Iterated exponentials of two numbers

Jerrold R. Griggs*

Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, MN 55455, USA
Current address: Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA

Received 9 August 1988

Abstract
Let $c \geq 0$ and let $n \in \mathbb{Z}^+$. The poset $A_n(c)$ of towers of exponentials of size $n$ of two numbers consists of the set $\{a, b\}^n$, ordered for $w, w' \in \{a, b\}^n$ by $w \leq_{A_n} w'$ if and only if for all values $b \geq a \geq c$, $T(w) \leq T(w')$ (as numbers), where for $w = w_1w_2 \cdots w_n$, $T(w)$ denotes the tower

$w_n \cdots w_2 w_1$

We explicitly describe the poset $A_n(c)$ for $c = c$ (it has width two for $n \geq 3$) and for $c = 3.6$ (it is a chain under reverse lexicographic order). The posets $A_n(c)$ are obtained for all $c$. Two notable consequences of this study are (i) for all $b \geq a > 0$, it is true that $b^a > a^b$; and (ii) the larger of any two towers of size $n$ in the same two numbers, each at least 3.6, can be determined by reverse lexicographic order, i.e., by comparing the highest place in the towers where they differ.

1. Introduction

A new partial order on the set $S_n$ of permutations of the set $\{1, \ldots, n\}$ has been introduced by Brunson [2]. It describes some inequalities that hold for towers of iterated exponentials of $n$ constrained variables. In general this tower order bears a similarity to the much-studied Bruhat order on $S_n$. In order to understand tower order, it is necessary to study related orders of towers of iterated exponentials of size $n$ consisting of just two variables with repeats allowed. This is the class of posets which we primarily investigate in the present article.

* Research supported in part by National Science Foundation grant no. DMS 87-01475.
Given $n$ variables $x_1, x_2, \ldots, x_n$, the iterated exponential or tower

$$x_n \ldots \ldots x_2 x_1$$

is evaluated in the conventional way, from top down. For typographical convenience we represent the tower above by $T(x_1 x_2 \cdots x_n)$. Brunson [2] observed that for all $x_2 \geq x_1 \geq e$, $T(x_1 x_2) \geq T(x_2 x_1)$, and that if $e$ is replaced by a smaller number this no longer holds in general. The permutation $x_1 x_2$ of $\{x_1, x_2\}$ can be thought of as being ordered above $x_2 x_1$. Motivated by this example, Brunson introduced for general $n$ what we shall call the tower order $T_n$, which is the set $S_n$ ordered as follows: Given a permutation $\pi \in S_n$, where we write

$$\pi = \pi(1)\pi(2) \cdots \pi(n),$$

$\pi(x)$ denotes the corresponding permutation $x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n)}$ of $\{x_1, \ldots, x_n\}$. The ordering $\leq_T$ in $T_n$ is then defined by $\sigma \leq_T \tau$ for $\sigma, \tau \in S_n$, if and only if $T(\sigma(x)) \leq T(\tau(x))$ for all valuations of the variables $x_i$ with $x_n \geq x_{n-1} \geq \cdots \geq x_1 \geq e$. To show that this is indeed a partial ordering, the antisymmetry must be established. But this is not too difficult [2, 5].

We see that $T_1$ is just the singleton poset $\{1\}$, while $T_2$ is the chain $21 \leq_T 12$. Brunson obtained $T_3$ and $T_4$, no easy task, since towers of size 3 are already difficult to work with. Several people, including this author, noticed the striking fact that for all $1 \leq n \leq 4$, $T_n$ is the dual of the Bruhat order on $S_n$, a poset of considerable interest [1]. Brunson put forth some conjectures about $T_n$, including that it is ranked for all $n$. As he was evidently unfamiliar with Bruhat order, he did not explicitly make the natural stronger conjecture that $T_n$ is the dual Bruhat order for all $n$.

It turns out that all of these conjectures fail for $T_3$, as was discovered independently by Griggs and Wachs [4] and by Strembridge [5]. Counterexamples were obtained by restricting the study of possible inequalities among towers using $\{x_1, \ldots, x_3\}$ to valuations of $x_3 \geq \cdots \geq x_1 \geq e$ in which at most two distinct values are assumed. Hence the tower order $T_n$ is closely related to the poset $A_n$ of towers of size $n$ on two symbols. Precisely stated, $A_n$ is the set $\{a, b\}^n$ of words of length $n$ in $a$ and $b$, ordered by $w \leq_A w'$ for $w, w' \in \{a, b\}^n$ if and only if $T(w) \leq T(w')$ for all $b \geq a \geq e$. Again it is straightforward to establish that this is a partial ordering. Thus $A_1$ is the chain $b \geq_A a$, while $A_2$ is the chain $bb \geq_A ab \geq_A ba \geq_A aa$. However, $A_3$ is no longer a chain since $aab \parallel_A bba$, i.e., they are not comparable. (See Fig. 1). Still, $A_n$ is close to being a chain. For $n \geq 3$ it has width two. After establishing several technical lemmas in Section 2, we shall obtain a complete description of $A_n$ in Section 3.

The domains for the variable sets used to define $T_n$ and $A_n$ contain the rather arbitrary lower bound $e$. In order to expand the scope of the study, we introduced the analogous posets for lower bounds $c \geq 0$. Define $T_n(c)$ to be the ordering of $S_n$ in which $\sigma \leq_T \tau$ whenever $T(\sigma(x)) \leq T(\tau(x))$ for all $x_n \geq \cdots \geq x_1 \geq c$. Similarly,
$A_n(c)$ is $\{a, b\}^n$ ordered by $w \leq_A w'$ whenever $T(w) \leq T(w')$ for all $b \geq a \geq c$ (recall that $b$ and $a$ are variables here, while $c$ is some given constant). In this notation, $T_n = T_n(e)$ and $A_n = A_n(e)$. Increasing $c$ strengthens the orderings $A_n(c)$ and $T_n(c)$ since it restricts the domain of definition.

