further show that $\eta_{\Pi}(Q^{(v)}(u))$ is an initial segment of $\Omega(\eta_{\Pi}(u))$, so that $C(\Omega(\eta_{\Pi}(u))) = C(\eta_{\Pi}(u))$, and they eventually yield the equality $C^\pm(\eta_{\Pi}(u))/\Gamma(\eta_{\Pi}(Q^{(v)}(u))) = [\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3]$. Analogous arguments proceeding from the above data finally show that $\eta_{\Pi}(\Omega^{(v)}(v))$ is an initial segment of $\Omega(\eta_{\Pi}(v))$, so that $C(\Omega(\eta_{\Pi}(v))) = C(\eta_{\Pi}(v))$, and, in the end, they produce the equality $C^\pm(\eta_{\Pi}(v))/\Gamma(\eta_{\Pi}(\Omega^{(v)}(v))) = [\mathcal{E}_1 \cup \mathcal{E}_2, \mathcal{E}_3 \cup \mathcal{E}_4]$. Hence we conclude that $C^\pm(\eta_{\Pi}(u))/\Gamma(\Omega(\eta_{\Pi}(u))) = [\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \cup \mathcal{E}_4]$ and $C^\pm(\eta_{\Pi}(v))/\Gamma(\Omega(\eta_{\Pi}(v))) = [\mathcal{E}_1 \cup \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4]$, and, as before, this difference together with the above expression for $C^\pm(\eta_{\Pi}(u))/\Gamma(\eta_{\Pi}(u))$ implies that $\eta_{\Pi}(Q^{(v)}(v))^+) \neq \Gamma(\eta_{\Pi}(u))$, as required.

Case 9. The last case we have to deal with is when the words $u,v$ with $(u,v) \notin \xi(X)$ satisfy all conditions in the definition of $\xi(X)$ except that $|C(u)| \geq 2$ but $0(u)$, $0(v) \notin \xi(X)$. Then let $\lambda \in \{1, \ldots, |C(u)|\}$ be the largest integer such that for all $\ell = 1, \ldots, \lambda$ the words $0^{\ell-1}(u)$, $0^{\ell-1}(v)$ satisfy all conditions in the definition of $\xi(X)$ except that $|C(0^{\ell-1}(u))| \geq 2$ but $0^\ell(u)$, $0^\ell(v) \notin \xi(X)$. Then the words $0^\ell(u)$, $0^\ell(v)$ already violate some other condition in the definition of $\xi(X)$. Note, however, that still $C(0^\ell(u)) = C(0^\ell(v))$. From the previous parts of this proof we know that then there exists a non-empty set $Z$ with $|Z| \leq 6$ and some substitution $\psi : C(0^2(u)) \cup C(0^2(v))' \to F'(Z)$ which is regular relatively to $S$ such that $(\psi(0^2(u)), \psi(0^2(v))) \notin \xi(Z)$. Furthermore, by our assumptions, we have $0(0^{\ell-1}(u)) = 0(0^{\ell-1}(v))$ and neither $0(0^{\ell-1}(u))$ nor $0(0^{\ell-1}(u))'$ occurs in the set $C(0^2(u))$. Thus we may choose a new element $s \notin Z$ and we can extend the substitution $\psi$ to get a substitution $\psi : Y \cup Y' \to F'(Z \cup \{s\})$ by the rules $y \mapsto s$ and $y' \mapsto s'$ for all $y \in Y \cup C(0^2(u))$. But then we clearly have either
\[ \bar{0}(\psi(u)) = \bar{0}(\psi(v)) = s \quad \text{or} \quad \bar{0}(\psi(u)) = \bar{0}(\psi(v)) = s', \quad \text{and, at the same time,} \quad 0(\psi(u)) = \psi(0^x(u)), \quad 0(\psi(v)) = \psi(0^x(v)), \quad \text{whence we get that} \quad (0(\psi(u)), 0(\psi(v))) \notin \xi(Z), \quad \text{so that} \quad (\psi(u), \psi(v)) \notin \xi(Z). \quad \square \]

5. Non-existence of finite basis of biidentities for \( OZ \)

In this section, we provide an infinite sequence of biidentities over our infinite set \( X \) of variables which are all satisfied in the e-variety \( OZ \) introduced earlier in this paper and which have the following remarkable property. From any set of biidentities valid in \( OZ \) and containing altogether only a finite number of variables, only finitely many biidentities of this sequence can be deduced within the e-variety \( (\mathcal{LRB})O\mathcal{LB} \). Consequently, it turns out that our e-variety \( OZ \) has no finite basis of biidentities already within \( (\mathcal{LRB})O\mathcal{LB} \). We have seen in the preceding section that \( OZ \) is finitely generated. The previous statement thus can be viewed as a statement about bases of biidentities of any finite semigroup generating \( OZ \). This provides us with examples of semigroups announced in the title of this paper. In addition, all words in all biidentities of the mentioned sequence will be composed only of letters of \( X \), so that they will actually form usual identities. Hence one can conclude that the finite semigroups so obtained also have no finite basis of identities in the usual sense.

Throughout this section, we will have to modify somewhat our common notation. Since most lowercase Roman letters will be utilized to denote various variables of \( X \) appearing in the mentioned sequence of identities, which is given below, in contrast to the previous sections, we will use here bold lowercase Greek letters to denote words of \( F'(X) \).

