EQUIVALENT CONDITIONS FOR HYPERBOLIC COORDINATES

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The existence of hyperbolic coordinates is equivalent to the pseudo orbits tracing property and expansiveness. Further equivalent conditions are also considered.

1. Results

Let $X$ be a compact metric space. $f : X \to X$ a homeomorphism. Any metric we are talking about in this note is considered to be compatible with a given topology of $X$.

**Theorem.** The following conditions are equivalent:

(i) $f$ has canonical coordinates and is expansive.

(ii) $f$ has hyperbolic coordinates with respect to some metric.

(iii) $f$ has the pseudo orbits tracing property and is $\lambda$-expansive with respect to some metric.

(iv) $f$ has the pseudo orbits tracing property and is expansive.

(v) $f$ has regular coordinates.

(vi) $(X, f)$ is a Smale space with respect to some metric.

This theorem covers and completes known results, see Bowen [2], Reddy [5], Ruelle [6] and the author [3]. We discuss this in Section 3. The concepts of canonical coordinates, hyperbolic coordinates, expansiveness and the pseudo orbits tracing property have been studied with regard to the Axiom A diffeomorphisms, see [1, 2, 5] for more information. The concept of a Smale space has appeared in [6] with regard to the thermodynamic formalism.

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New, however quite natural, are concepts of regular coordinates and $\lambda$-expansiveness, see [4] and [6, p. 127].

2. Definitions

For $x \in X$ and $\varepsilon > 0$ denote

$$B(x, \varepsilon) = \{x \in X: d(x, y) < \varepsilon\}, \quad B_\varepsilon = \{(x, y) \in X \times X: d(x, y) < \varepsilon\},$$

$$W^s_\varepsilon(x) = \{y \in X: d(f^n x, f^n y) < \varepsilon \text{ for all } n \geq 0\},$$

$$W^{u\varepsilon}(x) = \{y \in X: d(f^n x, f^n y) < \varepsilon \text{ for all } n = 0\},$$

$$W^s(x) = \{y \in X: d(f^n x, f^n y) \to 0, \text{ as } n \to \infty\},$$

$$W^u(x) = \{y \in X: d(f^n x, f^n y) \to 0, \text{ as } n \to -\infty\}.$$

$f$ has canonical coordinates if for each $\delta > 0$ there is $\varepsilon(\delta) > 0$ such that $d(x, y) < \varepsilon(\delta)$ implies $W^s_{\delta}(x) \cap W^u_{\delta}(y) \neq \emptyset$, see [1, 5].

$f$ has regular coordinates if there is a $\delta_0 \geq 0$ such that for each $\delta > 0$, $\delta \leq \delta_0$ there are $\varepsilon(\delta) > 0$ and a continuous function $[\quad, \quad]: B_{\varepsilon(\delta)} \to X$ such that:

$$W^s_{\delta}(x) \cap W^u_{\delta}(y) = \{[x, y]\} \quad \text{for any } (x, y) \in B_{\varepsilon(\delta)}$$ (1)

$$W^s_{\delta}(x) \subseteq W^s(x), \quad W^u_{\delta}(x) \subseteq W^u(x) \quad \text{for any } x \in X.$$

$f$ has hyperbolic coordinates if $f$ has canonical coordinates and there are $\nu \geq 0$, $0 < \lambda < 1$, $C \geq 0$ such that for any $x \in X$

$$W^s_{\nu}(x) \subseteq \{y \in X: d(f^n x, f^n y) \leq C \lambda^n d(x, y), \text{ for all } n \geq 0\},$$ (2)

$$W^u_{\nu}(x) \subseteq \{y \in X: d(f^n x, f^n y) \leq C \lambda^{-n} d(x, y), \text{ for all } n < 0\},$$ (3)

see [1, 5].

$f$ is expansive if there is $\varepsilon > 0$ such that $d(f^n x, f^n y) \leq \varepsilon$ for all integers $n$, implies $x = y$, see [1, 6].

$f$ is $\lambda$-expansive with $0 < \lambda < 1$ if there is $C > 0$, $\varepsilon > 0$ such that, for any natural $n$, $d(f^n x, f^n y) \leq \varepsilon$ with $|k| \leq n$ implies $d(x, y) \leq C \lambda^n d(x, y)$, see [4, 6, p. 127].

$f$ has the pseudo orbit tracing property if for each $\varepsilon > 0$ there is a $\delta > 0$ such that any $\delta$-pseudo orbit (i.e. a sequence $\{x_n\}_{n=-\infty}^\infty$ with $d(fx_n, x_{n+1}) < \delta$) is $\varepsilon$-traced (i.e. there is a point $x \in X$ with $d(f^n x, x_n) < \varepsilon$ for all integers $n$). See [2, 3, 4, 6].

$(X, f)$ is a Smale space if the following conditions hold:

(A) There are $\varepsilon > 0$, a continuous function $[\quad, \quad]: B_{\varepsilon} \to X$ such that for any $x, y, z \in X$:

$$[x, x] = x,$$

$$[[x, y], z] = [x, z], \quad [x, [y, z]] = [x, z],$$ (4)

$$f[x, y] = [fx, fy],$$ (5)

when the two sides of these relations are defined.
(B) For \( x \in X, \beta > 0 \) define
\[
V_{\beta}^u(x) = \{ y \in X: y = [x, y], d(x, y) \leq \beta \}
\]
\[
V_{\beta}^u(x) = \{ y \in X: y = [y, x], d(x, y) \leq \beta \}.
\]

One can see that for \( \beta \) small enough and for any \( x \in X \) the mapping \([ , ]: V_{\beta}^u(x) \times V_{\beta}^u(x) \to X \) is a homeomorphism onto its image [6, p. 126].