We shall be particularly interested in the posets $A_n(c)$. In Section 4 it is shown that for all $n$, $A_n(3.6)$ is a chain, the reverse lexicographic order on $\{a, b\}^n$. It is also observed there that $A_n(1)$ is in general a stronger ordering than the Boolean ordering $B_n$ obtained by considering the set of indices $i$ in $w = w_1w_2\cdots w_n \in \{a, b\}^n$ such that $w_i = b$, ordered by inclusion.

Section 5 contains a description of all posets $A_n(c)$, evolving from the weakest order at $c = 0$, to the strongest, the chain using reverse lexicographic order. In all, there are 8 different posets for $n = 3$. Some of the inequalities on towers of size three are rather surprising.

In Section 6 we recall from the paper of Stembridge [5] the elegant characterization of tower order $T_n$ in terms of projections into $A_n$. This holds more generally for $T_n(c)$, $c \geq e$. It follows that for any fixed $n$ the posets $T_n(c)$ are identical for all $c \geq 3.6$. Several problems for further investigations into $T_n(c)$ are proposed. The concluding section contains several general conjectures about the posets $A_n(c)$.

2. Fundamental lemmas

We obtain a series of facts that will be quite useful later on. We conclude the section with a rather technical result that is an essential tool in our study. This result can be regarded as the 'main lemma' in the paper.

**Lemma 2.1.** (i) The function $(\ln x)/x$ is strictly increasing on $[0, e]$ and strictly decreasing on $[e, \infty)$.

(ii) If $e \geq b > a \geq 0$, then $a^b < b^a$, while if $b > a \geq e$, then $a^b > b^a$.

**Proof.** Statement (i) follows by examining the derivative, and statement (ii) follows from (i) by comparing the logarithms of both sides. □
Lemma 2.2. The function \( c^x, x \geq 0 \), is strictly decreasing for all \( x \) if \( c < 1 \) and strictly increasing for all \( x \) if \( c > 1 \).

Proof. Examine the derivative of \( \ln c^x = x \ln c \).

Lemma 2.3. If \( b > a \geq 0 \) and \( x \geq 0 \), then \( a^x \approx b^x \), with equality if and only if \( x = 0 \).

Lemma 2.4. The function \( x^x \) is strictly decreasing on \([0, 1/e]\) and strictly increasing on \([1/e, \infty)\).

Lemma 2.5. The function \( T(xxx) = x^x \) is strictly increasing on \([0, \infty)\).

Proof. Examine the derivative of \( f(x) = \ln T(xxx) = x^x \ln x \). Then \( f'(x) = x^{x-1}(1 + x \ln x + x \ln^2 x) \), so it suffices to show that \( 1 + x \ln x + x \ln^2 x > 0 \) for all \( x > 0 \). By Lemma 2.4, \( x \ln x = \ln x^x \) is minimized at \( x = 1/e \), hence \( 1 + x \ln x + x \ln^2 x \geq 1 - (1/e) + x \ln^2 x > 0 \).

Lemma 2.6. The function \( T(xxx) \) for fixed \( c > 0 \) is decreasing for \( x < \exp(-1/c) \) and increasing for \( x \geq \exp(-1/c) \).

Lemma 2.7. If \( b \geq a \geq 3.6 \), then \( a^b/b^a \geq b/a \).

Proof. \( a^b/b^a \geq b/a \) in this range \( \iff a^{b+1} \geq b^{a+1} \iff \ln a/(a + 1) \geq \ln b/(b + 1) \). Therefore it suffices to prove that \( f(x) = \ln x / (x + 1) \) is decreasing for \( x \geq 3.6 \). One finds that \( f'(x) = 0 \) when \( x \) is the root, call it \( \beta \), of the equation \( \ln x = (x + 1)/x \). Then \( \beta = 3.591121477 \). For \( x > \beta \), \( f'(x) < 0 \), and the result follows.

Next we present the 'main lemma', a technical statement that is a surprisingly useful tool for reducing large towers (which are so difficult to handle) to smaller ones.

Lemma 2.8. Suppose \( p, q, r, u, v, w \geq 1 \) and

(i) \( v \geq e^{1/e} \),
(ii) \( p' \geq u^w \),
(iii) \( q'/v^w \geq r/w \).

Then

\[
\frac{T(p, q, r)}{T(u, v, w)} \geq \frac{T(p, r)}{T(u, w)}
\]

Proof. Let L.H.S. and R.H.S. denote the quantities on the left and right sides of the inequality that is to be proven. Then,

\[
\ln(\text{L.H.S.}) = (\ln p)q' - (\ln u)v^w = v^w((\ln p)(q'/v^w) - \ln u) \\
\geq v^w((\ln p)(r/w) - \ln u) = (v^w/w)((\ln p)r - (\ln u)w) \\
= (v^w/w)\ln(p'/u^w) \geq \ln(p'/u^w) = \ln(\text{R.H.S.}).
\]
Iterated exponentials of two numbers

3. The poset \( A_n \)

We are now ready to describe the poset \( A_n \) for general \( n \). The following theorem determines \( A_n \) by explicitly describing its covering relations. We say elements \( x \) and \( y \) of a poset \( P \), ordered by \( \geq P \), satisfy the condition that \( x \) covers \( y \), denoted \( x \geq_P y \), if \( x \geq P y \) and there is no element \( z \) such that \( x >_P z >_P y \).

