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On the hypergeometric matrix function

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Abstract

This paper deals with the study of the hypergeometric function with matrix arguments $F(A,B;C;z)$. Conditions for matrices A, B, C so that the series representation of the hypergeometric function be convergent for $|z|=1$ and satisfies a matrix differential equation are given. After the study of beta and gamma matrix functions, an integral representation of $F(A,B;C;z)$ is obtained for the case where B, C and $C-B$ are positive stable matrices with $BC=CB$.

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1. Introduction

Many special functions encountered in physics, engineering and probability theory are special cases of hypergeometric functions [17–19, 21]. Special matrix functions appear in the literature related to Statistics [1], Lie groups theory [11], and more recently in connection with matrix analogues of Laguerre, Hermite and Legendre differential equations and the corresponding polynomial families [12–14]. Apart from the close relationship with the well-known beta and gamma matrix functions, the emerging theory of orthogonal matrix polynomials [4–6] and its operational calculus suggest the study of hypergeometric matrix function.

The paper is organized as follows. Section 2 deals with the study of new properties of the beta and gamma matrix functions. We are mainly concerned with the matrix analog of the formula

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{1}$$

and may be regarded as a continuation of [16]. In Section 3 the Gauss hypergeometric matrix function $F(A,B;C;z)$ is introduced as a matrix power series. Conditions for the convergence on the boundary

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of the unit disc are treated. We also prove that if matrices B and C commute then $F(A,B;C;z)$ is a solution of the differential equation

$$z(1-z)W''-zAW'+W(C-z(B+I))-AWB=0.$$

If A is an arbitrary matrix in $C^{r\times r}$ and C is an invertible matrix whose eigenvalues are not negative integers then we prove that equation

$$z(1-z)W''-zAW'+W'(C+z(n-1)I)+nAW=0$$

has matrix polynomial solutions of degree n for all integer $n\geqslant 1$.

Finally in Section 4 an integral representation of the hypergeometric matrix function is given.

Throughout this paper for a matrix A in $C^{r\times r}$ its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. The 2-norm of A will be denoted by $\|A\|$ and it is defined by

$$\|A\|=\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2},$$

where for a y in $C^{r\times r}$, $\|y\|_2=(y^{\mathrm{T}}y)^{1/2}$ is the euclidean norm of y. Let us denote $\alpha(A)$ and $\beta(A)$ the real numbers

$$\alpha(A)=\max\{Re(z)\colon z\in\sigma(A)\},\qquad \beta(A)=\min\{\mathrm{Re}(z)\colon z\in\sigma(A)\}. \tag{2}$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A is a matrix in $C^{r\times r}$ with $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus [3, p.558], it follows that

$$f(A)g(A)=g(A)f(A). \tag{3}$$

The reciprocal gamma function denoted by $\Gamma^{-1}=1/\Gamma(z)$, is an entire function of the complex variable z. Then the image of $\Gamma^{-1}(z)$ acting on A, denoted by $\Gamma^{-1}(A)$ is a well defined matrix. Furthermore, if

$$A+nI \quad \text{is invertible for every integer } n\geqslant 0, \tag{4}$$

then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$, and one gets the formula

$$A(A+I)\cdots(A+(n-1)I)\Gamma^{-1}(A+nI)=\Gamma^{-1}(A),\quad n\geqslant 1, \tag{5}$$

see [10, p. 253]. Under condition (4), by (3), Eq. (5) can be written in the form

$$A(A+I)\cdots(A+(n-1)I)=\Gamma(A+nI)\Gamma^{-1}(A),\quad n\geqslant 1. \tag{6}$$

Taking into account the Pochhammer symbol or shifted factorial defined by

$$(z)_n=z(z+1)\cdots(z+n-1),\ n\geqslant 1,\ (z)_0=1,$$

by application of the matrix functional calculus to this function, for any matrix A in $C^{r\times r}$ one gets

$$(A)_n=A(A+I)\cdots(A+(n-1)I),\quad n\geqslant 1,\ (A)_0=I. \tag{7}$$

Taking into account the Schur decomposition of A, by [8, pp. 192–193] it follows that

$$\|e^{tA}\| \leqslant e^{t\alpha(A)} \sum_{k=0}^{r-1} \frac{(\|A\| r^{1/2} t)}{k!}, \quad t \geqslant 0. \tag{8}$$

2. On the beta and gamma matrix functions

Beta and gamma matrix functions are frequent in statistics [1, 11], Lie groups theory [11] and in the solution of matrix differential equations [12–14]. Beta function of two diagonal matrix arguments has been used in [1], and in [13] for the case where one of the two matrix arguments is a scalar multiple of the identity matrix.

