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Principal part of multi-parameter displacement functions ☆

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Abstract

In this paper we investigate planar polynomial multi-parameter deformations of Hamiltonian vector fields. We study first all coefficients in the development of the displacement function on a transversal to the period annulus. We show that they can be expressed through iterated integrals, whose length is bounded by the degree of the monomials.

A second result expresses the principal terms in the division of the displacement function in the Bautin ideal. More precisely, the principal terms in its division in a reduced basis of the Bautin ideal are given by iterated integrals. Our approach is algorithmic and generalizes Françoise algorithm for one-parameter families.

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1. Introduction

Small one-parameter polynomial deformations of integrable planar vector fields

$$\omega_\varepsilon = dF + \varepsilon\omega, \quad \varepsilon \in \mathbb{R}$$

and their displacement function $\Delta(t,\varepsilon) = \sum M_i(t)\varepsilon^i$ along $\gamma(t) \subset F^{-1}(t)$ have been extensively studied. Classical results express the term $M_1(t)$ as the abelian integral $M_1(t) = -\int_{\gamma(t)} \omega$. In [2] Françoise gives an algorithm for calculating the principal part (i.e. first non-zero term $M_i(t)$) of the displacement function of small one-parameter deformations of a Hamiltonian system under a condition (*). In [11] and [12] the algorithm was extended to an example where condition (*) was not satisfied. Next [4] Gavrilov (see also [5]) extends the algorithm to general one-parameter systems without the (*) condition and proves that in the one-parameter case the principal term is an iterated integral. The length of the integral is bounded by the order of the principal term of the displacement function. Finally, in [3], Gavrilov's algorithm is shown to work in general permitting to express any term $M_i(t)$ of the displacement function with the help of iterated integrals. A special example of the development of the displacement function for a multi-parameter family of quadratic vector fields in relationship with abelian integrals has been investigated in [18,8–10,15]. Here we want to extract from these studies a general approach.

Multi-parameter families can also be studied by choosing a one-parameter family in the region where the maximal number of limit cycles can appear (see for instance Roussarie [17] and Gavrilov [6] and references therein). The difficulty resides in the choice of the one-parameter family.

In this paper we study multi-parameter polynomial deformations of integrable systems given by

$$\Omega_\varepsilon = 0, \quad \text{where} \quad \Omega_\varepsilon = dF + \sum_{i=0}^{n} \varepsilon_i \omega_i, \quad \varepsilon = (\varepsilon_0, \dots, \varepsilon_n) \in \mathbb{R}^{n+1}. \tag{1.1}$$

We are interested in their displacement functions

$$\Delta(t,\varepsilon) = \sum M_\alpha(t)\varepsilon^\alpha \tag{1.2}$$

along a family of loops $\gamma(t) \subset F^{-1}(t)$ on a transversal Σ to the period annulus parametrized by the values of F. Here $\alpha = (\alpha_0, \dots, \alpha_n)$ and $\varepsilon^\alpha = \varepsilon_0^{\alpha_0} \cdots \varepsilon_n^{\alpha_n}$.

We consider the displacement function only on the interior of the period annulus (or possibly at a non-degenerate center or focus) where all functions are analytic. First, in Theorem 2.1, we give a generalization of the algorithm of Françoise calculating all terms $M_\alpha(t)$ in the above expression.

Next assume that $B = (\eta_0(\varepsilon), \dots, \eta_k(\varepsilon))$ is the Bautin ideal of the displacement function $\Delta(t,\varepsilon)$. (The name comes from the fact that the ideal B was calculated by Bautin in [1] for quadratic vector fields with a center-focus.) Then the displacement function can be written:

$$\Delta(t,\varepsilon) = \sum_{i=0}^{k} \eta_i(\varepsilon)\phi_i(t,\varepsilon) \tag{1.3}$$

with ϕ_i analytic on the regular part of orbits we consider (see e.g. [16]). We want to calculate the principal term of each ϕ_i, that is $\phi_i(t,0)$. We show in Theorem 2.2 that this can be done once we change the bases B for a reduced standard basis (see Definition 1).

2. Main results

Consider general linear families deforming integrable planar vector fields of the form (1.1). For $\beta = (\beta_0, \dots, \beta_n) \in \mathbb{N}_0^{n+1}$ put $|\beta| = \beta_0 + \dots + \beta_n$.