**Theorem 3.1.** The covering relations in \( A_n \) are as follows:

(i) \( bu \geq_A au \) for all \( u \in \{a, b\}^{n-1} \),
(ii) \( uab \geq_A uba \) for all \( u \in \{a, b\}^{n-2}, n \geq 2 \),
(iii) \( a^kbu \succeq_A b^kau \) for all \( u \in \{a, b\}^{n-k-1}, 1 \leq k \leq n-2 \), where, e.g., \( a^k \) denotes the word with \( k \) a's.

**Remarks.** From the theorem it follows easily by induction on \( n \) that for \( n \geq 3 \), \( A_n \) consists of two chains of size \( 2^{n-1} \) each: The 'lower' chain consists of elements of the form \( ua \), and the 'upper' chain contains elements of the form \( ub \). The elements in each chain are in reverse lexicographic order. Given a totally ordered set \( P \), this is a total order on \( P^n \) that we denote by \( \leq_L \): For \( w \neq w' \in P^n \), \( w \geq_L w' \) if there exists \( k \) such that \( w_j = w'_j \) for all \( j > k \) and \( w_k \geq_P w'_k \). For this case, the poset \( P \) is \( \{a, b\} \), ordered by \( b \geq_A a \). Further, it follows from the theorem that each element of the form \( uba \) in \( A_n \) is covered by the corresponding element \( uab \), and that no further coverings occur in \( A_n \). For example, see the poset \( A_3 \) in Fig. 1. We are now ready to prove the theorem.

**Proof.** The proof is by induction \( n \), with the cases \( n = 1 \) and \( n = 2 \) having been disposed of in Section 1. Assume for the remainder that \( n \geq 3 \). (The induction hypothesis is not required until the proof of Claim 3.4.).

There are two stages to the proof. In the first stage it is checked that the listed covering relations are valid order relations in \( A_n \) (tower inequalities), i.e., if we claim \( w \succeq_A w' \), we verify that the weaker assertion \( w \geq_A w' \) holds. In the second stage the noncovering relations are confirmed. The theorem will then be proved.

We begin the first stage, confirming that the claimed covering pairs in \( A_n \) are correctly ordered. Trivially, \( T(bu) \geq T(au) \) for all \( b \geq a \geq c \), so it follows that \( bu \geq_A au \), which confirms order relations of the first listed type. By Lemma 2.1, \( T(uab) \succeq T(uba) \) for all \( b \geq a \geq e \), and this confirms the ordering of pairs of the second listed type.

For this stage it remains to verify the third listed type of ordering which follows from this more general inequality.
Claim 3.2. For all \( b \preceq a \preceq e, \alpha \geq 2, \) and integer \( k \geq 0, \) \( T(a^k b a \alpha) \equiv T(b^k a a \alpha). \)

Proof of Claim 3.2. For \( k = 0, \) the claim reduces to \( T(b a \alpha) \equiv T(a a \alpha), \) which is immediate. For general \( k, \) it will hold if we prove that

\[
\frac{T(a^k b a \alpha)}{T(b^k a a \alpha)} \geq \frac{T(a^{k-1} b a \alpha)}{T(b^{k-1} a a \alpha)}, \quad \text{for } k \geq 1, \ b \succeq a \succeq e, \ \alpha \geq 2. \tag{1}
\]

We prove (1) by induction on \( k. \)

The basis case, \( k = 1, \) requires particular attention. In this case, taking logs, simplifying, and replacing \( b \) by \( x, \) yield the equivalent inequality \((\ln a)(x^\alpha + \alpha) \geq (\ln x)(a^\alpha + \alpha).\) The two sides are equal for \( x = a, \) so it suffices to show that the derivative with respect to \( x, \) with \( a \) fixed, is larger on the L.H.S. for \( x \preceq a. \) That is, it suffices for \( k = 1 \) to prove

\[
(\ln a)ax^{\alpha - 1} \geq \frac{1}{x} (a^\alpha + \alpha) \iff (\ln a)ax^\alpha \geq a^\alpha + \alpha, \quad \text{for } x \preceq a.
\]

Clearly, this holds if it is valid for \( x = a, \) which is equivalent to

\[ a^\alpha((\ln a)\alpha - 1) \succeq \alpha. \]

This inequality is tightest over \( a \succeq e \) when \( a = e, \) that is,

\[ e^\alpha(\alpha - 1) \succeq \alpha. \]

This holds easily for all \( \alpha \geq 2, \) so (1) holds when \( k = 1. \)

Now assume \( k \geq 2. \) We seek to apply the main lemma, 2.8, with \( p = q = a, \ r = T(a^{k-2} b a \alpha), \ u = v = b, \) and \( w = T(b^{k-2} a a \alpha). \) By induction (1) holds for \( k - 1, \) so that

\[
\frac{q'}{v^w} = \frac{T(a^{k-1} b a \alpha)}{T(b^{k-1} a a \alpha)} \geq \frac{T(a^{k-1} b a \alpha)}{T(b^{k-2} a a \alpha)} = \frac{r}{w}, \tag{2}
\]

which verifies hypothesis (iii) of the lemma. The L.H.S. of (2) is also equal to \( p'/u^w, \) while repeatedly applying (1) to the R.H.S. of (2) shows it is at least \( T(b a \alpha)/T(a a \alpha), \) which is at least 1. Therefore, \( p'/u^w \geq 1, \) which is hypothesis (ii) of the lemma. The remaining hypotheses hold trivially, so we may apply the lemma to conclude that (1) indeed holds for \( k. \) The claim then follows. \( \square \)

This completes the first stage of the proof.