In this section we address the extension to the matrix framework formula (1) that will be used in Section 4 to obtain an integral representation of the hypergeometric matrix function. For the sake of clarity in the presentation we state the following result recently proved in [16].

Theorem 1 (Jódar and Cortés [16]). *If M is a positive stable matrix in $C^{r\times r}$ and $n \geqslant 1$ is an integer, then*

$$\Gamma(M) = \lim_{n\to\infty} (n-1)!(M)_n^{-1} n^M.$$

In accordance with [16], if P and Q are positive stable matrices in $C^{r\times r}$ the beta function is well defined by

$$B(P,Q) = \int_0^1 t^{P-I}(1-t)^{Q-I}\,dt. \tag{9}$$

Hence one gets that if P and Q are commuting positive stable matrices then $B(P,Q)=B(Q,P)$, and commutativity is a necessary condition for the symmetry of the beta function, see [16].

Lemma 1. *Let P,Q be positive stable matrices in $C^{r\times r}$ such that $PQ=QP$ and satisfy the condition*

$$P+Q+mI \quad \text{is invertible for all integer } m \geqslant 0. \tag{10}$$

If $n \geqslant 0$ is an integer, then the following identies hold:

(i) $B(P,Q+nI)=(P+Q)_n^{-1}(Q)_n B(P,Q)$,

(ii) $B(P+nI,Q+nI)=(P)_n(Q)_n(P+Q)_{2n}^{-1} B(P,Q)$.

Proof. (i) For $n=0$ the equality is evident. Using that $PQ=QP$ and (9) for $m\geqslant 1$ it follows that

$$B(P,Q+mI) = \int_0^1 t^{P-I}(1-t)^{Q+(m-1)I}\,dt = \lim_{\delta\to 0}\int_\delta^{1-\delta} t^{P-I}(1-t)^{Q+(m-1)I}\,dt$$
$$= \lim_{\delta\to 0}\int_\delta^{1-\delta} t^{P+Q+(m-2)I}(1-t)^{Q+(m-1)I}t^{-(Q+(m-1)I)}\,dt = \lim_{\delta\to 0}\int_\delta^{1-\delta} u(t)v'(t)\,dt,$$

where

$$u(t)=(1-t)^{Q+(m-1)I}t^{-(Q+(m-1)I)}t^{P}; \qquad v'(t)=t^{P+Q+(m-2)I}.$$

Integrating by parts in the last integral one gets

$$\begin{aligned} B(P,Q+mI) = {} & \lim_{\delta\to 0}\,[(P+Q+(m-1)I)^{-1}(1-t)^{Q+(m-1)I}t^{P}]_{t=\delta}^{t=1-\delta} \\ & +\lim_{\delta\to 0}(P+Q+(m-1)I)^{-1}\int_{\delta}^{1-\delta}\{(Q+(m-1)I)(1-t)^{Q+(m-2)I}\,t^{P} \\ & +(Q+(m-1)I)(1-t)^{Q+(m-1)I}\,t^{P-I}\}\,\mathrm{d}t \\ = {} & (P+Q+(m-1)I)^{-1}(Q+(m-1)I)\int_{0}^{1}t^{P-I}(1-t)^{Q+(m-2)I}\,\mathrm{d}t \\ = {} & (P+Q+(m-1)I)^{-1}(Q+(m-1)I)\,B(P,Q+(m-1)I). \end{aligned}$$

Hence using an induction argument part (i) is established. In order to prove (ii) let us apply (i) taking $\hat{P}=P+nI,\ n\geqslant 1$. By (i) it follows that

$$B(\hat{P},Q+nI)=(\hat{P}+Q)_{n}^{-1}(Q)_{n}B(\hat{P},Q). \tag{11}$$

Since $PQ=QP$ we also have $\hat{P}Q=Q\hat{P}$ and $B(\hat{P},Q)=B(Q,\hat{P})$. By (11) it follows that

$$B(\hat{P},Q+nI)=(\hat{P}+Q)_{n}^{-1}(Q)_{n}\,B(Q,\hat{P}). \tag{12}$$

By (i) we have

$$B(Q,P+nI)=(Q+P)_{n}^{-1}(P)_{n}B(Q,P)=(Q+P)_{n}^{-1}(P)_{n}\,B(P,Q). \tag{13}$$