We define inductively (on the order $|\beta|$) a sequence of multivalued functions (f_β, g_β):

$$\begin{aligned} &g_{0,\dots,0} = 1, \\ &\omega_i = g_{e_i}\, dF + df_{e_i}, \\ &\sum_{\delta+e_i=\beta} g_\delta \omega_i = g_\beta\, dF + df_\beta, \quad |\beta| > 1. \end{aligned} \tag{2.4}$$

Here $e_i = (0, \dots, 1, \dots, 0) \in \mathbb{N}_0^{n+1}$, with 1 on the place of index $i = 0, \dots, n$.

Once the left-hand side is given, analytic functions f_β, g_β verifying the above system exist. Note however that in general they are multivalued. The functions g_β are obtained by integration of a one-form $\frac{d(\sum_{\delta+e_i=\beta} g_\delta\omega_i)}{dF}$ along trajectories γ_ε (1.1) starting from points of the transversal Σ parametrized by F. Next f_β are defined analogously by integration of $\sum_{\delta+e_i=\beta} g_\delta\omega_i - g_\beta\, dF$. In particular we can assume $f_\beta(t) = 0$. See [3] for more details.

Theorem 2.1. *The coefficients $M_\alpha(t)$ in the development (1.2) of the displacement function $\Delta(t, \varepsilon)$ are given by iterated integrals of length $\leqslant |\alpha|$. More precisely $M_\alpha(t)$ is equal to*

$$(-1)^{|\alpha|} \int_{\gamma_0(t)} \sum_{\beta+e_i=\alpha} g_\beta \omega_i \tag{2.5}$$

plus a linear combination of at most $|\alpha| + 1$ terms. Each of these terms is a product of derivatives of M_β of order at most $|\alpha| - 1$, with $|\beta| < |\alpha|$.

In particular if the displacement function $\Delta(t, \varepsilon)$ has no terms of order $|\beta| < |\alpha|$, then

$$M_\alpha(t) = (-1)^{|\alpha|} \int_{\gamma_0(t)} \sum_{\beta+e_i=\alpha} g_\beta \omega_i .$$

The second problem addressed in this paper is the problem of expressing principal terms $\phi_i(t, 0)$ in the division of the displacement function $\Delta(t, \varepsilon)$ in the Bautin ideal B. We will rather express analogous principal terms $\psi_i(t, \varepsilon)$ in the division of $\Delta(t, \varepsilon)$ with respect to a reduced standard basis of the Bautin ideal. Let $\prec$ be the negative degree reverse lexicographic order among monomials in $\mathbb{C}[\varepsilon]$:

$$\varepsilon^\alpha \prec \varepsilon^\beta \quad \text{if } |\alpha| > |\beta|, \quad \text{or} \quad |\alpha| = |\beta| \text{ and } \exists i \quad \alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}, \quad \alpha_i > \beta_i . \tag{2.6}$$

In the symbolic calculation program Singular this local order is denoted ds. Let $\mu \in \mathbb{C}[\varepsilon]$, we denote $L(\mu)$ the *leading monomial* of $\mu \neq 0$ i.e. the one which is “biggest” with respect to the order $\prec$ and call *leading coefficient* its coefficient $c(\mu)$. That is, the leading monomial of a polynomial μ is one of its non-zero monomials with lowest total degree. Among them one chooses the one with lowest power of ε_n. If there are several such monomials one makes the choice by looking at the power of ε_{n-1}, etc. Hence $\mu = c(\mu)L(\mu) +$ smaller terms with respect to $\prec$.

We consider the reduced standard basis $G = \{\mu_0(\varepsilon), \dots, \mu_\ell(\varepsilon)\}$ of the Bautin ideal (see Definition 1). Reduced standard bases always exist in the ring of formal power series $\mathbb{C}[[\varepsilon]]$ and are unique [7]. They exist in the polynomial ring if the localization of the quotient of $\mathbb{C}[\varepsilon]/B$ is finite dimensional. Computer packages like Singular are usually efficient in calculating the reduced standard bases G of polynomial ideals B. Even if the standard basis exists only in the ring of formal series, we will need only the beginning of this series which can be calculated. Note that ℓ does not necessarily coincides with k in (1.3). The displacement function Δ can be written:

$$\Delta(t,\varepsilon) = \sum_{i=0}^{\ell} \mu_i(\varepsilon)\psi_i(t,\varepsilon), \tag{2.7}$$

with ψ_i analytic functions. This follows from (1.3).