For the second stage, we must prove for every pair \( w, w' \in \{a, b\}^* \) that if \( w \) and \( w' \) are not ordered by the transitive closure of the order relations proved in the first stage above, then indeed \( w \parallel_A w' \) (which means they are unrelated in \( A_n). \)

To prove this, we must prove there exist values of \( a \) and \( b \) such that \( T(w) > T(w') \) and other values such that \( T(w) < T(w'). \) It turns out to be sufficient to consider valuations in which \( a = e. \) Rename \( b \) by \( x \) since we shall be taking derivatives. Typically, the relation between \( T(w) \) and \( T(w') \) will go one way when \( x \to \infty, \) and the other way when \( x \to e^+. \)
We first show that for all sufficiently large $x$, all elements $w \in \{e, x\}^n$ with $w_0 = x$ lie above all elements with $w_0 = e$. This is implied by the orderings in the first stage together with the following statement.

**Claim 3.3.** For all $n \geq 1$, $T(e^{n-1} x) > T(x^{n-1} e)$ for all sufficiently large $x$.

**Proof of Claim 3.3.** It is useful to first prove by induction that for $n \geq 2$,

$$T'(x^{n-1} c) - T(x^{n-1} c) T(x^{n-2} c) \cdots T(x c) c (\ln x)^{n-2} / x, \quad \text{as } x \to \infty, \tag{3}$$

where $T'(w)$ denotes the derivative of $T(w)$ with respect to $x$. For $n = 2$ it can be directly verified that (3) holds. For $n \geq 3$,

$$T(x^{n-1} e) = \exp (\ln x) (T(x^{n-2} e)),$$

so that

$$T'(x^{n-1} e) = T(x^{n-1} e) \left( \frac{1}{x} T(x^{n-2} e) + (\ln x) T'(x^{n-2} e) \right).$$

By induction, we may apply (3) to $T'(x^{n-2} e)$, which shows that $(\ln x) T'(x^{n-2} e)$ dominates $T(x^{n-2} e) / x$ as $x \to \infty$. Hence, $T'(x^{n-1} e) \sim T(x^{n-1} e) T'(x^{n-2} e) \ln x$, which by induction yields (3).

The claim itself is immediate for $n = 1$, and for $n = 2$, it follows from Lemma 2.1. Assume that $n \geq 3$ and that the claim holds for all smaller values. To show that $T(e^{n-1} x) > T(x^{n-1} e)$, it is equivalent to take the logarithm on both sides, $T(e^{n-2} x) > (\ln x) T(x^{n-2} e)$ and by taking derivatives on both sides, it suffices to show that

$$T'(e^{n-2} x) / ((\ln x) T(x^{n-2} e))' \to \infty, \quad \text{as } x \to \infty. \tag{4}$$

We have that $T'(e^{n-2} x) = T(e^{n-2} x) T(e^{n-3} x) \cdots T(e x)$, while by (3),

$$((\ln x) T(x^{n-2} e))' = (1 / x) T(x^{n-2} e) + (\ln x) T'(x^{n-2} e) \sim T(x^{n-2} e) T(x^{n-3} e) \cdots T(e x) e (\ln x)^{n-2} / x, \quad \text{as } x \to \infty.$$

Hence we have that

$$\frac{T'(e^{n-2} x)}{((\ln x) T(x^{n-2} e))'} \sim \frac{T(x^{n-2} e)}{T'(e^{n-3} x)} \frac{T(e^{n-3} x)}{T(x^{n-3} e)} \cdots \frac{T(e x)}{T(e)} \frac{1}{(\ln x)^{n-2}}.$$

The product of the last two terms, $x / (e (\ln x)^{n-2}) \to \infty$, while every other term in the product is greater than one for sufficiently large $x$, by applying the claim inductively. This proves (4), and the claim follows.

It remains to prove that for $n \geq 3$, towers of the lower chain of the form $u x e$ are larger than towers of the upper chain of the form $v e x$, for $a = e$ and sufficiently small $b = x > e$, whenever $u \geq_L v$ in $A_{n-2}$.

**Claim 3.4.** Let $n \geq 3$. If $u, v \in A_{n-2}$ and $u \geq_L v$, then for $a = e$ and $b = x$ in $u$ and $v$ it follows that

$$T'(u x e)|_{x=e} > T'(v e x)|_{x=e}.$$

Notice that $T(u x e) = T(v e x) = T(e^n)$ when $x = e$, so the claim implies that for sufficiently small $x$, $T(u x e) > T(v e x)$, as required to prove Theorem 3.1.
Proof of Claim 3.4. It is easily verified by induction on $n$ that for any $v \in \{e, x\}^{n-2}$, $T'(u_{vex})|_{x=e} = T'(u_{xe})|_{x=e}$. Therefore we may prove this equivalent inequality for applicable $u$ and $v$:

$$T'(u_{xe})|_{x=e} > T'(u_{xe})|_{x=e}. \quad (5)$$

(5) is to be proven by induction on $n$.

One type of covering $u \succeq_L v$ occurs when $u$ and $v$ are identical except for their first entry. Then if $a = e$ and $b = x$, there exists $w \in \{e, x\}^{n-1}$ such that $u_{xe} = xw$ and $v_{xe} = ew$. Think of $w$ as a function of $x$, $w(x)$, with $w(e) > 0$. Then we have

$$T'(xw(x)) = \exp((\ln x)w(x)) = T(ew(e))((w(x)/e) + (\ln x)w'(x)), \quad (6)$$

which evaluated at $e$ is

$$T'(xw(x))|_{x=e} = T(ew(e))((w(e)/e) + w'(e)), \quad (6)$$

while $T'(ew(x))|_{x=e} = T(ew(e))w'(e) < T'(xw)|_{x=e}$, and (5) follows in this case.