In order to introduce the promised sequence of identities, consider first, for any integer \( n \geq 1 \), the word

\[ \pi_n = gasbctdhx_1y_1x_2z_2y_2 \ldots x_nz_ny_ngctdasbh. \]

Next, for any integer \( n \geq 1 \) again, consider the words

\[ \nu_n = \pi_nfgabcdhx_1y_1x_2y_2 \ldots x_ny_ngadhf\pi_n \quad \text{and} \]

\[ \omega_n = \pi_ngcdabhx_1y_1x_2y_2 \ldots x_ny_ngadhf\pi_n. \]

Then the mentioned sequence will consist of identities

\[ \nu_n \sim \omega_n \quad \text{for all} \quad n = 5^k \quad \text{where} \quad k \geq 1 \quad \text{are arbitrary integers}. \]

Notice that, for any \( n \) as above, we have \( C(\nu_n) = C(\omega_n) \) and \( \bar{0}(\nu_n) = \bar{0}(\omega_n) = f = \bar{1}(\nu_n) = \bar{1}(\omega_n) \), \( 0(\nu_n) = 0(\omega_n) = \pi_n = 1(\nu_n) = 1(\omega_n) \), whence, according to Result 2.2, we get that

\[ \nu_n \rho(\mathcal{B}) \omega_n \quad \text{for all} \quad n \ \text{as above}. \]

Observe also that
\[ C^\pm(\pi_n)/\Gamma(\pi_n) = \left\{ \{ g^-, a^+, b^-, c^+, d^-, h^+ \}, \right. \\
\left. \{ a^-, s^+, s^-, b^+ \}, \{ c^-, t^+, t^- \}, \{ f^-, d^+ \}, \right\]
\[ \{ h^-, x_1^+ \}, \{ y_n^-, g^+ \}, \right\]
\[ \{ x_1^-, z_1^+ \}, \{ z_1^-, y_1^+ \}, \right\]
\[ \{ x_2^-, z_2^+ \}, \{ z_2^-, y_2^+ \}, \right\}
\[ \{ x_n^-, z_n^+ \}, \{ z_n^-, y_n^+ \} \right\}, \]

and that
\[ C^\pm(\nu_n)/\Gamma(\nu_n) = \left\{ \{ g^-, a^+, b^-, c^+, d^-, h^+ \}, \right. \\
\left. \{ a^-, b^+, s^-, s^+ \}, \{ c^-, d^+, t^-, t^+ \}, \right\]
\[ \{ f^+, h^-, x_1^+ \}, \{ y_n^-, g^+, f^- \}, \right\]
\[ \{ x_1^-, y_1^+, z_1^+ \}, \{ y_1^-, x_1^+ \}, \right\]
\[ \{ x_2^-, y_2^+, z_2^+ \}, \{ y_2^-, x_2^+ \}, \right\}
\[ \{ x_n^-, y_n^+, z_n^+ \}, \{ y_n^-, x_n^+ \} \right\}, \]

Further on, note that
\[ \sigma(\nu_n) = \pi_n f g a b c d h x_1 y_1 x_2 y_2 \cdots x_n y_n g a, \]
\[ \sigma(\omega_n) = \pi_n f g c d a b h x_1 y_1 x_2 y_2 \cdots x_n y_n g a. \]

At the same time, observe that
\[ C^\pm(\sigma(\nu_n))/\Gamma(\sigma(\nu_n)) = C^\pm(\sigma(\omega_n))/\Gamma(\sigma(\omega_n)) \]
\[ = \left\{ \{ g^-, a^+, b^-, c^+, d^-, h^+ \}, \right. \\
\left. \{ a^-, b^+, s^-, s^+ \}, \{ c^-, d^+, t^-, t^+ \}, \right\]
\[ \{ f^+, h^-, x_1^+ \}, \{ y_n^-, g^+, f^- \}, \right\]
\[ \{ x_1^-, y_1^+, z_1^+ \}, \{ y_1^-, x_1^+ \}, \right\]
\[ \{ x_2^-, y_2^+, z_2^+ \}, \{ y_2^-, x_2^+ \}, \right\}
\[ \{ x_n^-, y_n^+, z_n^+ \}, \{ y_n^-, x_n^+ \} \right\}. \]

Furthermore, for all \( i = 1, \ldots, n \), we have
\[ \sigma^+_{i+1}(\nu_n) = \pi_n f g a b c d h x_1 y_1 x_2 y_2 \cdots x_{n-i} y_n g a \quad \text{and} \]
\[ \sigma^+_{i+1}(\omega_n) = \pi_n f g c d a b h x_1 y_1 x_2 y_2 \cdots x_{n-i} y_n g a. \]
Moreover, we get that
\[ \mathcal{O}^{n+2}(\nu_n) = \pi_n f g a b c, \quad \mathcal{O}^{n+3}(\nu_n) = \pi_n f g a, \quad \text{and} \]
\[ \mathcal{O}^{n+2}(\omega_n) = \pi_n f g c d a, \quad \mathcal{O}^{n+3}(\omega_n) = \pi_n f g c. \]

In this connection, notice also that, for all \( i = 1, \ldots, n \), we have
\[ C^\pm(\mathcal{O}^{i+1}(\nu_n))/\Gamma(\mathcal{O}^{i+1}(\nu_n)) = C^\pm(\mathcal{O}^{i+1}(\omega_n))/\Gamma(\mathcal{O}^{i+1}(\omega_n)); \]
In particular, we have
\[ C^\pm(\mathcal{O}^{n+1}(\nu_n))/\Gamma(\mathcal{O}^{n+1}(\nu_n)) = C^\pm(\mathcal{O}^{n+1}(\omega_n))/\Gamma(\mathcal{O}^{n+1}(\omega_n)) \]
\[ = \left\{ \left\{ g^-, a^+, b^-, c^+, d^-, h^+ \right\}, \right. \]
\[ \left. \left\{ a^-, b^+, s^-, s^+ \right\}, \left\{ c^-, t^+, t^-, t^+ \right\}, \right. \]
\[ \left. \left\{ f^+, h^-, x^+_1 \right\}, \left\{ y^+_n, g^+, f^- \right\}, \right. \]
\[ \left. \left\{ x^+_1, z^+_1 \right\}, \left\{ z^+_1, y^+_1 \right\}, \left\{ y^+_1, x^+_2 \right\}, \right. \]
\[ \left. \left\{ x^+_2, z^+_2 \right\}, \left\{ z^+_2, y^+_2 \right\}, \left\{ y^+_2, x^+_3 \right\}, \ldots, \right. \]
\[ \left. \left\{ x^+_n, z^+_n \right\}, \left\{ z^+_n, y^+_n \right\} \right\}. \]