By (12) and (13) one gets

$$B(P+nI,Q+nI)=B(\hat{P},Q+nI)=(P+Q+nI)_{n}^{-1}(Q)_{n}(Q+P)_{n}^{-1}(P)_{n}B(P,Q).$$

By definition $(P+Q+nI)_{n}(P+Q)_{n}=(P+Q)_{2n}$ and by the last expression it follows that

$$B(P+nI,Q+nI)=(P+Q)_{2n}^{-1}(P)_{n}(Q)_{n}B(P,Q).$$

Hence the result is established. □

Part (ii) of Lemma 1 permits to extend the definition of the beta function of two not necessarily positive stable matrix arguments.

Definition 1. Let P, Q be commuting matrices in $C^{r\times r}$ such that for all integer $n\geqslant 0$ one satisfies the condition

$$P+nI, \quad Q+nI \quad \text{and} \quad P+Q+nI \quad \text{are invertible.} \tag{14}$$

Let $\alpha(P,Q)=\min\{\alpha(P),\alpha(Q),\alpha(P+Q)\}$ and let $n_0=n_0(P,Q)=[\,|\alpha(P,Q)|\,]+1$, where [ ] denotes the entire part function. Then we define $B(P,Q)$ by the formula

$$B(P,Q)=(P)_{n_0}^{-1}(Q)_{n_0}^{-1}(P+Q)_{2n_0}B(P+n_0I,Q+n_0I). \tag{15}$$

Remark 1. Note that previous definition agrees with the one given in [16], because by Lemma 1 formula (15) holds true for the case where P and Q are positive stable matrices.

Lemma 2. *Let $\hat{P},\hat{Q}$ be commuting matrices in $C^{r\times r}$ such that $\hat{P}$, $\hat{Q}$ and $\hat{P}+\hat{Q}$ are positive stable. Then*

$$B(\hat{P},\hat{Q})=\Gamma(\hat{P})\Gamma(\hat{Q})\Gamma^{-1}(\hat{P}+\hat{Q}).$$

Proof. By the hypothesis of positive stability of $\hat{P},\hat{Q}$ and the commutativity $\hat{P}\hat{Q}=\hat{Q}\hat{P}$ we can write

$$\Gamma(\hat{P})\Gamma(\hat{Q})=\left(\int_0^{\infty}\mathrm{e}^{-u}u^{\hat{P}-I}\,\mathrm{d}u\right)\left(\int_0^{\infty}\mathrm{e}^{-v}v^{\hat{Q}-I}\,\mathrm{d}v\right)=\int_0^{\infty}\int_0^{\infty}\mathrm{e}^{-u}u^{\hat{P}-I}\mathrm{e}^{-v}v^{\hat{Q}-I}\,\mathrm{d}u\,\mathrm{d}v.$$

Considering the change of variables $x=u/(u+v)$, $y=u+v$ in the above integral, the positive stability of $\hat{P}+\hat{Q}$ one gets $J(x,y)=y$ and

$$\begin{aligned}\Gamma(\hat{P})\Gamma(\hat{Q})&=\int_0^{\infty}\int_0^{1}\mathrm{e}^{-xy}(xy)^{\hat{P}-I}\mathrm{e}^{-y(1-x)}[y(1-x)]^{\hat{Q}-I}y\,\mathrm{d}x\,\mathrm{d}y\\&=\left(\int_0^{\infty}\mathrm{e}^{-y}y^{\hat{P}+\hat{Q}-I}\,\mathrm{d}y\right)\left(\int_0^{1}x^{\hat{P}-I}(1-x)^{\hat{Q}-I}\,\mathrm{d}x\right)\\&=\Gamma(\hat{P}+\hat{Q})B(\hat{P},\hat{Q}).\qquad\square\end{aligned}$$