It remains to consider the type of covering in which there exists $k \geq 1$ such that $u$ begins $a^x b$ while $v$ begins $b^x a$, but otherwise $u$ and $v$ are identical, i.e., if $a = e$ and $b = x$, there exists $w \in \{e, x\}^{n-k-1}$ such that $u_{xe} = e^x w$ and $v_{xe} = x^k w$. Since $w$ ends with $x e$, it is non-empty. We compare the derivatives of $T(e^x w)$ and $T(x^k w)$ at $x = e$. For each $k \geq 1$,

$$T'(e^x w)|_{x=e} = T'(e^{k+1} w(e)) T'(e^{k-1} x w)|_{x=e}. \quad (7)$$

Applying this equation repeatedly, along with (6) when $k$ reaches 0, yields $T'(u_{xe})|_{x=e}$:

$$T'(e^x w)|_{x=e} = T'(e^{k+1} w(e)) T'(e^{k} w(e)) \cdots T'(e w(e))((w(e)/e) + w'(e)). \quad (7)$$

Similarly, we have for $k \geq 1$ that

$$T'(x^k w)|_{x=e} = T'(e^{k+1} w(e))((T(e^k w(e))/e) + T'(x^{k-1} e w)|_{x=e}). \quad (8)$$

At $k = 0$, we have $T'(e w)|_{x=e} = w'(e) T'(e w(e))$, so that by applying (8) repeatedly we end up with $T'(u_{xe})|_{x=e}$:

$$T'(x^k w)|_{x=e} = T'(e^{k+1} w(e)) \cdots T'(e w(e)) w'(e)$$

$$+ \frac{1}{e} T'(e^{k+1} w(e)) \cdots T'(e w(e))$$

$$+ \frac{1}{e} T'(e^{k+1} w(e)) \cdots T'(e^2 w(e))$$

$$\vdots$$

$$+ \frac{1}{e} T'(e^{k+1} w(e)) T'(e w(e)) \quad (9)$$
It is to be shown that (7) > (9). After subtracting the \( w'(e) \) terms from each and removing any remaining common factors, it is equivalent to prove that

\[
a_{k-1}a_{k-2} \cdots a_0 > a_{k-1} \cdots a_1 + a_{k-1} \cdots a_2 + \cdots + a_{k-1} + 1,
\]

where \( a_i = T(e^i w(e)) \). (10)

For \( k = 1 \), (10) reduces to \( a_0 = w(e) > 1 \), which is true for our function \( w(x) \). Then suppose \( k > 1 \). Since \( w(e) > 2 \), it follows that \( a_i > 2 \) for all \( i \). Then

\[
a_{k-1} > 1 \Rightarrow a_{k-1}(a_{k-2} - 1) > 1 \quad \Leftrightarrow \quad a_{k-1}a_{k-2} > a_{k-1} + 1 \Rightarrow
\]

\[
a_{k-1}a_{k-2}(a_{k-3} - 1) > a_{k-1} + 1 \quad \Leftrightarrow \quad a_{k-1}a_{k-2}a_{k-3} > a_{k-1}a_{k-2} + a_{k-1} + 1,
\]

and so on, until (10) is finally derived. This in turns implies (5) for the third and last possible type of covering. Thus in every case, the claim holds. \( \square \)

The claim completes this stage, so the theorem is proven.

4. Further results for \( A_n(c) \)

The previous section obtained an explicit description of the posets \( A_n = A_n(e) \). In this section, two results about \( A_n(c) \) are given for other values of \( c \) and general \( n \). The first result is quite easy.

**Theorem 4.1.** The ordering of \( \{a, b\}^n \) given by \( A_n(1) \) is stronger than the Boolean ordering denoted \( w \leq_S w' \), which is defined to hold for \( w, w' \in \{a, b\}^n \) whenever \( \{i : w_i = b\} \subseteq \{i : w'_i = b\} \).

**Proof.** If \( w \leq_S w' \), it means that \( w' \) can be obtained from \( w \) by replacing some \( a \)'s in \( w \) by \( b \)'s. For any \( b \geq a \geq 1 \), this clearly forces \( T(w) \leq T(w') \), so that \( w \leq A w' \) in \( A_n(1) \). \( \square \)

As we shall discover in the next section, \( A_n(1) \) is isomorphic to the Boolean lattice \( B_n \) only for \( n = 1, 2 \). For \( n \geq 3 \), \( A_n(1) \) is a strictly stronger ordering.

Now we present one of the main results of our study.

**Theorem 4.2.** For \( n \geq 1 \), \( A_n(3.6) \) is a chain, the reverse lexicographic ordering of \( \{a, b\}^n \).

**Proof.** Since \( A_n(3.6) \) is stronger than \( A_n = A_n(e) \), it suffices to prove that \( T(a^{n-1} b) \geq T(b^{n-1} a) \) for all \( b \geq a \geq 3.6 \). For \( n = 1 \), this is immediate. For \( n = 2 \), it follows from Lemma 2.1, but we require for induction the stronger result from Lemma 2.7 that \( a^b / b^a \geq b / a \). More generally, we claim for all \( n \geq 2 \),

\[
\frac{T(a^{n-1} b)}{T(b^{n-1} a)} \geq \frac{T(a^{n-2} b)}{T(b^{n-2} a)}.
\]

(11)
For \( n \geq 3 \), this can be deduced from the ‘main lemma’ 2.8, by induction on \( n \). One takes \( p = q = a, r = a^{n-3}b, u = v = b, w = b^{n-3}a \). Hypothesis (iii) in Lemma 2.8 is simply (11) at \( n - 1 \) instead of \( n \). So (11) follows for all \( n \geq 2 \). Repeatedly applying (11), one finds that \( T(a^{n-1}b)/T(b^{n-1}a) \geq T(a^{n-2}b)/T(b^{n-2}a) \geq \ldots \geq b/a \), and since \( b/a \geq 1 \), it follows that \( T(a^{n-1}b) \geq T(b^{n-1}a) \). 