Theorem 2.2. *The principal term $\psi_i(t,0)$, $i = 0, \dots, \ell$, in the above decomposition of the displacement function with respect to a reduced standard basis G of the Bautin ideal are linear combinations of iterated integrals of length bounded by the degree of the leading monomial $L(\mu_i)$, $\mu_i \in G$.*

Remark 2.3. The proofs of Theorems 2.1 and 2.2 provide an algorithm for calculating principal terms $\psi_i(t,0)$ in the division of the displacement function $\Delta(t,\varepsilon)$ in a reduced standard basis of the Bautin ideal.

The interest in expressing principal parts $\psi_i(t,0)$ in the displacement function resides in the fact that the function

$$M(t,\varepsilon) = \sum_{i=0}^{\ell} \mu_i(\varepsilon)\psi_i(t,0) \tag{2.8}$$

(obtained by replacing $\psi_i(t,\varepsilon)$ by $\psi_i(t,0)$ in $\Delta(t,\varepsilon)$) is simpler than $\Delta(t,\varepsilon)$, but nevertheless often it has the *same qualitative properties* as the displacement function $\Delta(t,\varepsilon)$. The strongest sense of *same qualitative properties* is that the families $\Delta(t,\varepsilon)$ and $M(t,\varepsilon)$ have homeomorphic bifurcation diagram of their zeros.

This is true in particular if the functions $\psi_i(t,0)$ form a Chebyshev system and the family is linear in ε (i.e. $\mu_i(\varepsilon) = \varepsilon_i$). Some other cases when this is true are pinched family studied in [14] and the families appearing in [13]. It seems an interesting open problem to find conditions on μ_i and $\psi_i(t,0)$ under which the Chebyshev property of the system of functions $(\psi_0(t,0), \dots, \psi_\ell(t,0))$ implies that the families $\Delta(t,\varepsilon)$ and $M(t,\varepsilon)$ have homeomorphic bifurcation diagrams.

Note that in any case Chebyshev property of the functions $(\psi_0, \dots, \psi_\ell)$, if true, implies that the function $\Delta(t,\varepsilon)$ has at most ℓ zeros.

3. Study of some examples

Before proving Theorem 2.1, we study some examples.

Example 3.1. Consider a linear family $dF + \varepsilon_0\omega_0 + \varepsilon_1\omega_1 = 0$ and assume that the Bautin ideal is given by $B = (\varepsilon_0^2, \varepsilon_0\varepsilon_1, \varepsilon_1^2)$. The absence of linear terms in the displacement function implies that there exist analytic functions $g_{0,1}$, $f_{0,1}$, $g_{1,0}$, $f_{1,0}$ such that

$$\omega_0 = g_{1,0}\,dF + df_{1,0}, \qquad \omega_1 = g_{0,1}\,dF + df_{0,1}, \tag{3.9}$$

and the coefficients $M_\alpha(t)$ of order two in (1.2) are calculated by

$$M_{2,0}(t) = \int\limits_{\gamma(t,0)} g_{1,0}\omega_0, \qquad M_{0,2}(t) = \int\limits_{\gamma(t,0)} g_{0,1}\omega_1,$$

$$M_{1,1}(t) = \int\limits_{\gamma(t,0)} g_{0,1}\omega_0 + g_{1,0}\omega_1.$$

Moreover, these functions are univalued along the loop we consider.

We claim

$$\Delta(t,\varepsilon) = \varepsilon_0^2\Bigl(\int\limits_{\gamma(t,0)} g_{1,0}\omega_0 + \cdots\Bigr) + \varepsilon_0\varepsilon_1\Bigl(\int\limits_{\gamma(t,0)} (g_{1,0}\omega_1 + g_{0,1}\omega_0) + \cdots\Bigr) + \varepsilon_1^2\Bigl(\int\limits_{\gamma(t,0)} g_{0,1}\omega_1 + \cdots\Bigr). \tag{3.10}$$

Here and in the sequel three points $\cdots$ denote higher order terms.