But then, we get
\[ C^\pm(\mathcal{O}^{n+2}(\nu_n))/\Gamma(\mathcal{O}^{n+2}(\nu_n)) = \left\{ \left\{ g^-, a^+, b^-, c^+, d^-, h^+ \right\}, \right. \]
\[ \left. \left\{ a^-, b^+, s^-, s^+ \right\}, \left\{ c^-, t^+, t^-, t^+ \right\}, \right. \]
\[ \left. \left\{ f^+, h^-, x^+_1 \right\}, \left\{ y^+_n, g^+, f^- \right\}, \right. \]
\[ \left. \left\{ x^+_1, z^+_1 \right\}, \left\{ z^+_1, y^+_1 \right\}, \left\{ y^+_1, x^+_2 \right\}, \right. \]
\[ \left. \left\{ x^+_2, z^+_2 \right\}, \left\{ z^+_2, y^+_2 \right\}, \left\{ y^+_2, x^+_3 \right\}, \ldots, \right. \]
\[ \left. \left\{ x^+_n, z^+_n \right\}, \left\{ z^+_n, y^+_n \right\} \right\}. \]

while
\[ C^\pm(\mathcal{O}^{n+2}(\omega_n))/\Gamma(\mathcal{O}^{n+2}(\omega_n)) = \left\{ \left\{ g^-, a^+, b^-, c^+, d^-, h^+ \right\}, \right. \]
\[ \left. \left\{ a^-, b^+, s^-, s^+ \right\}, \left\{ c^-, t^+, t^-, t^+ \right\}, \right. \]
\[ \left. \left\{ f^+, h^-, x^+_1 \right\}, \left\{ y^+_n, g^+, f^- \right\}, \right. \]
\[ \left. \left\{ x^+_1, z^+_1 \right\}, \left\{ z^+_1, y^+_1 \right\}, \left\{ y^+_1, x^+_2 \right\}, \right. \]
\[ \left. \left\{ x^+_2, z^+_2 \right\}, \left\{ z^+_2, y^+_2 \right\}, \left\{ y^+_2, x^+_3 \right\}, \ldots, \right. \]
\[ \left. \left\{ x^+_n, z^+_n \right\}, \left\{ z^+_n, y^+_n \right\} \right\}. \]
At last, we obtain
\[
C^\pm (Q^{n+3}(u_n))/\Gamma(Q^{n+3}(u_n)) = C^\pm (Q^{n+3}(\omega_n))/\Gamma(Q^{n+3}(\omega_n))
\]
\[
= \{ \{ g^-, a^+, b^-, c^+, d^-, h^+ \},
\{ a^-, s^+, \{ s^-, b^+ \}, \{ e^-, i^+ \}, \{ i^-, d^+ \},
\{ f^+, h^-, x_i^+ \}, \{ y_n^+, g^+, f^- \},
\{ x_1^-, z_i^+, \{ z_1^-, y_i^+ \}, \{ y_1^-, x_2^+ \},
\{ x_2^-, z_2^+ \}, \{ z_2^-, y_2^+ \}, \{ y_2^-, x_3^+ \}, \ldots ,
\{ x_n^-, z_n^+ \}, \{ z_n^-, y_n^+ \} \} \}.
\]

For the sake of completeness, we also add that
\[
Q^{n+4}(u_n) = Q^{n+4}(\omega_n) = \pi_n f \quad \text{and}
\]
\[
Q^{n+5}(u_n) = Q^{n+5}(\omega_n) = \pi_n = 0(u_n) = 0(\omega_n).
\]

Thus, first of all, unfolding the inductive description of \(\rho((\L R B)\odot L B)\) given in Corollary 3.1, from \(\Gamma(Q^{n+2}(u_n)) \neq \Gamma(Q^{n+2}(\omega_n))\) we infer that
\[
(u_n, \omega_n) \notin \rho((\L R B)\odot L B) \quad \text{for all } n \text{ as above.}
\]

Furthermore, these considerations show that \(\varepsilon(u_n) = \varepsilon(\omega_n) = n + 5\), and it is also easy to deduce from the above formulas that
\[
[1, \ldots , n + 4]/\varepsilon(u_n) = [1, \ldots , n + 4]/\varepsilon(\omega_n)
\]
\[
= \{ [1], [2], \ldots , [n + 1], [n + 2, n + 3], [n + 4] \}.
\]

Hence, inspecting the definition of the relation \(\zeta(X)\) given in the third section of this paper and having the above formulas in mind, we may conclude that
\[
u_n \ \zeta(X) \ \omega_n \quad \text{for all } n \text{ as above.}
\]

This means that all identities of the sequence given at the beginning of this section are satisfied in the e-variety \(OZ\).

Before we can prove the main statement about this sequence of identities, which can be found in the subsequent proposition, we have to prepare an auxiliary device in advance.

**Lemma 5.1.** Let \(y \in F'(X)\) be any word, let \(\ell\) be any integer satisfying \(0 \leq \ell \leq |C(y)| - 1\), and let \(\psi : X \cup X' \to F'(Z)\) be any substitution which is regular relatively to \(\odot L B\). Then, for any word \(\alpha \in F'(Z) \cup [\square]\) and for any integer \(i \geq 1\) having the property that the word \(\odot^i(\alpha \psi(y))\) is a proper initial segment of \(\alpha \psi(0^i(y))\) and contains \(\alpha \psi(0^{i+1}(y))\odot (0^i(y))\) as its own initial segment, the mentioned word \(\odot^j(\alpha \psi(y))\) is of the form \(\alpha \psi(\odot^j(0^i(y)))\) for some integer \(j\) satisfying \(1 \leq j \leq \varepsilon(0^i(y)) - 1\).
Sketch of proof. The arguments verifying this assertion are very similar to those applied in the first paragraph of the proof of Lemma 3.5. Again, one has to confirm that proper initial segments of $\alpha \psi(0^i(\gamma))$ containing $\alpha \psi(0^{i+1}(\gamma) \tilde{0}(0^i(\gamma)))$ which are not of the form described above cannot appear among the words $\bigcup \psi(\alpha \psi(\gamma))$ for any $i \geq 1$. Any such initial segment either is of the form $\alpha \psi(\xi)$ for some proper initial segment $\xi$ of $0^i(\gamma)$ containing $0^{i+1}(\gamma) \tilde{0}(0^i(\gamma))$ and different from $\bigcup \psi(0^j(\gamma))$ for $j$ as given above, or else it cannot be obtained in that way. In both cases, having the definition of the operator $\bigcup$ in view, we come to a contradiction just as in the proof of Lemma 3.5 (in the first case, we again apply Lemma 3.4).