Of course, ‘3.6’ in the theorem above can be improved to \( \beta \approx 3.591 \). For any specific \( n \), it is likely that an even smaller value works. We show in the next section that for \( n = 3 \) a number \( \alpha_1 = 3.440 \) is best possible.

5. The posets \( A_n(c), n \leq 3 \)

It is perhaps hopeless to describe the posets \( A_n(c) \) in general. For \( n \) as little as 3 it is a considerable task to obtain them. In this section we present the complete evolution of the posets \( A_n(c), n \leq 3 \), beginning with large \( c \) (the strongest ordering) and decreasing down to \( c = 0 \) (the weakest ordering).

A straightforward application of Lemmas 2.1, 2.2, 2.3, and 2.4 verifies the Hasse diagrams for \( A_1(c) \) and \( A_2(c) \) shown in Fig. 2. We now concentrate on the posets \( A_3(c) \). There are 8 different posets altogether, shown in Fig. 3. The proof of their correctness is broken down into a series of propositions.

From Section 3 we know that \( A_3(c) \) is a chain except for one unrelated pair, \( aab \) and \( bba \). We consider this pair first.

**Proposition 5.1.** \( aab \preceq_A bba \) in \( A_3(c) \) if and only if \( c \preceq \alpha_1 \), where \( \alpha_1 \) is the root of the equation \( x \ln^2 x - x \ln x - 1 = 0 \), \( \alpha_1 = 3.439569 \).

**Proof.** We compare the logarithms of \( T(aab) \) and \( T(bba) \) with \( b \) replaced by \( x \). Taking derivatives, we compare \( a^x \ln^2 a \) with \( ax^{a-1} \ln x + x^{a-1} \). For \( x = a \), we have that the left side is larger whenever \( a^x \ln^2 a \geq a^x \ln a + a^x \), which holds if and only if \( a \geq \alpha_1 \). Thus, for any sufficiently small \( \epsilon > 0 \), if \( a = \alpha_1 - \epsilon \) and \( b \) is

![Fig. 2. The posets \( A_1(c) \) and \( A_2(c) \).](image)
sufficiently close to (but larger than) \( a \), we have \( T(aab) < T(bba) \), while by Section 3, \( T(aab) > T(bba) \) for \( b > a > 3.6 \). Thus \( aab \parallel bba \) if \( c < \alpha_1 \).

Next suppose \( x > a > \alpha_1 \). Notice that

\[
\frac{a \ln x + 1}{a \ln a + 1} < \frac{a \ln x}{a \ln a} = \frac{\ln x}{\ln a}.
\]

(12)

It follows that the right-side derivative above satisfies

\[
a x^{a-1} \ln x + x^{a-1} < x^{a-1} \left( \frac{\ln x}{\ln a} \right) + x^{a-1} \left( \frac{\ln x}{\ln a} \right) \ln a - a \ln^2 a,
\]

which is the derivative of the left side, where we used (12), the definition of \( \alpha_1 \), Lemma 2.1 (i), and Lemma 2.1 (ii). It follows that for \( b \geq a \geq a_1 \), \( T(aab) \geq T(bba) \), and the proposition is proven. \( \square \)
We have therefore established the posets $A_3(c)$ for $c \geq e$. When $c$ drops below $e$, several orderings are lost since it no longer holds that $a^b \geq b^a$ in general.

In particular, Lemma 2.1 implies the following proposition.

**Proposition 5.2.** $aab \geq_A aba$ in $A_3(c)$ if and only if $c \geq e$. Also, $bab \geq_A bba$ in $A_3(c)$ if and only if $c \geq e$.

We omit the details of the proofs of the following propositions since they are rather similar to the proof of 5.1.

**Proposition 5.3.** $abb \geq_A bba$ in $A_3(c)$ if and only if $c \geq \alpha_2$, where $\alpha_2$ is the root of the equation $x \ln^2 x - 1 = 0$, $\alpha_2 \approx 2.020747$.

**Proposition 5.4.** $aab \geq_A baa$ in $A_3(c)$ if and only if $c \geq \alpha_2$.

**Proposition 5.5.** $abb \geq_A bab$ in $A_3(c)$ if and only if $c \leq c_{\alpha_3}$, where $c_{\alpha_3}$ is the root of the equation $x\ln x - 1 = 0$, $c_{\alpha_3} = 1.763223$.

**Proposition 5.6.** $aba \geq_A baa$ in $A_3(c)$ if and only if $c \geq \alpha_3$.

**Proposition 5.7.** $abb \geq_A baa$ in $A_3(c)$ if and only if $c \geq c_{\alpha_4}$, where $c_{\alpha_4}$ is the root of the equation $x\ln x + x\ln x - 1 = 0$, $c_{\alpha_4} \approx 1.559134$.

The propositions above along with Theorem 4.1 suffice to prove the correctness of the Hasse diagrams in Fig. 3 for all $c \geq 1$ with just one exception, the surprising inequality $bab \geq_A aba$:

**Proposition 5.8.** For all $b > a \geq 0$, $b^{a^b} > a^{b^a}$.

**Proof.** This result will appear as a problem [3]. A proof due to Z. Füredi and the author has been submitted as well. This proof employs the usual techniques of taking logarithms and differentiating although one must be more clever than in the previous cases. □

Next we verify the orderings for $A_3(0)$ shown in Fig. 3.

**Proposition 5.9.** In $A_3(0)$ the following orderings hold:

- (i) $bbb \geq_A abb$,
- (ii) $bbb \geq_A bba$,
- (iii) $bba \geq_A aba$,
- (iv) $bba \geq_A aaa$,
- (v) $bab \geq_A aab$,
- (vi) $bab \geq_A aba$,
- (vii) $baa \geq_A aab$,
- (viii) $aab \geq_A aaa$. 