Indeed, consider the expression

$$\begin{aligned}&(1 - \varepsilon_0 g_{1,0} - \varepsilon_1 g_{0,1})(dF + \varepsilon_0\omega_0 + \varepsilon_1\omega_1)\\ &\quad = d(F + \varepsilon_0 f_{1,0} + \varepsilon_1 f_{0,1}) - \varepsilon_0^2 g_{1,0}\omega_0 - \varepsilon_0\varepsilon_1(g_{1,0}\omega_1 + g_{0,1}\omega_0) - \varepsilon_1^2 g_{0,1}\omega_1.\end{aligned}$$

Its integral along the deformed cycle $\gamma(t,\varepsilon)$ is zero. Hence

$$\begin{aligned}&\int\limits_{\gamma(t,\varepsilon)} d(F + \varepsilon_0 f_{1,0} + \varepsilon_1 f_{0,1})\\ &\quad = \varepsilon_0^2 \int\limits_{\gamma(t,\varepsilon)} g_{1,0}\omega_0 + \varepsilon_0\varepsilon_1 \int\limits_{\gamma(t,\varepsilon)} (g_{1,0}\omega_1 + g_{0,1}\omega_0) + \varepsilon_1^2 \int\limits_{\gamma(t,\varepsilon)} g_{0,1}\omega_1.\end{aligned}$$

The left-hand side gives $\Delta(t,\varepsilon) + o(|\varepsilon|^2)$. Indeed, the contribution after integration of each term $f_{i,j}$ above is $o(|\varepsilon|)$, as we calculate the difference of the value of $f_{i,j}$ at two points distant by $o(|\varepsilon|)$. We use here that $\Delta(t,\varepsilon) = o(|\varepsilon|)$. Multiplying by an extra ε_i gives the result. The quadratic terms of Δ are hence obtained by integration of the right-hand side for $\varepsilon = 0$.

Example 3.2. We consider the same linear family, but assume now that the Bautin ideal is given by $B = (\varepsilon_0^3, \varepsilon_1\varepsilon_0^2, \varepsilon_1^2\varepsilon_0, \varepsilon_1^3)$.

The Bautin ideal has no linear monomials so (3.9) is verified. Moreover, the absence of quadratic monomials implies that $\int_{\gamma(t,0)} g_{1,0}\omega_0 = 0$, $\int_{\gamma(t,0)}(g_{1,0}\omega_1 + g_{0,1}\omega_0) = 0$ and $\int_{\gamma(t,0)} g_{0,1}\omega_1 = 0$. Then there exist analytic functions $g_{2,0}$, $f_{2,0}$, $g_{1,1}$, $f_{1,1}$, $g_{0,2}$, $f_{0,2}$

$$\begin{aligned}&g_{1,0}\omega_0 = g_{2,0}\,dF + df_{2,0}, \qquad g_{1,0}\omega_1 + g_{0,1}\omega_0 = g_{1,1}\,dF + df_{1,1},\\ &g_{0,1}\omega_1 = g_{0,2}\,dF + df_{0,2}.\end{aligned} \tag{3.11}$$

We claim that the displacement function $\Delta(t,\varepsilon)$ is of the form

$$\Delta(t,\varepsilon)=\varepsilon_0^3\Bigl(-\int\limits_{\gamma(t,0)} g_{2,0}\omega_0+\cdots\Bigr)+\varepsilon_0^2\varepsilon_1\Bigl(-\int\limits_{\gamma(t,0)}(g_{2,0}\omega_1+g_{1,1}\omega_0)+\cdots\Bigr)$$
$$+\varepsilon_0\varepsilon_1^2\Bigl(-\int\limits_{\gamma(t,0)}(g_{1,1}\omega_1+g_{0,2}\omega_0)+\cdots\Bigr)+\varepsilon_1^3\Bigl(-\int\limits_{\gamma(t,0)} g_{0,2}\omega_1+\cdots\Bigr). \tag{3.12}$$

To calculate the coefficients of ε_0^3, $\varepsilon_0^2\varepsilon_1$, $\varepsilon_0\varepsilon_1^2$, ε_1^3, we consider

$$\bigl(1-\varepsilon_0 g_{1,0}-\varepsilon_1 g_{0,1}+\varepsilon_0^2 g_{2,0}+\varepsilon_0\varepsilon_1 g_{1,1}+\varepsilon_1^2 g_{0,2}\bigr)(dF+\varepsilon_0\omega_0+\varepsilon_1\omega_1).$$

Using (3.9) and (3.11), we simplify the expression and get

$$d\bigl(F+\varepsilon_0 f_{1,0}+\varepsilon_1 f_{0,1}-\varepsilon_0^2 f_{2,0}-\varepsilon_0\varepsilon_1 f_{1,1}-\varepsilon_1^2 f_{0,2}\bigr)$$
$$+\varepsilon_0^3 g_{2,0}\omega_0+\varepsilon_0^2\varepsilon_1(g_{2,0}\omega_1+g_{1,1}\omega_0)+\varepsilon_0\varepsilon_1^2(g_{1,1}\omega_1+g_{0,2}\omega_0)+\varepsilon_1^3 g_{0,2}\omega_1. \tag{3.13}$$

We integrate the above expression along the trajectory $\gamma(t,\varepsilon)$ of the deformed vector field. Observe that this integral is identically equal to zero. On the other hand the displacement function Δ is by definition the integral of dF. The assumption that the displacement function has no linear or quadratic monomials implies that by integrating the first line in (3.13), up to quartic monomials, we also obtain the displacement function $\Delta(t,\varepsilon)$.