Now we are ready to state and prove the principal property of our sequence of identities $v_n \sim w_n$ where $n$ are the integers specified above.

**Proposition 5.2.** Let $\Phi$ be any set of biidentities valid in $OZ$ and containing at most $m$ variables for some positive integer $m$. That is, $\Phi$ is a subset of the relation $\varsigma(Y)$ for some finite subset $Y \subseteq X$ satisfying $|Y| = m$. Then, for any integer $n \geq m$, the biiidentity $v_n \sim w_n$ is not a consequence of the biidentities of $\Phi$ within the e-variety $(LRB)OLB$, that is, $(v_n, w_n) \notin \rho([\Phi]_{(LRB)OLB}, X)$.

**Proof.** Suppose, on the contrary, that $v_n \rho([\Phi]_{(LRB)OLB}) w_n$. Since, as we have seen, $(v_n, w_n) \notin \rho((LRB)OLB)$ by Theorem 1.5 and the remark after it, there exist $\ell \geq 1$ and words $\sigma_1, \ldots, \sigma_\ell, \tau_1, \ldots, \tau_\ell \in F'(X)$ such that

$$v_n \rho((LRB)OLB) \sigma_1, \tau_1 \rho((LRB)OLB) \sigma_2, \ldots, \tau_{\ell-1} \rho((LRB)OLB) \sigma_\ell, \tau_\ell \rho((LRB)OLB) w_n,$$

and for each $i = 1, \ldots, \ell$ there exist words $\alpha_i, \beta_i \in F'(X) \cup \{\square\}$, a biidentity $\gamma_i = \delta_i$ in $\Phi$ and a substitution $\psi_i : Y \cup Y' \rightarrow F'(X)$ which is regular relatively to the e-variety $OZ$ such that

$$[\sigma_i, \tau_i] = [\alpha_i \psi_i(\gamma_i) \beta_i, \alpha_i \psi_i(\delta_i) \beta_i].$$

Since $\gamma_i, \varsigma(X) \delta_i$ for all $i$, $\varsigma(X)$ is a biiinvariant congruence on $F'(X)$, and we have $\rho(OZ, X) \subseteq \rho((LRB)OLB, X) \subseteq \varsigma(X)$, in accordance with the notes after the definition of biiinvariant congruences, we hence obtain that

$$v_n \varsigma(X) \sigma_i \varsigma(X) \tau_i \varsigma(X) w_n$$

for all $i = 1, \ldots, \ell$. Thus, by the definition of the relation $\varsigma(X)$ and the remark after it, from the previous notes in this section concerning the words $v_n$ and $w_n$ we get that, for all $i$ as before, we have $\epsilon(v_n) = \epsilon(\sigma_i) = \epsilon(\tau_i) = \epsilon(w_n) = n + 5$ and

$$\{1, \ldots, n + 4\}/\sim(\sigma_i) = \{1, \ldots, n + 4\}/\sim(\tau_i)$$

$$= \{1, \ldots, n + 4\}/\sim(w_n)$$
\[= \{1, \ldots, n+4\}/\sigma_\omega(n)\]

\[= \{1, 2, \ldots, n+1, [n+2, n+3, [n+4]\},\]

and hence, in particular, also

\[\Gamma(Q^{n+1}(u_i)) = \Gamma(Q^{n+1}(u_i)) = \Gamma(Q^{n+1}(u_i)) = \Gamma(Q^{n+1}(u_i)) \text{ and} \]

\[\Gamma(Q^{n+3}(u_i)) = \Gamma(Q^{n+3}(u_i)) = \Gamma(Q^{n+3}(u_i)) = \Gamma(Q^{n+3}(u_i)).\]

That is, for all \(i = 1, \ldots, \ell\) again, we have

\[C^\pm(Q^{n+1}(u_n))/\Gamma(Q^{n+1}(u_n)) = C^\pm(Q^{n+1}(u_n))/\Gamma(Q^{n+1}(u_n))\]

\[= C^\pm(Q^{n+1}(u_n))/\Gamma(Q^{n+1}(u_n)) = C^\pm(Q^{n+1}(u_n))/\Gamma(Q^{n+1}(u_n))\]

\[= \{a^- , b^+, s^-, s^+, c^-, d^+, t^-, t^+, \}

\[\{f^+, h^-, x^+_1\}, \{y^-, g^+, f^+\}, \]

\[\{x_1^-, x_1^+\}, \{z_1^-, y_1^+\}, \{y_1^-, x_2^+\}, \]

\[\{x_2^-, z_2^+\}, \{z_2^-, y_2^+\}, \{y_2^-, x_3^+\}, \ldots, \{x_n^-, z_n^+\}, \{z_n^-, y_n^+\}\}\]

and

\[C^\pm(Q^{n+3}(u_n))/\Gamma(Q^{n+3}(u_n)) = C^\pm(Q^{n+3}(u_n))/\Gamma(Q^{n+3}(u_n))\]

\[= C^\pm(Q^{n+3}(u_n))/\Gamma(Q^{n+3}(u_n)) = C^\pm(Q^{n+3}(u_n))/\Gamma(Q^{n+3}(u_n))\]

\[= \{a^- , s^+, s^-, b^+, c^-, d^+, t^-, t^+, \}

\[\{f^+, h^-, x^+_1\}, \{y^-, g^+, f^+\}, \]

\[\{x_1^-, x_1^+\}, \{z_1^-, y_1^+\}, \{y_1^-, x_2^+\}, \]

\[\{x_2^-, z_2^+\}, \{z_2^-, y_2^+\}, \{y_2^-, x_3^+\}, \ldots, \{x_n^-, z_n^+\}, \{z_n^-, y_n^+\}\}\]

But this means that each of the partitions \(C^\pm(Q^{n+2}(u_i))/\Gamma(Q^{n+2}(u_i))\) and

\(C^\pm(Q^{n+2}(u_i))/\Gamma(Q^{n+2}(u_i))\) for all \(i = 1, \ldots, \ell\), must arise from the partition