We assume for such \( \beta \) and any \( x \in X \)
\[
d(f^n y, f^n z) \leq C \lambda^{-n} d(y, z), \quad \text{if } y, z \in V_{\delta}^u(x), \quad n > 0,
\]
\[
d(f^n y, f^n z) \leq C \lambda^{-n} d(y, z), \quad \text{if } y, z \in V_{\delta}^u(x), \quad n \leq 0
\]
where \( C \geq 1, 0 < \lambda < 1 \) do not depend on \( \delta \) and \( x \). See [6] (\( C = 1 \) therein).

It is easy to see that the concepts of canonical coordinates, regular coordinates, the pseudo orbits tracing property and expansiveness do not depend on the metric used.

3. Proof of Theorem

We proceed as follows

(i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v) \( \Rightarrow \) (i); and then (vi) \( \Rightarrow \) (iii) and (ii) \( \Rightarrow \) (vi).

(i) \( \Rightarrow \) (ii); this is a result of Reddy [5].

(ii) \( \Rightarrow \) (iii); the pseudo orbits tracing property was proved by Bowen [2]. See also [4], \( \lambda \)-expansiveness was proved in [4].

(iii) \( \Rightarrow \) (iv); it is obvious that \( \lambda \)-expansiveness is expansiveness.

(iv) \( \Rightarrow \) (v); it is a result from [3].

(v) \( \Rightarrow \) (i); let \( \varepsilon = \min(\delta_0, \varepsilon(\delta_0)) \) where \( \delta_0 \) and \( \varepsilon(\delta_0) \) are given in the definition of regular coordinates. Let
\[
d(f^n x, f^n y) < \varepsilon \quad \text{for all } n.
\]
Then \( (x, y) \in B_{\varepsilon(\delta_0)} \), so \( W_{\delta_0}^u(x) \cap W_{\delta_0}^u(y) = [x, y] \). Moreover, from (8) we have \( x \in W_{\delta_0}^u(x) \cap W_{\delta_0}^u(y) \) and \( y \in W_{\delta_0}^u(x) \cap W_{\delta_0}^u(y) \), hence \( x = y \) and \( f \) is expansive.

(ii) \( \Rightarrow \) (vi); since (ii) \( \Rightarrow \) (i) and (i) does not depend on the particular metric we can assume the following:

There is \( \delta_0 \geq 0, C \geq 1, 0 < \lambda < 1 \) such that for each \( \delta \leq \delta_0 \) there are \( \varepsilon = \varepsilon(\delta) > 0 \) and a continuous function \([ , ]: B_{\varepsilon} \to X \) such that conditions (1), (2), (3) hold true (with \( \varepsilon = \delta \)).

Let \( \eta \leq \delta_0 \) be chosen in such a way that \( d(x, y) \leq \eta \) implies \( d(fx, fy) \leq \delta_0 \). Put \( \varepsilon = \varepsilon(\min(\delta_0/2, \eta)) \). We prove that the function \([ , ]: B_{\varepsilon} \to X \) satisfies the condition (A). Since \( x \in W_{\varepsilon}^s(x) \cap W_{\varepsilon}^u(x) \) for any \( \delta > 0 \) then \( x = [x, x] \). To prove the first formula in (4) assume that \( x, y, z \in X \) with \( (x, y) \in B_{\varepsilon} \), \( ([x, y], z) \in B_{\varepsilon} \), \( (x, z) \in B_{\varepsilon} \). Denote \( c = [[x, y], z] \). Since \( [x, y] \in W_{\delta_0/2}^u(x) \) and \( c \in W_{\delta_0/2}^u([x, y]) \) then by the triangle inequality \( d(f^n x, f^n c) \leq d(f^n x, f^n [x, y]) + d(f^n [x, y], f^n c) \leq \delta_0 \) for all \( n \geq 0 \), hence \( c \in W_{\delta_0}^u(x) \). On the other hand \( c \in W_{\delta_0}^u([x, y]) \subseteq W_{\delta_0}^u(z) \). By the condition (1), \( c = [x, z] \).
The proof of the second formula is the same. To prove (5) assume that \((x, y) \in B_r\).
Since \([x, y] \in W^s_{\delta_0/2}(x) \subset W^s_{\delta_0}(x)\), then for all \(n \geq 0\) we have \(d(f^n[x, y], f^nfx) = d(f^{n+1}[x, y], f^{n+1}x) \leq \delta_0\) what means that \(f[x, y] \in W^s_{\delta_0}(fx)\). Since \([x, y] \in W^u_{\eta}(y)\) then for all \(n \leq 0\) \(d(f^n[x, y], f^ny) \leq \eta\). In particular \(d([x, y], y) \leq \eta\) so by the choice of \(\eta\), \(d(f^n[x, y], f^ny) \leq \delta_0\) for all \(n \leq 0\), hence \(f[x, y] \in W^u_{\delta_0}(fy)\). By the condition (1), \(f[x, y] = [fx, fy]\).

We prove the condition (6) for \(\beta \leq \epsilon\), the proof of (7) is the same. Let \(y, z \in V_{\beta}(x)\). Then \(y = [x, y] \in W^s_{\delta_0/2}(x), z = [x, z] \in W^s_{\delta_0/2}(x)\) hence by the triangle inequality \(d(f^ny, f^nz) \leq \delta_0\) for all \(n \leq 0\), hence \(z \in W^s_{\delta_0}(y)\) and now (6) follows from (2).

References