Proof. (vi) is the previous proposition. Lemma 2.3 implies (i), (iii), (v), and (vii). Lemma 2.2 implies (ii) and (viii). Lemma 2.6 implies (iv).

One more ordering holds if $c$ is not too small.

**Proposition 5.10.** In $A_3(c)$, $bbb \succeq_A aab$ if and only if $c \geq \alpha_5$, where $\alpha_5$ is defined to be the maximum value $a$ such that there exists $b \in (a, 1)$ with $T(bbb) = T(aab)$, $\alpha_5 = 0.025099$.

**Proof.** It is surprising that an $a$ exists at all such that for some $b > a$, $T(bbb) = T(aab)$, for any $a$, $T(bbb) > T(aab)$ for $b$ sufficiently close to and greater than $a$. But consider, e.g., $a = 0.025$. If $b > 1$, say, then $T(bbb) > T(aab)$ as usual, while if $b = 0.74$, say, then $T(bbb) < T(aab)$. By continuity, $T(bbb) = T(aab)$ for $a = 0.025$ and some $b \in (0.74, 1)$.

Application of Lemma 2.6 with $c = b$ shows that $\alpha_5 < 1/e$, while continuity guarantees it exists. Consider the function $h_a(x) = \ln(T(xxx)/T(aax))$, where $a \in (0, 1/e)$. Then $h_a(a) = 0$ while $h_a(1) > 0$. For $a = \alpha_5$, there exists $\gamma \in (\alpha_5, 1)$ such that $h_a(\gamma) = 0$. It must be that for $a = \alpha_5$, $h_a(\gamma) = 0$, or else we could increase $a$ slightly and slightly adjust $\gamma$ while still satisfying $h_a(b) = 0$, contradicting the definition of $\alpha_5$. Thus setting $a = \alpha_5$ and $x = \gamma$ must solve this system of equations:

\begin{align*}
  x^x \ln x - a^x \ln a &= 0, \\
  x^x \left( \ln^2 x + \ln x + \frac{1}{x} \right) - a^x \ln^2 a &= 0.
\end{align*}

A comparison of (13) and (14) yields

\begin{equation}
  \ln a = \ln x + 1 + \frac{1}{x \ln x}.
\end{equation}

Substituting for $a$ in (13) gives an equation in a single variable $x$. After simplification, we find the following equation in $x$.

\begin{equation}
  \ln x - \left( \ln x + 1 + \frac{1}{x \ln x} \right) \exp \left( x + \frac{1}{\ln x} \right) = 0.
\end{equation}

A numerical investigation of (16) determined that it has just one root in $(0, 1)$, so that $\gamma$ is this root, and $\alpha_5$ is determined by (15):

$\gamma \approx 0.731507$ and $\alpha_5 = 0.025099$.

To complete the verification of the diagrams for $A_3(c), c < 1$, it remains to eliminate orderings from $A_3(1)$ that fail in every $A_3(c), c < 1$. For such unordered pairs of elements, a valuation of $a$ and $b$ in one direction follows from the ordering in $A_3(1)$, so an example of an ordering in the other direction suffices.
Proposition 5.11. For every $c$, $0 \leq c < 1$, and for every inequality listed, there exists $b > a > c$ such that the inequality holds:

(i) $T(bab) > T(bbb)$,
(ii) $T(baa) > T(bbb)$,
(iii) $T(baa) > T(bab)$,
(iv) $T(baa) > T(bba)$,
(v) $T(aaa) > T(abb)$,
(vi) $T(aab) > T(abb)$,
(vii) $T(aba) > T(abb)$,
(viii) $T(aaa) > T(aba)$.

Proof. Inequalities (i), (iv), and (viii) hold for $b > 1 > a > c$ by Lemmas 2.3 and 2.2. Inequalities (ii) and (v) hold for $1 > b > a > c$ by Lemmas 2.4 and 2.2. Inequalities (iii) and (vii) hold for $b > 1 > a > c$ by Lemma 2.2. applied twice. Finally, (vi) follows from (v) and Proposition 5.9. □

The proposition above gives the desired orderings, so completes our proof of the following theorem.

Theorem 5.12. For $c \geq 0$, the Hasse diagram of the poset $A_3(c)$ is given in Fig. 3.

6. Consequences and conjectures for the tower order, $T_n(c)$

In the article that introduced the tower order $T_n = T_n(e)$ on $S_n$, Brunson [2] conjectured, in effect, that $T_n(e)$ is the dual of the (strong) Bruhat order on $S_n$, a well-known poset. He proved this for $n \leq 4$. In general, $T_n(e)$ is stronger than the dual Bruhat order. Griggs and Wachs [4] and Stembridge [5] obtained counterexamples for $n \geq 5$ by approaches that are similar to each other. In each case, projections of $T_n(e)$ into $A_n = A_n(e)$ are the key tool. Stembridge noticed that in fact these projections characterize $T_n(e)$. We begin by reviewing this work.

For $1 \leq i < n$, the projection $\phi_i : S_n \to \{a, b\}^n$ is induced by sending $1, \ldots, i$ to $a$ and $i + 1, \ldots, n$ to $b$. Here is the projection theorem.

Projection Theorem 6.1 [5]. (i) If $c \geq 0$ and if $\sigma, \tau \in S_n$ are such that in $T_n(c)$, $\sigma \leq_T \tau$, then for all $i$, $\phi_i(\sigma) \leq_A \phi_i(\tau)$ in $A_n(c)$.

(ii) If $c \geq e$ and if $\sigma, \tau \in S_n$ are such that for all $i$, $\phi_i(\sigma) \leq_A \phi_i(\tau)$ in $A_n(c)$, then $\sigma \leq_T \tau$ in $T_n(c)$.