Theorem 2. *Let P, Q be commuting matrices in $C^{r\times r}$ satisfying the condition* (14) *for all integer $n\geqslant 0$. Then*

$$B(P,Q)=\Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q).$$

Proof. Let $n_0=n_0(P,Q)$ be defined as in Definition 1 so that

$$B(P,Q)=(P)_{n_0}^{-1}(Q)_{n_0}^{-1}(P+Q)_{2n_0}B(P+n_0I,Q+n_0I),$$

where $P+n_0I$ and $Q+n_0I$ are positive stable. By (5) we can write

$$\begin{aligned}\Gamma(P)&=\Gamma(P+n_0I)(P+(n_0-1)I)^{-1}\cdots(P+I)^{-1}P^{-1},\\\Gamma(Q)&=\Gamma(Q+n_0I)(Q+(n_0-1)I)^{-1}\cdots(Q+I)^{-1}Q^{-1},\\\Gamma(P+Q)&=\Gamma(P+Q+2n_0I)(P+Q+(2n_0-1)I)^{-1}\cdots(P+Q+I)^{-1}(P+Q)^{-1}.\end{aligned}$$

As $PQ=QP$ from the last equalities it follows that

$$\begin{aligned}&\Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q)\\&\quad=\Gamma(P+n_0I)\Gamma(Q+n_0I)\Gamma^{-1}(P+Q+2n_0I)(P+(n_0-1)I)^{-1}\cdots\\&\qquad(P+I)^{-1}P^{-1}(Q+(n_0-1)I)^{-1}\cdots(Q+I)^{-1}Q^{-1}\\&\qquad\times(P+Q+(2n_0-1)I)\cdots(P+Q+I)(P+Q)\\&\quad=\Gamma(P+n_0I)\Gamma(Q+n_0I)\Gamma^{-1}(P+Q+2n_0I)(P)_{n_0}^{-1}(Q)_{n_0}^{-1}(P+Q)_{2n_0}.\end{aligned}\tag{16}$$

Since $P+Q+2n_0I$, $P+n_0I$ and $Q+n_0I$ are positive stable matrices, by Lemma 2 one gets

$$\Gamma(P+n_0I)\Gamma(Q+n_0I)\Gamma^{-1}(P+Q+2n_0I)=B(P+n_0I,Q+n_0I) \tag{17}$$

and by Lemma 1(ii) one gets

$$B(P+n_0I,Q+n_0I)=(P)_{n_0}(Q)_{n_0}(P+Q)_{2n_0}^{-1}B(P,Q). \tag{18}$$

By (16)–(18) it follows that

$$\Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q)=B(P,Q). \qquad \square$$

3. On the hypergeometric matrix function

Hypergeometric matrix function ${}_0F_1(-;A,z)$ has been recently introduced in [15] in connection with Laguerre matrix polynomials and in this section we deal with the hypergeometric matrix function $F(A,B;C;z)$ that is defined by

$$F(A,B;C;z)=\sum_{n\geqslant 0}\frac{(A)_n(B)_n(C)_n^{-1}}{n!}z^n \tag{19}$$

for matrices A, B, C in $C^{r\times r}$ such that

$$C+nI \text{ is invertible for all integer } n\geqslant 0. \tag{20}$$

In an analogous way to [15] it is easy prove the convergence of (19) for $|z|<1$. Now we study the conditions so that $F(A,B;C;z)$ converges for $|z|=1$.

Theorem 3. *Let A,B C be positive stable matrices in $C^{r\times r}$ such that*

$$\beta(C)>\alpha(A)+\alpha(B). \tag{21}$$

Then the series (19) *is absolutely convergent for* $|z|=1$.

Proof. By hypothesis (21), there exists a positive number δ such that

$$\beta(C)-\alpha(A)-\alpha(B)=2\delta. \tag{22}$$

Let us write

$$\begin{aligned}
&n^{1+\delta}\left(\frac{1}{n!}(A)_n(B)_n(C)_n^{-1}\right)\\
&=\frac{n^{1+\delta}}{n!}\cdot\frac{(n-1)!\,n^A\,n^{-A}\,(A)_n}{(n-1)!}\cdot\frac{(n-1)!\,n^B\,n^{-B}\,(B)_n}{(n-1)!}\cdot(C)_n^{-1}\,n^C\,n^{-C}\\
&=\frac{n^{1+\delta}}{n}\left(\frac{n^{-A}\,(A)_n}{(n-1)!}\right)n^A\left(\frac{n^{-B}\,(B)_n}{(n-1)!}\right)n^B\left((n-1)!\,(C)_n^{-1}\,n^C\right)n^{-C}.
\end{aligned}$$