\(C^\pm(Q^{n+1}(u_n))/\Gamma(Q^{n+1}(u_n)) = C^\pm(Q^{n+1}(u_n))/\Gamma(Q^{n+1}(u_n))\) either by splitting the class \([a^-, b^+, s^-, s^+]\) into \([a^-, s^+], [s^-, b^+]\) or by splitting the class \([c^-, d^+, t^-, t^+]\) into \([c^-, t^+], [t^-, d^+]\). That is, looking at the previous notes in this section again, we see that each of these partitions must be equal either to \(C^\pm(Q^{n+2}(u_\omega))/\Gamma(Q^{n+2}(u_\omega))\) or to \(C^\pm(Q^{n+2}(u_n))/\Gamma(Q^{n+2}(u_n))\). Furthermore, by Corollary 3.1, from the relations displayed at the beginning of this proof we get that
We intend to show that such a situation is not possible. Thus suppose that the partition \( Y \) such that other conditions just named, the second partition \( C \), and there exist words \( \Gamma(\sigma) \) of the same form, that is, we will arrive at the equality \( \ell \). Hence we may infer that there exists \( \tau \) such that \( i_{\sigma} \in \{1, \ldots, \ell\} \) such that \( \Gamma(\sigma_{i_{\sigma}}) = \Gamma(\tau_{i_{\tau}}) \) and \( \Gamma(\tau_{i_{\tau}}) = \Gamma(\omega_{n}) \).

These equalities yield, of course, the equalities of the underlying partitions.

The findings of the previous part of this proof can be summarized as follows. If \( \nu_{n} \rho([\Phi]_{\mathcal{C}RB} \omega_{n} \mathcal{L}B \mathcal{L}B) \) then there exist words \( \sigma, \tau \in F'(X) \) satisfying

\[
\nu_{n} \xi(X) \sigma \xi(X) \tau \xi(X) \omega_{n}
\]
such that

\[
C^{\pm}(\Omega^{n+2}(\sigma))/\Gamma(\Omega^{n+2}(\sigma)) = \{ [g^{-}, a^{+}, b^{+}, c^{+}, d^{-}, h^{+}] , \ldots , [f^{+}, h^{-}, x_{1}^{+}] , [y_{2}^{+}, y_{3}^{+}, \ldots] \},
\]

and there exist words \( \alpha, \beta \in F'(X) \cup \{ \square \} \), a bidequity \( \gamma = \delta \) in \( \Phi \) and a substitution \( \psi : Y \cup Y' \to F'(X) \) which is regular relatively to \( \mathcal{O}LB \) such that \( \{ \sigma, \tau \} = \{ \alpha \psi(\gamma) \beta, \alpha \psi(\delta) \beta \} \).

Clearly, we may assume that

\[
\sigma = \alpha \psi(\gamma) \beta \quad \text{and} \quad \tau = \alpha \psi(\delta) \beta.
\]

We intend to show that such a situation is not possible. Thus suppose that the partition \( C^{\pm}(\Omega^{n+2}(\sigma))/\Gamma(\Omega^{n+2}(\sigma)) \) is of the above form. We will see that then, assuming the other conditions just named, the second partition \( C^{\pm}(\Omega^{n+2}(\tau))/\Gamma(\Omega^{n+2}(\tau)) \) will be of the same form, that is, we will arrive at the equality \( \Gamma(\Omega^{n+2}(\sigma)) = \Gamma(\Omega^{n+2}(\tau)) \). This contradiction will refute the supposition that \( \nu_{n} \rho([\Phi]_{\mathcal{C}RB} \omega_{n} \mathcal{L}B \mathcal{L}B) \).
Remember once again that, from the definition of \( \zeta(X) \), we know that the relations \( \nu_j \zeta(X) = \zeta(X) \tau \zeta(X) \omega_n \) imply, in particular, the equalities \( \nu_j \zeta(X) = \zeta(X) \omega_n = \nu_j = \zeta(X) \omega_n = n + 5 \) and \( \{1, \ldots, n + 4\} / \nu_j \zeta(X) = \{1, \ldots, n + 4\} / \zeta(X) \omega_n \) = \{1\}, \{2\}, \ldots, \{n + 1\}, \{n + 2, \ldots, n + 3\}, \{n + 4\} \), and, consequently, also the equalities \( \Gamma(\Omega(j)) = \Gamma(\Omega(j + 1)) = \Gamma(\Omega(j + 2)) \) for all \( j = 0, 1, 2, \ldots, n + 1 \) and for \( j = n + 3, n + 4 \). Recall also that, by our assumption, we have \( \Gamma(\Omega(n + 2)(\sigma)) = \Gamma(\Omega(n + 2)(\omega_n)) \). Then we need to distinguish the following three cases:

**Case I.** Assume that the word \( \Omega(n + 2)(\sigma) \) is a proper initial segment of the word \( \sigma \). Then, of course, \( \Omega(n + 2)(\sigma) \subset \Omega(n + 1)(\sigma) \) is an initial segment of \( \sigma \). Hence, owing to the equalities \( \Gamma(\Omega(n + 2)(\sigma)) = \Gamma(\Omega(n + 1)(\sigma)) = \Gamma(\Omega(n + 1)(\tau)) \), it is evident that \( \Omega(n + 2)(\tau) \subset \Omega(n + 2)(\sigma) \subset \Omega(n + 1)(\sigma) \), so that \( \Omega(n + 2)(\tau) \subset \Omega(n + 2)(\sigma) \). Thus we have \( \Gamma(\Omega(n + 2)(\sigma)) = \Gamma(\Omega(n + 2)(\tau)) \), as desired.