4. Proof of Theorem 2.1

Proof. Let Σ be a transversal to a period annulus parametrized by F. Let $\gamma_\varepsilon(t)$ be an orbit of (1.1) starting at a point in $\Sigma\cap F^{-1}(t)$. Observe that then the integral of the expression

$$\Bigl(\sum_{0\leqslant|\beta|<|\alpha|}(-1)^{|\beta|}g_\beta\varepsilon^\beta\Bigr)\Bigl(dF+\sum_{i=0}^n\varepsilon_i\omega_i\Bigr) \tag{4.14}$$

along $\gamma_\varepsilon(t)$ vanishes for all (t,ε).

We develop (4.14), recalling that $g_{0,\dots,0}=1$:

$$dF+\sum_{i=0}^n\varepsilon_i\omega_i+\sum_{1\leqslant|\beta|<|\alpha|}(-1)^{|\beta|}\varepsilon^\beta g_\beta\,dF+\sum_{i=0}^n\sum_{1\leqslant|\beta|<|\alpha|}(-1)^{|\beta|}\varepsilon^{\beta+e_i}g_\beta\omega_i.$$

We simplify it using the two relations of (2.4). Then after renaming the variable $\beta+e_i$ by β in the last sum we get

$$dF+\sum_{1\leqslant|\beta|<|\alpha|}(-1)^{|\beta|}\varepsilon^\beta df_\beta-(-1)^{|\alpha|}\sum_{|\beta|=|\alpha|}\varepsilon^\beta\sum_{\delta+e_i=\beta}g_\delta\omega_i. \tag{4.15}$$

Recalling that the integral of the above expression along $\gamma_\varepsilon(t)$ vanishes, we get

$$\Delta_\varepsilon(t)=(-1)^{|\alpha|}\sum_{|\beta|=|\alpha|}\varepsilon^\beta\sum_{\delta+e_i=\beta}\int\limits_{\gamma_\varepsilon(t)}g_\delta\omega_i-\sum_{1\leqslant|\beta|<|\alpha|}(-1)^{|\beta|}\varepsilon^\beta f_\beta\bigl(P_\varepsilon(t)\bigr). \tag{4.16}$$

Here

$$P_\varepsilon(t)=t+\Delta_\varepsilon(t)=t+\sum\varepsilon^\beta M_\beta(t) \tag{4.17}$$

is the first return map. As f_β is multivalued, in (4.16) and in the sequel, by $f_\beta(P_\varepsilon(t))$ we mean the value that f_β takes when continuing it analytically along $\gamma_\varepsilon(t)$ starting with $f_\beta(t)=0$.

Recall that we assume in the construction (2.4) that $f_\beta(t)=0$. It follows that

$$M_\alpha(t)=(-1)^{|\alpha|}\sum_{\delta+e_i=\alpha}\int_{\gamma_0(t)} g_\delta\omega_i-\sum_{1\leqslant|\beta|<|\alpha|}\sum_{\delta+\beta=\alpha}(-1)^{|\beta|}\frac{1}{\delta!}\frac{\partial^\delta}{\partial\varepsilon^\delta}\bigg|_{\varepsilon=0} f_\beta\bigl(P_\varepsilon(t)\bigr). \tag{4.18}$$

It remains to show that the derivatives of $f_\beta(P_\varepsilon(t))$, $|\beta|<|\alpha|$, with respect to ε at 0 can be replaced by linear combinations of some derivatives of the functions M_δ. For this we look again at (4.16), truncating the development at orders higher than ε^β, for $|\beta|\geqslant|\alpha|$. This means that we neglect the first sum in (4.16) and use (4.17). This gives

$$M_\beta(t)=\sum_{\delta+\gamma=\beta}(-1)^{|\gamma|}\frac{1}{\delta!}\frac{\partial^\delta}{\partial\varepsilon^\delta}\bigg|_{\varepsilon=0} f_\gamma\bigl(P_\varepsilon(t)\bigr),$$

which is a regular triangular system of equations expressing f_β through M_γ starting by $f_\beta=M_\beta$, for $|\beta|=1$. Solving recursively these equations one shows that for any β, f_β can be expressed by a linear combination of products of derivatives of M_δ, for $|\delta|\leqslant|\beta|$.