Or

$$n^{1+\delta}\left(\frac{(A)_n(B)_n(C)_n^{-1}}{n!}\right)=n^{\delta}\left(\frac{n^{-A}(A)_n}{(n-1)!}\right)n^{A}\left(\frac{n^{-B}(B)_n}{(n-1)!}\right)n^{B}\left((n-1)!\,(C)_n^{-1}\,n^{C}\right)n^{-C}. \tag{23}$$

By (8), taking into account that $\alpha(-C)=-\beta(C)$ we can write

$$\begin{aligned}
&\|n^{A}\|\,\|n^{B}\|\,\|n^{-C}\|\\
&\leqslant n^{\alpha(A)+\alpha(B)-\beta(C)}\left\{\sum_{j=0}^{r-1}\frac{(\|A\|\,r^{1/2}\,\ln n)^{j}}{j!}\right\}\left\{\sum_{j=0}^{r-1}\frac{(\|B\|\,r^{1/2}\,\ln n)^{j}}{j!}\right\}\left\{\sum_{j=0}^{r-1}\frac{(\|C\|\,r^{1/2}\,\ln n)^{j}}{j!}\right\}\\
&\leqslant n^{-2\delta}\left\{\sum_{j=0}^{r-1}\frac{(\max\{\|A\|,\|B\|,\|C\|\}\,r^{1/2}\,\ln n)^{j}}{j!}\right\}^{3}.
\end{aligned} \tag{24}$$

By (22)–(24) and Theorem 1, for $|z|=1$, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} n^{1+\delta}\left\|\frac{(A)_n(B)_n(C)_n^{-1}z^{n}}{n!}\right\|\\
&\leqslant n^{\delta}\lim_{n\to\infty}\left\|\frac{n^{-A}(A)_n}{(n-1)!}\right\|\,\|n^{A}\|\,\left\|\frac{n^{-B}(B)_n}{(n-1)!}\right\|\,\|n^{B}\|\,\|(n-1)!(C)_n^{-1}n^{C}\|\,\|n^{-C}\|\\
&\leqslant \lim_{n\to\infty}\|\Gamma^{-1}(A)\|\,\|\Gamma^{-1}(B)\|\,\|\Gamma(C)\|n^{-\delta}\left\{\sum_{j=0}^{r-1}\frac{(r^{1/2}\max\{\|A\|,\|B\|,\|C\|\})^{j}}{j!}(\ln n)^{j}\right\}^{3}\\
&=\|\Gamma^{-1}(A)\|\,\|\Gamma^{-1}(B)\|\,\|\Gamma(C)\|\cdot 0=0,
\end{aligned}$$

because

$$\lim_{n\to\infty} n^{-\delta}(\ln n)^{k}=0,\quad \text{for } k\geqslant 0 \text{ integer.}$$

Since

$$\lim_{n\to\infty} n^{1+\delta}\left\|\frac{(A)_n(B)_n(C)_n^{-1}z^{n}}{n!}\right\|=0,\quad |z|=1,$$

by the comparison theorem of numerical series of positive numbers one concludes the absolute convergence of series (19). □

We conclude this section showing that under certain conditions $F(A,B;C;z)$ satisfies a matrix differential equation of bilateral type.

Theorem 4. *Suppose that C is a matrix in $C^{r\times r}$ satisfying* (20) *and* $CB=BC$. *Then $F(A,B;C;z)$ is the solution of*

$$z(1-z)W''-zAW'+W'(C-z(B+I))-AWB=0,\quad 0\leqslant|z|<1 \tag{25}$$

satisfying $F(A,B;C;0)=I$.