**Case II.** Assume that the word \( \alpha \psi(\gamma) \) is an initial segment of the word \( \Omega(n + 2)(\sigma) \). This means that \( \Omega(n + 2)(\sigma) = \alpha \psi(\gamma) \eta \) for some (possibly empty) initial segment \( \eta \) of \( \beta \). Let \( z = \Omega(\Omega(n + 1)(\sigma)) \). Then \( \eta z \) is an initial segment of \( \beta \) and \( \Gamma(\Omega(n + 1)(\sigma)) = \Gamma(\alpha \psi(\gamma) \eta z) \). Since \( \gamma \zeta(X) \delta \), and therefore \( \alpha \psi(\gamma) \eta \zeta(X) \alpha \psi(\delta) \eta z \zeta(X) \alpha \psi(\delta) \eta \), using the definition of \( \zeta(X) \), we hence get that \( \Gamma(\alpha \psi(\gamma) \eta z) = \Gamma(\alpha \psi(\delta) \eta z) \). As we also have \( \Gamma(\Omega(n + 1)(\sigma)) = \Gamma(\Omega(n + 1)(\tau)) \), these equalities altogether show that \( \Gamma(\Omega(n + 1)(\tau)) = \Gamma(\alpha \psi(\delta) \eta z) \). But this means that \( \Omega(n + 2)(\tau) = \Omega(\alpha \psi(\delta) \eta z) \). Once again, as above, \( \gamma \zeta(X) \delta \) also yields \( \alpha \psi(\gamma) \eta \zeta(X) \alpha \psi(\delta) \eta \zeta(X) \alpha \psi(\delta) \eta \), which entails the equality \( \Gamma(\alpha \psi(\gamma) \eta) = \Gamma(\alpha \psi(\delta) \eta) \). Since \( \Gamma(\alpha \psi(\gamma) \eta) \subseteq \Gamma(\alpha \psi(\delta) \eta) \), we obtain in this way that \( \Gamma(\alpha \psi(\delta) \eta) \subseteq \Gamma(\alpha \psi(\delta) \eta) \). This fact together with the above expression for \( \Omega(n + 2)(\tau) \) makes it possible to conclude that \( \Omega(n + 2)(\tau) = \alpha \psi(\delta) \eta \). Hence, using the previous equality once more, we come to the conclusion that we again have \( \Gamma(\Omega(n + 2)(\sigma)) = \Gamma(\Omega(n + 2)(\tau)) \), as desired.

**Case III.** Assume that the word \( \Omega(n + 2)(\sigma) \) is a proper initial segment of the word \( \alpha \psi(\gamma) \) and that it contains the word \( \alpha \) as its own initial segment. Let \( \lambda \geq 0 \) be the greatest integer for which \( \Omega(n + 2)(\sigma) \) is a proper initial segment of the word \( \alpha \psi(0^{\lambda}(\gamma)) \). Note that then \( \lambda \leq \lvert C(\gamma) \rvert - 1 \). Then we need again to distinguish two subcases:

**Subcase III.1.** Assume that the word \( \Omega(n + 2)(\sigma) \) is a proper initial segment of the word \( \alpha \psi(0^{\lambda + 1}(\gamma) 0^{\lambda}(\tau)) \) containing the word \( \alpha \psi(0^{\lambda}(\gamma)) \) as its own initial segment. This means that \( \Omega(n + 2)(\sigma) = \alpha \psi(0^{\lambda + 1}(\gamma)) \eta \) for some (possibly empty) initial segment \( \eta \) of \( \psi(0^{\lambda}(\tau)) \). Let \( z = \Omega(\Omega(n + 1)(\sigma)) \). Then \( \eta z \) is an initial segment of \( \psi(0^{\lambda}(\tau)) \). Since, by the definition of \( \zeta(X) \), \( \gamma \zeta(X) \delta \) yields \( \psi(0^{\lambda}(\tau)) = \psi(0^{\lambda}(\delta)) \), we see that \( \eta \gamma \) is also an initial segment of \( \psi(0^{\lambda}(\delta)) \). Moreover, we then have the equality \( \Gamma(\Omega(n + 1)(\sigma)) = \Gamma(\alpha \psi(0^{\lambda}(\gamma)) \eta z) \). Furthermore, if \( \lambda \leq \lvert C(\gamma) \rvert - 2 \) then \( \gamma \zeta(X) \delta \) also implies \( \psi(0^{\lambda}(\gamma)) \zeta(X) \alpha \psi(0^{\lambda}(\gamma)) \eta z \zeta(X) \alpha \psi(0^{\lambda}(\delta)) \eta z \), while if \( \lambda = \lvert C(\gamma) \rvert - 1 \) then, of course, \( \psi(0^{\lambda}(\gamma)) = \psi(0^{\lambda}(\delta)) \). From these facts, using the properties of \( \zeta(X) \) again, we obtain the relations \( \alpha \psi(0^{\lambda + 1}(\gamma)) \eta \zeta(X) \alpha \psi(0^{\lambda}(\gamma)) \eta \) and \( \alpha \psi(0^{\lambda}(\gamma)) \eta \zeta(X) \alpha \psi(0^{\lambda}(\delta)) \eta z \), whence, as before, we infer the equalities \( \Gamma(\alpha \psi(0^{\lambda}(\gamma)) \eta) = \Gamma(\alpha \psi(0^{\lambda}(\delta)) \eta) \) and \( \Gamma(\alpha \psi(0^{\lambda}(\gamma)) \eta z) = \Gamma(\alpha \psi(0^{\lambda}(\delta)) \eta z) \). The latter equality, the previous equality of this kind and the fact that \( \Gamma(\Omega(n + 1)(\sigma)) = \Gamma(\Omega(n + 1)(\tau)) \) together show that \( \Gamma(\Omega(n + 1)(\tau)) = \Gamma(\alpha \psi(0^{\lambda}(\delta)) \eta z) \), which means that \( \Omega(n + 2)(\tau) = \Omega(\alpha \psi(0^{\lambda + 1}(\gamma)) \eta z) \). In addition, since \( \Gamma(\alpha \psi(0^{\lambda}(\gamma)) \eta z \subseteq \Gamma(\alpha \psi(0^{\lambda}(\gamma)) \eta z) \), from the mentioned equalities we obtain that \( \Gamma(\alpha \psi(0^{\lambda}(\gamma)) \eta z \subseteq \Gamma(\alpha \psi(0^{\lambda}(\delta)) \eta z) \), which together with the previous expression for \( \Omega(n + 2)(\tau) \) allows to conclude that \( \Omega(n + 2)(\tau) = \alpha \psi(0^{\lambda + 1}(\delta)) \eta z \). Hence, using
one of the above equalities once more, we again see that $\Gamma(\mathbb{O}^{n+2}(\sigma)) = \Gamma(\mathbb{O}^{n+2}(\tau))$, as required.