The claim of the theorem now follows from (4.18). Expression (2.5) is an iterated integral of length at most $|\alpha|$. The additional terms will be given by iterated integrals of lower orders. □

5. Standard bases

Let us consider the ring $\mathbb{C}[\varepsilon]$ of complex polynomials in $\varepsilon=(\varepsilon_0,\dots,\varepsilon_n)$ with the negative degree reverse lexicographic order $\prec$ defined in (2.6). It corresponds to the order option *ds* in the computer algebra system Singular.

Definition 1. *Let $G\subset\mathbb{C}[\varepsilon]$. Let $L(G)$ be the set of leading monomials $\{L(\nu)|\nu\in G\setminus\{0\}\}$.*

(i) *A finite set $G=\{\mu_0,\dots,\mu_\ell\}\subset I$ is a* standard basis of an ideal I *if the set $L(G)$ generates the ideal $L(I)$.*
(ii) *A standard basis G is interreduced if $0\notin G$ and $L(g)\nmid L(f)$ for any two $f\neq g\in G$.*
(iii) *An interreduced standard basis G is reduced if for any $g\in G$ the leading coefficient is* 1 *and no monomial of the tail $g-L(g)$ of g is in the ideal of leading monomials $L(G)$.*

Standard basis for any order can be calculated using computer packages such as Singular (see Algorithm 1.7.8 in [7]). Given a standard basis G, an interreduced standard basis is obtained by eliminating superfluous terms $g\in G$ such that $L(f)|L(g)$ for some $f\neq g\in G$ (see Remark 1.6.3(2) in [7]).

Hence, interreduced standard basis always exists for any ordering. In order to obtain a reduced standard basis one has to reduce the tail of any term in an interreduced standard basis. This is based on a division algorithm applied to the tail of g. Note the difficulty in local orderings (such as *ds* considered here). Consider the ideal I generated by $x-x^2$ and the function $f=x$. In order to prove that f belongs to the ideal I in the ring of formal power series $\mathbb{C}[[x]]$, one can perform an infinite division $f=(x-x^2)\sum_{i=0}^\infty x^i$. Of course from algorithmic point of view this is not satisfactory. One prefers to work in the localization. Starting from a polynomial ideal, denote $\mathrm{Loc}_\prec\mathbb{C}[\varepsilon]$ the *localization of* $\mathbb{C}[\varepsilon]$ given by $\mathrm{Loc}_\prec\mathbb{C}[\varepsilon]=S_\prec^{-1}\mathbb{C}[\varepsilon]$, where

$S_{\prec} = \{1 + \nu \mid \nu = 0 \text{ or } \nu \in \mathbb{C}[\varepsilon] \setminus \{0\} \text{ and } 1 \succ L(\nu)\}$ is the set of unities. The above division is seen as $(1 - x)f = (x - x^2)$, with $(1 - x)$ a unity.

Starting with and interreduced standard basis G in $\mathbb{C}[\varepsilon]$ and reducing the tail of all terms leads to a reduced standard basis. The reduction uses weak normal forms of functions in the localization $\mathrm{Loc}_{\prec}\mathbb{C}[\varepsilon]$ (see Definition 1.6.5 and Algorithm 1.7.6 in [7]).

The principal advantage of standard basis with respect to any basis is that the algorithm of division of an element with respect to a standard basis works well contrary to the general multivariate case when using any bases.

6. Proof of Theorem 2.2

Put $\alpha! = \alpha_0! \cdots \alpha_n!$. Denote $\partial^\alpha \Delta$ the partial derivative $\frac{\partial^{|\alpha|}\Delta}{\partial\varepsilon_0^{\alpha_0}\cdots\partial\varepsilon_n^{\alpha_n}}$ and $\partial^\alpha\Delta(0)$ its value for $\varepsilon = 0$. For $\Delta \in \mathbb{C}(t)[\varepsilon]$, the derivative $\partial^\alpha\Delta(0)$ belongs to $\mathbb{C}(t)$.