Proof. By the hypothesis $CB=BC$ we can write

$$F_n=\frac{(A)_n(B)_n(C)_n^{-1}}{n!}=\frac{(A)_n(C)_n^{-1}(B)_n}{n!};\qquad W(z)=F(A,B;C;z)=\sum_{n\geqslant 0}F_n z^n,\quad |z|<1. \tag{26}$$

Since $W(z)$ is a power series convergent for $|z|<1$, it is termwise differentiable in this domain and

$$W'(z)=\sum_{n\geqslant 1}nF_n z^{n-1},\quad W''(z)=\sum_{n\geqslant 2}n(n-1)F_n z^{n-2},\quad |z|<1.$$

Hence

$$\begin{aligned}
&z(1-z)W''(z)-zA\,W'(z)+W'(z)(C-z(B+I))-A\,W(z)B\\
&\quad=\sum_{n\geqslant 2}n(n-1)F_n z^{n-1}-\sum_{n\geqslant 2}n(n-1)F_n z^n-A\sum_{n\geqslant 1}nF_n z^n+\sum_{n\geqslant 1}nF_n Cz^{n-1}\\
&\qquad-\sum_{n\geqslant 1}nF_n(B+I)z^n-\sum_{n\geqslant 0}AF_nBz^n\\
&\quad=\sum_{n\geqslant 2}\{n(n+1)F_{n+1}-n(n-1)F_n-nAF_n+(n+1)F_{n+1}C-nF_n(B+I)-AF_nB\}z^n\\
&\qquad+2F_2z-AF_1z+F_1C+2F_2Cz-F_1(B+I)z-AF_0B-AF_1Bz=0.
\end{aligned}$$

By equating the coefficients of each power z^n and taking into account that $F_0=I$ one gets

$$\begin{aligned}
&z^0:\quad F_1C-AB=0,\\
&z^1:\quad 2F_2-AF_1+2F_2C-F_1(B+I)-AF_1B=2F_2(I+C)-AF_1(I+B)-F_1(B+I)=0,\\
&\vdots\qquad\vdots\\
&z^n:\quad n(n+1)F_{n+1}-n(n+1)F_n-nAF_n-nF_nB-nF_n+(n+1)F_{n+1}C-AF_nB\\
&\qquad\quad=F_{n+1}(n+1)(nI+C)-(A+nI)F_n(B+nI)=0,
\end{aligned}$$

because from (26) one gets

$$F_{n+1}=\frac{(A+nI)F_n(B+nI)(C+nI)^{-1}}{n+1},\quad n\geqslant 0.$$

Hence $W(z)=F(A,B;C;z)$ is the only solution of (25) satisfying $W(0)=I$, [2, p. 287]. □

Corollary 1. *Let C be a matrix in $C^{r\times r}$ satisfying* (20), *let A be an arbitrary matrix in $C^{r\times r}$ and let n be a positive integer. Then equation*

$$z(1-z)W''-zAW'+W'(C+z(n-1)I)+nAW=0 \tag{27}$$

has matrix polynomial solutions of degree n.

Proof. Let $B=-nI$. By Theorem 4 the function $W(z)=F(A,-nI;C;z)$ satisfies Eq. (25) for $B=-nI$, but $(B)_{n+j}=0$ for $j\geqslant 1$.

Hence

$$W(z)=F(A,-nI;C;z)=\sum_{k=0}^{n}\frac{(A)_k(-nI)_k(C)_k^{-1}}{k!}z^k$$

is a matrix polynomial solution of degree n of Eq. (27). □

4. An integral representation

We begin this section considering the binomial function of a matrix exponent. If y is a complex number with $|y|<1$ and b is a complex number then $(1-y)^b=\exp(b\log(1-y))$, where *Log* is the principal branch of the logarithm function, see [20, p. 72]. The Taylor expansion of $(1-y)^{-a}$ about $y=0$ is given by [22, p. 73]

$$(1-y)^{-a}=\sum_{n\geqslant 0}\frac{(a)_n}{n!}y^n,\quad |y|<1,\ a\in C. \tag{28}$$

Now we consider the function of complex variable a defined by (28). Let $f_n(a)$ be the function defined by

$$f_n(a)=\frac{(a)_n}{n!}y^n=\frac{a(a+1)\cdots(a+n-1)}{n!}y^n,\quad a\in C, \tag{29}$$

for a fixed complex number y with $|y|<1$. It is clear that f_n is an holomorphic function of variable a defined in the complex plane. Given a closed bounded disc $D_R=\{a\in C\colon |a|\leqslant R\}$, one gets

$$|f_n(a)|\leqslant\frac{(|a|)_n|y|^n}{n!}\leqslant\frac{(R)_n|y|^n}{n!},\quad n\geqslant 0,\ |a|\leqslant R.$$

Since

$$\sum_{n\geqslant 0}\frac{(R)_n|y|^n}{n!}<+\infty,$$

by the Weierstrass theorem for the convergence of holomorphic functions [20] it follows that

$$g(a)=\sum_{n\geqslant 0}\frac{(a)_n}{n!}y^n=(1-y)^{-a}$$

is holomorphic in C. By application of the holomorphic functional calculus, [3], for any matrix A in $C^{r\times r}$, the image of g by this functional calculus, acting on A yields

$$(1-y)^{-A}=g(A)=\sum_{n\geqslant 0}\frac{(A)_n}{n!}y^n,\quad |y|<1, \tag{30}$$

where $(A)_n$ is given by (7).