Subcase III.2. Assume that the word $\mathbb{O}^{n+2}(\sigma)$ is a proper initial segment of the word $\alpha \psi(0^2(y))$ containing the word $\alpha \psi(0^{n+1}(y)0^2(y))$ as its own initial segment. Note beforehand that this is the key subcase of this entire discussion. The word $\mathbb{O}^{n+2}(y)$ is itself an initial segment of the whole word $\sigma$. We claim that this word $\alpha \psi(0^2(y))$ is actually an initial segment of $\mathbb{O}(\sigma)$. Suppose, on the contrary, that this is not the case, so that the word $\mathbb{O}(\sigma)$ is a proper initial segment of $\alpha \psi(0^2(y))$. But this means that, for all $j = 1, \ldots, n + 2$, the words $\mathbb{O}^j(\sigma)$ are proper initial segments of $\alpha \psi(0^2(y))$ containing $\alpha \psi(0^{n+1}(y)0^2(y))$ as an initial segment. Since these words $\mathbb{O}^j(\sigma)$ must clearly be equal to $\mathbb{O}^j(\alpha \psi(0^2(y)))$ for suitable integers $k \geq 1$, from Lemma 5.1 we know that all words $\mathbb{O}^j(\sigma)$ for $j = 1, \ldots, n + 2$ must be of the form $\alpha \psi(0^k(0^2(y)))$ for some integers $k$ satisfying $1 \leq k \leq \varepsilon(0^2(y)) - 1$. However, since $\mathbf{v} \in F(Y)$ and $|Y| = m$, we have $|\mathbf{v}| \leq m$, and as $\varepsilon(0^2(y)) < 1$, we see that $\varepsilon(0^2(y)) < m + 1$. Hence we also get that $\varepsilon(0^2(y)) \leq m + 1$. This shows that the range of possible values of the exponent $k$ contains at most $m$ numbers, while we have to produce $n + 2$ words $\mathbb{O}^j(\sigma)$ in the way just described. But we have assumed that $m \leq n$. The contradiction thus obtained verifies that the word $\alpha \psi(0^2(y))$ is indeed an initial segment of $\mathbb{O}(\sigma)$. This fact entails that $\Gamma(\alpha \psi(0^2(y))) \subseteq \Gamma(\mathbb{O}(\sigma))$. In addition, we have already noted earlier, among others, that $\Gamma(\mathbb{O}(\sigma)) = \Gamma(\mathbb{O}(\mathbf{v}))$, whence we get that $\Gamma(\alpha \psi(0^2(y))) \subseteq \Gamma(\mathbb{O}(\mathbf{v}))$. Furthermore, we see that $\mathbb{O}(\mathbb{O}^{n+2}(\sigma)) \subseteq \mathbb{O}(\alpha \psi(0^2(y)))$, and $\mathbb{O}(\mathbb{O}^{n+1}(\sigma)) = \mathbb{O}(\mathbb{O}(\mathbf{v}))$, which claims follow from the facts that $\mathbb{O}(\mathbb{O}^{n+2}(\mathbf{v})) = b$ and $\mathbb{O}(\mathbb{O}^{n+1}(\mathbf{v})) = d$, from our earlier notes and assumptions saying that $\mathbb{O}(\mathbb{O}^{n+1}(\sigma)) = \Gamma(\mathbb{O}(\mathbf{v}))$. From one of the formulas appearing formerly in this section we know that $\mathbb{O}(\mathbf{v}) \not\subseteq \mathbb{O}(\mathbf{v})$. Consequently, we obtain that $\mathbb{O}(\mathbb{O}^{n+2}(\sigma)) \not\subseteq \mathbb{O}(\mathbb{O}(\mathbf{v}))$. Since, as we have seen above, we have $\Gamma(\alpha \psi(0^2(y))) \subseteq \Gamma(\mathbb{O}(\mathbf{v}))$, we may conclude that $\mathbb{O}(\mathbb{O}^{n+2}(\sigma)) \not\subseteq \mathbb{O}(\alpha \psi(0^2(y)))$. In addition, from the assumptions about the word $\mathbb{O}^{n+2}(\sigma)$ in this subcase and from the fact that $\varepsilon(\sigma) = n + 5$ it follows that $\neg(\neg(\mathbb{O}^{n+2}(\sigma)) = \mathbb{O}(\alpha \psi(0^2(y)))$ and $\mathbb{O}(\mathbb{O}^{n+2}(\sigma)) = \mathbb{O}(\mathbb{O}^{n+2}(\alpha \psi(0^2(y))))$ for some integer $\phi$ such that $1 \leq \phi < \varepsilon(\alpha \psi(0^2(y))) - 1$. Thus the previous conclusion may be rewritten in the form $\mathbb{O}(\mathbb{O}^{n+2}(\alpha \psi(0^2(y)))) \not\subseteq \mathbb{O}(\mathbb{O}^{n+2}(\alpha \psi(0^2(y)))) \not\subseteq \mathbb{O}(\alpha \psi(0^2(y)))$. For the definition of $\varepsilon(\alpha \psi(0^2(y)))$, this means that $\phi = \varepsilon(\alpha \psi(0^2(y)))$ for some integer $\zeta$ such that $1 \leq \zeta < \varepsilon(\alpha \psi(0^2(y)))$. Furthermore, we have $\gamma(\zeta(X)) \delta$, whence, as before, we get that $\alpha \psi(0^2(y)) \subseteq \alpha \psi(0^2(\delta))$. This gives the equality $\Gamma(\alpha \psi(0^2(y))) = \Gamma(\alpha \psi(0^2(\delta)))$. In addition, using the definition of $\zeta(X)$ and having the above property of $\phi$ in mind, from the previous relation we deduce also the equality $\Gamma(\mathbb{O}^{n+2}(\alpha \psi(0^2(y)))) = \Gamma(\mathbb{O}^{n+2}(\alpha \psi(0^2(\delta))))$. Moreover, we have $\Gamma(\mathbb{O}^{n+2}(\alpha \psi(0^2(y)))) \subseteq \Gamma(\alpha \psi(0^2(y)))$, since, by the assumptions about the word $\mathbb{O}^{n+2}(\sigma)$, the word $\mathbb{O}^{n+2}(\mathbb{O}^{n+2}(\alpha \psi(0^2(y))))$ is an initial segment of $\mathbb{O}^{n+2}(\alpha \psi(0^2(y)))$. Hence, using the already mentioned fact that $\Gamma(\mathbb{O}^{n+1}(\sigma)) = \Gamma(\mathbb{O}(\mathbf{v}))$ (together with the first of the previous two equalities, we conclude that we also have $\Gamma(\mathbb{O}^{n+1}(\mathbf{v})) \subseteq \Gamma(\alpha \psi(0^2(y)))$. As $\varepsilon(\tau) = n + 5$, this entails that $\mathbb{O}^{n+2}(\tau) = \mathbb{O}^{n+2}(\alpha \psi(0^2(y)))$ for some integer $\phi'$ such that $1 \leq \phi' < \varepsilon(\alpha \psi(0^2(y))) - 1$. Now we intend to show that $\phi' = \phi$. We know that $\Gamma(\sigma) = \Gamma(\tau)$, which ensures that both partitions
C^\pm(\Omega^{n+2}(\sigma))/\Gamma(\Omega^{n+2}(\sigma)) and C^\pm(\Omega^{n+2}(\tau))/\Gamma(\Omega^{n+2}(\tau)) have the same number of classes. Having the above expressions for \Omega^{n+2}(\sigma) and \Omega^{n+2}(\tau) in view, however, we see that this means that the partitions C^\pm(\Omega^\upnu(\alpha \psi(0^\upnu(\gamma))))/\Gamma(\Omega^\upnu(\alpha \psi(0^\upnu(\gamma)))) and C^\pm(\Omega^\upnu(\alpha \psi(0^\upnu(\delta))))/\Gamma(\Omega^\upnu(\alpha \psi(0^\upnu(\delta)))) have the same number of classes. In this situation, the above equality \Gamma(\alpha \psi(0^\upnu(\gamma))) = \Gamma(\alpha \psi(0^\upnu(\delta))) assures that \varphi' = \varphi, as claimed. Therefore \Omega^{n+2}(\tau) = \Omega^\upnu(\alpha \psi(0^\upnu(\delta))), and, consequently, the equality \Gamma(\Omega^\upnu(\alpha \psi(0^\upnu(\gamma)))) = \Gamma(\Omega^\upnu(\alpha \psi(0^\upnu(\delta))))), which has also been obtained above, says that we again have \Gamma(\Omega^{n+2}(\sigma)) = \Gamma(\Omega^{n+2}(\tau)), as required. □