The first statement in the theorem is merely a consequence of our terminology, so the main content of the theorem lies in its second statement. Stembridge proved it more generally in the setting of words in $\{1, \ldots, n\}$, i.e., repeats are allowed. We have checked that statement (ii) in this more general setting fails for
small enough $c$ when $n = 3$. It may then be that statement (ii) also fails in general in its original setting if $c$ is sufficiently small. Still, it would be interesting to consider the following assertion.

**Conjecture 6.2.** For every $c \geq 0$ and $\sigma, \tau \in S_n$, $\sigma \leq_T \tau$ in $T_n(c)$ if and only if for all $i$, $\phi_i(\tau) \leq_A \phi_i(\tau)$ in $A_n(c)$.

The original proof breaks down for $c < e$.

For $n \geq 5$, $T_n(e)$ fails to be ranked [4–5], unlike the Bruhat order on $S_n$. However, as Stembridge observed, the explicit description of $A_n(e)$ in Theorem 4.1 shows that it is self-dual under interchanging $a$'s and $b$'s, so that the projection theorem has the following corollary for $T_n(e)$, which we conjecture holds more generally for $T_n(c)$.

**Corollary 6.3 [5].** For all $n$, the poset $T_n(e)$ is self-dual.

**Conjecture 6.4.** For all $c$ and $n$, the poset $T_n(c)$ is self-dual.

Next consider the tower order for large $c$. Theorem 4.2 shows that $A_n(c)$ is simply the reverse lexicographic order on $\{a, b\}^n$ for all $c \geq 3.6$ and all $n$. We can lift this information up via the projection theorem to obtain the following result about tower order.

**Theorem 6.5.** For all $c \geq 3.6$ the poset $T_n(c)$ is determined by its projections into the reverse lexicographic order on $\{a, b\}^n$.

It follows immediately for any given $n$ that the posets $T_n(c)$ are identical for all $c \geq 3.6$ and that these posets are self-dual. It is easily checked that for $n \leq 3$, $T_n(3.6)$ is the Bruhat order on $S_n$, the same as $T_n(e)$. For $n = 4$, several orderings hold in $T_4(3.6)$ that are not true for $T_4(e)$, i.e., for the dual of Bruhat order on $S_4$. They are all obtained by adding to the dual of Bruhat order the ordering $3214 >_T 2341$ and applying transitivity. As a consequence of this extra relation among two elements that have the same rank in Bruhat order, it follows that $T_4(3.6)$ is not ranked.

**Example.** It must be true from the theorem that

$$T(4, e\pi, 15, 1000) > T(15, 4, e\pi, 1000) > T(e\pi, 1000, 15, 4).$$

On the other hand, the larger of the two numbers $T(15, 4, e\pi, 1000)$ and $T(4, 1000, 15, e\pi)$ cannot be determined by our methods alone.

The case $c = 1$ would be interesting to determine in general. We saw in Section 1 that $T_2(1)$ is an antichain, i.e., totally unordered. For $n = 3$, we have determined $A_3(1)$. Applying the first statement from the projection theorem, it is easily verified that $T_3(1)$ is also an antichain.
Conjecture 6.6. For all $n$, $T_n(1)$ is an antichain.

7. Conjectures for $A_n(c)$

Although it would be too much to ask for an explicit description of the posets $A_n(c)$ in general, it may be possible to determine some general properties. The posets $A_n(e)$ and $A_n(3.6)$ are self-dual and ranked for all $n$, as are all posets $A_3(c)$ for $c \geq 1$. Unfortunately, $A_3(0)$ is neither self-dual, nor ranked. One would suspect that if $c$ is not too small, then $A_n(c)$ will behave nicely.

Conjecture 7.1. For all $c \geq 1$ and all $n$, $A_n(c)$ is self-dual and ranked.

For $n \geq 3$, the poset $A_n(1)$ is not isomorphic to the Boolean algebra $B_n$, although $A_n(1)$ contains $B_n$, i.e., strengthens it. The poset $A_3(1)$ is still ranked by the number of b's in the word, just like the Boolean algebra. One might hope this holds in general.

Conjecture 7.2. For all $n$, the poset $A_n(1)$ is ranked by the number of b's.

Because of the first statement in the Projection Theorem 6.1, our last conjecture above implies our earlier conjecture 6.6 that in general the tower order $T_n(1)$ is an antichain.

Finally let us consider the threshold above which $A_n(c)$ is a chain. Define $y_n$ to be the smallest $c$ such that $A_n(c)$ is a chain. We have proven that $y_1 = 0$, $y_2 = e \approx 2.718282$, and $y_3 = \alpha_1 = 3.439569$. Clearly $y_{n+1} \geq y_n$ for all $n$. We proved in Theorem 4.2 the general upper bound $y_n < \beta = 3.59121$. It is possible to study the case $n = 4$ and show that $y_3 < y_4 < \beta$. So we propose the following statement to conclude the paper.

Conjecture 7.3. The sequence $y_1, y_2, \ldots$ is strictly increasing and converges to $\beta = 3.591121$.

Acknowledgements

The author is grateful to Michelle Wachs and Zoltán Füredi for their continuing interest in this project. The author is indebted to Prof. Wachs for pointing out an error in an earlier version of the manuscript. This research project was initiated during a visit to the Mathematics Department, University of Miami, Coral Gables, Florida, USA. Thanks are due to Martin Aigner and Rudolf Wille for organizing the meeting 'Kombinatorik geordneter Mengen', April 24–30, 1988, at the Mathematisches Forschungsinstitut Oberwolfach, West
Iterated exponentials of two numbers

Germany, where this paper was presented. Finally, the author wishes to acknowledge the hospitality of the Institute for Mathematics and its Applications during his stay January–May, 1988.

References