Note that

$$\begin{aligned} \partial^\alpha \varepsilon^\beta(0) &= \alpha! \quad \text{if } \alpha = \beta, \\ &= 0 \quad \text{if } \alpha \neq \beta. \end{aligned} \tag{6.19}$$

Let G be a reduced standard basis of B. This means in particular that all the leading monomials are distinct. We can then suppose that the terms of G are ordered by decreasing order of the leading monomials: $G = (\mu_0, \ldots, \mu_\ell)$, with $L(\mu_i) \succ L(\mu_j)$, if $i < j$. Let $\alpha^i = (\alpha_0^i, \ldots, \alpha_n^i)$ be the multiexponent of the leading monomial of μ_i: $L(\mu_i) = \varepsilon^{\alpha^i}$. More generally for any monomial $T(\mu_i)$ of μ_i the multiindex $\alpha(T(\mu_i))$ denotes the exponents of the monomial $T(\mu_i)$: $T(\mu_i) = C(T(\mu_i))\varepsilon^{\alpha(T(\mu_i))}$, where $C(T(\mu_i))$ denotes the coefficient of the term $T(\mu_i)$.

We calculate the derivatives:

$$\partial^{\alpha^i}\Delta(0) = \sum_{j=0}^{\ell} \partial^{\alpha^i}(\mu_j\psi_j)(0), \quad i = 0, \ldots, \ell. \tag{6.20}$$

Lemma 6.1.

(1) $\partial^{\alpha^i}(\mu_i\psi_i)(0) = \alpha^i!\psi_i(0)$.
(2) *For $j > i$, $\partial^{\alpha^i}(\mu_j\psi_j)(0) = 0$.*
(3) *For $j < i$, $\partial^{\alpha^i}(\mu_j\psi_j)(0) = \sum * \partial^{\alpha^i - \alpha(T(\mu_j))}\psi_j$. The sum is taken over all monomials $T(\mu_j)$ of μ_j whose all components of $\alpha^i - \alpha(T(\mu_j))$ are non-negative and $*$ denotes a non-specified constant.*
(4) $\partial^{\alpha^i}\Delta(0) = (\alpha^i)!\psi_i(0) + \sum_{j<i}\partial^{\alpha^i}(\mu_j\psi_j)(0)$.

Proof. The first claim is straightforward.

For the second claim note first that $L(\mu_i) \succ L(\mu_j)$ by the choice of order in the elements of the reduced standard basis. But $L(\mu_j) \succ$ any monomials of μ_j. Hence by transitivity $L(\mu_i) \succ T(\mu_j)$, for any monomial $T(\mu_j)$ of μ_j. The claim now follows from (6.19).

For the third expression, note first that the monomials $T(\mu_j)$ which are not of the above form will not contribute, because some powers will survive derivation and on substituting $\varepsilon = 0$ we will get 0.

The last expression is straightforward. □

Proposition 6.2. *Let G be a reduced standard basis of the Bautin ideal of the displacement function Δ with respect to the negative degree reverse lexicographic order $\prec$. Let $\Delta \in \mathbb{C}(t)[\varepsilon]$ belong to the ideal generated by G as in (2.7). The principal part $\psi_i(t,0)$, of each of the functions ψ_i, $i = 0, \ldots, \ell$ in (2.7) is a linear combination of derivatives of the function Δ. The order of the derivative is bounded by the degrees of the leading monomials of the standard basis.*

Proof. By Lemma 6.1, we get a lower triangular system of equations for $\psi_i(t,0)$ with non-zero diagonal terms. Next using (4) of Lemma 6.1, terms under the diagonal can be expressed as derivatives of the function Δ. This is proved by induction as in (4) of Lemma 6.1. Hence they are previously expressed with the help of the displacement function Δ. Their derivatives which appear in (3) of Lemma 6.1 can hence also be expressed through derivatives of Δ. □

Proof of Theorem 2.2. The claim follows by putting together Proposition 6.2 and Theorem 2.1. □

Remark 6.3. Note that Proposition 6.2 and Theorem 2.2 are not true without any condition on the basis of the Bautin ideal. Here the hypothesis that we divide the displacement function in a reduced standard basis can be seen as a kind of *minimality* condition assuring the independence of the elements of the basis. As a trivial counter-example to generalizations assume that $\Delta(t,\varepsilon) = \varepsilon\psi(t,\varepsilon)$, $\varepsilon \in \mathbb{C}$, with $\psi(t,0) \neq 0$. The reduced standard basis is then $G = \{\varepsilon\}$. Of course we can take a redundant basis $B = (\varepsilon, \varepsilon^2)$ and write $\Delta(t,\varepsilon) = \varepsilon\phi_1(t,\varepsilon) + \varepsilon^2\phi_2(t,\varepsilon)$. Then $\phi_2(t,0)$ is arbitrary, and cannot be expressed through derivatives of the displacement function $\Delta(t,\varepsilon)$.