Suppose that B and C are matrices in $C^{r\times r}$ such that

$$BC=CB, \tag{31}$$

$$C,\ B \quad \text{and} \quad C-B \text{ are positive stable.} \tag{32}$$

By (5), (7) and (32) one gets

$$\begin{aligned}(B)_n(C)_n^{-1} &= \Gamma^{-1}(B)\Gamma(B+nI)\Gamma^{-1}(C)\,\Gamma(C+nI)^{-1}\\ &= \Gamma^{-1}(B)\Gamma(B+nI)\Gamma^{-1}(C+nI)\Gamma(C)\\ &= \Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C-B)\Gamma(B+nI)\Gamma^{-1}(C+nI)\Gamma(C).\end{aligned} \tag{33}$$

By (32) and Lemma 2 it follows that

$$\int_0^1 t^{B+(n-1)I}(1-t)^{C-B-I}\,\mathrm{d}t=B(B+nI,C-B)=\Gamma(C-B)\Gamma(B+nI)\Gamma^{-1}(C+nI), \tag{34}$$

and by (33)–(34) one gets

$$(B)_n(C)_n^{-1}=\Gamma^{-1}(B)\Gamma^{-1}(C-B)\left(\int_0^1 t^{B+(n-1)I}(1-t)^{C-B-I}\,\mathrm{d}t\right)\Gamma(C). \tag{35}$$

Hence, for $|z|<1$ we can write

$$\begin{aligned}F(A,B;C;z) &= \sum_{n\geqslant 0}\frac{(A)_n(B)_n(C)_n^{-1}}{n!}z^n\\ &= \sum_{n\geqslant 0}\frac{(A)_n\Gamma^{-1}(B)\Gamma^{-1}(C-B)}{n!}\left(\int_0^1 t^{B+(n-1)I}(1-t)^{C-B-I}\,\mathrm{d}t\right)\Gamma(C)z^n\\ &= \sum_{n\geqslant 0}\left(\int_0^1\frac{(A)_n\Gamma^{-1}(B)\Gamma^{-1}(C-B)t^{B+(n-1)I}(1-t)^{C-B-I}\Gamma(C)z^n}{n!}\,\mathrm{d}t\right).\end{aligned} \tag{36}$$

Now we are interested in the permutation of the series and the integral in the last expression. To legitimate this operation let us consider the sequence of matrix functions defined by

$$S_n(t)=\frac{(A)_n\Gamma^{-1}(B)\Gamma^{-1}(C-B)t^{B+(n-1)I}(1-t)^{C-B-I}\Gamma(C)z^n}{n!},\quad 0\leqslant t\leqslant 1,$$

and note that for $0<t<1$ and $n\geqslant 0$ one gets

$$\|S_n(t)\|\leqslant\frac{(\|A\|)_n\,\|\Gamma^{-1}(B)\|\,\|\Gamma^{-1}(C-B)\|\,\|\Gamma(C)\|\,\|t^{B-I}\|\,\|(1-t)^{C-B-I}\|\,|z|^n}{n!}. \tag{37}$$

By (8) it follows that

$$\|t^{B-I}\|\,\|(1-t)^{C-B-I}\|$$
$$\leqslant t^{\alpha(B)-1}(1-t)^{\alpha(C-B)-1}\left(\sum_{j=0}^{r-1}\frac{(\|B-I\|\,r^{1/2}\ \ln t)^j}{j!}\right)\left(\sum_{j=0}^{r-1}\frac{(\|C-B-I\|\,r^{1/2}\ \ln t)^j}{j!}\right)$$

and taking into account that for $0<t<1$ one gets $\ln t<t<1$ and $\ln(1-t)<1-t<1$, from the above expression one gets

$$\|t^{B-I}\|\,\|(1-t)^{C-B-I}\|\leqslant \Lambda \quad t^{\alpha(B)-1}(1-t)^{\alpha(C-B)-1},\quad 0<t<1, \tag{38}$$

where

$$\Lambda=\left(\sum_{n=0}^{r-1}\frac{(\|B-I\|r^{1/2})^j}{j!}\right)\left(\sum_{n=0}^{r-1}\frac{(\|C-B-I\|r^{1/2})^j}{j!}\right). \tag{39}$$