Since, as we have seen before, all identities \upsilon_n = \omega_n of the infinite sequence introduced in this section are satisfied in the e-variety \mathcal{OZ}, and, according to the previous proposition, only finitely many members of this sequence may be consequences of any finite set of biidentities valid in \mathcal{OZ} within the e-variety (\mathcal{LRB})\mathcal{OLB}, we are led to the conclusion that the e-variety \mathcal{OZ} has no finite basis of biidentities within (\mathcal{LRB})\mathcal{OLB}. Of course, this means that \mathcal{OZ} also has no finite basis of biidentities within any larger e-variety of orthodox semigroups, such as \mathcal{OLB} or \mathcal{O}, for instance. Naturally, the same statements hold true for bases of biidentities of any finite semigroup generating \mathcal{OZ}.

Next we turn our attention to ordinary semigroup identities. Given any finite set of identities which are satisfied in any finite semigroup that generates \mathcal{OZ} as an e-variety, and which are therefore valid in the e-variety \mathcal{OZ} itself, from the previous proposition we know that the identities \upsilon_n = \omega_n, beyond a finite number of exceptions, are not consequences of this set of identities within the class of all orthodox semigroups. So much the more, these identities cannot be consequences of this set of identities within the class of all semigroups. Hence it follows that the finite generators of \mathcal{OZ} also cannot have any finite basis of identities in the usual sense.

We conclude our considerations in this paper with the following remark concerning the lattice of all sub-e-varieties of the e-variety \mathcal{OLB} itself. We have already noted in the second section of this paper that there is a complete lattice congruence on that lattice whose classes are the intervals [A, A\mathcal{OLB}], where A runs over the lattice of all varieties of bands. Consider first the variety \mathcal{LRB} of all left regular bands and the interval [\mathcal{LRB}, (\mathcal{LRB})\mathcal{OLB}]. Remember that we have assembled our sequence of identities in this section of the identities \upsilon_n = \omega_n for all n = 5^k where k \geq 1 are arbitrary integers. Since |C(\upsilon_n)| = |C(\omega_n)| = 3n + 9, and as 5^{k+1} > 3 \cdot 5^k + 9 for all k \geq 1, it follows from Proposition 5.2 that, for any given integer \ell \geq 1, the set of identities \mathcal{E}_\ell = \{\upsilon_n = \omega_n: n = 5^k, k = 1, \ldots, \ell\} has the property that the other identities of our sequence are not consequences of this set \mathcal{E}_\ell within (\mathcal{LRB})\mathcal{OLB}. Hence it is clear that these sets of identities \mathcal{E}_\ell for \ell = 1, 2, \ldots, when considered within (\mathcal{LRB})\mathcal{OLB}, determine a strictly descending infinite chain of sub-e-varieties of (\mathcal{LRB})\mathcal{OLB}. Moreover, we have also noted in this section that we have \upsilon_n \rho(B) \omega_n for all n as above, which means that all sub-e-varieties of (\mathcal{LRB})\mathcal{OLB} just described contain all bands of (\mathcal{LRB})\mathcal{OLB}, and hence they include the whole variety \mathcal{LRB}. In this way, we obtain infinitely many e-varieties in the interval [\mathcal{LRB}, (\mathcal{LRB})\mathcal{OLB}]. Dual arguments show that the same can be done with the variety \mathcal{RRB} of all right regular bands, so that the same conclusion applies also to the interval [\mathcal{RRB}, (\mathcal{RRB})\mathcal{OLB}]. Furthermore, if we take any variety A of bands such that \mathcal{LRB} \subseteq A and if we replace in the above considerations the e-variety (\mathcal{LRB})\mathcal{OLB}
with $AOLB$, we see that the same statement holds true for the interval $[A, AOLB]$. Dual arguments again show that the same can be done with any variety $A$ of bands such that $RRB \subseteq A$. In all, we see that, for any band variety $A$ containing either all left regular bands or all right regular bands, the interval $[A, AOLB]$ is infinite.

References