7. Example to Proposition 6.2

Example 7.1. Consider the quadratic family unfolding the Hamiltonian triangle case

$$\begin{aligned} &dF + \varepsilon_0\big(y^2\,dx - x^2\,dy\big) + \varepsilon_1 y^2\,dy + \varepsilon_2 x^2\,dx \\ &\quad + \varepsilon_3(-1 + x + y)y\,dx + \varepsilon_4(1 - x - y)x\,dy = 0. \end{aligned} \tag{7.21}$$

This family appears as the versal unfolding of the Hamiltonian triangle center. It was studied among other in [18,8,15]. Note that the family (7.21) can be written

$$dF + \sum_{i=0}^{4} \varepsilon_i\omega_i = 0$$

with

$$\begin{aligned} &F = xy(1 - x - y), \qquad \omega_0 = y^2\,dx - x^2\,dy, \qquad \omega_1 = y^2\,dy, \\ &\omega_2 = x^2\,dx, \qquad \omega_3 = (-1 + x + y)y\,dx, \qquad \omega_4 = (1 - x - y)x\,dy. \end{aligned} \tag{7.22}$$

As proved in [18] (see also [15]) the Bautin ideal of the family (7.21) is

$$B = \big(\varepsilon_0, \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4, (\varepsilon_3 - \varepsilon_4)(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4), (\varepsilon_1 + \varepsilon_2)(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4)\big). \tag{7.23}$$

In fact in [18] and [15], the displacement function is studied by parameterizing a transversal by a first integral $F_{\varepsilon_3,\varepsilon_4}$ and not with the initial F. Moreover, an integrating factor appears. Multiplying by this integrating factor the family is no longer linear. We cannot hence apply the algorithm of Theorems 2.1 and 2.2 as it stands. We believe that the algorithm can be adapted to include this case. Here this example serves only to illustrate the division in the reduced standard basis.

Calculations using the command std with negative degree reverse lexicographic order in Singular (command ds) give

$$G = \{\varepsilon_0, \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4, \varepsilon_1\varepsilon_2\varepsilon_4 + \varepsilon_2^2\varepsilon_4, \varepsilon_2\varepsilon_3\varepsilon_4 - \varepsilon_2\varepsilon_4^2\} = \{\mu_0, \mu_1, \mu_2, \mu_3\}. \tag{7.24}$$

Note that reduced standard basis G is simpler than the initial basis of the Bautin ideal B (7.23). We can hence decompose the displacement function

$$\Delta(t, \varepsilon) = \sum_{i=0}^{3} \mu_i(\varepsilon)\psi_i(t, \varepsilon).$$

We calculate $\psi_i(t, 0)$. Consider the leading monomial $L(\mu_i)$ of each μ_i with respect to the negative degree reverse lexicographic order. We have:

$$L(\mu_0) = \varepsilon_0, \qquad L(\mu_1) = \varepsilon_1\varepsilon_3, \qquad L(\mu_2) = \varepsilon_1\varepsilon_2\varepsilon_4, \qquad L(\mu_3) = \varepsilon_2\varepsilon_3\varepsilon_4. \tag{7.25}$$

We claim that

$$\psi_0(t, 0) = \frac{\partial\Delta(t, \varepsilon)}{\partial\varepsilon_0}\bigg|_{\varepsilon=0}, \qquad \psi_1(t, 0) = \frac{\partial^2\Delta(t, \varepsilon)}{\partial\varepsilon_1\partial\varepsilon_3}\bigg|_{\varepsilon=0},$$
$$\psi_2(t, 0) = \frac{\partial^3\Delta(t, \varepsilon)}{\partial\varepsilon_1\varepsilon_2\varepsilon_4}\bigg|_{\varepsilon=0} - \frac{1}{2}\frac{\partial^3\Delta(t, \varepsilon)}{\partial\varepsilon_1^2\partial\varepsilon_3}\bigg|_{\varepsilon=0},$$
$$\psi_3(t, 0) = \frac{\partial^3\Delta(t, \varepsilon)}{\partial\varepsilon_2\partial\varepsilon_3\partial\varepsilon_4}\bigg|_{\varepsilon=0} - \frac{1}{2}\frac{\partial^3\Delta(t, \varepsilon)}{\partial\varepsilon_1\partial\varepsilon_3^2}\bigg|_{\varepsilon=0}.$$

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