Let S be the sum of the convergent numerical series

$$S=\sum_{n\geqslant 0}\frac{(\|A\|)_n|z|^n}{n!},\quad |z|<1 \tag{40}$$

and note that by (37)–(40) one gets

$$\sum_{n\geqslant 0}\|S_n(t)\|\leqslant\varphi(t)=L\Lambda St^{\alpha(B)-1}(1-t)^{\alpha(C-B)-1},\quad 0<t<1, \tag{41}$$

where

$$L=\|\Gamma^{-1}(B)\|\,\|\Gamma^{-1}(C-B)\|\,\|\Gamma(C)\|.$$

Since $\alpha(B)>0$, $\alpha(C-B)>0$, the function $\varphi(t)=L\Lambda S\,t^{\alpha(B)-1}(1-t)^{\alpha(C-B)-1}$ is integrable and

$$\int_0^1\varphi(t)\,\mathrm{d}t=L\Lambda SB(\alpha(B),\alpha(C-B)).$$

By the dominated convergence theorem [7, p. 83], the series and the integral can be permuted in (36) an using (31) we can write

$$F(A;B;C;z)=\int_0^1\left\{\sum_{n\geqslant 0}\left(\frac{(A)_n(tz)^n}{n!}\right)t^{B-I}(1-t)^{C-B-I}\right\}\mathrm{d}t\,\Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C). \tag{42}$$

Now by (30) one gets

$$\sum_{n\geqslant 0}\frac{(A)_n(tz)^n}{n!}=(1-tz)^{-A},\quad |z|<1,\ 0<t<1, \tag{43}$$

and (42) takes the form

$$F(A,B;C;z)=\int_0^1(1-tz)^{-A}t^{B-I}(1-t)^{C-B-I}\,\mathrm{d}t\,\Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C). \tag{44}$$

Summarizing, the following result has been established:

Theorem 5. *Let A,B and C be matrices in $C^{r\times r}$ such that $CB=BC$ and $C,B,C-B$ are positive stable. Then for $|z|<1$ it follows that*

$$F(A,B;C;z)=\left(\int_0^1 (1-tz)^{-A}t^{B-I}(1-t)^{C-B-I}\,\mathrm{d}t\right)\Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C). \tag{45}$$

The following corollary is a consequence of Theorem 5.

Corollary 2. *Let A, B and C be matrices in $C^{r\times r}$ and let $\hat{\alpha}(B,C)=\min\{\alpha(B),\alpha(C),\alpha(C-B)\}$ and $n_1=n_1(B,C)=[|\hat{\alpha}(B,C)|]+1$, where [ ] denotes the entire part function. Suppose that $BC=CB$, and*

$$\begin{aligned}&\sigma(B)\subset C\sim\{-n;n\geqslant n_1,\ n\ \text{integer}\},\\ &\sigma(C-B)\subset C\sim\{-n;n\geqslant n_1,\ n\ \text{integer}\},\\ &\sigma(C)\subset C\sim\{-2n;n\geqslant n_1,\ n\ \text{integer}\}.\end{aligned} \tag{46}$$

Then for $|z|<1$ one gets

$$\begin{aligned}&F(A,B+n_1I;C+2n_1I;z)\\ &\quad=\left[\int_0^1 (1-tz)^{-A}t^{B+(n_1-1)I}(1-t)^{C-B+(n_1-1)I}\,\mathrm{d}t\right]\Gamma^{-1}(B+n_1I)\Gamma^{-1}(C-B+n_1I)\Gamma(C+2n_1I).\end{aligned}$$

Proof. Consider matrices A, $\hat{B}=B+n_1I$, $\hat{C}=C+2n_1I$ and note that $\hat{C}$, $\hat{B}$ and $\hat{C}-\hat{B}=C-B+n_1I$ are both positive stable. The result is now a consequence of Theorem 5. □

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